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ASYMPTOTIC ANALYSIS IN ELASTICITY PROBLEMS ON THIN PERIODIC STRUCTURES

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Abstract. Thin periodic structures depend on two interrelated small geometric parameters ε and $h(\varepsilon)$ which control the thickness of constituents and the cell of periodicity. We study homogenisation of elasticity theory problems on these structures by method of asymptotic expansions. A particular attention is paid to the case of critical thickness when $\lim_{\varepsilon\to 0} h(\varepsilon)\varepsilon^{-1}$ is a positive constant. Planar grids are taken as a model example.

1. Introduction.

Figure 1 displays a periodic grid F^h composed of strips of width $2h > 0$. The periodicity cell $\square = [-\frac{1}{2}, \frac{1}{2})^2$ is shown by a dashed line. As $h \to 0$, the thin grid F^h transforms into an infinitely thin grid $F^0 = F$ (see Fig. 1), which is called a singular grid. This is only a model example. The results presented below are valid for fairly general grids without symmetries as rich as in the present one.

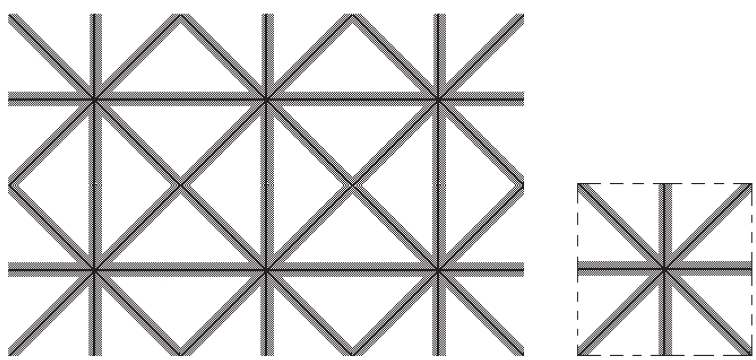

Figure 1.

Let $\mu^h (h \geq 0$ and $\mu^0 = \mu)$ denote a measure supported by F^h proportional to a two-dimensional Lebesgue measure (for $h > 0$) or a one-dimensional Lebesgue

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measure (for $h = 0$), and normalized by the condition

$$\int_{\square} d\mu^h = 1.$$

It is clear that $\mu^h \rightharpoonup \mu$ as $h \downarrow 0$. A contraction maps F^h $(h > 0)$ into the ε-periodic grid $F^h_\varepsilon = \varepsilon F^h$, which supports the measure μ^h_ε, $\mu^h_\varepsilon(B) = \varepsilon^2 \mu^h(\varepsilon^{-1}B)$ for any Borel set $B \subset \mathbb{R}^2$. Furthermore, we relate the parameters ε and h in such a manner that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

It was found in [13], that the elastic properties of periodic structures manifest a qualitative difference depending on the value of the limit $\lim\limits_{\varepsilon} h(\varepsilon)/\varepsilon$, and according to this, thin structures were classified as

(i) sufficiently thick structures if $h(\varepsilon)/\varepsilon \to \infty$;
(ii) structures of critical thickness if $h(\varepsilon)/\varepsilon \to \theta > 0$;
(iii) sufficiently thin structures if $h(\varepsilon)/\varepsilon \to 0$.

Let $A = \{a_{ijsp}\}$ be a symmetric and positive definite elasticity tensor:

$$a_{ijsp} = a_{jisp} = a_{spij}, \quad A\xi \cdot \xi \geq c_0 \xi \cdot \xi, \quad c_0 > 0.$$

In the case of an isotropic tensor, we have

$$\begin{gathered} A\xi = k\xi + k_1 E tr\xi, \quad tr\xi = \xi_{11} + \xi_{22}, \\ E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{12} & \xi_{22} \end{pmatrix}, \quad k > 0, \quad k_1 \geq 0. \end{gathered} \tag{1}$$

A bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary is associated with a perforated domain $\Omega_{\varepsilon,h} = \Omega \cap F^h_\varepsilon$ and the space $W_{\varepsilon,h}$ that is the closure of $C_0^\infty(\Omega)^2$ with respect to the norm

$$\left(\int_{\Omega_{\varepsilon,h}} (|\varphi|^2 + |e(\varphi)|^2) dx \right)^{\frac{1}{2}},$$

where $e(\varphi) = \frac{1}{2}\{\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i}\}$ is the strain tensor or symmetric gradient of the vector φ.

Consider the problem

$$u^{\varepsilon,h} \in W_{\varepsilon,h}, \quad \int_{\Omega_{\varepsilon,h}} Ae(u^{\varepsilon,h}) \cdot e(\varphi) dx = \int_{\Omega_{\varepsilon,h}} f \cdot \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega)^2, \tag{2}$$

where $f \in C^\infty(\bar{\Omega})^2$. This is a generalized or variational statement of the boundary value problem for the elasticity system in $\Omega_{\varepsilon,h}$:

$$-div Ae(u^{\varepsilon,h}) = f \quad \text{in} \quad \Omega_{\varepsilon,h}, \quad u^{\varepsilon,h}|_{\partial\Omega \cap F^h_\varepsilon} = 0, \quad Ae(u^{\varepsilon,h}) \cdot n|_{\partial F^h_\varepsilon \cap \Omega} = 0,$$

with the clamping condition on $\partial\Omega \cap F^h_\varepsilon$ and with no stress condition on the remaining part of the boundary of $\Omega_{\varepsilon,h}$. Here n is the outward normal to the boundary.

The following Korn inequality is valid, see [15]:

$$\int_{\Omega_{\varepsilon,h}} |\varphi|^2 dx \leq C_0 \left(1 + \left(\frac{\varepsilon}{h} \right)^2 \right) \int_{\Omega_{\varepsilon,h}} |e(\varphi)|^2 dx \quad \forall \varphi \in C_0^\infty(\Omega)^2, \quad C_0 = const(\Omega, F). \tag{3}$$

For thin structures F_ε^h of the first two types (i) and (ii) inequality (3) provides the boundedness of the solutions $u^{\varepsilon,h}$:

$$\limsup_{\varepsilon\to 0} \frac{1}{|\Omega_{\varepsilon,h}|} \int_{\Omega_{\varepsilon,h}} (|u^{\varepsilon,h}|^2 + |e(u^{\varepsilon,h})|^2)dx < \infty.$$

Our goal is to investigate the asymptotic behavior of the sequences $u^{\varepsilon,h}$, $e(u^{\varepsilon,h})$ when $\varepsilon \to 0$, in other words, to prove homogenization principle. The most complicated is the case of critical thickness. We begin with it. The result can be formulated and proved by means of the two-scale convergence in a variable L^2-spaces on $\Omega\cap F_\varepsilon^h$ introduced by Zhikov in [13]. This approach (involving the two-scaled convergence) is the most general, for, it serves under the minimal regularity assumptions (on the domain Ω, elasticity tensor A and right-side function f in the equation) and can be adapted to the nonlinear case.

Homogenization for elasticity problems on a plane periodic grid of the critical thickness was obtained by the method of the two-scale convergence in [16] (as for three-dimensional structures, see [11]). In the present paper we reject this approach and return to the classical method of asymptotic expansions. We prove a corrector theorem that refines the result of [16].

At first we begin with asymptotic expansion

$$u^{\varepsilon,h}(x) = u_0(x) + \varepsilon u_1^h(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2^h(x, \frac{x}{\varepsilon}) + \varepsilon^3 u_3^h(x, \frac{x}{\varepsilon}) + ...$$

as if $h>0$ does not depend on ε. This expansion is common for scalar problems and elasticity problems in classical perforated domains, see [2]. Functions $u_i^h(x,\cdot)$ are found as solutions of certain periodic problems on F^h. When we try to take here the limit as $h\to 0$, there appears difference between scalar and vector problems.

In scalar case the above expansion is uniform in h, and passing to the limit can be performed in the component u_i^h itself, thanks to some uniform estimates. Nothing depends on how $h(\varepsilon)\to 0$.

On contrary, in elasticity problems passing to the limit is possible only in the products $\varepsilon^i u_i^h$ and the relation between $h(\varepsilon)$ and ε is essential. For instance, if $h = \theta\varepsilon$, then $\theta^2\varepsilon^2 u_2^h \to \chi \not\equiv 0$ in some sence. So the term χ must be included in the zero approximation.

Really, we have the following result in the critical case.

Theorem 1.1. *There is the convergence*

$$\frac{1}{|\Omega_{\varepsilon,h}|} \int_{\Omega_{\varepsilon,h}} |u^{\varepsilon,h}(x) - u_0(x) - \chi(x, \frac{x}{\varepsilon})|^2 dx \to 0. \tag{4}$$

Here $u_0(x)$ is the solution to the usual homogenized problem in the domain Ω, see (13), and $\chi(x,\cdot)$ is the solution of some periodic equation on the singular structure F. So, initially $\chi(x,\cdot)$ is found on F, while for integration in (4) it must be defined on F^h. To this end, we extend χ to F^h in some natural fashion.

For convergence with gradient, the more complicated relation holds.

Theorem 1.2. *There is the convergence*

$$\frac{1}{|\Omega_{\varepsilon,h}|} \int_{\Omega_{\varepsilon,h}} |e(u^{\varepsilon,h}(x) - u_0(x) - \chi(x, \frac{x}{\varepsilon}) - \varepsilon u_1(x, \frac{x}{\varepsilon}) - \varepsilon g(x, \frac{x}{\varepsilon}))|^2 dx \to 0. \tag{5}$$

Here, there are correctors for both terms of the zero approximation. The term $\varepsilon u_1(x, \frac{x}{\varepsilon})$ is a classical corrector, i.e. $u_1(x, y) = e_{ij}(u_0(x))N^{ij}(y)$ and the vector $N^{ij}(y)$ is the solution of canonical cell problem. The second corrector $\varepsilon g(x, y)$ is related to zero term $\chi(x, y)$, namely, $e_y(g){=}{-}e_x(\chi)$ on F. For example, on horizontal link $I \subset F$ we have

$$-e_x(\chi)|_I = \begin{pmatrix} 0 & \alpha(x, y_1) \\ \alpha(x, y_1) & \beta(x, y_1). \end{pmatrix} \Longrightarrow g(x, y) = y_2(2\alpha(x, y_1), \beta(x, y_1)).$$

The result on the homogenization of the problem (1.2) close to ours in its formulation was obtained in [3] by unfolding method related to the two-scale convergence of Nguetseng and Allaire [8], [1]. The homogenization of elasticity problems on thin periodic rod structures was studied for the first time in the works of G.P. Panasenko (see e.g. [9] and given there references), who used to this end the method of asymptotic expansions which differs from ours. The main result in [9] also has a different form. The problem considered there is set on the torus, in absence of the boundary formal expansions of arbitrary high order are constructed and their closeness to the exact solution is discussed.

Few words about organisation of this paper. The derivation of homogenisation theorems is rather complicated. To make it more transparent, we first sketch it explaining the main ideas. Sections 2-4 are devoted to the direct proof of this limit result. Here, we unfold the brief survey [12]. Auxiliary material concerning thin structures F^h is located in Appendices. In order to give a self-contained exposition, we prefer not to make references to previous papers [16], [12], [17] but reproduce most of the necessary results from them, with proof.

2. **Asymptotic expansions.** In this Section we formulate some preliminary homogenization result for the problem (2), see Th.2.3, and sketch its proof, demonstrating main ideas of our method.

1°. We introduce the divergence operator with respect to μ^h_ε. Given a matrix $\Phi(x)$ and a vector $b(x)$ from $L^2(\Omega_{\varepsilon,h})$, the equality $div\Phi = b$ (with respect to μ^h_ε) means, by definition, that the integral identity

$$\int\limits_\Omega \Phi \cdot e(\varphi) d\mu^h_\varepsilon = -\int\limits_\Omega b \cdot \varphi d\mu^h_\varepsilon \quad \forall \varphi \in C^\infty_0(\Omega)^2 \tag{6}$$

holds true. Now, problem (2) can be written as the equation

$$u^{\varepsilon,h} \in W_{\varepsilon,h}, \quad -div Ae(u^{\varepsilon,h}) = f \quad \text{(with respect to } \mu^h_\varepsilon). \tag{7}$$

Define the divergence operator with respect to the periodic measure μ^h. For $\Phi(y)$, $b(y) \in L^2_{per}(\square, d\mu^h)$, where $\Phi(y)$ is a matrix and $b(y)$ is a vector, the equality $div_y\Phi = b$ (with respect to μ^h) means that the integral identity

$$\int\limits_\square \Phi \cdot e(\varphi) d\mu^h = -\int\limits_\square b \cdot \varphi d\mu^h \quad \forall \varphi \in C^\infty_{per}(\square)^2 \tag{8}$$

holds true.

For a symmetric matrix $\Phi(x, y)$ $(y = \varepsilon^{-1}x)$ that is smooth with respect to x and 1-periodic with respect to y, we use the formula

$$div\Phi = div_x\Phi + \varepsilon^{-1} div_y\Phi. \tag{9}$$

Here, $div\Phi$ is the divergence with respect to μ_ε^h, $div_y\Phi$ is the divergence with respect to μ^h for a fixed x, and $div_x\Phi$ is the ordinary divergence (i.e., $div_x\Phi = \frac{\partial\Phi_{ij}}{\partial x_j}$). Equality (9) means that the integral identity

$$-\int_\Omega \Phi\cdot e(\varphi)d\mu_\varepsilon^h = \int_\Omega div_x\Phi\cdot\varphi d\mu_\varepsilon^h + \varepsilon^{-1}\int_\Omega b\cdot\varphi d\mu_\varepsilon^h, \quad b = div_y\Phi$$

holds for any $\varphi\in C_0^\infty(\Omega)^2$. Its verification is straightforward due to (6), (8) in the case of factorization $\Phi(x,y) = \varphi(x)\psi(y)$ which is sufficient for us. When $h(\varepsilon)\equiv 0$ and $\mu_\varepsilon^0 = \mu_\varepsilon$ is a measure supported on ε-periodic singular structure $F_\varepsilon = F_\varepsilon^0$, the relation (9) was proved in [13, Sect.18]. For thin structures, when $h > 0$, the proof is similar.

$2°$. An approximation to the exact solution of problem (7) is sought in the form

$$U(x,y) = u_0(x) + \varepsilon u_1^h(x,y) + \varepsilon^2 u_2^h(x,y) + \varepsilon^3 u_3^h(x,y), \quad y = \varepsilon^{-1}x, \tag{10}$$

where $u_0(x)$ and $u_i^h(x,y)$ $(i = 1,2,3)$ are smooth functions of x and one-periodic functions of $y\in F^h$.

It follows from (10) and (9) that

$$e(U) = e_x(u_0) + e_y(u_1^h) + \varepsilon(e_x(u_1^h) + e_y(u_2^h)) + \varepsilon^2(e_x(u_2^h) + e_y(u_3^h)) + \varepsilon^3 e_x(u_3^h) \tag{11}$$

$$\begin{gathered} divAe(U) = \varepsilon^{-1}[div_yA(e_x(u_0) + e_y(u_1^h))]+ \\ +[div_yA(e_x(u_1^h) + e_y(u_2^h)) + div_xA(e_x(u_0) + e_y(u_1^h))]+ \\ +|\varepsilon[div_yA(e_x(u_2^h) + e_y(u_3^h)) + div_xA(e_x(u_1) + e_y(u_2^h))]+ \\ +\varepsilon^2 div_xA(e_x(u_2^h) + e_y(u_3^h)) + \varepsilon^3 divAe_x(u_3^h). \end{gathered} \tag{12}$$

Making the residual of U in Eq. (7) small, we obtain problems for determining u_0, u_i^h, $i = 1,2,3$. The function $u_0(x)$ solves the homogenized problem

$$-div_xA^{hom}e(u_0) = f \text{ in } \Omega, \quad u_0|_{\partial\Omega} = 0, \tag{13}$$

where

$$A^{hom}\xi\cdot\xi = \inf_{\varphi\in C_{per}^\infty(\square)^2}\int_\square A(\xi + e(\varphi))\cdot(\xi + e(\varphi))d\mu. \tag{14}$$

The tensor A_h^{hom} is defined by a formula similar to (14):

$$A_h^{hom}\xi\cdot\xi = \inf_{\varphi\in C_{per}^\infty(\square)^2}\int_\square A(\xi + e(\varphi))\cdot(\xi + e(\varphi))d\mu^h. \tag{15}$$

It is well known [13, Sect. 16] that $A_h^{hom}\to A^{hom}$.

The functions $u_i^h(x,\cdot)$ $(i = 1,2,3)$ are solutions to the cell problems

$$u_1^h(x,\cdot)\in H_{per}^1(\square, d\mu^h)^2 \quad div_yA(e_x(u_0) + e_y(u_1^h)) = 0; \tag{16}$$

$$\begin{gathered} u_2^h(x,\cdot)\in H_{per}^1(\square, d\mu^h)^2, \quad -div_yA(e_x(u_1^h) + e_y(u_2^h)) = \\ div_xA(e_x(u_0) + e_y(u_1^h)) - div_xA_h^{hom}e_x(u_0); \end{gathered} \tag{17}$$

$$\begin{gathered} u_3^h(x,\cdot)\in H_{per}^1(\square, d\mu^h)^2, \\ -div_yA(e_x(u_2^h) + e_y(u_3^h)) = div_xA(e_x(u_1) + e_y(u_2^h)) + g^h(x), \end{gathered} \tag{18}$$

where $g^h(x) = -\int_\square div_xA(e_x(u_1^h) + e_y(u_2^h))d\mu^h$.

Cell problems are solvable not uniquely. We consider only solutions with zero mean value over the circle $B^h = \{x : |x| < h\}$ that may be properly estimated, see. Sect. 3.3.

$3°$. Taking u_0 and u_i^h from above problems, we obtain (see (12))

$$\begin{gathered} divAe(U) - div_x A_h^{hom} e_x(u_0) = \\ = \varepsilon^3 divAe_x(u_3^h) + \varepsilon^2 div_x A(e_x(u_2^h) + e_y(u_3^h)) - \varepsilon F^h(x). \end{gathered} \tag{19}$$

Via $e_x(u_0)$, involved in Eq. (16), the solution u_1^h is a function of x, which is treated as a parameter. The solutions u_2^h and u_3^h are also functions of x. Importantly, the functions $u_i^h(x,y)$ have the sum structure

$$\sum a_j(x) b_j^h(y), \quad a_j \in C^\infty(\bar{\Omega}), \quad b_j^h \in L^2(Y, d\mu^h)^2 \tag{20}$$

where $a_j(x)$ are expressed in terms of $u_0(x)$, and $b_j^h(y)$ are solutions to canonical cell problems.

It follows from (19) that

$$-\, divAe(u^{\varepsilon,h} - U) = r^h(x, \varepsilon^{-1}x) + divR^h(x, \varepsilon^{-1}x), \tag{21}$$

where $r^h(x,y) = \varepsilon^2 div_x A(e_x(u_2^h) + e_y(u_3^h)) - \varepsilon F^h(x) - div_x(A_h^{hom} - A^{hom}) e_x(u_0)$ and $R^h(x,y) = \varepsilon^3 Ae_x(u_3^h)$.

In what follows, we say that $v^{\varepsilon,h}(x) \in L^2(\Omega_{\varepsilon,h}) \equiv L^2(\Omega, d\mu_\varepsilon^h)$ *converges (strongly) to zero in* $L^2(\Omega_{\varepsilon,h})$*:* $v_{\varepsilon,h} \to 0$ *in* $L^2(\Omega_{\varepsilon,h})$*, if* $\int\limits_\Omega |v^{\varepsilon,h}|^2 d\mu_\varepsilon^h \to 0$.

0ur goal is to prove convergence in $L^2(\Omega_{\varepsilon,h})$:

$$\begin{gathered} u^{\varepsilon,h}(x) - U(x, \varepsilon^{-1}x) \to 0, \\ e(u^{\varepsilon,h}(x) - U(x, \varepsilon^{-1}x)) \to 0. \end{gathered} \tag{22}$$

To this end, starting from energy estimates for $v^{\varepsilon,h} \in W_{\varepsilon,h}$, $-divAe(v^{\varepsilon,h}) = g_0 + divg$ (with respect to μ_ε^h), we have to prove, first of all, the convergence of the functions appearing on the right-hand side of (21): $r^h(x, \varepsilon^{-1}x) \to 0$ and $R^h(x, \varepsilon^{-1}x) \to 0$ in $L^2(\Omega_{\varepsilon,h})$. Since $r^h(x,y)$ and $R^h(x,y)$ have structure (20), the last relations follow from

$$\int\limits_\square |r^h(x,y)|^2 d\mu^h(y) \to 0, \quad \int\limits_\square |R^h(x,y)|^2 d\mu^h(y) \to 0 \quad \forall x, \tag{23}$$

thanks to

The mean value property (see [13, Sect. 12]. *Suppose that* $a^h \in L^1_{per}(\square, d\mu^h)$, $a^h \geq 0$*, and* $\lim\limits_{h\to 0} \int\limits_\square a^h d\mu^h = \alpha < \infty$*. Then*

$$\lim_{\varepsilon\to 0} \int\limits_\Omega \varphi(x) a^h(\varepsilon^{-1}x) d\mu_\varepsilon^h = \alpha \int\limits_\Omega \varphi(x) dx, \quad \varphi \in C(\bar{\Omega}).$$

$4°$. Now we explain how to prove (23). Solutions $u_i^h(x,\cdot)$ have to be analyzed in detail, because their L^2-estimates due to the equations, when based only on Korn's inequality, are crude: $u_i^h(x,\cdot) \sim h^{-i}$, whence $r^h \sim 1$ and $R^h \sim 1$. However, fine properties of the equations (16)-(18), we are going to reveal, imply the desired relations.

Lemma 2.1. *For any* x*, the following convergence holds in* $L^2(\square, d\mu^h)$*:*

$$\begin{gathered} hu_1^h \to 0, \quad h^3 u_3^h \to 0, \quad h^2 u_2^h \to v \neq 0, \\ he_y(u_2^h) \rightharpoonup 0, \quad h^2(e_x(u_2^h) + e_y(u_3^h)) \to 0. \end{gathered} \tag{24}$$

This lemma is really crucial in our considerations. Its proof is given in Sect. 3.4. Lemma 2.1 makes use of the convergence in the variable space $L^2(\square, d\mu^h)$, see [13, 17] and Appendix B.

Definition 2.2. We say that $v^h \in L^2(\square, d\mu^h)$ is bounded if $\limsup_{h\to 0} \int_\square |v^h|^2 d\mu^h < \infty$.

Bounded in $L^2(\square, d\mu^h)$ sequence v^h
(i) converges weakly to an element $v \in L^2(\square, d\mu) : v^h \rightharpoonup v$, if $\int_\square v^h \varphi d\mu^h \to \int_\square v\varphi d\mu \ \forall \varphi \in C^\infty_{per}(\square)$;
(ii) converges strongly to $v \in L^2(\square, d\mu) : v^h \to v$, if $\int_\square v^h w^h d\mu^h \to \int_\square vw d\mu$ whenever $w^h \rightharpoonup w$.

5°. In addition to the difficulty concerning the right-hand side of (21), there is another difficulty, namely, the difference $u^{\varepsilon,h} - U(x, \varepsilon^{-1}x)$ does not satisfy the zero conditions on $\partial\Omega$. This difficulty is resolved by localization thanks to structural features of limits of functions being in (24), see Sect.4.

Finally, we come to

Theorem 2.3. *Let $u^{\varepsilon,h}$ be a solution to problem (2.2) and the function $U(x, \varepsilon^{-1}x)$ be defined by (2.5) and (2.8)-(2.13). Then (2.17) holds.*

Theorems 1.1 and 1.2 become the corollaries of Th.2.3, see their derivation in Sect.4. Homogenization result for structures with other type of thickness - not critical - is also given in Sect.4.

3. Passing to the limit in cell problems.

3.1. **About method.** Our main goal is to prove Lemma 2.1 and to this end we need to pass to the limit in the cell problems (16)-(18). Preliminary, we, first, develop necessary analysis in variable L^2-spaces on thin structures, and, second, investigate the general periodic elasticity problem on thin grid F^h.

Asymptotic analysis on thin structures has been subject matter of a huge number of publications, among which we mention the monographs by Ciarlet [4], Nazarov [5], and Panasenko [10], where further references can be found. We would like to distinguish the interesting works of Nazarov and Slutskii [6, 7] devoted to elasticity problems on rod junctures, which have most of all influenced our approach to these problems. In [17] non-periodic elasticity problem (even nonlinear) on thin structure related to connected finite graph F of generic geometry was studied and limit equation on F was derived. In Appendices we reproduce for periodic case basic results from [17]. The solution u^h of the elasticity problem on a thin structure F^h has certain properties of *boundedness*, say, hu^h, $e(u^h)$ on each rod $I^h \subset F^h$ are bounded in variable L^2-norm. It seems natural to consider, first of all, an arbitrary sequence of vector-valued functions u^h with these properties of boundedness. We derive "structural" theorems of the following type: the limit of every such sequence satisfies conjugation conditions at the nodes, as well as some other requirements whose combination determines the "energy space" of the limit problem on F. After this, passing to the limit in the integral identity (on the basis of some "density theorems") yields a problem on the graph F. As a corollary we extract all the relations (24) when we take into account the specific features of each cell problem.

3.2. **Structural Theorems for limit of vector-functions.** We are going to introduce two spaces of vector-valued functions on the limit structure F. They will naturally appear in forthcoming structural theorems for limit of periodic vector-fields convergent in $L^2(\Box, d\mu^h)$ under some boundedness conditions. Here and hereafter often, we make no difference in notation of L^2-spaces of scalar- or vector-valued functions if there is no confusion. In this Section we consider only sequences which are bounded or convergent in $L^2(\Box, d\mu^h)$, so we say in this situation simply that sequences are bounded or convergent.

On every link I from F we choose (from two alternatives) a right pair consisting of a longitudinal and a transverse unit vectors τ, ν. Note that $\tau = (\tau^1, \tau^2)$.

1°. The *space* $\mathcal{T}$ is the set of all 1-periodic vector-fields u defined on F and satisfying the following conditions:

(i) (tangentiality) $u \cdot \nu = 0$ everywhere outside the nodes;
(ii) (smoothness) $u \cdot \tau \in H^1(I)$ for every link I;
(iii) (conjugation at the nodes) for every node O, there is a vector C such that

$$(u - C) \cdot \tau_j|_O = 0, \quad j = 1, 2, \ldots, m, \tag{25}$$

where $I_1, ..., I_m$ are all links joining the node O with directional unit vectors $\tau_1, ..., \tau_m$.

The norm in $\mathcal{T}$ is defined as the sum of H^1-norms on each link I from $F \cap \Box$.

Let us clarify the conjugation condition (25). Consider two m-dimensional vectors, composed of corresponding coordinates of directional vectors τ_j:

$$\tau^{(1)} = (\tau_1^1, \ldots, \tau_m^1), \quad \tau^{(2)} = (\tau_1^2, \ldots, \tau_m^2).$$

Then the conjugation condition (25) is equivalent to the condition

$$(u \cdot \tau_1, u \cdot \tau_2, \ldots, u \cdot \tau_m)|_O \in L(\tau^{(1)}, \tau^{(2)}), \tag{26}$$

where $L(\tau^{(1)}, \tau^{(2)})$ is the linear hull of $\tau^{(1)}$, $\tau^{(2)}$ in $\mathbb{R}^m$. Hence, it is easy to see that in the case of $m = 2$ condition (25) is always satisfied.

The following proposition gives another representation of the space $\mathcal{T}$.

Theorem 3.1 (Density theorem)**.** *The longitudinal components of the vectors from $C^\infty_{per}(\Box)^2$ are dense in the space $\mathcal{T}$.*

This theorem can be obtained as a corollary from a result about Sobolev spaces of the elasticity theory on singular structures [13, Lemma 6.1]. For the convenience of the reader, we give a somewhat different proof in Appendix E.

Theorem 3.2. *Suppose that the sequence $u^h \in H^1_{per}(\Box, d\mu^h)^2$ is such that $u^h_\tau = (u^h \cdot \tau)\tau$, $e(u^h)$ are bounded and $u^h_\tau \rightharpoonup u$. Then $u \in \mathcal{T}$.*

Some remark about the longitudinal component $u^h_\tau = (u^h \cdot \tau)\tau$. It can be found uniquely on each h-rod I^h, related to the link I. But on F^h, in the neighborhood of nodes, we deal with the bundle of such rods. So at the points, where two or more h-rods intersect, we sum longitudinal components, defined on each h-rod.

For scalar-valued functions there is a simpler analogue of Theorem 3.2.

Theorem 3.3. *Suppose that the sequence $u^h \in H^1_{per}(\Box, d\mu^h)$ is such that u^h, ∇u^h are bounded and $u^h \rightharpoonup u$. Then, $u|_I \in H^1(I)$ at every link I and u is continuous at every node. Moreover, the convergence of u^h to u is strong, $u^h \to u$.*

2°. The *space* $\mathcal{N}$ is the set of all 1-periodic vector-fields v defined on F and satisfying the following conditions:

(i) (transversality) $v \cdot \tau = 0$ everywhere outside the nodes;
(ii) (smoothness) $v \cdot \nu \in H^2(I)$ for every link I;
(iii) (clamping) $v = 0$ at every node of F;
(iv) (conjugation at the nodes) for every node O

$$(v \cdot \nu_i)'|_O = (v \cdot \nu_j)'|_O, \quad i, j = 1, 2, \dots, m, \tag{27}$$

where $I_1, ..., I_m$ are all links joining the node O and the prime denotes the longitudinal derivative,

$$(v \cdot \nu)' = \frac{d(v \cdot \nu)}{d\tau}.$$

The norm in $\mathcal{N}$ is defined as the sum of H^2-norms on each link I from $F \cap \Box$.

Obviously, the derivative $\frac{d(v \cdot \nu)}{d\tau} = (v \cdot \nu)'$ does not depend on the choice of the vector τ. Similarly, the vector $\nu(v \cdot \nu)''$ does not depend on the choice of τ and is denoted by v''. Thus, we can define, in an invariant manner, odd derivatives of the projection, $(v \cdot \nu)', (v \cdot \nu)''', \dots$, as well as even derivatives of the vector, $v'', v'''', \dots$. The situation is similar with the longitudinal vector u, for which we can give invariant definitions of $(u \cdot \tau)'$, u''.

Remark 1. Properties (i)-(iv) imply that in a neighborhood of the node, the vector v can be well-approximated by rigid displacements

$$t\omega(x), \quad \omega(x) = (-x_2, x_1), \quad t \in \mathbb{R}^1. \tag{28}$$

Indeed, suppose that the node coincides with the origin. We have on the link I_j

$$v(x) \cdot \nu_j = v_j(s), \quad s = x \cdot \tau_j, \quad 0 \le s \le 1,$$

According to (ii)-(iv),

$$v_j \in H^2(0, 1), \quad v_j(0) = 0, \quad v_j'(0) = t, \quad j = 1, 2, \dots, m.$$

Hence, using the Taylor formula, we get

$$v(x) \cdot \nu_j = ts + \int_0^s (s - \theta) v_j''(\theta)\, d\theta = t\omega(x) \cdot \nu_j + O(|s|^{3/2}),$$

since $ts = t(x \cdot \tau_j) = t\omega(x) \cdot \nu_j$.

Theorem 3.4. *Let hu^h, u^h_τ, $e(u^h)$ be bounded and $hu^h \rightharpoonup v$. Then $v \in \mathcal{N}$. Moreover, the convergence of hu^h to v is strong, $hu^h \to v$.*

The derivation of structral theorems is given in Appendix D.

3.3. **General periodic problem.** Suppose that the vector f^h and the matrix G^h are bounded in $L^2(\Box, d\mu^h)$. We examine the periodic elasticity equation

$$u^h \in H^1_{per}(\Box, d\mu^h)^2 \quad - div_y A(G^h + e_y(u^h)) = hf^h. \tag{29}$$

To study (29) we need uniform in h Korn inequalities for periodic functions on F^h. Such inequalities are proved in Appendix A.

Lemma 3.5. *(i) Equation (29) is solvable* $\Longleftrightarrow \int_\Box f^h d\mu^h = 0$.

(ii) Its solution satisfies the estimate

$$\int_\Box |e(u^h)|^2 d\mu^h \le C \int_\Box (|G^h|^2 + |f^h|^2) d\mu^h, \quad C = const(F, A), \tag{30}$$

and is unique if it is subjected to the following orthogonality condition

$$\int_{B^h} u^h dx = 0, \qquad where\ B^h = \{x : |x| < h\}. \tag{31}$$

The proof of this Lemma is straightforward due to inequality (56) and we omit it.

Suppose that

$$(f^h \cdot \nu)\nu \rightharpoonup f_\nu, \quad G^h \cdot \eta \rightharpoonup G_\tau, \quad G^h \cdot \sigma^h \rightharpoonup G_\nu, \tag{32}$$

where

$$\eta = \tau \times \tau = \{\tau^i \tau^j\}, \quad \sigma^h(y) = -\beta^h(y)\eta, \tag{33}$$

and $\beta^h(y) = h^{-1}(y \cdot \nu)$ on each h-strip I^h unless y belongs to the intersection of two or more h-strips, where we assume that $\beta^h(y)$ and $\sigma^h(y)$ are zero.

The tensor A is associated with unidimensinal tensor (scalar) $\rho(y) = (A^{-1}\eta\cdot\eta)^{-1}$ defined on the singular grid F. In the isotropic case (see (6)), the calculations give the constant $\rho(y) \equiv \hat{k} = \frac{k(k+2k_1)}{k+k_1}$.

Consider the solution to Eq. (29) subject to the orthogonality condition (31). From (30) and (56), (57) we derive readily the boundedness of the sequences $u^h_\tau = (u^h \cdot \tau)\tau, hu^h, e(u^h)$ in $L^2(\Box, d\mu^h)$.

Theorem 3.6. *Under assumptions (3.8) there is the convergence*

$$hu^h \to v, \quad A(G^h + e_y(u^h)) \rightharpoonup \rho(G_\tau + (u\cdot\tau)')\eta \quad in \quad L^2(\Box, d\mu^h), \tag{34}$$

where limit functions u and v are the solutions of the following problems

$$u \in \mathcal{T}, \quad \int_\Box \rho(u' + G_\tau \tau)\cdot\varphi' d\mu = 0 \quad \forall\varphi \in \mathcal{T}, \tag{35}$$

$$v \in \mathcal{N}, \quad \int_\Box \rho(\frac{1}{3}v'' + G_\nu \nu)\cdot\psi'' d\mu = \int_\Box f_\nu \cdot \psi d\mu \quad \forall\psi \in \mathcal{N}. \tag{36}$$

The proof of Th. 3.6 is rather technical and is given in Appendix E. Here we discuss only solvability of the limit equations.

Lemma 3.7. *(i) Problem (36) is uniquely sobvable.*

(ii) Problem (35) is solvable, but not uniquely. In any case, $G_\tau \equiv 0 \Longrightarrow u'|_I \equiv 0$ for every link I.

Proof. In order to obtain (i), let us verify that the expression $(\int_\Box |u''|^2\, d\mu)^{1/2}$ defines an equivalent norm on $\mathcal{N}$. To this end, it suffices to have the estimate

$$\int_\Box (|u|^2 + |u'|^2)\, d\mu \le k \int_\Box |u''|^2\, d\mu, \quad u \in \mathcal{N}, \quad k = const(F),$$

which is established by method *ex contrario* with the help of compactness arguments.

Assertion (ii) is proved similarly. ◻

3.4. **Taking the limit in cell problems.** We are in a position to prove Lemma 2.1. To this end we use Th. 3.6. and the main work is checking of assumptions (32) for Eq. (16)-(18).

1°**.** Begin with equation for u_1^h. Since $\xi = e_x(u_0(x))$ is independant on y, and hence $\xi \cdot \sigma^h(y) \rightharpoonup 0$ (see $(33)_2$), the relations (34), (35) take form

$$\begin{array}{ll} hu_1^h \to 0, & A(\xi + e_y(u_1^h)) \rightharpoonup \rho(\xi \cdot \eta + (u_1 \cdot \tau)')\eta \\ u_1 \in \mathcal{T}, & \int\limits_{\square} \rho(u_1' + (\xi \cdot \eta)\tau) \cdot \varphi' d\mu = 0 \quad \forall \varphi \in \mathcal{T}. \end{array} \tag{37}$$

It can be shown also that

$$e_x(hu_1^h) \to 0, \quad \nabla_x e_x(hu_1^h) \to 0. \tag{38}$$

2°**.** From (17), we have the following equation for $v_2^h \equiv hu_2^h$

$$\begin{array}{l} v_2^h \in H^1_{per}(\square, d\mu^h)^2, \; -div_y A(e_x(hu_1^h) + e_y(v_2^h)) = hf^h, \\ f^h = f + div_x(A^{hom} - A^{hom}_h)e_x(u_0) + div_x A(e_x(u_0) + e_y(u_1^h)) = f + f_1^h + f_2^h. \end{array} \tag{39}$$

Our aim is to prove the limit relations

$$hv_2^h \to v \not\equiv 0, \quad e_y(v_2^h) \rightharpoonup 0, \tag{40}$$

$$v \in \mathcal{N}, \quad \frac{1}{3} \int\limits_{\square} \rho v'' \cdot \psi'' d\mu = \int\limits_{\square} f \cdot \psi d\mu \quad \forall \psi \in \mathcal{N}. \tag{41}$$

By Th. 3.6, for $(40)_1$ coupling with (41), it is enough (see $(38)_1$) to deduce that

$$f_1^h \to 0; \quad f_2^h(x,\cdot) \to t(x,\cdot), \text{ where } t(x,\cdot)\cdot\nu = 0. \tag{42}$$

But $(42)_1$ follows from the convergence of tensors $A_h^{hom} \to A^{hom}$. To derive $(42)_2$ take into account the structure of limit in $(37)_2$. Namely, $A(e_x(u_0)+e_y(u_1^h))(x,\cdot) \rightharpoonup a(x,\cdot)\eta$ for some scalar function $a(x,y)$, where

$$\eta = \tau \times \tau = \begin{pmatrix} (\tau^1)^2 & \tau^1\tau^2 \\ \tau^1\tau^2 & (\tau^2)^2 \end{pmatrix}.$$

Hence, $div_x(a(x,y)\eta) = \frac{\partial a}{\partial x_1}\tau^1\tau + \frac{\partial a}{\partial x_2}\tau^2\tau$ is a tangential vector, and we obtain $(42)_2$. The relation $(40)_1$, (41) are proved.

Turn to $(40)_2$. In the case of the problem (39), because of $(38)_1$, the limit equation (35) contains $G_\tau = \lim\limits_{h\to 0} e_x(hu_1^h)\cdot\eta = 0$. So by Lemma 3.7 (ii),

$$A(e_x(hu_1^h) + e_y(v_2^h)) \rightharpoonup 0 \Longrightarrow e_y(v_2^h) \rightharpoonup 0.$$

We have completed the proof of (40). In other terms, relations $(24)_3$, $(24)_4$ are verified.

It can be shown also that

$$\nabla_x e_y(hu_2^h) \rightharpoonup 0; \quad e_x(h^2 u_2^h) \text{ is bounded.} \tag{43}$$

Moreover,

$$e_x(h^2u_2^h) \to z(x,\cdot), \text{ where } z(x,\cdot)\cdot\eta \equiv 0. \tag{44}$$

The last structural property is determined by structure of the limit vector v from $(24)_3$. Namely,

$$\begin{array}{c} v(x,y) = \sum a_j(x)b_j(y), \quad b_j \in \mathcal{N}, \quad a_j \text{ is a scalar function } \Longrightarrow \\ z(x,y) = e_x(v(x,y)) = \sum \frac{1}{2}(\nabla_x a_j(x) \times b_j(y) + b_j(y) \times \nabla_x a_j(x)) \perp \eta. \end{array}$$

For example, on the horisontal link, denoting $b_j = b$, $a_j = a$, we have

$$b(y) = \beta(y)\nu, \quad \nu = (0,1), \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow$$

$$e_x(a(x)b(y)) = \beta(y)e_x(a(x)\nu) = \beta(y)\begin{pmatrix} 0 & \frac{1}{2}\frac{\partial a}{\partial x_1} \\ \frac{1}{2}\frac{\partial a}{\partial x_1} & \frac{1}{2}\frac{\partial a}{\partial x_2} \end{pmatrix} \perp \eta.$$

3°. From (18) we deduce equation for $v_3^h \equiv h^2u_3^h$. Namely,

$$\begin{aligned} &v_3^h \in H^1_{per}(\square, d\mu^h)^2,\ -divA(y)(e_x(h^2u_2^h) + e_y(v_3^h)) = hf^h, \\ &f^h = div_x A(e_x(hu_1^h) + e_y(hu_2^h)) + hg^h(x). \end{aligned} \tag{45}$$

Here $f^h \rightharpoonup 0$ due to $(38)_2$, $(43)_1$. Hence, (see also $(43)_2$) by Lemma 7.5, $e_y(v_3^h)$ and hv_3^h are bounded, and thus the integral identity for v_3^h with the test function v_3^h implies the convergence

$$\int_\square A(e_x(h^2u_2^h) + e_y(v_3^h)) \cdot e_y(v_3^h) d\mu^h \to 0. \tag{46}$$

Thanks to the structural property (44) and the relation $f^h \rightharpoonup 0$, we have $G_\nu \equiv 0$ and $f_\nu \equiv 0$ in the limit equation (36). From here, $hv_3^h \to 0$ and we obtain the convergence $(24)_3$.

It remains to deduce only the last relation in (24). Again recall the structural condition (44) for the matrix z which implies some recovering property we are going to present. In order to prove it, we use *natural extension* b_h of $b \in L^2(\square, d\mu)$ to the support of the measure μ^h. Here we give its

Definition 3.8. Extend given function $b \in L^2(\square, d\mu)$, defined on F, to the support of the measure μ^h, that is F^h, as follows. Take a separate h-rod $I^h \subset F^h$, corresponding to the link $I \subset F$, and let the extension of $b|_I$ to I^h be constant in the transverse direction. In the neighborhood of nodes, at the points where two or more h-rods intersect, sum these extensions. The result is denoted by b_h and is called *natural extension*. We have convergence $b_h \to b$, see Appendix B.

Lemma 3.9. *Let symmetric matrix $z \in L^2(\square, d\mu)$ and $z \cdot \eta = 0$. Then there exists a sequence $w^h \in H^1_{per}(Y, d\mu^h)^2$, such that $w^h \to 0$ and $e_y(w^h) \to z$ in $L^2(\square, d\mu^h)$.*

Proof. We can use $w^h(y) = h\zeta^h(y)\beta^h(y)k_h(y)$, where $\beta^h(y)$ is defined in Section 3.3 after (33); $\zeta^h(y) \in C^\infty_{per}(Y)$ is such that $\zeta^h = 0$ in the $2h$-neighborhood of the nodes, $\zeta^h = 1$ outside the $4h$-neighborhood of the nodes and $|h\nabla\zeta^h| < 2$; k_h is the natural extension on F^h of some vector k (calculated from z on each link of F). For example, on the horizontal link I (with the longitudinal coordinate y_1), we have $z|_I = \begin{pmatrix} 0 & \alpha \\ \alpha & \gamma \end{pmatrix}$, where $\alpha, \gamma \in L^2(Y, d\mu)$. Then, $k|_I = (2\alpha, \gamma)$ and, on the corresponding h-strip, $k_h|_{I^h}(y) = (2\alpha(y_1), \gamma(y_1))$, $w^h|_{I^h}(y) = \zeta^h(y)(2y_2\alpha(y_1), y_2\gamma(y_1))$. □

Verification of the desired properties for w^h is straightforward and we omit it.

Return to the proof of $(24)_5$. To this end, test Eq. (45) with vector w^h from Lemma 3.9, corresponding to the matrix z from (44). Then

$$\int_\square A(e_x(h^2u_2^h) + e_y(v_3^h)) \cdot e_y(w^h) d\mu^h = \int_\square hf^h \cdot w^h d\mu^h \to 0,$$

and, because of the equality $e_y(w^h) = z_h + o(1) = e_x(h^2u_2^h) + o(1)$ in $L^2_{per}(\square, d\mu^h)$, we obtain

$$\lim_{h\to 0}\int\limits_\square A(e_x(h^2u_2^h) + e_y(v_3^h)) \cdot e_x(h^2u_2^h)d\mu^h = 0.$$

In combination with (46) this implies

$$\lim_{h\to 0}\int\limits_\square A(e_x(h^2u_2^h) + e_y(v_3^h) \cdot (e_x(h^2u_2^h)) + e_y(v_3^h))d\mu^h = 0 \Longrightarrow (24)_5.$$

Lemma 2.1 is proved.

4°. Now examine the relations (37). For model grid, or even for every grid composed of lines, the solution of Eq. $(37)_2$ is trivial, i.e. $u_1' = 0$. Hence, in addition to $(37)_1$ we have

$$e_y(u_1^h) \to p(y,\xi), \quad p(y,\xi) = \rho(\xi\cdot\eta)A^{-1}\eta - \xi \perp \eta. \tag{47}$$

At first, we can speak only about weak convergence in (47). Applying Lemma 3.9 to the matrix p, we find the vector $w^h = w^h(\cdot,\xi)$, such that

$$w^h \to 0, \quad e_y(w^h) \to p \text{ in } L^2(\square, d\mu^h), \tag{48}$$

and test with w^h Eq. (16), setting $\xi = e(u_0(x))$. Hence,

$$\int\limits_\square Ae_y(u_1^h)\cdot e_y(w^h)d\mu^h = -\int\limits_\square A\xi\cdot e_y(w^h)d\mu^h \Longrightarrow \int\limits_\square Ap\cdot pd\mu = -\int\limits_\square A\xi\cdot pd\mu. \tag{49}$$

Here taking the limit is possible, since $e_y(u_1^h) \rightharpoonup p$ and $e_y(w^h) \to p$. On the other hand, from the energy equality for (16), we deduce

$$\int\limits_\square Ae_y(u_1^h) \cdot e_y(u_1^h)d\mu^h = -\int\limits_\square A\xi \cdot e_y(u_1^h)d\mu^h \to -\int\limits_\square A\xi \cdot pd\mu.$$

In combination with (49), this leads to the relation

$$\lim_{h\to 0}\int\limits_\square Ae_y(u_1^h) \cdot e_y(u_1^h)d\mu^h = \int\limits_\square Ap \cdot pd\mu,$$

and $(47)_1$ is thereby true (see the strong convergence criterion in Appendix B).

We conclude our considerations of u_1^h by equality (see (47), (48))

$$\begin{gathered} e_y(u_1^h) = e_y(w^h(y,\xi)) + o(1) = e_y(\bar{u}_1^h(x,y)) + o(1) \text{ in } L^2(\square, d\mu^h), \\ \bar{u}_1^h(x,y) = w^h(y,\xi)|_{\xi=e(u_0(x))}, \end{gathered} \tag{50}$$

where the vector w^h is calculated from the matrix p, given in (47), according to the rule described in proof of Lemma 3.9.

4. **Justification of asymptotic expansions.**

4.1. **Proof of Theorem 2.3.** sketched in Sect.2.2 is performed here in several steps.

1°. First we point out some properties of convergence in $L^2(\Omega_{\varepsilon,h})$.

Lemma 4.1. *Suppose $b^h \to b$, $c^h \rightharpoonup c$ in $L^2(\square, d\mu^h)$. Then the following relations hold:*

(i) $\lim\limits_{\varepsilon\to 0}\int\limits_\Omega \varphi(x)b^h(\frac{x}{\varepsilon})c^h(\frac{x}{\varepsilon})d\mu_\varepsilon^h = \int\limits_\Omega\int\limits_\square \varphi(x)b(y)c(y)d\mu(y)dx, \quad \varphi\in C(\bar\Omega);$

(ii) $\lim_{\varepsilon\to 0}\int_\Omega v^{\varepsilon,h}(x)(b^h(\frac{x}{\varepsilon})-b_h(\frac{x}{\varepsilon}))d\mu^h_\varepsilon=0,$
where $v^{\varepsilon,h}$ is bounded in $L^2(\Omega,d\mu^h_\varepsilon)$ and b_h is a natural extension of b on F^h.

To prove assertions (i),(ii) use the mean value property and the definition of convergence in $L^2(\square,d\mu^h)$.
2°. Pass to the direct proof of the theorem 2.3. Take the function

$$v^{\varepsilon,h}\in W_{\varepsilon,h},\quad -divAe(v^{\varepsilon,h})=r^h(x,\varepsilon^{-1}x)+divR^h(x,\varepsilon^{-1}x),$$

where $r^h(x,y)$, $R^h(x,y)$ are the same as in (21). Due to the energy estimate, we have

$$\limsup_{\varepsilon\to 0}\int_\Omega(|v^{\varepsilon,h}|^2+|e(v^{\varepsilon,h})|^2)d\mu^h_\varepsilon=0.$$

Obviously, if $\bar u^{\varepsilon,h}=U-v^{\varepsilon,h}$, then $u^{\varepsilon,h}-\bar u^{\varepsilon,h}$ satisfies Eq. (7) with zero rightside function. Hence,

$$\int_\Omega Ae(u^{\varepsilon,h}-\bar u^{\varepsilon,h})\cdot e(u^{\varepsilon,h}-\tilde u^{\varepsilon,h})d\mu^h_\varepsilon=0,$$

where $\tilde u^{\varepsilon,h}(x)=u_0(x)+\psi(x)\tilde U-v^{\varepsilon,h}(x)$, $\tilde U=\varepsilon u_1^h+\varepsilon^2u_2^h+\varepsilon^3u_3^h$, $\psi\in C_0^\infty(\Omega)$, $0\le\psi\le 1$ and $u^{\varepsilon,h}-\tilde u^{\varepsilon,h}$ can really serve as a test function for (7). Since

$$u^{\varepsilon,h}-\tilde u^{\varepsilon,h}=(u^{\varepsilon,h}-\bar u^{\varepsilon,h})+(\bar u^{\varepsilon,h}-\tilde u^{\varepsilon,h}),$$
$$\tilde u^{\varepsilon,h}-\bar u^{\varepsilon,h}=(\psi-1)\tilde U,\quad e(\tilde u^{\varepsilon,h}-\bar u^{\varepsilon,h})=(\psi-1)e(\tilde U)+(\nabla\psi\times\tilde U)^s,$$

where $(\nabla\psi\times\tilde U)^s=\frac12(\nabla\psi\times\tilde U+\tilde U\times\nabla\psi)$, it follows that

$$\int_\Omega Ae(u^{\varepsilon,h}-\bar u^{\varepsilon,h})\cdot e(u^{\varepsilon,h}-\bar u^{\varepsilon,h})d\mu^h_\varepsilon=$$

$$=\int_\Omega Ae(u^{\varepsilon,h}-\bar u^{\varepsilon,h})\cdot(\nabla\psi\times\tilde U)d\mu^h_\varepsilon+\int_\Omega Ae(u^{\varepsilon,h}-\bar u^{\varepsilon,h})\cdot e(\tilde U)(\psi-1)d\mu^h_\varepsilon=$$

$$=I(\varepsilon)+II(\varepsilon).$$

Here and hereafter, we use matrix equality $S\cdot B=S\cdot B^s$, where S is symmetric and $B^s=\frac12(B+B^T)$.

It remains to examine terms $I(\varepsilon),II(\varepsilon)$.
3°. Starting from boundedness of $e(u^{\varepsilon,h}-\bar u^{\varepsilon,h})$ in $L^2(\Omega,d\mu^h_\varepsilon)$, we deduce that

$$|II(\varepsilon)|^2\le C\int_\Omega(\psi-1)^2|e(\tilde U)|^2d\mu^h_\varepsilon\equiv C\int_\Gamma\Phi^h(x,\frac{x}{\varepsilon})d\mu^h_\varepsilon,$$

where $\Gamma=supp(1-\psi)\cap\Omega$. Covering Γ with quadrates $\varepsilon\square_i$ of area ε^2, we split the latter integral into the sum and obtain the estimate

$$|II(\varepsilon)|\le C\sum_i\int_{\varepsilon\square_i}\Phi^h(x,\frac{x}{\varepsilon})d\mu^h_\varepsilon=C\sum_i\varepsilon^2\int_{\square_i}\Phi^h(\varepsilon y,y)d\mu^h(y)\le C'|\Gamma|\lambda<\delta$$

for arbitrary small δ. Here, $\limsup_{h\to 0}\int_\square\Phi^h(x,y)d\mu^h(y)\le\lambda\quad\forall x$ and Lebesque measure $|\Gamma|$ of the set Γ may be done arbitrary small by appropriate choice of ψ.

4°**.** Split the term $I(\varepsilon)$ into the following sum

$$I(\varepsilon) =$$

$$= \int\limits_{\Omega} Ae(u^{\varepsilon,h} - \bar{u}^{\varepsilon,h}) \cdot (\nabla\psi \times (\varepsilon u_1^h + \varepsilon^3 u_3^h))d\mu_\varepsilon^h - \int\limits_{\Omega} Ae(\bar{u}^{\varepsilon,h}) \cdot (\nabla\psi \times \varepsilon^2 u_2^h)d\mu_\varepsilon^h +$$

$$+ \int\limits_{\Omega} Ae(u^{\varepsilon,h}) \cdot (\nabla\psi \times \varepsilon^2 u_2^h)d\mu_\varepsilon^h = I_1(\varepsilon) - I_2(\varepsilon) + I_3(\varepsilon).$$

From boundedness of $e(u^{\varepsilon,h} - \bar{u}^{\varepsilon,h})$ in $L^2(\Omega, d\mu_\varepsilon^h)$ and $(24)_1$, $(24)_2$, we have

$$|I_1(\varepsilon)|^2 \leq C \int\limits_{\Omega} |\nabla\psi \times (\varepsilon u_1^h + \varepsilon^3 u_3^h)|^2 d\mu_\varepsilon^h \to 0.$$

To study the term $I_2(\varepsilon)$ make the following simplification

$$I_2(\varepsilon) = \int\limits_{\Omega} A(e_x(u_0) + e_y(u_1^h)) \cdot (\nabla\psi \times \varepsilon^2 u_2^h)d\mu_\varepsilon^h + o(1)$$

using properties of constituents of $\bar{u}^{\varepsilon,h}$, and note that

$$A(e_x(u_0) + e_y(u_1^h))(x,\cdot) \rightharpoonup a(x,\cdot)\eta, \quad (\nabla\psi \times h^2 u_2^h)^s(x,\cdot) \to (\nabla\psi \times v)^s \equiv z, \quad (51)$$

by virtue of $(37)_3$, $(24)_2$,. Here, $v \in \mathcal{N}$, therefore $z \perp \eta$ (see derivation of (44)) and

$$\lim_{\varepsilon\to 0} I_2(\varepsilon) = \int\limits_{\Omega}\int\limits_{\square} a(x,y)\eta \cdot z dx d\mu = 0.$$

By Lem. 4.1 (ii) and relation $(4.1)_2$

$$I_3(\varepsilon) = \int\limits_{\Omega} Ae(u^{\varepsilon,h}) \cdot (\nabla\psi \times \varepsilon^2 u_2^h)d\mu_\varepsilon^h = \int\limits_{\Omega} Ae(u^{\varepsilon,h}) \cdot z_h d\mu_\varepsilon^h + o(1).$$

Applying Lemma 3.9 to the matrix z, find vector $w^h(x,y)$, such that $w^h(x,\cdot) \to 0$, $e_y(w^h(x,\cdot)) \to z(x,\cdot)$ in $L^2(\square, d\mu^h)$. Moreover, being finite in Ω, $w^h(x, \frac{x}{\varepsilon})$ can be taken to test (7). Consequently,

$$I_3(\varepsilon) = \int\limits_{\Omega} Ae(u^{\varepsilon,h}) \cdot e_y(w^h)d\mu_\varepsilon^h + o(1) = \int\limits_{\Omega} Ae(u^{\varepsilon,h}) \cdot e(\varepsilon w^h)d\mu_\varepsilon^h + o(1) =$$

$$= \varepsilon \int\limits_{\Omega} f \cdot w^h d\mu_\varepsilon^h + o(1) \Longrightarrow I_3(\varepsilon) \to 0.$$

Gathering the results of sections $2° - 4°$, we come to the limit relation (22).

4.2. **Approximations by means of solutions to the problems on singular structure.** In this Section we define similar to (10) expansions in terms of solutions to periodic problems on the singular grid F that are extended in the natural or special manner to the thin grid F^h. We aim at different type approximations. Recall that $\lim\limits_{\varepsilon\to 0} h(\varepsilon)/\varepsilon = \theta$.

1°. Simplifying, by virtue of $(24)_1$-$(24)_3$, the expansion $U(x,\cdot)$ from (10), one "descends" to the singular structure F in the following way

$$U(x,\cdot) = u_0(x) + \theta^{-2} v_h(x,\cdot) + o(1) \quad in \quad L^2(\square, d\mu^h). \quad (52)$$

Here, $v(x,\cdot)$ is the solution of Eq. (36) and $v_h(x,\cdot)$ denotes its natural extension to F^h (see definition 3.8). The simplified expansion (52) ensures L^2-approximation only to the solution $u^{\varepsilon,h}$ but not to its gradient. As a result, we derive Th.1.1.

2°. By properties (24) of the gradients of u_i^h, we derive from $(22)_2$ the weak two-scale convergence

$$e(u^{\varepsilon,h}(x)) \stackrel{2}{\rightharpoonup} e(u_0(x)) + z(x,y), \tag{53}$$

where $z(x,y) = p(y,\xi)|_{\xi=e(u_0(x))}$, $p(y,\xi)$ is defined in (47).

Recall [13, 17] that convergence $v^{\varepsilon,h}(x) \stackrel{2}{\rightharpoonup} v(x,y)$ means

$$\int_\Omega v^{\varepsilon,h}(x)\varphi(x)b(\frac{x}{\varepsilon})d\mu_\varepsilon^h \to \int_\Omega\int_\Box v(x,y)\varphi(x)b(y)dxd\mu(y)$$

for every $\varphi \in C_0^\infty(\Omega)$ and $b \in C_{per}^\infty(\Box)$. The relation (53) is well-known and was proved earlier in [13] by method of two-scale convergence.

3°. A "singular" analoque of (10) for approximation of $u^{\varepsilon,h}$ with gradient (see (22)) involves the same, as in (52), functions $u_0(x)$ and $v(x,\cdot)$ and, in addition to them, two correctors, which stem from the terms u_1^h and u_3^h of (10):

$$\bar{U}(x,y) = u_0(x) + \varepsilon\bar{u}_1^h(x,y) + \theta^{-2}v^h(x,y) + \varepsilon\theta^{-2}g^h(x,y). \tag{54}$$

One can calculate both correctors through the same procedure starting from the given matrices orthogonal to η at each link of F. Here, $\bar{u}_1^h(x,y)$ is defined (see (50)) in terms of matrix $p(x,y)$, given in (47), and $g^h(x,\cdot)$ is defined in terms of matrix $e_x(v(x,\cdot))$ (see Lemma 3.9).

Finally, $v^h(x,\cdot)$ denotes in (54) the special extension to F^h of $v(x,\cdot)$, defined on F as the solution of Eq. (36). Let us describe this extension. Firstly, one finds approxmation $\tilde{v}^h \in \mathcal{N}$, such that $\tilde{v}^h \to v$ in $\mathcal{N}$, $\tilde{v}^h$ coincides with rigid displacement in h-neighborhood of each node and with v outside $5h$-neighborhood of nodes. Secondly, one extends $\tilde{v}^h$ from each link I of grid F to the corresponding h-rod I^h of F^h according to the Lemma C.6. Further details and justification are left out.

We can conclude our considerations. If in (54) we set $\chi(x,y) = \theta^{-2}v(x,y)$ and omit symbols of extension, we come to (5).

4.3. **About structures sufficiently thick and thin.** According to the classification of thin periodic structures there are three types of them, see Sect.1. Before, we have studied the most complicated structures of critical thickness.

On the sufficiently thin structures, instead of (2), we consider the following problem

$$u^{\varepsilon,h} \in W_{\varepsilon,h}, \quad \int_{\Omega_{\varepsilon,h}} Ae(u^{\varepsilon,h})\cdot e(\varphi)dx = \frac{h}{\varepsilon}\int_{\Omega_{\varepsilon,h}} f\cdot\varphi dx \quad \forall\varphi \in C_0^\infty(\Omega), \tag{55}$$

where $f \in C^\infty(\bar{\Omega})^2$. By the method of the two-scale convergence, as in [16], or by the method of asymptotic expansions, as above, one can prove

Theorem 4.2. *Let $h(\varepsilon)/\varepsilon \to 0$. Then for the solution of (55) there is the convergence*

$$\frac{1}{|\Omega_{\varepsilon,h}|}\int_{\Omega_{\varepsilon,h}} |\frac{h}{\varepsilon}u^{\varepsilon,h}(x) - \chi_h(x,\frac{x}{\varepsilon})|^2dx \to 0,$$

where $\chi(x,\cdot)$ is the solution of periodic problem

$$\chi(x,\cdot) \in \mathcal{N}, \quad \frac{1}{3}\int_\Box \rho\chi''(x,\cdot)\cdot\psi''d\mu = \int_\Box f(x)\cdot\psi d\mu \quad \forall\psi \in \mathcal{N},$$

and $\chi_h(x,\cdot)$ in the natural extension of $\chi(x,\cdot)$ on F^h.

For sufficiently thick structure we consider the problem (2) and derive from Th.2.3 the following result.

Theorem 4.3. *Let $h(\varepsilon)/\varepsilon \to \infty$. Then for the solution of the problem (2) there is the convergence*

$$\frac{1}{|\Omega_{\varepsilon,h}|}\int\limits_{\Omega_{\varepsilon,h}} |u^{\varepsilon,h}(x)-u_0(x)|^2dx \to 0,$$

where $u_0(x)$ is the solution of homogenized problem (13).

5. **Appendix A. Korn inequelities for periodic functions on thin structures.** Consider plane periodic grid F^h of thickness $h{\geq}0$, $\square{=}[0,1)^2$ is a periodicity cell. So $F^0{=}F$ is 1-periodic graph. Begin with description of thin grids. If F is not composed of only lines, we make the following convention how to construct F^h using F. For this purpose, we elongate each link I in both sides by $\frac{h}{2}$ so that a segment I' of length $|I|+h$ will be obtained. Construct a strip I^h of width h with middle line I'. The union of all these strips I^h defines the grid F^h. Note that each node of the singular grid F belongs to the grid F^h, together with the circle of radius $\frac{h}{2}$, by construction.

Now we are going to introduce rigid grids. We try ascribe the mark (or label) $k=0,1,\dots$ to each node of the grid F, proceeding by induction. Consider in F any subgraph F', generated by two noncollinear lines, and assign zero label only to the nodes of F'. Let all the nodes to which one can ascribe the marks $k\leq N$ be defined. Then we ascribe the mark $N+1$ to the yet unmarked node if one can go out from it along two noncollinear links into the neighboring nodes that already have marks. Graph F is called *rigid* if for some pair of lines chosen at the beginning the above labelling procedure labells all the nodes of F.

From now on, F is assumed to be rigid. Suppose also that origin coinsides with "zero" node.

Theorem 5.1. *The following inequlities are valid for $u\in C^\infty_{per}(\square)^2$:*

$$h^2\int\limits_{\square\cap F^h}(|u|^2+|\nabla u|^2)dx \leq C_0\int\limits_{\square\cap F^h}|e(u)|^2dx,\quad C_0=const(F), \tag{56}$$

$$\int\limits_{I^h}|u\cdot\tau|^2dx\leq C_1\int\limits_{\square\cap F^h}|e(u)|^2dx,\quad C_1=const(F), \tag{57}$$

where

$$\int\limits_{B^h}udx=0,\quad B^h=\{x:\ |x|<\frac{h}{2}\}, \tag{58}$$

I^h is arbitrary h-rod in F^h and τ is the longitudinal orth on it.

We only sketch the proof of this theorem. First, the "tangential" inequality (57) is establisted for the strips I^h containing the circle B^h from (58), and to this end we use the method of contraction of variable strip I^h to fixed rectangle. Then we spread inequality (57) step by step to other strips of F^h following the proceedure of labelling the nodes in rigid graph F. At the same time we derive the inequality

$$\frac{1}{h}\int\limits_{B_i^h} |u|^2 dx \le c_0 \int\limits_{\square\cap F^h} |e(u)|^2 dx, \quad c_0 = const(F), \quad \text{whenever} \quad \int\limits_{B^h} u dx = 0. \tag{59}$$

Here and hereafter, B_i^h denotes the circle of radius $\frac{h}{2}$ centered at arbitrary node in $\square \cap F$.

Similarly the following inequalities

$$\int\limits_{\square\cap F^h} |u|^2 dx \le c_1 (\int\limits_{\square\cap F^h} |\nabla u|^2 dx + \sum_i \frac{1}{h}\int\limits_{B_i^h} |u|^2 dx), \quad c_1 = const(F), \tag{60}$$

$$\int\limits_{\square\cap F^h} |\nabla u|^2 dx \le c_2 (\frac{1}{h^2}\int\limits_{\square\cap F^h} |e(u)|^2 dx + \sum_i \frac{1}{h}\int\limits_{B_i^h} |u|^2 dx), \quad c_2 = const(F), \tag{61}$$

can be obtained. Finally, the estimates (59)-(61) imply (56).

Following the derivation of non-periodic analogs of (59)-(61) in [15, Sect.2], one can restore omitted details of the proof of Th.A.1.

6. **Appendix B. L^2-convergence in a variable space.** Let us make some remarks about convergence of sequence $u^h \in L^2(\square, d\mu^h)$ see Definition 2.2.

1°. Here we consider arbitrary plane graph F and related thin grid F^h. We assume that each h-rod I^h of F^h has the corresponding link I of F as the midline. Measures μ^h and μ, supported on F^h and F, respectively, are defined in Sect. 1.

Note the following properties of convergence in $L^2(\square, d\mu^h)$ [13, 17]:

(i) any bounded sequence contains a weakly convergent subsequence;

(ii) $u^h \to u \iff u^h \rightharpoonup u$ and $\int\limits_{\square} |u^h|^2 d\mu^h \to \int\limits_{\square} |u|^2 d\mu$ (strong convergence criterion).

Example 1. For given function $b \in L^2(\square, d\mu)$ consider its natural extension $b_h \in L^2(\square, d\mu)$, see definition 3.8. The natural extension preserves the mean value: $\int\limits_{\square} b_h d\mu^h = \int\limits_{\square} b\, d\mu$. Hence, simple calculations show that

$$b_h \rightharpoonup b, \quad \int\limits_{\square} |b_h|^2 d\mu^h \to \int\limits_{\square} |b|^2 d\mu.$$

Therefore, see (ii), $b_h \to b$ (strong convergence of natural extension).

2°. Definition 2.2 is general and admits arbitrary 1-periodic Borel measure $\mu^h \rightharpoonup \mu$. In our case, when μ^h is related to the thin grid F^h, it is possible to introduce the convergence in the variable space $L^2(\square, d\mu^h)$ otherwise. We shall do this in two steps.

Step 1. Start with the convergence on a single rod I^h. For the sake of clarity, we discuss the case of a horizontal rod $I^h = I \times [-h, h]$, $I = [0, 1]$.

We say that a sequence $u^h \in L^2(I^h)$ is *bounded* if $h^{-1} \int\limits_{I^h} |u^h|^2\, dx \le C$, where C does not depend on h. A bounded sequence u^h is said to be *weakly convergent* to

$u \in L^2(I)$, $u^h \rightharpoonup u$, if the sequence of transverse averages

$$\bar{u}^h = \bar{u}^h(x_1) = \frac{1}{2h} \int\limits_{-h}^{h} u(x_1, x_2)\, dx_2$$

is weakly convergent to u in $L^2(I)$.

We say that u^h is *strongly convergent* to u, $u^h \to u$, if $\lim\limits_{h\to 0} \frac{1}{h} \int\limits_{I^h} |u^h - u|^2\, dx = 0$.

Example 2. If u^h has the following structure: $u^h(x_1, x_2) = b\left(x_1, \frac{x_2}{h}\right)$, where $b \in L^2(I\times[-1,1])$, then $u^h \rightharpoonup u = u(x_1) = \frac{1}{2} \int\limits_{-1}^{1} b(x_1, x_2)\, dx_2$ and the strong convergence is observed only if $b = b(x_1)$ is independent of x_2.

Step 2. It is natural to untroduce

Definition 6.1. *We say that a sequence $u^h \in L^2(\square, d\mu^h)$ is bounded, if the restrictions $u^h|_{I^h}$ on every rod I^h are bounded in the above sense. Bounded sequence $u^h \in L^2(\square, d\mu^h)$ weakly (or strongly) converges to $u \in L^2(\square, d\mu)$, if $u^h|_{I^h} \rightharpoonup u|_I$ (or $u^h|_{I^h} \to u|_I$) on every rod I^h.*

Proposition 1. *The convergence in the sense of Definition 2.2 is equivalent to the convergence in the sense of Definition 6.1.*

See proof in [17].

7. **Appendix C. Asymptotic analysis on a single rod.** 1°. Consider more closely the convergence of functions v^h on $I^h = I \times [-h, h]$ to a function v on the segment I. At first, note some basic properties of this convergence, introduced in Appendix B.

Lemma 7.1. *Let $v^h \in H^1(I^h)$ and suppose that v^h, $\frac{\partial v^h}{\partial x_1}$ are bounded. Then, we have* (*up to a subsequence*)

$$v^h \rightharpoonup v, \quad \frac{\partial v^h}{\partial x_1} \rightharpoonup \frac{\partial v}{\partial x_1}, \quad \textit{where} \ \ v \in H^1(I).$$

Moreover, the convergence of the transverse averages $\bar{v}^h$ is uniform on I.

Proof. Since

$$\frac{\partial \bar{v}^h}{\partial x_1} = \frac{1}{2h} \int\limits_{-h}^{h} \frac{\partial}{\partial x_1} v^h(x_1, x_2)\, dx_2, \tag{62}$$

it suffices to refer to the simplest properties of the Sobolev space $H^1(I)$. $\square$

Lemma 7.2. *If v^h, ∇v^h are bounded, then v^h is compact with respect to strong convergence.*

Proof. From (62), it follows that $\bar{v}^h$ is bounded in $H^1(I)$, and therefore, it is compact in $L^2(I)$. By the Poincaré inequality, we have

$$\int\limits_{-h}^{h} |v^h(x_1, x_2) - \bar{v}^h(x_1)|^2\, dx_2 \le ch^2 \int\limits_{-h}^{h} \left|\frac{\partial v^h}{\partial x_2}\right|^2\, dx_2,$$

$$\frac{1}{2h} \int\limits_{I^h} |v^h - \bar{v}^h|^2\, dx \le ch \int\limits_{I^h} \left|\frac{\partial v^h}{\partial x_2}\right|^2\, dx \le c_1 h,$$

and now the compactness of v^h becomes evident. □

2°. Now take on a horizontal rod I^h a sequence of vector-fields u^h such that

$$e(u^h), \quad u_1^h, \quad hu_2^h, \quad h\nabla u^h \text{ are bounded.} \tag{63}$$

It can be assumed (see Lemmas 7.1, 7.2) that over a subsequence

$$u_1^h \rightharpoonup u_1, \quad hu_2^h \to u_2, \quad u_1, u_2 \in H^1(I). \tag{64}$$

Lemma 7.3. *The component u_2 belongs to $H^2(I)$ and there is strong convergence*

$$u_1^h(x_1, hx_2) \to u_1(x_1) - x_2 \frac{\partial u_2}{\partial x_1}(x_1) \text{ in } L^2(Q), \quad Q = I \times [-1, 1]. \tag{65}$$

Proof. Changing the variables $y_1 = x_1, \quad y_2 = h^{-1}x_2, \quad v_1 = u_1, \quad v_2 = hu_2$, for a vector-field $v^h = v^h(y)$ defined on a fixed rectangle $Q = I \times [-1, 1]$, we have, by virtue of (63),

$$\int_Q |v^h|^2\, dy \le C, \quad \int_Q \left[\left(\frac{\partial v_1^h}{\partial y_1}\right)^2 + h^{-2}\left(\frac{\partial v_1^h}{\partial y_2} + \frac{\partial v_2^h}{\partial y_1}\right)^2 + h^{-4}\left(\frac{\partial v_2^h}{\partial y_2}\right)^2 \right] dy \le C,$$

Without loss of generality, we can assume that $v^h \rightharpoonup v$ in $H^1(Q)^2$. Now, we get $v_2 = v_2(y_1)$, $\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} = 0$, whence

$$v_1 = -y_2 \frac{\partial v_2}{\partial y_1} + g(y_1), \quad g(y_1) = \frac{1}{2}\int_{-1}^{1} v_1\, dy_2,$$

$$\frac{1}{2}\int_{-1}^{1} y_2 v_1\, dy_2 = -\frac{1}{2}\int_{-1}^{1} y_2^2 \frac{\partial v_2}{\partial y_1}\, dy_2 = -\frac{1}{3}\frac{\partial v_2(y_1)}{\partial y_1},$$

and finally, $v_2 \in H^2(I)$. Let us verify the relations $g = u_1$, $v_2 = u_2$, see (64). For $\alpha \in C_0^\infty(I)$, we have

$$\int_I \overline{u}_1^h \alpha\, dx_1 = \frac{1}{2h}\int_{I^h} u_1^h \alpha\, dx = \frac{1}{2}\int_Q v_1^h \alpha\, dy \to \frac{1}{2}\int_Q v_1 \alpha\, dy = \int_0^1 g(x_1)\alpha(x_1)\, dx_1,$$

which yields the convergence $\bar{u}_1^h \rightharpoonup g$ in $L^2(I)$ and the relation $u_1 = g$. The equality $u_2 = v_2$ is established in a similar way. The lemma is proved. □

Lemma 7.4. *Under assumptions (62), (63), the following relations hold:*

$$u_1^h - u_1 + \frac{x_2}{h}\frac{\partial u_2}{\partial x_1} \to 0, \quad \frac{x_2}{h} u_1^h \rightharpoonup -\frac{1}{3} u_2', \quad \frac{x_2}{h}\frac{\partial u_1^h}{\partial x_1} \rightharpoonup -\frac{1}{3} u_2''. \tag{66}$$

Proof. Relation (65) implies $(66)_1$. It suffices to prove only $(66)_2$, since the last statement will then follow from Lemma 4.6 applied to the sequence $v^h = \frac{x_2}{h} u_1^h$.

For $\alpha \in C_0^\infty(I)$, using strong convergence $(66)_1$, we find that for $h \to 0$,

$$\int_0^1 \left(\frac{1}{2h}\int_{-h}^{h} \frac{x_2}{h} u_1^h\, dx_2\right)\alpha(x_1)\, dx_1 = \frac{1}{2h}\int_{I^h} \frac{x_2}{h} u_1^h \alpha\, dx =$$

$$= \frac{1}{2h}\int\limits_{I^h} \frac{x_2}{h} u_1(x_1)\alpha(x_1)\,dx - \frac{1}{2h}\int\limits_{I^h} \Big(\frac{x_2}{h}\Big)^2 \frac{\partial u_2}{\partial x_1}(x_1)\alpha(x_1)\,dx_1 + o(1) =$$
$$= -\frac{1}{3}\int\limits_{I} \frac{\partial u_2}{\partial x_1}\alpha\,dx_1 + o(1),$$

since $\int\limits_{-1}^{1} y_2\,dy_2 = 0, \quad \int\limits_{-1}^{1} y_2^2\,dy_2 = \frac{2}{3}$. The lemma is proved. □

3°. Now we restrict the Eq.(29) to the single rod I^h, let it be horizontal at first, and study the behavior of its solution. Thus, no conditions are imposed on the edges of I^h, and only integral identity in $\mathbb{R}^2$

$$\int A(G^h + e(u^h)) \cdot e(\varphi) d\mu^h = \int h f^h \cdot \varphi d\mu \tag{67}$$

holds for smooth vectors φ vanishing near the edges. Here

$$d\mu^h = \frac{d\,x}{2h}|_{I^h} \rightharpoonup \mu = d\,x_1|_I \quad \text{ and } f^h, G^h \text{ are bounded in } L^2(I^h, d\mu^h).$$

Since the Korn inequality is absent in this case, we require for the solution, in addition to (67), some properties from (63),(64).

Lemma 7.5. *Suppose that* $e(u^h)$ *is bounded and* $u_1^h \rightharpoonup u_1$, $G_{11}^h \rightharpoonup G_{11}$ *in* $L^2(I^h, d\mu^h)$. *Then*

$$A(G^h + e(u^h)) \rightharpoonup \rho(G_{11} + \frac{\partial u_1^h}{\partial x_1})\eta, \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. One can assume that $e(u^h) \rightharpoonup p$, where $p_{11} = \frac{\partial u_1}{\partial x_1}$, see Lemma 7.1, and $G^h \rightharpoonup G$. Passing to the limit in (67) yields

$$\int A(G + p) \cdot e(\varphi)\,d\mu = 0.$$

Take $\varphi = (2\alpha(x_1)x_2, \beta(x_1)x_2)$ with arbitrary $\alpha, \beta \in C_0^\infty(I)$. Since $e(\varphi)|_I = \begin{pmatrix} 0 & \alpha \\ \alpha & \beta \end{pmatrix}$, we derive

$$A(G+p) = a\eta \implies (G+p)\cdot\eta = aA^{-1}\eta\cdot\eta \implies G_{11} + \frac{\partial u_1}{\partial x_1} =$$
$$= aA^{-1}\eta\cdot\eta \implies a = \rho(G_{11} + \frac{\partial u_1}{\partial x_1}),$$

and the proof is complete. □

In order to examine the behavior of the transverse component u_2^h, we need special extensions (to the strip I^h) of transverse vector-fields defined on the segment I.

Lemma 7.6. *For a smooth vector-field* $g(x_1) = (0, \psi(x_1))$ *defined on the closed segment* I, *there is an extension* g^h *to the strip* I^h *such that*

(i) $\quad Ae(g^h) = -x_2\rho\psi''(x_1)\eta + O(h^2)$,

(ii) $\quad g_1^h = -x_2\psi'(x_1) + O(h^2)$,

(iii) $\quad g_2^h = \psi + O(h^2)$.

Proof. Set

$$g^h(x) = \left(-x_2\psi'(x_1) - \gamma\psi''(x_1)x_2^2, \psi(x_1) - \alpha\psi''(x_1)\frac{x_2^2}{2}\right), \tag{68}$$

where α, γ are constants to be determined. Calculations show that

$$-e(g^h) = \begin{pmatrix} \psi''(x_1)x_2 & \gamma\psi''(x_1)x_2 \\ \gamma\psi''(x_1)x_2 & \alpha\psi''(x_1)x_2 \end{pmatrix} + O(h^2) = \psi''(x_1)x_2 \begin{pmatrix} 1 & \gamma \\ \gamma & \alpha \end{pmatrix} + O(h^2).$$

Let us require that the matrix $A\begin{pmatrix} 1 & \gamma \\ \gamma & \alpha \end{pmatrix}$ be proportional to $\eta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We have

$$A\begin{pmatrix} 1 & \gamma \\ \gamma & \alpha \end{pmatrix} = c\eta \implies \begin{pmatrix} 1 & \gamma \\ \gamma & \alpha \end{pmatrix} = cA^{-1}\eta \implies c = (A^{-1}\eta\cdot\eta)^{-1} = \rho.$$

Thus, the constants α, γ are determined in a unique way and all the properties (i)-(iii) are ensured. □

Lemma 7.7. *Suppose that solution of Eq. (67) satisfies conditions (63), (64) and* $f_2^h \rightharpoonup f_2$, $-\frac{x_2}{h}G_{11}^h \rightharpoonup G_\nu$. *Then*

$$\int_I \rho(\frac{1}{3}u_2'' + G_\nu)\psi''d\,x_1 = \int_I f_2\psi d\,x_1 \quad \forall\psi \in C_0^\infty(I).$$

Proof. Find for a vector-field $g = (0, \psi(x_1))$ the extension g^h constructed in Lemma 7.6 and test Eq. (67) with $\varepsilon = g^h$. Transforming both sides of the identity

$$\int A(G^h + e(u^h))\cdot e(g^h)\,d\mu^h = \int (G^h + e(u^h))\cdot Ae(g^h)\,d\mu^h =$$

$$= -h\int \rho(G_{11}^h + \frac{\partial u_1^h}{\partial x_1})\frac{x_2}{h}\psi''d\mu^h + O(h^2),$$

$$\int hf^h\cdot g^h\,d\mu^h = h\int f_2^h\psi\,d\mu^h + O(h^2)$$

and dividing them by h, we obtain

$$-\int \rho(G_{11}^h + \frac{\partial u_1^h}{\partial x_1})\frac{x_2}{h}\psi''(x_1)\,d\mu^h = \int f_2^h\psi\,d\mu^h + O(h).$$

Passing to the limit as $h \to 0$, thanks to $(66)_3$ and conditions on f^h and G^h, completes the proof. □

4°. In conclusion we formulate some of the above results on a rod of an arbitrary direction and, to this end, use notation from the Sect.3.2. Consider vector-field u^h such that $e(u^h)$ is bounded and

$$u_\tau^h = (u^h\cdot\tau)\tau \rightharpoonup u, \quad hu^h \to v. \tag{69}$$

The general variant of relations $(66)_2$, $(66)_3$ is following:

$$\beta^h u^h \rightharpoonup -\frac{1}{3}(v\cdot\nu)', \quad \beta^h e(u^h)\cdot\eta \rightharpoonup -\frac{1}{3}(v\cdot\nu)''. \tag{70}$$

For a vertical rod with $\tau = (0,1)$, $\nu = (-1,0)$, we have $\beta^h(x) = -\frac{x_1}{h}$ and

$$\frac{x_1}{h}u_2^h \rightharpoonup -\frac{1}{3}\frac{\partial u_1}{\partial x_2}. \tag{71}$$

The results for the Eq.(67) on the rod I^h (see Lemmas 7.5, 7.7) are summarized in

Lemma 7.8. *Under assumptions (32), for the solution of Eq.(67) we have convergences (69) and in addition them the limit relation*

$$A(G^h + e_y(u^h)) \rightharpoonup \rho(G_\tau + (u\cdot\tau)')\eta.$$

The limit functions from (69) satisfy integral identities

$$\int \rho((u\cdot\tau)' + G_\tau)\varphi' d\,\mu = 0,$$

$$\int \rho(\frac{1}{3}(v\cdot\nu)'' + G_\nu)\varphi'' d\,\mu = \int f_\nu \cdot \nu\,\varphi\,d\mu$$

for $\varphi \in C^\infty(\mathbb{R}^2)$ vanishing near endpoints of I.

8. **Appendix D. Proof of structural theorems.** Structural theorems are valid in the case of arbitrary connected graph F, and in order to prove them, it is enough to consider only a bundle of segments $I_1, ..., I_m$, joining the node O. This structure of F will be assumed in the sequel throughout this section.

1°. Let us prove Theorem 3.2. Since the inclusion $u\cdot\tau_j \in H^1(I_j)$ follows from the corresponding results for a segment (see Appendix C), we have to verify the conjugation condition (25) or equivalent condition (26). To this end, take an arbitrary vector

$$b \in L(\tau^{(1)}, \tau^{(2)})^\perp, \quad \text{i.e.} \quad b = (b_1, \cdots, b_m), \quad \sum_{i=1}^m b_i\tau_i = 0. \tag{72}$$

Denote by Q_h the union of all "short" (i.e., of length $4h$) h-rods issueing from the node O, and let Γ_i $(i = 1, \ldots, m)$ be the outer edges of these rods (see Fig.2 corresponding to the case $m = 3$). Assume that the node O coincides with the origin.

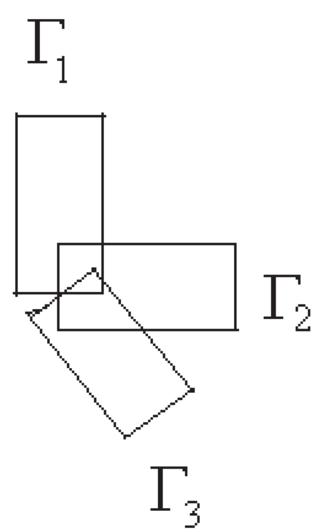

FIGURE 2.

Consider the Neumann problem

$$w \in H^1(Q_h)^2, \quad \operatorname{div} e(w) = 0 \text{ in } Q_h, \quad e(w)n|_{\partial Q_h} = g, \tag{73}$$

$$g = \begin{cases} b_i\tau_i & \text{on } \Gamma_i \ (i = 1, 2, \ldots, m), \\ 0 & \text{on the rest of } \partial Q_h, \end{cases}$$

where n is the unit outward normal to ∂Q_h. This problem has a solution, since the function g satisfies the condition

$$\int\limits_{\partial Q_h} g(x) \cdot r(x)\, d\sigma = 0, \tag{74}$$

where $r(x)$ is an arbitrary rigid displacement on the plane, $r(x) = c + t\omega$, $\omega(x) = (-x_2, x_1)$, $c \in \mathbb{R}^2$, $t \in \mathbb{R}^1$. Indeed,

$$\int\limits_{\partial Q_h} g{\cdot}r\, d\sigma = \sum_i \Bigl(\int\limits_{\Gamma_i} b_i\tau_i{\cdot}c\, d\sigma + tb_i \int\limits_{\Gamma_i} \tau_i{\cdot}\omega\, d\sigma \Bigr) = 2hc{\cdot}\sum b_i\tau_i + t\sum b_i \int\limits_{\Gamma_i} \nu_i{\cdot}x\, dx = 0,$$

where we have used condition (72) and the relation $\tau_i \cdot \omega = \nu_i \cdot x$.

The solution of problem (73) is determined to within a rigid displacement. Let us prove the following estimate for the solution w orthogonal to all rigid displacements on Q_h:

$$\int\limits_{Q_h} |e(w)|^2\, dx \le Ch^2. \tag{75}$$

From the integral identity for the solution of problem (73), we have

$$\int\limits_{Q_h} |e(w)|^2\, dx = \int\limits_{\partial Q_h} e(w)n \cdot w\, d\sigma = \sum_i \int\limits_{\Gamma_i} b_i\tau_i \cdot w\, d\sigma_i \le Ch^{1/2} \Bigl(\int\limits_{\partial Q_h} |w|^2\, d\sigma \Bigr)^{1/2}.$$

Now, in order to obtain the estimate (75), it suffices to use the inequality for the trace

$$\int\limits_{\partial Q_h} |w|^2\, d\sigma \le hC \int\limits_{Q_h} |\nabla w|^2\, dx$$

and also the Korn inequality

$$\int\limits_{Q_h} |\nabla w|^2\, dx \le C \int\limits_{Q_h} |e(w)|^2\, dx.$$

Due to the Gauss formula,

$$\frac{1}{2h} \int\limits_{Q_h} e(u^h) \cdot e(w)\, dx = \sum_i b_i \frac{1}{2h} \int\limits_{\Gamma_i} u^h \cdot \tau_i\, d\sigma.$$

The left-hand side of this relation tends to zero as $h \to 0$, since we have the estimate (75) and the sequence $e(u^h)$ is bounded. The right-hand side contains transverse averages converging to $u{\cdot}\tau_i|_O$. Thus, we find that the vector $(u{\cdot}\tau_1, u{\cdot}\tau_2, \ldots, u{\cdot}\tau_m)|_O$ is orthogonal to the vector b. Since $b \in L(\tau^{(1)}, \tau^{(2)})^\perp$ is arbitrary, it follows that (26) holds. Theorem 3.2 is proved.

The scalar version of this theorem, namely, Theorem 3.3 is simpler. Its proof can be obtained following the same method, with great simplifications, for instance, one may restrict oneself to two rods while proving coinsideness of values at the node attained by function along the different rods.

2°. Now let us prove Theorem 3.4. We have to verify only conditions at the node, since the other statements follow from the results for a single rod. By Th.3.3, our assumptions imply continuity and transversality of the vector v at the same time, which is possible only, if $v|_O = 0$ and clamping condition is checked. It remains to show conjugation condition (27). It suffices to consider two noncollinear links joined at the node O. For simplicity, let it be segments I_1, I_2 issuing from the origin in the directions of the coordinate axes. Therefore, $\tau = (1, 0)$, $\nu = (0, 1)$ and $\tau = (0, 1)$, $\nu = (-1, 0)$ on I_1 and I_2, respectively, and $u \cdot \nu|_{I_1} = u_2(x_1, 0)$, $\quad u \cdot \nu|_{I_2} = -u_1(0, x_2)$, and we have to verify the relation

$$\frac{\partial u_2}{\partial x_1}(x_1, 0)\Big|_0 + \frac{\partial u_1}{\partial x_2}(0, x_2)\Big|_0 = 0. \tag{76}$$

The domain Q_h (introduced in Sect. 1°) corresponding to this case is a "small angle" with the edges Γ_1, Γ_2 (see Fig.3). In Q_h, consider the Neumann problem (73) with

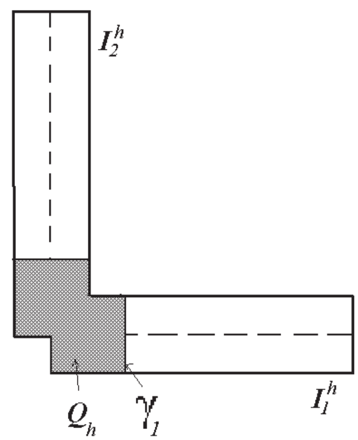

FIGURE 3.

$$g = \begin{cases} \frac{x_2}{h}\tau_1 & \text{on } \Gamma_1, \\ \frac{x_1}{h}\tau_2 & \text{on } \Gamma_2, \\ 0 & \text{on the rest of } \partial Q_h. \end{cases}$$

It is easy to check that the vector-field g satisfies condition (74) ensuring the solvability of problem (73). In the same way as in Sect. 1, we establish the estimate (75) for the solution w orthogonal to all rigid displacements on Q_h.

By the Gauss formula, we have

$$\frac{1}{2h}\int\limits_{Q_h} e(u^h) \cdot e(w)\, dx = \frac{1}{2h}\int\limits_{\Gamma_1} \frac{x_2}{h} u_1^h\, d\sigma + \frac{1}{2h}\int\limits_{\Gamma_2} \frac{x_1}{h} u_2^h\, d\sigma.$$

The left-hand side of this equality tends to zero, since we have the estimate (75) and the sequence $e(u^h)$ is bounded. Let us find the limit of each term on the right-hand side. From $(66)_2$, we get $z_1^h \equiv \frac{x_2}{h} u_1^h \rightharpoonup -\frac{1}{3}\frac{\partial u_2}{\partial x_1}$. Since the sequence $\frac{\partial z_1^h}{\partial x_1}$ is bounded, the convergence of the transverse averages $\bar{z}_1^h \to -\frac{1}{3}\frac{\partial u_2}{\partial x_1}$ is uniform on I_1

(see Lemma 7.1). Hence, we deduce that

$$\lim_{h\to 0}\frac{1}{2h}\int\limits_{\Gamma_1}\frac{x_2}{h}u_1^h\,d\sigma = -\frac{1}{3}\frac{\partial u_2}{\partial x_1}\Big|_O .$$

In a similar way, we obtain the following relation, see (70):

$$\lim_{h\to 0}\frac{1}{2h}\int\limits_{\Gamma_2}\frac{x_1}{h}u_2^h\,d\sigma = -\frac{1}{3}\frac{\partial u_1}{\partial x_2}\Big|_O ,$$

and therefore, (76) holds. The theorem is proved.

9. **Appendix E. Passing to the limit in the general periodic elasticity problem.** 1°. Throughout this Section suppose that F is a rigid graph. This condition ensures Korn inequalities (56), (57) that allow, under assumption (32), to derive for solution u^h of Eq. (29) boundedness of sequences u^h_τ, hu^h, $h\nabla u^h$, $e(u^h)$ in $L^2(\square, d\mu^h)^2$. One can extract (see Appendix B and Th.3.2, Lem. 7.5) the subsequence of solutions, denoted again as u^h, such that $hu^h \to v$, $\quad u^h_\tau \rightharpoonup u$. Let us identify limit functions u and v as solutions of the problems (35), (36).
2°. We begin with function v that is the element of $\mathcal{N}$, by Th. 3.4. Derivation of equation (3.12) requires some density and extension assertions for the space $\mathcal{N}$.

Denote by D the set of all vectors in $\mathcal{N}$ that are smooth outside the neighborhood of nodes and linear on the links near each node. It is easy to see that D is dense in $\mathcal{N}$, see Remark in Sect.3.2.

Lemma 9.1. *Any vector-field $\psi \in D$ admits an extension $\psi^h = \psi^h(x)$ to the grid F^h, such that*
(i) $e(\psi^h) = 0$ in a neighborhood of every node;
(ii) $Ae(\psi^h) = -h\rho(\psi\cdot\nu)''\beta^h\eta + O(h^2)$;
(iii) $\psi^h = \psi + O(h)$, $\quad \psi^h_\tau = -h\beta^h(\psi\cdot\nu)'\tau + O(h^2)$.

Proof. On every separate rod, we extend ψ with the help of Lemma 7.6 and its variant for the rod arbitrary directed, then check whether the extensions coincide on the intersection of the rods. It suffices to consider the case of two rods, one horizontal and another vertical, with the node being the origin. Starting with

$$\psi|_{I_1} = (0, a_1(x_1)), \quad 0\le x_1\le 1, \quad \psi|_{I_2} = (a_2(x_2), 0), \quad 0\le x_2\le 1,$$

where the functions a_1, a_2 are linear near zero, and the conjugation conditions ensure that

$$a_2(0) = a_1(0) = 0, \quad -a_2'(0) = a_1'(0) = t,$$

take $\psi^h|_{I_1^h}{=}(-x_2a_1'{-}\gamma_1a_1''x_2^2, a_1{-}\alpha_1a_1''\frac{1}{2}x_2^2), \psi^h|_{I_2^h}{=}(a_2{-}\alpha_2a_2''\frac{1}{2}x_1^2, -x_1a_2'{-}\gamma_2a_2''\frac{1}{2}x_1^2)$. We have near the node $\psi^h|_{I_1^h}{=}\psi^h|_{I_2^h}{=}(-x_2t, tx_1)$, i.e. ψ^h coinsides with rigid displacement (28).

The lemma is proved. □

Turn to the direct proof of (36). To this end, test Eq. (29) with the function $\psi^h(x)$ that is the extension to F^h of arbitrary $\psi(x){\in}D$ constructed in Lemma 9.1. This implies

$$h\int\limits_{F^h} f^h\cdot\psi^h\,dx = \int\limits_{F^h} A(e(u^h)+G^h)\cdot e(\psi^h)\,dx = \int\limits_{F^h}(e(u^h)+G^h)\cdot Ae(\psi^h)\,dx.$$

Let us transform both sides of this relation, using the properties of involved functions. Then,

$$-h \int_{F^h} (e(u^h) + G^h) \cdot \eta\beta^h(\psi \cdot \nu)'' \, dx = \; = h \int_{F^h} (f^h \cdot \nu)\nu \cdot \psi \, dx + O(h^3).$$

Dividing this expression by h^2 and passing to the limit as $h \to 0$ (use assumptions (32) on f^h, G^h and convergence $(70)_2$ on every h-rod) lead to the identity (36) with the test function $\psi \in D$. Recalling that D is dense in $\mathcal{N}$, we obtain the desired result. One part of the Th.3.6 is proved.
3°. In order to prove another part of Th. 3.6, test Eq. (29) with function $\psi \in C^\infty_{per}(\Box)^2$ and pass to the limit in it as $h \to 0$. By Th.3.2, Lemma 7.8 and Th.3.1, we derive that the limit function u belongs to $\mathcal{T}$ and satisfies integral identity (35) for test functions φ, which are the longitudinal components of the vectors from $C^\infty_{per}(\Box)^2$ and, thus, by density arguments for any element $\varphi \in \mathcal{T}$.

It remains to give

Proof of Theorem 3.1. Denote by $\mathcal{T}'$ the closure in $\mathcal{T}$ of longitudinal components of vector-fields in $C^\infty_{per}(\Box)^2$. Since the convergence in the $\mathcal{T}$-norm implies uniform convergence, the function $u \in \mathcal{T}'$ inherits the property (25) from the sequence $u_\delta \in C^\infty_{per}(\Box)^2$ converging to u. Therefore, $\mathcal{T}' \subset \mathcal{T}$. Suppose that

$$\exists u \in \mathcal{T} \setminus \mathcal{T}' \colon \; (u, \varphi) \equiv \int (u' \cdot \varphi' + u \cdot \varphi) \, d\mu = 0 \quad \forall \varphi \in \mathcal{T}'. \tag{77}$$

We can assume that $C = 0$ in condition (25) for the function u. Moreover, let the origin coincide with the node O. Take $\chi_h(x) = \chi(h^{-1}|x|)$, where $\chi(t) = 0$ for $t < \frac{1}{2}$, $\chi(t) = 1$ for $t > 1$. Then, $\chi_h u \in \mathcal{T}'$ and by the Cauchy inequality we have $|u(x)|^2 = O(h)$ for $|x| < h$. Hence, taking $\varphi = \chi_h u$ in (77), we find that

$$\int \chi_h(|u'|^2 + |u|^2) \, d\mu = -\int u' \cdot u\chi_h' \, d\mu \equiv J(h),$$

where

$$|J(h)|^2 \leq \int_{|x| \leq h} |u'|^2 \, d\mu \cdot Ch^{-2} \int_{|x| \leq h} |u|^2 \, d\mu = o(1) \text{ as } h \to 0.$$

Passing to the limit, we get $(u, u) = 0$, and therefore, $u \equiv 0$, i.e., $\mathcal{T} = \mathcal{T}'$. The theorem is proved.

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REFERENCES

[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., **23** (1992), 1482–1518.

[2] N. S. Bakhvalov and G. P. Panasenko, "Averaging Processes in Periodic Media," Mathematical problems in the mechanics of composite materials, Nauka, Moscow 1984; English transl., Kluwer, Dordrecht 1989

[3] J. Casado-Diaz and M. Luna-Laynez, *Homogenization of the anisotropic heterogeneous linearized elasticity system in thin reticulated structures*, Proceedings of the Royal Soc. of Edinbourgh, **134A** (2004), 1041–1083.

[4] P. G. Ciarlet, *Mathematical elasticity. II. Theory of plates*, Amsterdam: Elsevier, Studies in Math. and Applications, **27** (1997).

[5] S. A. Nazarov, "Asymptotic Analysis of Thin Plates and Rods," Vol. *I.* Decreasing Dimension and Integral Estimates, [[9]in Russian], Nauchn. Kniga, Novosibirsk (2002).

[6] S. A. Nazarov and A. S. Slutskii, *Arbitrary planar anisotropic beam systems*, Trudy Mat. Inst. Steklov, **236** (2002), 234–261; English transl. in Proc. Steklov Inst. Math., **66** (2002).

[7] S. A. Nazarov and A. S. Slutskii, *Asymptotic analysis of an arbitrary spatial system of thin rods*, Trudy St. Peterb. Mat. Ob-va, **10** (2004), 63–117.

[8] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal, **20** (1989), 608–623.

[9] G. P. Panasenko, *Asymptotic solutions of the elasticity theory system of equations for lattice and skeletal structures*, Math.Sb., **183** (1992), 89–112; English transl. in Russian Acad. Sci. Sbornic Math., **75** (1993), 85–110.

[10] G. P. Panasenko, "Multi-scale Modelling for Structures and Composites," Kluwer Academic Press, Dordrecht, 2004.

[11] S. E. Pastukhova, *Homogenization of elasticity problems on periodic box and rod frames of critical thickness*, Sovrem. Mat. Ee Prilozh., **12** (2004), 51–98; English transl. in Journal of Math. Sciences, 2004, 61–96

[12] S. E. Pastukhova, *Correctors in the homogenization of elasticity problems on thin structures*, Dokl. Ross. Akad. Nauk, **401** (2005), 21–25; English transl. in Dokl Math., (2005)

[13] V. V. Zhikov, *Homogenization of elasticity problems on singular structures*, Izv. Ross. Akad. Nauk Ser. Mat., **66** (2002), 81–148; English transl. in Izv. Mat., **66** (2002).

[14] V. V. Zhikov, *Two-scale convergence*, Trudy. Sem. im. I.G. Petrovskogo, **23** (2002), 149–187; English transl. in Journal of Math. Sciences, (2003).

[15] V. V. Zhikov and S. E. Pastukhova, *Korn inequalities on thin periodic structures*, Networks and Heterogeneous Media, **4** (2009), 153–175.

[16] V. V. Zhikov and S. E. Pastukhova, *Homogenization of elasticity problems on periodic networks of critical thickness*, Mat.Sb., **194** N5 (2003), 61–96; English transl. in Sb. Math., **194** (2003).

[17] V. V. Zhikov and S. E. Pastukhova, *Derivation of limit equations of the elasticity theory on thin nets*, Trudy. Sem. im. I.G. Petrovskogo, **25** (2006) 55–97; English transl. in Journal of Math. Sciences, (2006).

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