NETWORKS AND HETEROGENEOUS MEDIA
doi:10.3934/nhm.2009.4.537
©American Institute of Mathematical Sciences
Volume **4**, Number **3**, September **2009**
pp. **537–575**

ROBUSTNESS OF SQUARE NETWORKS

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(Communicated by Roberto Natalini)

Abstract. The topic of security often enters in many real world situations. In this paper we focus on security of networks on which it is based the delivery of services and goods (e.g. water and electric supply networks) the transfer of data (e.g. web and telecommunication networks), the movement of transport means (e.g. road networks), etc... We use a fluid dynamic framework, networks are described by nodes and lines and our analysis starts from an equilibrium status: the flows are constant in time and along the lines. When a failure occurs in the network a shunt changes the topology of the network and the flows adapt to it reaching a new equilibrium status. The question we consider is the following: is the new equilibrium satisfactory in terms of achieved quality standards? We essentially individuate, for regular square networks, the nodes whose breakage compromises the quality of the flows. It comes out that networks which allow circular flows are the most robust with respect to damages.

1. **Introduction.**

In this paper we deal with the security issue of facing a quite common emergency situation: either a human action or a natural event can damage the network where data or services flow from sources to destinations. Then the supplied flux can be strongly modified with severe negative effects. In particular we consider flows on networks (either water and electrical supply, or data or road traffic networks) and the problem of dealing with breakage at some of their nodes. For a service delivery company the analysis of equilibrium solutions of flows on networks when a failure occurs is a key factor in order to guarantee a sufficiently good standard of the service delivery, even in an emergency situation. In other words, for a healthy network, the company is able to ensure a certain outflow (delivered quantities to the final customers) through the supply of a certain inflow (inlet quantities from sources.) If a node is damaged, the same outflow may not be guaranteed. In this case the company delivers the best possible outflow in the new situation by inletting the necessary inflow.

Our analysis is based on a fluid dynamic model consisting, on each arc of the network, of a single conservation law:

$$\rho_t + f(\rho)_x = 0, \qquad \rho \in [0, \rho_{max}],$$

where ρ represents the density, ρ_{max} its maximal value, while the flux f is determined by a function which is usually assumed to be concave. Then, the dynamics at nodes is determined assuming conservation of mass and maximization of the

2000 *Mathematics Subject Classification.* Primary: 90B18; Secondary: 35L65.
Key words and phrases. Flows on Networks, Equilibria, Security.

through flux, see section 2 for details.

There are many alternative models for flows on networks. A complete account of the existing literature is beyond the scope of the paper, however some overview can be found, for instance, in [1, 17, 21] for data networks, in [7, 8, 13, 19, 22] for road traffic, in [2, 12, 16] for supply chains, in [15, 18, 23] for water supply and in [3, 9] for gas supply. The issue of equilibria on regular (not damaged) networks was tackled by the author in [20]. Therein the definition of equilibrium was given: it is a solution, on the whole network, which is constant in time. We assume that the flux f admits a unique maximum $\sigma \in [0, \rho_{max}]$ and vanishes at extreme points, i.e. $f(0) = f(\rho_{max}) = 0$. A consequence is that there exist shocks with sonic, i.e. zero, velocity. Thus an equilibrium may well exhibit an infinite number of shocks inside each line. Still we are interested in the flux and density values that equilibria takes at nodes. The former are, in fact, constant on each line and the latter are called the equilibria values. In this paper we use the techniques and outcomes of [20] and go further: we focus on square networks and obtain the following main results:

- We give an explicit description of the equilibrium solutions for data networks, before and after a node breakage.
- For data networks, before and after some node failure occurs, we give a map assigning a unique equilibrium for each set of inflows of the network.

Each network is represented by a collection of lines, modeled by real intervals, and nodes at which lines intersect. To determine the set of equilibria values, we consider the space of flux values as variables. Thus we have one unknown for each line, while to be in equilibrium the set of unknowns must satisfy a linear relation at each node (i.e. total incoming flux equals total outgoing flux.) Thus the dimension of the space of equilibria values is readily computed (see Proposition 1.) To each vector of equilibria values, there corresponds at least one equilibrium solution. However, we are more interested in those solutions, whose density is constant along each line. The fulfillment of this property implies that one more constraint should be satisfied at nodes. More precisely, one defines *bad* and *good* values for incoming and outgoing lines at a node depending if the density is lower or greater than the value σ of maximum flux (see Definition 2.2.) Then, only some combinations of *bad* and *good* values at a node are possible.

Most of the paper then focuses on the analysis of equilibrium solutions, with constant densities along lines, for square networks of the following type:

- Circular Manhattan, with oriented lines but flow in any direction;
- Oriented Manhattan with oriented lines and data flowing always up and right;
- Full Manhattan: with non-oriented flows (modeled by a couple of lines for each edge of the network.)

The analysis of equilibria is carried out in the following way. First, we determine the possible types of equilibria at each node, depending on the combination of *good* and *bad* values at the incident lines. Then, we deduce the types for networks consisting of N nodes and for the same networks when a node failure occurs. Finally we compute the equilibrium solution in both cases. In particular we study the relation between inflow and outflow and the effect of a node breakage on such relation. We only consider damages at the interior nodes since boundary nodes represent either sources or destinations of the network. Moreover, from mathematical point of view,

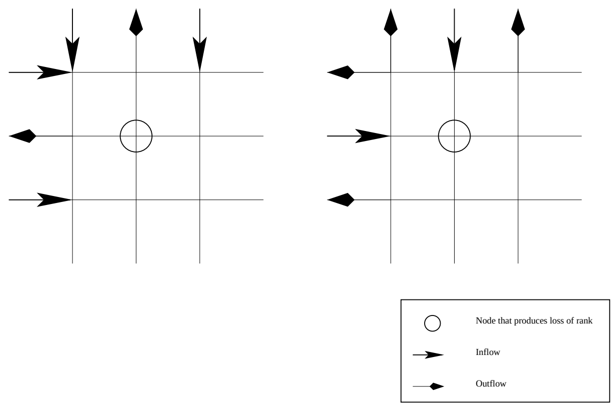

FIGURE 1. The nodes at corners of the Circular Network with incoming and outgoing flows as shown in the picture, and eventually rotated, are critical

a damage at boundary nodes would simply mean looking at the map $inflow \mapsto outflow$ restricted to a subspace either in the domain or in the codomain.

Definition 1.1. We say that a node is *critical* if its breakage entail a loss of rank of the mapping $inflow \mapsto outflow$. If a node is not critical we will say that it is stable. A network with no critical nodes will be called robust.

We then characterize the critical nodes for the three types of networks and prove that

Theorem 1.2. **C.M.N.:** *the critical nodes of the Circular Manhattan Network lie on the corners whose topology is described by Figure 1.*
O.M.N.: *the critical nodes of the Oriented Manhattan Network lie on the anti–diagonals of maximal length as shown in Figure 2.*
F.M.N.: *the Full Manhattan Network is robust.*

The paper is organized as follows. In section 2, we summarize the model we use (see [11]) and the basic definitions for equilibrium solutions (see [20].)
In section 3 we introduce some useful notation and report some results from the theory of matrix equations. In section 4 we illustrate the square networks we deal with and the main tools to investigate equilibria. Then sections 5, 6 and 7 are dedicated to the analysis of the equilibrium solutions in the Circular, Oriented and Full Manhattan networks, respectively.

2. **Modeling of flows on networks.** A network is formed by a finite collection of lines and nodes, each element (either data packet or car or fluid volume element) is seen as a particle on the network. We assume conservation of particles and get the following simple model consisting of a single conservation law:

$$\rho_t + f(\rho)_x = 0, \tag{1}$$

where ρ is the density, v is the velocity and $f(\rho) = v\rho$ is the flux.

We model a network by a finite set of intervals $I_i = [a_i, b_i] \subset \mathbb{R}, i = 1, ..., L, a_i < b_i$, on which we consider the equation (1). Hence the datum is given by a finite set of functions ρ_i defined on $[0, +\infty[\times I_i$.

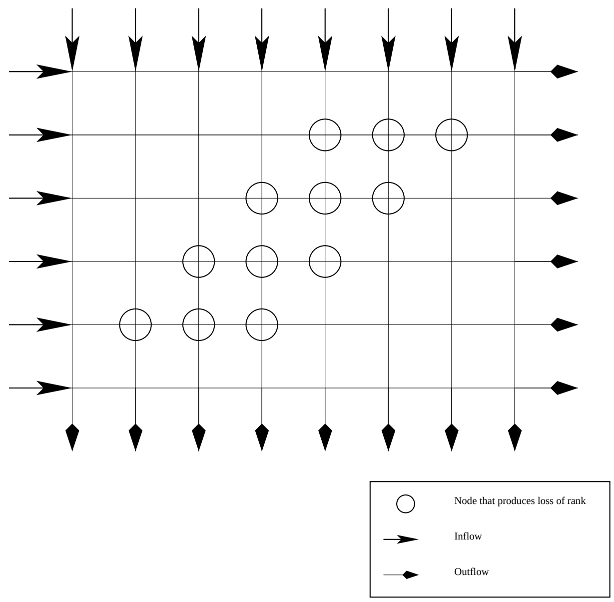

FIGURE 2. The nodes of the Oriented Manhattan Network on the anti–diagonal of maximal length are critical

We assume that the lines are connected by some nodes. Each node J is given by a finite number of incoming lines and a finite number of outgoing lines, thus we identify J with $((i_1,...,i_m),(j_1,...j_n))$ where the first m-tuple indicates the set of incoming lines and the second n-tuple indicates the set of outgoing lines. Each line can be incoming line at most for one node and outgoing at most for one node. Hence the complete model is given by a couple $(\mathcal{I},\mathcal{J})$, where $\mathcal{I}=\{I_i : i=1,...,L\}$ is the collection of lines and $\mathcal{J}$ is the collection of nodes. We set N to be the cardinality of $\mathcal{J}$.

In order to consider complex networks, one needs a way of solving dynamics at nodes in which many lines intersect. We assume the following:

RA The particles travel through the node so as to maximize the flux.

A key role is played by Cauchy problems with initial data constant on each line called Riemann problems at the node. In order to determine unique solutions to Riemann problems, some additional parameters are introduced, called respectively priority parameters and traffic distribution parameters. The theory for this model is developed in [11].

On each line I_i the evolution is given by equation (1) and we assume that the flux f is a strictly concave function (with $f(0)=f(\rho_{max})=0$), thus there exists a unique $\sigma\in[0,\rho_{max}]$ such that $f'(\sigma)=0$ and is the maximum of f over $[0,\rho_{max}]$. For notational simplicity, we assume, without loss of generality, that $\rho_{max}=1$.

Definition 2.1. We let $\tau:[0,1]\to[0,1]$ be the map such that $f(\rho)=f(\tau(\rho))$ and $\tau(\rho)\neq\rho$ if $\rho\neq\sigma$. Thus τ sends ρ to the other density value with the same flux (and $\tau(\sigma)=\sigma$.)

For a simple network formed of a single node with m incoming and n outgoing lines, once the quantities flowing from initial to final nodes are assigned, the final equilibrium as function of the traffic distribution (and priority) parameters can be computed as follows.

We have only m priority parameters $p \in\,]0,1[$ and n traffic distribution parameters $\alpha \in\,]0,1[$. We denote with $\rho_\varphi(t,x)$, $\varphi = 1,\dots,m$, and $\rho_\psi(t,x)$, $\psi = m+1,\dots,m+n$, the traffic densities, respectively, on the incoming lines and on the outgoing ones and by $(\rho_{\varphi,0}, \rho_{\psi,0})$ the initial data. Since the speed of waves must be negative on incoming lines and positive on outgoing ones, we want to determine a unique $(m+n)$-tuple $(\hat\rho_1, ..., \hat\rho_{m+n}) \in [0,1]^{m+n}$ such that

$$\hat\rho_\varphi \in \begin{cases} \{\rho_{\varphi,0}\} \cup\,]\tau(\rho_{\varphi,0}),1]\,, & \text{if } 0 \le \rho_{\varphi,0} < \sigma, \\ [\sigma,1]\,, & \text{if } \sigma \le \rho_{\varphi,0} \le 1, \end{cases} \tag{2}$$

$\varphi = 1,...,m$, and

$$\hat\rho_\psi \in \begin{cases} [0,\sigma], & \text{if } 0 \le \rho_{\psi,0} \le \sigma, \\ \{\rho_{\psi,0}\} \cup [0,\tau(\rho_{\psi,0})[\,, & \text{if } \sigma < \rho_{\psi,0} \le 1, \end{cases} \tag{3}$$

$\psi = m+1,...,m+n$, and on each incoming line I_φ, $\varphi = 1,...,m$, the solution consists of the single wave $(\rho_{\varphi,0}, \hat\rho_\varphi)$, while on each outgoing line I_ψ, $\psi = m+1,...,m+n$, the solution consists of the single wave $(\hat\rho_\psi, \rho_{\psi,0})$.

Define $\gamma_\varphi^{\max}$ and $\gamma_\psi^{\max}$ as follows:

$$\gamma_\varphi^{\max} = \begin{cases} f(\rho_{\varphi,0}), & \text{if } \rho_{\varphi,0} \in [0,\sigma[, \\ f(\sigma), & \text{if } \rho_{\varphi,0} \in [\sigma,1]\,, \end{cases} \quad \varphi = 1,\dots,m, \tag{4}$$

and

$$\gamma_\psi^{\max} = \begin{cases} f(\sigma), & \text{if } \rho_{\psi,0} \in [0,\sigma], \\ f(\rho_{\psi,0}), & \text{if } \rho_{\psi,0} \in\,]\sigma,1]\,, \end{cases} \quad \psi = m+1,\dots,m+n. \tag{5}$$

The quantities $\gamma_\varphi^{\max}$ and $\gamma_\psi^{\max}$ represent the maximum flux that can be obtained by a single wave solution on each line. In order to maximize the number of particles through the node over incoming and outgoing lines we define

$$\Gamma = \min\{\Gamma_{in}, \Gamma_{out}\}\,,$$

where $\Gamma_{in} = \sum_{\varphi=1}^m \gamma_\varphi^{\max}$ and $\Gamma_{out} = \sum_{\psi=m+1}^{m+n} \gamma_\psi^{\max}$. One can easily see that, to solve the Riemann problem, it is enough to determine the fluxes $\hat\gamma_\varphi = f(\hat\rho_\varphi), \varphi = 1,\dots,m$, and $\hat\gamma_\psi = f(\hat\rho_\psi), \psi = m+1,\dots,m+n$. Let us determine $\hat\gamma_\varphi, \varphi = 1,\dots,m$. We have to distinguish two cases:

I: $\Gamma_{in} = \Gamma$,
II: $\Gamma_{in} > \Gamma$.

In the first case we set $\hat\gamma_\varphi = \gamma_\varphi^{\max}, \varphi = 1,\dots,m$. Let us analyze the second case in which we use the priority parameters $p_1,\dots,p_m$ where $0 < p_\varphi < 1$ and $\sum_{\varphi=1}^m p_\varphi = 1$. Not all particles can enter the node, so let C be the amount of particles that can go through. Then $p_\varphi C$ particles come from the φ–st incoming line. Consider the space $(\gamma_1,\dots,\gamma_m)$ and denote by P the point with coordinates $\gamma_\varphi = p_\varphi\Gamma$. Now the final fluxes should belong to the region:

$$\Omega = \left\{(\gamma_1,\dots,\gamma_m) : 0 \le \gamma_\varphi \le \gamma_\varphi^{\max}, \varphi = 1,\dots,m\right\}.$$

We distinguish two cases:

a) P belongs to Ω,
b) P is outside Ω.

In the first case we set $(\hat{\gamma}_1, \ldots, \hat{\gamma}_m) = P$, while in the second case we set $(\hat{\gamma}_1, \ldots, \hat{\gamma}_m) = Q$, with $Q = proj(P)$ where $proj$ is some projection on Ω. From the choice of this projection the analysis and the choice of the parameters $p_1, \ldots, p_m$ can be very different. The most natural projection to take is the projection on a convex set (see [21].)

Let us now determine $\hat{\gamma}_\psi, \psi = m+1, \ldots, m+n$. As for the incoming lines we have to distinguish two cases :

I: $\Gamma_{out} = \Gamma$,
II: $\Gamma_{out} > \Gamma$.

In the first case $\hat{\gamma}_\psi = \gamma_\psi^{\max}, \psi = m+1, \ldots, m+n$. Let us determine $\hat{\gamma}_\psi$ in the second case in which we use the traffic distribution parameters $\alpha_{m+1}, \ldots, \alpha_{m+n}$ where $\alpha_\psi \in]0,1[$ and $\sum_{\psi=m+1}^{n} \alpha_\psi = 1$. Since not all particles can go on the outgoing lines, we let C be the amount that goes through. Then $\alpha_\psi C$ particles go on the outgoing line I_ψ. Consider the space $(\gamma_{m+1}, \ldots, \gamma_{m+n})$ and denote by P the point with coordinates: $\gamma_\psi = \alpha_\psi \Gamma$.

Now the final fluxes should belong to the region:

$$\Omega = \left\{ (\gamma_{m+1}, \ldots, \gamma_{m+n}) : 0 \le \gamma_\psi \le \gamma_\psi^{\max}, \psi = m+1, \ldots, m+n \right\}.$$

We distinguish two cases:

a) P belongs to Ω
b) P is outside Ω.

In the first case we set $(\hat{\gamma}_{m+1}, \ldots, \hat{\gamma}_{m+n}) = P$, while in the second case we set $(\hat{\gamma}_{m+1}, \ldots, \hat{\gamma}_{m+n}) = Q$, where $Q = proj(P)$.

The solution to the Riemann Problem $((\hat{\rho}_1, \ldots, \hat{\rho}_m), (\hat{\rho}_{m+1}, \ldots, \hat{\rho}_{m+n}))$ is computed from the equilibria fluxes $((\hat{\gamma}_1, \ldots, \hat{\gamma}_m), (\hat{\gamma}_{m+1}, \ldots, \hat{\gamma}_{m+n}))$ by taking the unique solution of equations $\hat{\gamma}_\varphi = f(\hat{\rho}_\varphi),\ \varphi = 1, \ldots, m$ and $\hat{\gamma}_\psi = f(\hat{\rho}_\psi),\ \psi = m+1, \ldots, m+n$ such that conditions (2) and (3) are satisfied.

Definition 2.2. A component of the solution at one node, $\hat{\rho}_\varphi,\ \varphi = 1, \ldots, m$, is

bad: if $\hat{\rho}_\varphi \in [0, \sigma[$;
good: if $\hat{\rho}_\varphi \in [\sigma, 1]$;

and a component of the solution, $\hat{\rho}_\psi,\ \psi = m+1, \ldots, m+n$ is

bad: if $\hat{\rho}_\psi \in]\sigma, 1]$;
good: if $\hat{\rho}_\psi \in [0, \sigma]$.

Remark 1. We notice that if there exists one index $\bar{\varphi}$ for which $\hat{\rho}_{\bar{\varphi}}$ is *good* then, for all $\psi = m+1, \ldots, m+n$, it must be $\hat{\rho}_\psi$ *bad*. Indeed $\hat{\rho}_{\bar{\varphi}}$ *good* means that $\Gamma_{in} > \Gamma = \Gamma_{out}$ and $\hat{\gamma}_{\bar{\varphi}} < \gamma_{\bar{\varphi}}^{max}$. Therefore, for all ψ, $\hat{\gamma}_\psi = \gamma_\psi^{max}$ and $\hat{\rho}_\psi$ is *bad*.
The converse also holds: if an index $\bar{\psi}$ exists such that $\hat{\rho}_{\bar{\psi}}$ is *good* then, for all $\varphi = 1, \ldots, m$, it must be that $\hat{\rho}_\varphi$ *bad*.

We are now interested in solutions over the whole network $(\mathcal{I}, \mathcal{J})$, not only on solutions at one single node. More precisely we are interested in equilibrium solutions.

Definition 2.3. An equilibrium is a solution $\rho(t, x) = (\rho_1, \ldots, \rho_L)$ (recall that L is the cardinality of $\mathcal{I}$), which is constant in time. We also assume that $\rho(t, \cdot)$ is BV, thus we can define, for every $i = 1, \ldots, L$, the values $\rho_i^- = \lim_{x \to a_i} \rho(t, x)$ and $\rho_i^+ = \lim_{x \to b_i} \rho(t, x)$.

Since ρ is a solution then

$$\sum_{\varphi=1}^{m} f(\rho_{j_\varphi}) = \sum_{\psi=m+1}^{m+n} f(\rho_{j_\psi}), \tag{6}$$

is satisfied at each node $J_j \in \mathcal{J}$, $j = 1, \ldots, N$. In (6) we have denoted by $\rho_{j_1}, \ldots, \rho_{j_m}, \rho_{j_{m+1}}, \ldots, \rho_{j_{m+n}}$ the densities along the m incoming lines $I_{j_1}, \ldots, I_{j_m}$ and the n outgoing lines $I_{j_{m+1}}, \ldots, I_{j_{m+n}}$ at node J_j.
We distinguish two cases

- **i:** there exists $i = 1, \ldots, L$, such that $\rho_i^- \neq \rho_i^+$. In this case, $\rho_i^+ = \tau(\rho_i^-)$ and the fluxes $\gamma_i = f(\rho_i^\pm)$, are anyhow constant in time and along the whole line I_i;
- **ii:** for all $i = 1, \ldots, L$, $\rho_i^+ = \rho_i^-$ and we call this value ρ_i.

Definition 2.4. Let $\rho = (\rho_1, \ldots, \rho_L)$ be an equilibrium for the network $(\mathcal{I}, \mathcal{J})$, satisfying **ii**. We say that $\rho_1, \ldots, \rho_L$ are the *values of the equilibrium*. Moreover, if ρ_i is of type τ_i, with $\tau_i \in \{bad, good\}$, then we say that $T = (\tau_1, \ldots, \tau_L)$ is the *equilibrium type*.

In section 4 we focus on square networks, describe the possible equilibria types and give a brief description of the system that one has to solve to find possible equilibria values over the whole network, while, in sections 5, 6 and 7 we explicit the equilibria values in the specific cases of the Circular, Oriented and Full Manhattan networks. The equilibria values are found by solving a matrix equation. In next section 3 we describe the matrix notation that will be used and some basic results on the solution of Sylvester (matrix) equations.

3. Notations and standard results.

We denote by $I(n)$ the identity matrix of order n, by $L(n)$ the sub–diagonal matrix of order n:

$$L(n) = \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix},$$

by $I(n)^{odd}$ the sparse matrix with

$$\begin{cases} (I(n)^{odd})_{ii} = 1 & \text{if } i \text{ is odd} \\ (I(n)^{odd})_{ii} = 0 & \text{if } i \text{ is even} \\ (I(n)^{odd})_{ij} = 0 & \text{for } \ i \neq j, \end{cases}$$

$I(n)^{even} = I(n) - I(n)^{odd}$ and $A(n) = L(n) - I(n)$. We also denote by $J(n)$ the antidiagonal matrix

$$J(n) = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & & & \\ & 1 & & \\ 1 & & & 0 \end{bmatrix}$$

and by 0 any sparse matrix.

For a generic matrix M we denote by M^T its transpose, and by $colspace(M)$, (resp. $rowspace(M)$) the matrix whose columns (resp. rows) are a base for the vector space generated by the columns (resp. rows) of M.

3.1. **The sign function of matrices.** The sign function of a $n \times n$ matrix M may be defined as

$$sign(M) = \frac{1}{\pi i} \int_{\gamma} (zI(n) - M)^{-1} dz - I(n),$$

where γ is a simple closed curve in the complex plane enclosing all eigenvalues of M with positive real part. A simple method for computing $sign(M)$ is to take the canonical Jordan decomposition $M = RJR^{-1}$, denote by D the diagonal part of J, $D = diag(d_1, \ldots, d_n)$, and by $S = diag(s_1, \ldots, s_n)$ the diagonal matrix whose entries s_i are the signs of the real parts of the d_i's, $\Re(d_i)$, i.e.

$$\begin{cases} s_i = 1 & \text{if } \Re(d_i) > 0 \\ s_i = -1 & \text{otherwise.} \end{cases}$$

Then $sign(M) = RSR^{-1}$. Clearly, for a stable matrix M, since all the eigenvalues have negative real part, we get $S = -I(n)$ and $sign(M) = -RI(n)R^{-1} = -I(n)$.

Another method for computing $sign(M)$ is the Newton iteration: if M has no pure imaginary eigenvalues then $sign(M) = \lim_{k\to\infty} M_k$ where

$$\begin{cases} M_0 = M \\ M_{k+1} = \frac{1}{2}(M_k + M_k^{-1}). \end{cases}$$

In the following for a sequence M_k of matrices we denote by M_∞ its limit, i.e. $M_\infty = \lim_{k\to\infty} M_k$.

3.2. **Products of matrices.** We denote by $\otimes$, the Kronecker product of two matrices. If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix:

$$\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

The Kronecker product is bilinear and associative. Moreover the transposition operation is distributive over the Kronecker product, i.e. $(A \otimes B)^T = A^T \otimes B^T$.
The Hadamard product, denoted by $\circ$, is the componentwise multiplication of two matrices of equal size. If A and B are two $m \times n$ matrices then $A \circ B$ is the $m \times n$ matrix with $(A \circ B)_{ij} = A_{ij}B_{ij}$.
The vectorization of a matrix A is formed by stacking the columns of A into a single column vector denoted by $vec(A)$.

3.3. **Matrix equation.** The matrix equation $AXB = C$ can be conveniently represented by means of the Kronecker product:

$$(B^T \otimes A)vec(X) = vec(AXB) = vec(C).$$

Then one can solve a matrix equation by transforming it into a system of linear equations.

3.4. **Sylvester equations.** A particular case of matrix equations is the Sylvester equation which is a matrix equation of the type:

$$AX + XB + C = 0, \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C, X \in \mathbb{R}^{n\times m}$. X is the sought-after solution. The square $(n+m)\times(n+m)$ matrix

$$H = \begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}$$

is put in diagonal form by the similarity transformation induced by

$$Q = \begin{bmatrix} I(n) & X \\ 0 & -I(m) \end{bmatrix}.$$

That is

$$Q^{-1}HQ = H = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix}. \tag{8}$$

Using the sign function of H and (8) we derive the following expression for the solution X of equation (7):

$$\frac{1}{2}(sign(H) + I(n+m)) = \begin{bmatrix} 0 & X \\ 0 & I(m) \end{bmatrix}. \tag{9}$$

Computing the Newton iteration of H we get

$$\begin{cases} H_0 = H \\ H_k = \begin{bmatrix} A_k & C_k \\ 0 & -B_k \end{bmatrix}, \end{cases}$$

where

$$\begin{array}{ll} A_0 = A, & A_{k+1} = \frac{1}{2}(A_k + A_k^{-1}), \\ B_0 = B, & B_{k+1} = \frac{1}{2}(B_k + B_k^{-1}), \\ C_0 = C, & C_{k+1} = \frac{1}{2}(C_k + A_k^{-1}C_k B_k^{-1}). \end{array}$$

Moreover, if A and B are stable matrices, we get

$$sign(H) = \begin{bmatrix} -I(n) & C_\infty \\ 0 & I(m) \end{bmatrix}$$

and, finally, by equation (9), the sought-after solution is:

$$X = \frac{1}{2}C_\infty.$$

4. **Manhattan type networks.** Recall that a network is actually an oriented graph (see [20].) In this paper we will treat the cases of square networks for which an embedding on the plane exists such that its image gives rise to a square tiling of a limited region of the plane.

In particular we will consider square networks with orientations as in Figure 3 which we call respectively (from top to bottom) *Circular Manhattan*, *Oriented Manhattan*, and *Full Manhattan*.

More precisely we fix two integers s and t and consider a square network, called $s \times t$ network, comprised of st nodes arranged on s lines and t columns. Due to the regularity of the networks we are considering, the nodes of the whole network can be seen as if they were the nodes of a square tiling and number them in a matricial way, $J_{11}, \ldots, J_{1t}, \ldots, J_{s1}, \ldots, J_{st}$. Moreover for the Oriented and the Circular Manhattan Networks, we have $m = 2$ incoming and $n = 2$ outgoing lines for each node, while for the Full Manhattan Network, we have $m = 4$ incoming and $n = 4$ outgoing lines for each node. Therefore the cardinality of $\mathcal{J}$ is $N = st$ and the cardinality of $\mathcal{I}$ is $L = (m+n)st$.

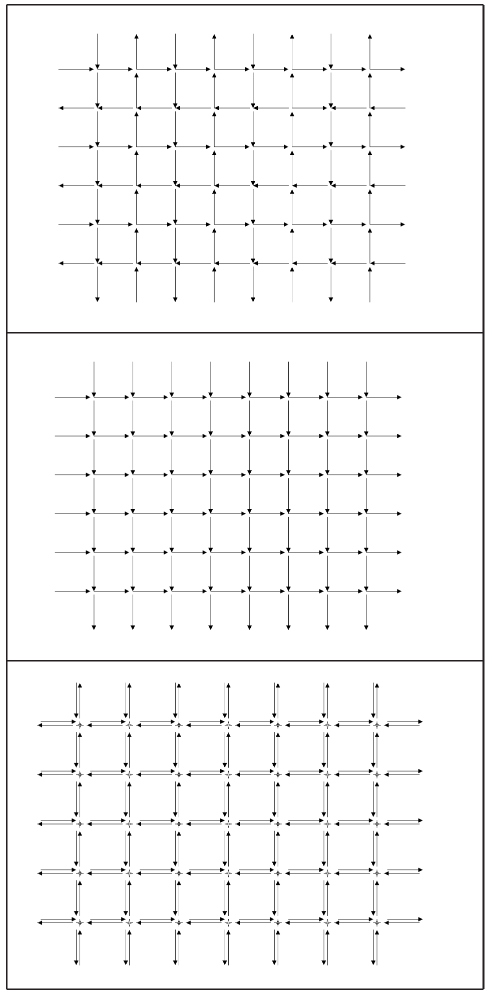

FIGURE 3. From top to bottom. Figure 1.1. The network with square tiling and alternated directions so that circular paths are possible. Figure 1.2. The network with square tiling and directions such that the overall flow goes from north to south and from west to east. Figure 1.3. The network with square tiling. The flow among two adjacent nodes is possible in both directions.

In [20] we got the following.

Proposition 1. *Consider networks* $(\mathcal{I}, \mathcal{J})$ *with* $N = st$ *the cardinality of* $\mathcal{J}$ *and* $L = (m+n)st$ *the cardinality of* $\mathcal{I}$*. The set of equilibrium values is a* $L - N = (m+n-1)st$ *dimensional subspace of* $\mathbb{R}^{(m+n)st}$.

Proposition 1 tells the following. At the equilibrium, on each line $I \in \mathcal{I}$, there is a flow among two adjacent nodes which is constant in time and along the line I (L variables.) But these L variables are subject to the N constraints (equation (6) written for each node J) which say that the total flux incoming from incoming lines at each node must be equal to the total flux departing through the outgoing lines from the same node.

This result is based only on the topological property of the network. We next see that the equilibrium type imposes more constraints on the set of equilibrium values thus giving a tighter estimate of its dimension.

To this purpose we consider an equilibrium at one node of a square network, and describe componentwise its type *good* or *bad*. The equilibrium at one node J is a $(n+m)$–vector. We then have that the possible equilibrium types at one node are one among the followings (we use the short notations b for *bad* and g for *good*):

I**:** $((b,b),(b,b))$,
$II0$**:** $((b,b),(g,g))$, $II1.1$: $((b,b),(b,g))$, $II1.2$: $((b,b),(g,b))$,
$III0$**:** $((g,g),(b,b))$, $III1.1$: $((b,g),(b,b))$, $III1.2$: $((g,b),(b,b))$,

for the oriented and circular cases and

I**:** $((b,b,b,b),(b,b,b,b))$
$II0$**:** $((b,b,b,b),(g,g,g,g))$,
$II1.1$**:** $((b,b,b,b),(b,g,g,g))$, $II1.2$: $((b,b,b,b),(g,b,g,g))$,
$II1.3$: $((b,b,b,b),(g,g,b,g))$, $II1.4$: $((b,b,b,b),(g,g,g,b))$,
$II2.1.2$**:** $((b,b,b,b),(b,b,g,g))$, $II2.1.3$: $((b,b,b,b),(b,g,b,g))$,
$II2.1.4$: $((b,b,b,b),(b,g,g,b))$, $II2.2.3$: $((b,b,b,b),(g,b,b,g))$,
$II2.2.4$: $((b,b,b,b),(g,b,g,b))$, $II2.3.4$: $((b,b,b,b),(g,g,b,b))$,
$II3.1.2.3$**:** $((b,b,b,b),(b,b,b,g))$, $II3.1.2.4$: $((b,b,b,b),(b,b,g,b))$,
$II3.1.3.4$: $((b,b,b,b),(b,g,b,b))$, $II3.2.3.4$: $((b,b,b,b),(g,b,b,b))$,
$III0$**:** $((g,g,g,g),(b,b,b,b))$
$III1.1$**:** $((b,g,g,g),(b,b,b,b))$, $III1.2$: $((g,b,g,g),(b,b,b,b))$,
$III1.3$: $((g,g,b,g),(b,b,b,b))$, $III1.4$: $((g,g,g,b),(b,b,b,b))$,
$III2.1.2$**:** $((b,b,g,g),(b,b,b,b))$, $III2.1.3$: $((b,g,b,g),(b,b,b,b))$,
$III2.1.4$: $((b,g,g,b),(b,b,b,b))$, $III2.2.3$: $((g,b,b,g),(b,b,b,b))$,
$III2.2.4$: $((g,b,g,b),(b,b,b,b))$, $III2.3.4$: $((g,g,b,b),(b,b,b,b))$,
$III3.1.2.3$**:** $((b,b,b,g),(b,b,b,b))$, $III3.1.2.4$: $((b,b,g,b),(b,b,b,b))$,
$III3.1.3.4$: $((b,g,b,b),(b,b,b,b))$, $III3.2.3.4$: $((g,b,b,b),(b,b,b,b))$,

for the full case.

Definition 4.1. We denote by $\mathcal{M} = \{I, II0, II1.h, \dots II(n-1).h_1.\cdots.h_{n-1}, III0, III1.h, \dots III(n-1).h_1.\cdots.h_{n-1}\}$, where either $n=2$ or $n=4$, the set of all possible equilibria types at one node.

Definition 4.2. We denote by

$$\widehat{\mathcal{N}} = \{T_{ij} \in \mathcal{M},\ i=1,\dots,s,\ j=1,\dots,t\} = \mathcal{M}^{st},$$

the set of all possible equilibria types over the whole network.

Clearly the cardinality of $\widehat{\mathcal{N}}$ is the cardinality of $\mathcal{M}$ to the power st. However, as it will be more clear in a while, not all the elements of $\widehat{\mathcal{N}}$ may arise.
Indeed, by Definition 2.3, two kinds of equilibria over the whole network may be considered. In case **ii** the following compatibility rule must be satisfied:

H: if a line I_i is incoming for some node J_1 and outgoing for some other node J_2 then the following holds. Whenever $\hat{\rho}_i$ is of type *bad* for J_1 then it must be of type *good* for J_2 and viceversa.

Rule **H** gives rise to a compatibility relation among equilibria at adjacent nodes. Such compatibility relation in turns determines the subset $\mathcal{N} \subset \widehat{\mathcal{N}}$ of admissible equilibrium states for the whole network.

This fact does not hold for case **i**. Indeed a shock wave along the line I_i may transform the density $\hat{\rho}_i$ into $\tau(\hat{\rho}_i)$ while keeping constant the flux $\hat{\gamma}_i$. Therefore $\hat{\rho}_i$ of type *good* or *bad* as incoming for J_1 does not influence the type of $\hat{\rho}_i$ as outgoing density for J_2. No compatibility relation among adjacent nodes can be deduced hence the set of possible equilibrium states for the whole network is $\mathcal{N} = \widehat{\mathcal{N}}$.

We have shown the following:

Proposition 2. *In case we consider equilibria* **i** *then* $\mathcal{N} = \widehat{\mathcal{N}}$. *If otherwise we consider equilibria* **ii.** *then* $\mathcal{N} \underset{\neq}{\subset} \widehat{\mathcal{N}}$.

In sections 5, 6 and 7 we will consider the equilibria **ii.** and give a characterization of $\mathcal{N}$.

Now we want to see what are the additional constraints imposed by the equilibrium type at one node.

Proposition 3. *Let* $\alpha_1, \ldots, \alpha_n$ *the real positive numbers with* $\sum_{i=1}^n \alpha_i = 1$ *and* $p_1, \ldots, p_m$ *the real positive numbers with* $\sum_{i=1}^n p_i = 1$, *associated to the Riemann solver at the node* J.
Assume first that node J *is of type* $II0$ *then*

$$\begin{array}{l} \gamma_\phi = \gamma_\phi^{max}, \ \phi = 1, \ldots, n, \\ (\gamma_{n+1}, \ldots, \gamma_{2n}) = P \end{array}$$

where $P = (\alpha_1 \Gamma_{in}, \ldots, \alpha_n \Gamma_{in})$.
If the node J *is of type* $II\nu.h_1.h_2.\ldots.h_\nu$ *then*

$$\begin{array}{l} \gamma_\phi = \gamma_\phi^{max}, \ \phi = 1, \ldots, n, \\ \gamma_\psi = \gamma_\psi^{max}, \ \psi = h_1, \ldots, h_\nu \\ (\gamma_{n+1}, \ldots, \gamma_{2n}) = proj(P) \end{array}$$

where $P = (\alpha_1 \Gamma_{in}, \ldots, \alpha_n \Gamma_{in})$.
If the node J *is of type* $III0$ *then*

$$\begin{array}{l} \gamma_\psi = \gamma_\psi^{max}, \ \phi = n+1, \ldots, 2n, \\ (\gamma_1, \ldots, \gamma_n) = P \end{array}$$

where $P = (p_1 \Gamma_{out}, \ldots, p_n \Gamma_{out})$.
If finally the node J *is of type* $III\nu.h_1.h_2.\ldots.h_\nu$ *then*

$$\begin{array}{l} \gamma_\psi = \gamma_\psi^{max}, \ \psi = n+1, \ldots, 2n, \\ \gamma_\phi = \gamma_\phi^{max}, \ \phi = h_1, \ldots, h_\nu, \\ (\gamma_1, \ldots, \gamma_n) = proj(P) \end{array}$$

where $P = (p_1 \Gamma_{out}, \ldots, p_n \Gamma_{out})$.

Notice that if a node is of type I there is no constraint on γ_ϕ and γ_ψ. For all other cases, with $n = 2$, we have

$$\begin{cases} \gamma_\phi = \gamma_\phi^{max}, \phi = 1, 2, \\ \gamma_3 = \gamma_3^{max}, \\ \gamma_4 = \Gamma - \gamma_3, \end{cases} \qquad \text{for the node } II1.1;$$

$$\begin{cases} \gamma_\phi = \gamma_\phi^{max}, \phi = 1, 2, \\ \gamma_4 = \gamma_4^{max}, \\ \gamma_3 = \Gamma - \gamma_4, \end{cases} \qquad \text{for the node } II1.2;$$

$$\begin{cases} \gamma_\phi = \gamma_\phi^{max}, \phi = 1, 2, \\ \gamma_3 = \gamma_3^{max}, \\ \gamma_4 = \alpha\gamma_3, \end{cases} \qquad \text{for the node } II0;$$

$$\begin{cases} \gamma_\psi = \gamma_\psi^{max}, \phi = 3, 4, \\ \gamma_1 = \gamma_1^{max}, \\ \gamma_2 = \Gamma - \gamma_1, \end{cases} \qquad \text{for the node } III1.1;$$

$$\begin{cases} \gamma_\psi = \gamma_\psi^{max}, \psi = 3, 4, \\ \gamma_2 = \gamma_2^{max}, \\ \gamma_1 = \Gamma - \gamma_2, \end{cases} \qquad \text{for the node } III1.2;$$

$$\begin{cases} \gamma_\psi = \gamma_\psi^{max}, \phi = 3, 4, \\ \gamma_1 = \gamma_1^{max}, \\ \gamma_2 = p\gamma_2, \end{cases} \qquad \text{for the node } III0.$$

where $0 < \alpha, p < 1$ In the next sections we do the following:

- **1):** From the possible types at nodes we deduce what are the possible types for the whole network, i.e. we give a qualitative description of the equilibrium solutions on the whole network. This is done both for non damaged networks (actually we report the result contained in [20]) and for networks with a damaged node.
- **2):** The qualitative description of the equilibrium solution on the network allows us to impose the additional constraints of Proposition 3 thus to give a quantitative description of the equilibrium solutions on the whole network either damaged or not.

Definition 4.3. We call inlines the lines incoming in the network which are not outgoing for any node and the outlines the lines outgoing from the network which are not incoming for any node. The inflows and the outflows are the fluxes along the inlines and the outlines respectively.

Point **2)** is actually the main contribution of this paper together with the following.

- **3):** We give exact description of the outflows as function of the inflows both in damaged and undamaged network.

4.1. **Notations for flows.** We tackle the above described points **2)** and **3)** by modelling the constraints as a matrix equation. Then we need further notations for the involved matrices. At a node J_{ij} there are m incoming and m outgoing lines with either $m = 2$ (for oriented and circular manhattan cases) or $m = 4$ (for the full manhattan case.) If the first case applies, the incoming and outgoing fluxes along these lines are denoted respectively by $incoming^l_{ij}$ and $outgoing^l_{ij}$, $l = h, v$ (where h and v stand for horizontal and vertical.) In the case with $m = 4$ we fix an order of the lines adjacent to the node and denote the incoming and outgoing fluxes along these lines by $incoming^l_{ij}$ and $outgoing^l_{ij}$ where $l = 1, 2, 3, 4$. Then given a $s \times t$ network, for each l we consider the $s \times t$ matrices E^l and O^l with $E^l_{ij} = incoming^l_{ij}$ and $O^l_{ij} = outgoing^l_{ij}$, $i = 1, \ldots, s, j = 1, \ldots, t\}$.

5. Circular Manhattan networks.

5.1. Qualitative solutions for circular Manhattan networks.

Following rule **H** we get the following tables of possible types for a 3×3 undamaged network (see [20] for details):

$II0$	$II0, II2$	$II0, II1$
$II0, II1$	$II0$	$II0, II2$
$II0, II2$	$II0, II1$	J_{33}

J_{11}	$III0, III1$	$III0, III2$
$III0, III2$	$III0$	$III0, III1$
$III0, III1$	$III0, III2$	$III0$

.

which indicate which are the possible equilibria at each node J_{ij}, for $i = 1, 2, 3$ and $j = 1, 2, 3$. The type of nodes J_{33} in the first table and J_{11} in the second one can be uniquely determined by the equilibrium type at the adjacent nodes. However writing all possible types of these two nodes is not crucial to our purposes. Indeed gluing many 3×3 subnetworks to form a $s \times t$ network gives the following:

Theorem 5.1. *We have*

$$\begin{aligned} \mathcal{N} = \quad & \{\{J_{ij} = II0,\ i = 2 \ldots, s-1, j = 2, \ldots, t-1\}, \\ & \{J_{ij} = III0,\ i = 2 \ldots, s-1, j = 2, \ldots, t-1\}\} \end{aligned}$$

that is the equilibria on undamaged Circular Manhattan Network are given by either type $II0$ or type $III0$ apart at most the first line and column and the last line and column.

We now consider a damaged network.

Theorem 5.2. *If a failure occurs at any interior node of the network, then the equilibrium is of the same type of the undamaged network.*

Proof. The statement holds because the interior nodes are all of type $II0$ or $III0$. □

In the next section we consider networks where the nodes are all either of type II0 or of type III0.

5.2. **Exact solutions for undamaged circular Manhattan networks.** For a $s \times t$ network we have the following $4st$ flux variables: $incoming^h_{ij}$, $incoming^v_{ij}$, $outgoing^h_{ij}$, $outgoing^v_{ij}$, with $i = 1, \ldots, s$ and $j = 1, \ldots, t$. Next we describe the constraints. Equations (10) and (17) say that the flux is constant along lines connecting adjacent nodes: for nodes all of type $II0$ we write the incoming flows in terms of outgoing flows while for nodes all of type $III0$ we write outgoing flows in terms of incoming. These equations correspond to $s(t-1) + t(s-1)$ constraints. Equation (6) is written for each node in (11). Finally equations (12) and (18) describe the constraints of a node of being of type $II0$ or $III0$ and allow to write $outgoing^v$ in terms of $outgoing^h$ and $incoming^v$ in terms of $incoming^h$, respectively. Finally we get a total of $4st - s - t$ constraints in $4st$ variables.

For a network with nodes all of type $II0$ we have

$$\begin{cases} incoming^h_{ij} = outgoing^h_{is(i,j)}, \\ incoming^v_{ij} = outgoing^v_{t(i,j)j} \end{cases} \tag{10}$$

for all $i = 1, \ldots, s$, $j = 1, \ldots, t$, where

$$s(i,j) = \begin{cases} j-1 & \text{if } i \text{ is odd} \\ j+1 & \text{if } i \text{ is even} \end{cases} \qquad t(i,j) = \begin{cases} i-1 & \text{if } j \text{ is odd} \\ i+1 & \text{if } j \text{ is even,} \end{cases}$$

and

$out =$

$$\left[\begin{array}{c|cccc|c}
0 & outgoing^v_{12} & 0 & outgoing^v_{14} & \cdots & outgoing^h_{1t} \\ \hline
outgoing^h_{21} & & & & & 0 \\
0 & & & & & outgoing^h_{3t} \\
outgoing^h_{41} & & 0 & & & 0 \\
0 & & & & & outgoing^h_{5t} \\
\vdots & & & & & \vdots \\ \hline
 & 0 & outgoing^v_{s3} & 0 & \cdots &
\end{array}\right],$$

is the outflow matrix and $vec(out)$ is the outflow vector.

Moreover constraint (6) gives, for $i = 1, \ldots, s$ and $j = 1, \ldots, t$,

$$incoming^h_{ij} + incoming^v_{ij} = outgoing^h_{ij} + outgoing^v_{ij} \tag{11}$$

and by the condition imposed by the type equilibrium $II0$,

$$outgoing^v_{ij} = \alpha\, outgoing^h_{ij}. \tag{12}$$

Substituting equations (10) and (12) into (11) we get

$$outgoing^h_{i\ s(i,j)} + \alpha outgoing^h_{t(i,j)\ j} = (1+\alpha) outgoing^h_{ij}, \tag{13}$$

and, in matrix form, recalling that $O^h_{ij} = outgoing^h_{ij}$:

$$O^h_{i\ s(i,j)} + \alpha O^h_{t(i,j)\ j} = (1+\alpha) O^h_{ij} \tag{14}$$

for $i = 1, \ldots, s$ and $j = 1, \ldots, t$.

Equations (14) can be written as a matrix equation as follows:

$$(1+\alpha)O^h = \\ \alpha \left(L(s) O^h\, I(t)^{odd} + L(s)^T O^h\, I(t)^{ev} \right) + \left(I(s)^{odd} O^h\, L(t)^T + I(s)^{ev} O^h\, L(t) \right) + in,$$

where

$in =$

$$\left[\begin{array}{c|cccc|c} \begin{array}{c} incoming^h_{11} \\ + \\ incoming^v_{11} \end{array} & 0 & incoming^v_{13} & 0 & \cdots & \\ \hline \begin{array}{c} 0 \\ incoming^h_{31} \\ 0 \\ incoming^h_{51} \\ \vdots \end{array} & & & 0 & & \begin{array}{c} incoming^h_{2t} \\ 0 \\ incoming^j_{4t} \\ 0 \\ \vdots \end{array} \\ \hline & incoming^v_{s2} & 0 & incoming^v_{s4} & \cdots & \end{array}\right]$$

is the inflow matrix and $vec(in)$ is the inflow vector. We get the following matrix equation

$$\begin{array}{l} \alpha\,(I(s)-L(s))\,O^h\,I(t)^{odd} + \alpha\left(I(s)-L(s)^T\right)O^h\,I(t)^{ev} + \\ +I(s)^{odd}O^h\left(I(t)-L(t)^T\right) + I(s)^{ev}O^h\,(I(t)-L(t)) - in = 0 \end{array}$$

which, recalling that $A(n) = -I(n) + L(n)$, becomes:

$$\begin{array}{l} \alpha\left(A(s)\,O^h\,I(t)^{odd} + A^T(s)O^h\,I(t)^{ev}\right) + \\ +I(s)^{odd}O^h\,A^T(t) + I(s)^{ev}O^h\,A(t) + in = 0. \end{array}$$

We get that O^h is the solution of the above equation if

$$Kvec(O^h) + vec(in) = 0 \tag{15}$$

where

$$K = \alpha\left(I(t)^{odd}\otimes A(s) + I(t)^{ev}\otimes A^T(s)\right) + \left(A(t)\otimes I(s)^{odd} + A^T(t)\otimes I(s)^{ev}\right). \tag{16}$$

Solving the linear system (15) we get the matrix O^h. Such solution is unique if K is non singular.

Proposition 4. *K is non singular and there exists a unique solution O^h.*

Proof.

$$I(t)^{odd}\otimes A(s) + I(t)^{ev}\otimes A^T(s) = diag(A(s), A^T(s), A(s), A^T(s), \ldots)$$

and

$$A(t)\otimes I(s)^{odd} + A^T(t)\otimes I(s)^{ev} = \left[\begin{array}{cccc} -I(s) & I(s)^{ev} & & 0 \\ I(s)^{odd} & -I(s) & \ddots & \\ 0 & \ddots & \ddots & I(s)^{ev} \\ & & I(s)^{odd} & -I(s) \end{array}\right].$$

Then

$$K = \left[\begin{array}{cccc} A(s)-I(s) & I(s)^{ev} & & 0 \\ I(s)^{odd} & A^T(s)-I(s) & \ddots & \\ 0 & \ddots & \ddots & I(s)^{ev} \\ & & I(s)^{odd} & A'(s)-I(s) \end{array}\right],$$

where $A'(s) = A^T(s)$ if t is even or $A'(s) = A$ if t is odd. Since K is irreducible (i.e. there exists no permutation of rows and columns that put K in triangular form, or, equivalently, the induced oriented graph of K is strongly connected) and with diagonal dominance then it is non singular. □

The above proposition says that for each vector $vec(in)$ there exists a unique equilibrium solution over the whole network. That is, for each desired equilibrium solution $vec(O^h)$, there exists a unique $vec(in)$ that produces such equilibrium: $vec(O^h) = K^{-1}vec(in)$. Next we answer to the more interesting following question (from the point of view of a service delivery company.) Find the map H mapping inflows into outflows, and, given a desired outflow vector $vec(out)$, say if an inflow vector $vec(in)$ exists that produces such an outflow, i.e. if H is non singular.

To this purpose, we introduce the following matrices:

$$(\widetilde{in})_{ij} = \begin{cases} 1 & \text{if } i \text{ is odd and } j = 1 \\ 1 & \text{if } i \text{ is even and } j = t \\ 1 & \text{if } j \text{ is odd and } i = 1 \\ 1 & \text{if } j \text{ is even and } i = s \\ 0 & \text{otherwise,} \end{cases}$$

$$(\widetilde{out})_{ij} = \begin{cases} 1 & \text{if } i \text{ is even and } j = 1 \\ 1 & \text{if } i \text{ is odd and } j = t \\ 1 & \text{if } j \text{ is even and } i = 1 \\ 1 & \text{if } j \text{ is odd and } i = s \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\widetilde{in}$ and $\widetilde{out}$ have zero elements if and only if the corresponding elements of in and out are zero and 1's otherwise. Otherwise we can say that the $\widetilde{in}$ and $\widetilde{out}$ are such that $\widetilde{in} \circ in = in$ and $\widetilde{out} \circ out = out$. Therefore

$$vec(out) = diag(vec(\widetilde{out}))vec(O)$$

and

$$vec(in) = diag(vec(\widetilde{in}))vec(E).$$

We aim at finding the relation among outflows and inflows, that is we want to write the not zero components of $vec(out)$ as function of the not zero components of $vec(in)$ which we denote by $\widehat{vec(out)}$ and $\widehat{vec(in)}$ respectively. Then, the sought-after relation is

$$\widehat{vec(out)} + H\widehat{vec(in)} = 0,$$

with

$$H = RK^{-1}C,$$

where

$$C = colspace(diag(vec(\widetilde{in}))),$$
$$R = rowspace(diag(vec(\widetilde{out}))).$$

The spaces generated by $diag(vec(\widetilde{in}))$ and $diag(vec(\widetilde{out}))$ have dimension $\rho = s + t - 2$, if both s and t are even, or $\rho = s + t - 1$ otherwise. Then R is a $\rho \times st$ matrix, C is a $st \times \rho$ matrix and H is a square matrix of order $\rho \times \rho$.

Proposition 5. *If $\alpha \neq 1$ then H is full rank. Hence, for each inflow $\widehat{vec(in)}$ there exists a unique outflow $\widehat{vec(out)}$ and the mapping is invertible.*

Proof. H is non singular if and only if $det(C^{\perp}KR^{\perp}) \neq 0$, where now $C^{\perp}$ is matrix of order $\hat{\rho} \times st$ and $R^{\perp}$ is a matrix of order $st \times \hat{\rho}$, with $\hat{\rho} = st - \rho$. By some lengthy computations we get that $det(C^{\perp}KR^{\perp}) = (\alpha^2 - 1)^{\hat{\rho}/2}$. Therefore, if $\alpha \neq 1$, we get that H is full rank. □

For a network with all nodes of type $III0$ we have the following relations:

$$\begin{aligned} outgoing^h_{ij} &= incoming^h_{i\hat{s}(i,j)}, \\ outgoing^v_{ij} &= incoming^v_{\hat{t}(i,j)j} \end{aligned} \tag{17}$$

for all $i = 1, \ldots, s$, $j = 1, \ldots, t$, where

$$\hat{s}(i,j) = \begin{cases} j+1 & \text{if } i \text{ is odd} \\ j-1 & \text{if } i \text{ is even} \end{cases} \quad \hat{t}(i,j) = \begin{cases} i+1 & \text{if } j \text{ is odd} \\ i-1 & \text{if } j \text{ is even,} \end{cases}$$

and $vec(in)$ and $vec(out)$ are, as for the previous case, the inflow and outflow vectors.

For nodes of type $III0$, it holds the following

$$incoming^v_{ij} = p\ incoming^h_{ij}. \tag{18}$$

Substituting equations (17) and (18) into (11) we get

$$incoming^h_{i\ \hat{s}(i,j)} + p\, incoming^h_{\hat{t}(i,j)\ j} = (1+p) incoming^h_{ij}. \tag{19}$$

In matrix form, recalling that $E^h_{ij} = incoming^h_{ij}$, we have:

$$E^h_{i\ \hat{s}(i,j)} + pE^h_{\hat{t}(i,j)\ j} = (1+p)E^h_{ij} \tag{20}$$

for $i = 1, \ldots, s$ and $j = 1, \ldots, t$.

We rewrite system (20) as a matrix equation:

$$\begin{aligned} &(1+p)E^h = \\ &p\left(L(s)^T E^h\, I(t)^{odd} + L(s) E^h\, I(t)^{ev}\right) + \left(I(s)^{odd} E^h\, L(t) + I(s)^{ev} E^h\, L(t)^T\right)\right) + out \end{aligned}$$

and get the following:

$$\begin{aligned} &p\left(I(s) - L(s)^T\right) E^h\, I(t)^{odd} + p\left(I(s) - L(s)\right) E^h\, I(t)^{ev} + \\ &+ I(s)^{odd} E^h \left(I(t) - L(t)\right) + I(s)^{ev} E^h \left(I(t) - L(t)^T\right) - out = 0, \end{aligned}$$

i.e.:

$$p\left(A^T(s)\, E^h\, I(t)^{odd} + A(s) E^h\, I(t)^{ev}\right) + I(s)^{odd} E^h\, A(t) + I(s)^{ev} E^h\, A^T(t) + out = 0.$$

We get that E^h is the solution of the above equation if

$$K' vec(E^h) + vec(out) = 0 \tag{21}$$

where

$$K' = p\left(I(t)^{odd} \otimes A^T(s) + I(t)^{ev} \otimes A(s)\right) + \left(A^T(t) \otimes I(s)^{odd} + A(t) \otimes I(s)^{ev}\right).$$

Solving the linear system (21) we get the matrix E^h. We notice that $K' = K^T$ where K is the Kronecker product defined in equation (16). Similarly we wave the following:

Proposition 6. *K' is non singular and there exists a unique solution E^h.*

Now we can say that for each outflow vector $vec(out)$ there exists a unique equilibrium solution E^h. As for the previous case we aim at finding a relation among outflows and inflows. Using the same notations we get:

$$\widehat{vec(in)} + H' \widehat{vec(out)} = 0,$$

where now

$$H' = C^T (K')^{-1} R^T = C^T K^{-T} R^T = (RK^{-1}C)^T = H^T,$$

and a similar result to that of Proposition 5:

Proposition 7. *If $p \neq 1$ then H' is full rank. Hence, for each outflow $\widehat{vec(out)}$ there exists a unique inflow $\widehat{vec(in)}$ and the mapping is invertible.*

5.3. **Exact solutions for the circular Manhattan network with damaged node at position lk.** The interior nodes of the network are indexed lk with $2 \leq l \leq s-1$ and $2 \leq k \leq t-1$. Assume that there is a breakage at an interior node lk. After some time a new equilibrium is reached where the flux, along lines departing from and arriving to node lk, is annihilated. In order to describe this situation we set $outgoing^h_{ij} = outgoing^v_{ij} = incoming^h_{ij} = outgoing^v_{ij} = 0$, for some appropriate indices ij, in the equations (10),(17),(11),(12),(18) in previous section.

Therefore the number of variables is now $4st-4$ and the number of constraints is $4st-s-t-2$ equations for a total of $s+t+2$ degrees of freedom.

Consider first the case of a network with nodes all of type $II0$. In place of constraint (12) and in order to describe the fact that some variables are zero we set the following relation among unknowns:

$$\alpha^h \circ O^h + \alpha^v \circ O^v = 0, \tag{22}$$

where

$$\alpha^h_{ij} = \begin{cases} 0 & \text{if } j = k \text{ and } i = t(l,k) \\ \alpha & \text{otherwise} \end{cases}$$

and

$$\alpha^v_{ij} = \begin{cases} 0 & \text{if } i = l \text{ and } j = s(l,k) \\ -1 & \text{otherwise.} \end{cases}$$

As for the undamaged network case, we have the following:

$$\begin{aligned} &O^h + O^v = \\ &\left(L(s)O^v\, I(t)^{odd} + L(s)^T O^v\, I(t)^{ev}\right) + \left(I(s)^{odd} O^h\, L(t)^T + I(s)^{ev} O^h\, L(t)\right) + in. \end{aligned} \tag{23}$$

From equation (22) we get:

$$diag(vec(\alpha^h))vec(O^h) + diag(vec(\alpha^v))vec(O^v) = 0 \tag{24}$$

and, from equation (23), we get:

$$\begin{aligned} &\left(A(s)O^v\, I(t)^{odd} + A^T(s)O^v\, I(t)^{ev}\right) + \\ &+ \left(I(s)^{odd}O^h\, A^T(t) + I(s)^{ev}O^h\, A(t)\right) + in = 0, \end{aligned} \tag{25}$$

hence

$$\begin{aligned} &\left(I(t)^{odd} \otimes A(s) + I(t)^{ev} \otimes A^T(s)\right) vec(O^v) + \\ &\left(A(t) \otimes I(s)^{odd} + A^T(t) \otimes I(s)^{ev}\right) vec(O^h) + vec(in) = 0. \end{aligned} \tag{26}$$

By writing

$$T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right],$$

where

$$\begin{aligned} T_1 &= \left(I(t)^{odd} \otimes A(s) + I(t)^{ev} \otimes A^T(s)\right), \\ T_2 &= \left(A(t) \otimes I(s)^{odd} + A^T(t) \otimes I(s)^{ev}\right), \\ T_3 &= diag(vec(\alpha^v)), \\ T_4 &= diag(vec(\alpha^h)), \end{aligned}$$

$w = [vec(O^v), vec(O^h)]^T$ and $z = [vec(in), 0]^T$, we get the following compact form for equations (24) and (26):

$$Tw + z = 0.$$

Now T_1 is invertible hence

$$vec(O^v) = -T_1^{-1}vec(in) - T_1^{-1}T_2vec(O^h)$$

and

$$\begin{array}{l} 0 = T_3vec(O^v) + T_4vec(O^h) = \\ -T_3T_1^{-1}vec(in) - (T_3T_1^{-1}T_2 - T_4)vec(O^h) = \\ -T_3T_1^{-1}vec(in) + (T|T_1)vec(O^h), \end{array}$$

where $(T|T_1) = T_4 - T_3T_1^{-1}T_2$ is the Schur complement of T_1 in T. Hence

$$\begin{array}{l} vec(O^h) = S^h vec(in), \\ vec(O^v) = S^v vec(in), \end{array}$$

where

$$\begin{array}{l} S^h = (T|T_1)^{-1}T_3T_1^{-1} \\ S^v = -T_1^{-1} - T_1^{-1}T_2S^h. \end{array}$$

By substituting the expression of S^h in S^v we get:

$$\begin{array}{l} S^v = -T_1^{-1} - T_1^{-1}T_2S^h = -T_1^{-1} - T_1^{-1}T_2(T|T_1)^{-1}T_3T_1^{-1} = \\ -T_1^{-1} - ((T|T_1)T_2^{-1}T_1)^{-1}T_3T_1^{-1} = -T_1^{-1} - (T_4T_2^{-1}T_1 - T_3)^{-1}T_3T_1^{-1} = \\ -(I(st) + (T|T_2)^{-1}T_3))T_1^{-1} = -(T|T_2)^{-1}(T_4T_2^{-1}T_1)T_1^{-1} = \\ -(T|T_2)^{-1}T_4T_2^{-1}, \end{array}$$

where $(T|T_2) = -(T_3 - T_4T_2^{-1}T_1)$ is the Schur complement of T in T_2.

Proposition 8. *S^h and S^v have corank* 1.

Proof. S^h has corank 1 since T_3 is a diagonal matrix with 1 null entry and S^v has corank 1 since T_4 is a diagonal matrix with 1 null entry. □

Now we are interested only in the rows and columns of S^h and S^v relative to the outflows and inflows, i.e. we compute $H^h = RS^hC$ and $H^v = RS^vC$ and, by some computations we get the following:

Proposition 9. *For s, t even and*

$$(l, k) \in \{(2, 2), (s-1, t-1), (2, t-1), (s-1, 2)\}$$

for s, t odd and

$$(l, k) \in \{(2, 2), (s-1, t-1)\},$$

for s even and t odd and

$$(l, k) \in \{(2, 2), (s-1, 2)\},$$

for s odd and t even and

$$(l, k) \in \{(2, 2), (2, t-1)\},$$

$H^v = \alpha H^h$ and H^h and H^v have corank 1.

Assume now that the nodes are all of type $III0$. In place of constraint (18) and in order to describe the fact that some variables are zero we set the following relation among unknowns:

$$p^h \circ E^h + p^v \circ E^v = 0, \tag{27}$$

where

$$p_{ij}^h = \begin{cases} 0 & \text{if } j = k \text{ and } i = \hat{t}(l, k) \\ p & \text{otherwise} \end{cases}$$

and

$$p^v_{ij} = \begin{cases} 0 & \text{if } i = l \text{ and } j = \hat{s}(l,k) \\ -1 & \text{otherwise.} \end{cases}$$

As for the undamaged network we have the following:

$$E^h + E^v = \big(L(s)^T E^v\, I(t)^{odd} + L(s)E^v\, I(t)^{ev}\big) + \big(I(s)^{odd}E^h\, L(t) + I(s)^{ev}E^h\, L(t)^T\big)\big) + out. \tag{28}$$

From equation (27) we get:

$$diag(vec(p^h))vec(E^h) + diag(vec(p^v))vec(E^v) = 0 \tag{29}$$

and, from equation (28), we get:

$$\begin{aligned}&\big(A^T(s)E^v\, I(t)^{odd} + A(s)E^v\, I(t)^{ev}\big) + \\ &+ \big(I(s)^{odd}E^h\, A(t) + I(s)^{ev}E^h\, A^T(t)\big)\big) + out = 0,\end{aligned} \tag{30}$$

hence

$$\begin{aligned}&\big(I(t)^{odd} \otimes A^T(s) + I(t)^{ev} \otimes A(s)\big)\, vec(E^v) + \\ &\big(A^T(t) \otimes I(s)^{odd} + A(t) \otimes I(s)^{ev}\big)\, vec(E^h) + vec(out) = 0.\end{aligned} \tag{31}$$

By writing

$$T' = \left[\begin{array}{c|c} T'_1 & T'_2 \\ \hline T'_3 & T'_4 \end{array}\right],$$

where

$$\begin{aligned} T'_1 &= T_1^T \\ T'_2 &= T_2^T \\ T'_3 &= diag(vec(p^v)), \\ T'_4 &= diag(vec(p^h)), \end{aligned}$$

$w' = [vec(E^v), vec(E^h)]^T$ and $z' = [vec(out), 0]^T$, we get the following compact form for equations (29) and (31):

$$T'w' + z' = 0,$$

hence

$$\begin{aligned} vec(E^h) &= S^h vec(out), \\ vec(E^v) &= S^v vec(out), \end{aligned}$$

where now

$$\begin{aligned} S^h &= (T'|T_1^T)^{-1}T'_3T_1^{-T} \\ S^v &= -T_1^{-T} - T_1^{-T}T_2^T S^h = -(T'|T_2)^{-1}T'_4T_2^{-1}. \end{aligned}$$

As for the case $II0$ we get the following:

Proposition 10. *S^h and S^v have corank* 1.

Now we are interested only in the rows and columns of S^h and S^v relative to the outflows and inflows, i.e. we compute $H^h = C^T S^h R^T$ and $H^v = C^T S^v R^T$ and, by some computations we get the following:

Proposition 11. *For s, t even and*

$$(l,k) \in \{(2,2), (s-1,t-1), (2,t-1), (s-1,2)\}$$

for s, t odd and

$$(l,k) \in \{(2,2), (s-1,t-1)\},$$

for s even and t odd and

$$(l,k) \in \{(2,2), (s-1,2)\},$$

for s odd and t even and

$$(l,k) \in \{(2,2),(2,t-1)\},$$

$H^v = pH^h$ *and* H^h *and* H^v *have corank* 1.

The proof of Theorem 1.2, **C.M.N.** easily follows:

Proof. (**Theorem 1.2, C.M.N.**) The nodes lk listed in Propositions 9 and 11 are those for which a breakage entails a loss of rank of the mappings H^h and H^v, hence of the mapping *inflow* $\mapsto$ *outflow*. Therefore the cited nodes are critical for the Circular Manhattan Network. Moreover, these nodes lie at the corners of the network as shown in Figure 1. □

6. Oriented Manhattan networks.

6.1. Qualitative solutions for oriented Manhattan networks.

Following rule **H** we get the followings.

Proposition 12. *If node J_{11} is of type $I, III0, III1, III2$ then the network is described by the following table:*

$I, III0, III1, III2$	$III0, III2$	$III0, III2$	$\cdots$
$III0, III1$	$III0$	$III0$	$\cdots$
$III0, III1$	$III0$	$III0$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$

Proposition 13. *If node J_{st} is of type $I, II0, II1, II2$ then the network is described by the following table:*

$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\cdots$	$II0$	$II0$	$II0, II1$
$\cdots$	$II0$	$II0$	$II0, II1$
$\cdots$	$II0, II2$	$II0, II2$	$I, II0, II1, II2$

Proposition 14. *Let J_{ij} be a node of type I. Then*

- *all nodes J_{il}, $l = j+1, \ldots, t$ are either of type $III0$ or of type $III2$,*

- *all nodes* $J_{il},\ l = 1, \ldots, j-1$ *are either of type* $II0$ *or of type* $II2$*,*
- *all nodes* $J_{kj},\ k = i+1, \ldots, s$ *are either of type* $III0$ *or of type* $III1$*,*
- *all nodes* $J_{kj},\ k = 1, \ldots, i-1$ *are either of type* $II0$ *or of type* $II1$*.*

Corollary 1. *Let* J_{ij} *be a node of type* I*. Then*

- *all nodes* J_{kl}*, with* $k = 1, \ldots, j-1$ *and* $l = 1, \ldots, i-1$ *are of type* $II0$*;*
- *all nodes* J_{kl}*, with* $k = j+1, \ldots, t$ *and* $l = i+1, \ldots, s$ *are of type* $III0$*.*
- *if exists* J_{kl} *of type* I *it must necessary be either* $k > i, l > j$ *or* $k < i, l < j$*.*

Corollary 2. *The number of nodes of type* I *in a* $s \times t$ *network is at most* $min(s,t)$*, the minimum among* s *and* t*.*

Collecting the above propositions and corollaries we have the following:

Theorem 6.1. *A* $s \times t$ *network is described by* $4st$ *parameters* $incoming^h_{ij}$*,* $incoming^v_{ij}$*,* $otugoing^h_{ij}$*,* $outgoing^v_{ij}$*,* $i = 1, \ldots, s, j = 1, \ldots, t$*, together with* $4st - s - t - \omega$ *constraints, where* $\omega \le min(s,t)$ *is the number of nodes of type* I*.*

Proof. The constraints are st for the equations (11), $st - \omega$ for the additional constraints (see Remark 1) at the nodes, $2st - s - t$ for the equilibrium equation $incoming^h_{ij} = outgoing^h_{i\ j-1}$, $incoming^v_{ij} = outgoing^v_{i-1\ j}$. □

In the rest of this section we only consider networks of type $II0$ or $III0$. For a damaged network we have the following:

Theorem 6.2. *If a failure occurs at any interior node of the network with all nodes of type* $II0$ *or* $III0$*, then the new equilibrium is of the same type of the original one.*

Proof. The statement holds since we are assuming that nodes are all of type $II0$ or $III0$. □

6.2. **Exact solutions for undamaged oriented Manhattan networks.** For a $s \times t$ network we have $4st$ flux variables: $incoming^h_{ij}$, $incoming^v_{ij}$, $outgoing^h_{ij}$, $outgoing^v_{ij}$, with $i = 1, \ldots, s$ and $j = 1, \ldots, t$.

Next we describe the constraints. Equations (32) and (35) say that the flux is constant along lines connecting adjacent nodes: for nodes all of type $II0$ we write the incoming flows in terms of outgoing flows while for nodes all of type $III0$ we write outgoing flows in terms of incoming. These equations correspond to $s(t-1) + t(s-1)$ constraints. Moreover, as for the Circular Manhattan Network constraints, (11), (12) and (18), hold. Finally we get a total of $4st - s - t$ constraints in $4st$ variables.

For a network with all nodes of type $II0$ we have:

$$\begin{aligned} incoming^h_{ij} &= outgoing^h_{ij-1}, \\ incoming^v_{ij} &= outgoing^v_{i-1\ j} \end{aligned} \tag{32}$$

for all $i = 1, \dots, s$, $j = 1, \dots, t$, where

$out =$

$$\left[\begin{array}{c|ccc|c}
0 & 0 & \cdots & 0 & outgoing^h_{1t} \\ \hline
0 & & & & outgoing^h_{2t} \\
\vdots & & 0 & & \vdots \\
0 & & & & outgoing^h_{s-1\,t} \\ \hline
outgoing^v_{s1} & outgoing^v_{s2} & \cdots & outgoing^v_{s-1\,t} & \begin{array}{c} outgoing^h_{st}+ \\ outgoing^v_{st} \end{array}
\end{array}\right],$$

is the outflow matrix and $vec(out)$ is the outflow vector. Moreover we have that equations (11) and (12) hold for all $i = 1, \dots, s$ and $j = 1, \dots, t$.

Substituting equations (32) and (12) into (11) we get

$$outgoing^h_{i\ j-1} + \alpha outgoing^h_{i-1\ j} = (1+\alpha) outgoing^h_{ij}. \tag{33}$$

Recalling that $O^h_{ij} = outgoing^h_{ij}$, we write equations (33) as a Sylvester equation as follows:

$$(1+\alpha)O^h - \alpha L(s) O^h - O^h L(t)^T - in = 0,$$

where the inflow matrix is now

$in =$

$$\left[\begin{array}{c|ccc|c}
\begin{array}{c} incoming^h_{11}+ \\ incoming^v_{11} \end{array} & incoming^v_{12} & \cdots & incoming^v_{1\,t-1} & incoming^v_{1t} \\ \hline
incoming^h_{21} & & & & 0 \\
\vdots & & 0 & & \vdots \\
incoming^h_{s-1\,1} & & & & 0 \\ \hline
incoming^h_{s1} & 0 & \cdots & 0 & 0
\end{array}\right].$$

We rewrite the above matrix equation as:

$$\alpha(I(s) - L(s))O^h + O^h(I(t) - L(t)^T) - in = 0$$

and, by denoting $A = \alpha A(s)$ and $B = A(t)^T$:

$$\alpha A\, O^h + O^h\, B + in = 0. \tag{34}$$

Now if O^h solves the above Sylvester equation (34) then

$$O^h = \frac{1}{2}\mathcal{O}_\infty.$$

where

$$\mathcal{O}_0 = in,$$
$$\mathcal{O}_{k+1} = \tfrac{1}{2}(\mathcal{O}_k + A_k^{-1}\mathcal{O}_k (A^T)_k^{-1}).$$

Consider now the vectorization of equation (34):

$$K vec(O^h) + vec(in) = 0,$$

where

$$K = \big(I(s) \otimes A + B^T \otimes I(t)\big) = \begin{bmatrix} A - I(s) & & & 0 \\ I(s) & A - I(s) & \ddots & \\ 0 & \ddots & \ddots & 0 \\ & & I(s) & A - I(s) \end{bmatrix}$$

is a non singular matrix. As we did for the Circular Manhattan Network we introduce the following matrices:

$$(\widetilde{in})_{ij} = \begin{cases} 1 & \text{for} \;\; j = 1 \\ 1 & \text{for} \;\; i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$(\widetilde{out})_{ij} = \begin{cases} 1 & \text{for} \;\; j = 1 \\ 1 & \text{for} \;\; 1 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We aim at finding a relation among outflows and inflows, that is we want to write the non zero components of $vec(out)$ as function of the non zero components of $vec(in)$ which we denote by $\widehat{vec(out)}$ and $\widehat{vec(in)}$ respectively. Then

$$\widehat{vec(out)} + H\widehat{vec(in)} = 0$$

with

$$H = RK^{-1}C,$$

where now

$$\begin{aligned} C &= colspace(diag(vec(\widetilde{in}))), \\ R &= rowspace(diag(vec(\widetilde{out}))). \end{aligned}$$

We observe that the spaces generated by $diag(vec(\widetilde{in}))$ and $diag(vec(\widetilde{out}))$ have dimension $\rho = s + t - 1$ then R is a $\rho \times st$ matrix, C is a $st \times \rho$ matrix and H is a square matrix of order $\rho \times \rho$.

Proposition 15. *H has rank $max(s, t)$.*

Proof. The inverse of K is

$$K^{-1} = \begin{bmatrix} (A - I(s))^{-1} & & & \\ -(A - I(s))^{-2} & (A - I(s))^{-1} & & 0 \\ \vdots & \ddots & \ddots & \\ (-1)^{t+1}(A - I(s))^{-t} & \cdots & -(A - I(s))^{-2} & (A - I(s))^{-1} \end{bmatrix},$$

and

$$H = RK^{-1}C = \left[\begin{array}{ccc|cccccc} a_{s1}^1 & \cdots & a_{ss}^1 & 0 & & & & \\ a_{s1}^2 & \cdots & a_{ss}^2 & a_{s1}^1 & 0 & & & \\ a_{s1}^3 & \cdots & a_{ss}^3 & a_{s1}^2 & a_{s1}^1 & \ddots & & \\ \vdots & & \vdots & \vdots & & \ddots & 0 & \\ a_{s1}^{t-1} & \cdots & a_{ss}^{t-1} & a_{s1}^{t-2} & \cdots & & a_{s1}^1 & 0 \\ \hline a_{11}^t & \cdots & a_{1s}^t & a_{11}^{t-1} & a_{11}^{t-2} & \cdots & a_{11}^2 & a_{11}^1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{s1}^t & \cdots & a_{ss}^t & a_{s1}^{t-1} & a_{s1}^{t-2} & \cdots & a_{s1}^2 & a_{s1}^1 \end{array}\right],$$

where we have used the short notation $a^j = (-1)^{j+1}(A - I(s))^{-j}$. Consider the block matrix decomposition

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix},$$

with $H_1 \in \mathbb{R}^{(t-1)\times s}$, $H_2 \in \mathbb{R}^{(t-1)\times t-1}$, $H_3 \in \mathbb{R}^{s\times s}$ and $H_4 \in \mathbb{R}^{s\times(t-1)}$. Then, recalling the definition of $J(n)$ given in section 3: $J(s+t-1)HJ(s+t-1) = H^T$,

$$\begin{bmatrix} 0 & J(s) \\ J(t-1) & -J(t-1)H_1H_3^{-1} \end{bmatrix} \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} -H_3^{-1}H_4J(t-1) & J(s) \\ J(t-1) & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & J(s)H_3J(s) \\ J(t-1)(H_2 - H_1H_3^{-1}H_4)J(t-1) & 0 \end{bmatrix} =$$

$$\begin{bmatrix} J(s)H_3 & 0 \\ 0 & J(t-1)(H_2 - H_1H_3^{-1}H_4) \end{bmatrix} \begin{bmatrix} 0 & J(s) \\ J(t-1) & 0 \end{bmatrix}$$

and H has rank $h = rank(H_3) + rank(H_2 - H_1H_3^{-1}H_4)$.

Now we observe that

$$H_2 = e_s^T a^1 e_1 L(s) + e_s^T a^2 e_1 L^2(s) + \cdots = \sum_{k=1}^{t-2} e_s^T a^k e_1 L(s)^k.$$

On the other hand

$$H_1H_3^{-1}H_4 = \begin{bmatrix} e_s^T a^1 \\ \hline e_s^T a^2 \\ \hline \vdots \\ \hline e_s^T a^{t-1} \end{bmatrix} a^{-t} \left[\, a^{t-1}e_1 \,\middle|\, a^{t-2}e_1 \,\middle|\, \cdots \,\middle|\, a^1 e_1 \,\right] =$$

$$\begin{bmatrix} e_s^T a^1 \\ \hline e_s^T a^2 \\ \hline \vdots \\ \hline e_s^T a^{t-1} \end{bmatrix} \left[\, a^{-1}e_1 \,\middle|\, a^{-2}e_1 \,\middle|\, \cdots \,\middle|\, a^{-t+1} e_1 \,\right] =$$

$$\begin{bmatrix} e_s^T e_1 & e_s^T a^{-1} e_1 & \cdots & e_s^T a^{-t+2} e_1 \\ e_s^T a^1 e_1 & e_s^T e_1 & \cdots & e_s^T a^{-t+3} e_1 \\ \vdots & \ddots & & \\ e_s^T a^{t-2} e_1 & \cdots & e_s^T a^1 e_1 & e_s^T e_1 \end{bmatrix}.$$

We then have that $H_1H_3^{-1}H_4 = H_2 + T$ where T is an upper triangular matrix of order $k = t - s$. Indeed $e_s^T a^{-j} e_1 = e_s^T (A - I(s))^j e_1$ where $A = \alpha(-I(s) + L(s))$. Now $L(s)^{s-1} = e_s e_1^T$ and $L(s) = 0$. Therefore $e_s^T (A - I(s))^j e_1 \neq 0$ if and only if $j \geq s - 1$ and $k = (t-2) - (s-1) + 1 = t - s$. If $t \leq s$ then $k = 0$. Then, if $t > s$ $rank(H) = rank(H_3) + k = s + (t - s) = t$. If otherwise $t \leq s$ then $rank(H) = rank(H_3) = s$. Finally we have that $rank(H) = max(s, t)$. □

Consider now a network with all nodes of type $III0$. We have the following relations.

$$\begin{aligned} outgoing^h_{ij} &= incoming^h_{ij+1}, \\ outgoing^v_{ij} &= incoming^v_{i+1\ j} \end{aligned} \tag{35}$$

for all $i = 1, \ldots, s$, $j = 1, \ldots, t$, where *out* and *in* are respectively the outflow and the inflow matrices.

Moreover we have that equations (11) and (18) hold for all $i = 1, \ldots, s$ and $j = 1, \ldots, t$. The number of equations is then $s(t-1)+(s-1)t+2st = 4st-s-t$.

Substituting equations (35) and (18) into (11) we get

$$incoming^h_{i\ j+1} + p\, incoming^h_{i+1\ j} = (1+p) incoming^h_{ij}. \tag{36}$$

In matrix form, recalling that $E^h_{ij} = incoming^h_{ij}$. We get the following matrix equation:

$$(1+p)E^h - pL(s)^T E^h - E^h L(t) - out = 0.$$

We rewrite the above matrix equation as:

$$p(I(s) - L(s)^T)E^h + E^h(I(t) - L(t)) - out = 0$$

and, by denoting $A = pA(s)$ and $B = A(t)^T$:

$$A^T E^h + E^h B^T + out = 0. \tag{37}$$

Now if E^h solves the above Sylvester equation (37) then

$$E^h = \frac{1}{2}\mathcal{E}_\infty,$$

where

$$\begin{aligned} \mathcal{E}_0 &= out, \\ \mathcal{E}_{k+1} &= \tfrac{1}{2}(\mathcal{E}_k + A_k^{-T}\mathcal{E}_k B_k^{-T}). \end{aligned}$$

Consider now the vectorization of equation (37):

$$K' vec(E) + vec(out) = 0,$$

where

$$K' = \big(I(s) \otimes A^T + B \otimes I(t)\big) = K^T$$

is non singular. In this case we have

$$\widehat{vec(in)} + H' \widehat{vec(out)}$$

with

$$H' = R^T K^{-T} C^T = (CK^{-1}R)^T = H^T.$$

Then

Proposition 16. *H' has rank $max(s,t)$.*

6.3. **Exact solutions for the oriented Manhattan network with damaged node at position lk.** Also for the Oriented Manhattan Network we consider a breakage at an interior node lk, i.e. $2 \leq l \leq s-1$ and $2 \leq k \leq t-1$. After some time a new equilibrium is reached where the flux, along lines departing from and arriving to node lk, is annihilated. In order to describe this situation we set $outgoing^h_{ij} = outgoing^v_{ij} = incoming^h_{ij} = outgoing^v_{ij} = 0$, for some appropriate indices ij, in the equations (32),(35),(11),(12),(18) in previous section.

As in the circular case with damaged node we have $4st-4$ variables and $4st-s-t-2$ equations for a total of $s+t+2$ degrees of freedom.

Consider a network with all nodes of type $II0$. Consider first the case of a network with nodes all of type $II0$. In place of constraint (12) and in order to describe the fact that some variables are zero we set the following relation among unknowns:

$$\alpha^h \circ O^h + \alpha^v \circ O^v = 0, \tag{38}$$

where

$$\alpha^h_{ij} = \begin{cases} 0 & \text{if } j = k \text{ and } i = l-1 \\ \alpha & \text{otherwise} \end{cases}$$

and

$$\alpha^v_{ij} = \begin{cases} 0 & \text{if } i = l \text{ and } j = k-1 \\ -1 & \text{otherwise,} \end{cases}$$

and

$$O^h + O^v = L(s)O^v + O^h\, L(t)^T + in. \tag{39}$$

From equation (38) we get:

$$diag(vec(\alpha^h))vec(O^h) + diag(vec(\alpha^v))vec(O^v) = 0 \tag{40}$$

and, from equation (39), we get:

$$A(s)O^v + O^h\, A^T(t) + in = 0, \tag{41}$$

hence:

$$(I(t) \otimes A(s))\, vec(O^v) + (A(t) \otimes I(s))\, vec(O^h) + vec(in) = 0. \tag{42}$$

By writing

$$T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right],$$

where

$$\begin{aligned} T_1 &= (I(t) \otimes A(s))\,, \\ T_2 &= (A(t) \otimes I(s))\,, \\ T_3 &= diag(vec(\alpha^v)), \\ T_4 &= diag(vec(\alpha^h)), \end{aligned}$$

$w = [vec(O^v), vec(O^h)]^T$ and $z = [vec(in), 0]^T$, we get the following compact form for equations (40) and (42):

$$Tw + z = 0.$$

Now T_1 is invertible hence

$$vec(O^v) = -T_1^{-1} vec(in) - T_1^{-1} T_2 vec(O^h),$$

and

$$\begin{aligned} 0 &= T_3 vec(O^v) + T_4 vec(O^h) = \\ &-T_3 T_1^{-1} vec(in) - (T_3 T_1^{-1} T_2 - T_4) vec(O^h) = \\ &-T_3 T_1^{-1} vec(in) + (T|T_1) vec(O^h) \end{aligned}$$

where $(T|T_1) = T_4 - T_3 T_1^{-1} T_2$ is the Schur complement of T_1 in T. Hence

$$\begin{aligned} vec(O^h) &= S^h vec(in), \\ vec(O^v) &= S^v vec(in), \end{aligned}$$

where

$$\begin{aligned} S^h &= (T|T_1)^{-1} T_3 T_1^{-1} \\ S^v &= -T_1^{-1} - T_1^{-1} T_2 S^h = -(T|T_2)^{-1} T_4 T_2^{-1}. \end{aligned}$$

Similarly to the case $II0$ of the circular manhattan case we have the following proposition (see Proposition 8):

Proposition 17. *S^h and S^v have corank* 1*.*

Now we are interested only in the rows and columns of S^h and S^v relative to the outflows and inflows, i.e. we compute $H^h = RS^hC$ and $H^v = RS^vC$.

By some computations we get the following:

Proposition 18. *If (l,k) is such that $min(s,t)+1 \le l+k \le max(s,t)+1$ then H^h and S^v have rank $max(s,t)-1$. Otherwise $rank(H^h) = rank(H^v) = max(s,t)$.*

Consider now a network with all nodes of type $III0$. In place of constraint (18) and in order to describe the fact that some variables are zero we set the following relation among unknowns:

$$p^h \circ E^h + p^v \circ E^v = 0, \tag{43}$$

where

$$p^h_{ij} = \begin{cases} 0 & \text{if } j = k \text{ and } i = l+1 \\ p & \text{otherwise} \end{cases}$$

and

$$p^v_{ij} = \begin{cases} 0 & \text{if } i = l \text{ and } j = k+1 \\ -1 & \text{otherwise,} \end{cases}$$

and

$$E^h + E^v = L(s)^T E^v + E^h L(t) + out. \tag{44}$$

From the equation (43) we get:

$$diag(vec(p^h))vec(E^h) + diag(vec(p^v))vec(E^v) = 0 \tag{45}$$

and, from equation (44), we get:

$$A(s)^T E^v + E^h A(t) + out = 0, \tag{46}$$

hence

$$\big(I(t) \otimes A^T(s)\big)\, vec(E^v) + \big(A(t)^T \otimes I(s)\big)\, vec(E^h) + vec(out) = 0. \tag{47}$$

By writing

$$T' = \left[\begin{array}{c|c} T'_1 & T'_2 \\ \hline T'_3 & T'_4 \end{array}\right],$$

where

$$\begin{aligned} T'_1 &= T_1^T, \\ T'_2 &= T_2^T, \\ T'_3 &= diag(vec(p^v)), \\ T'_4 &= diag(vec(p^h)), \end{aligned}$$

$w' = [vec(E^v), vec(E^h)]^T$ and $z' = [vec(out), 0]^T$, we get the following compact form for equations (45) and (47):

$$T'w' + z' = 0,$$

hence

$$\begin{aligned} vec(E^h) &= S^h vec(out) \\ vec(E^v) &= S^v vec(out) \end{aligned}$$

where now

$$\begin{aligned} S^h &= (T'|T_1^T)^{-1} T'_3 T_1^{-T}, \\ S^v &= -T_1^{-T} - T_1^{-T} T_2^T S^h = -(T'|T_2)^{-1} T'_4 T_2^{-1}. \end{aligned}$$

Similarly to the case $III0$ of the circular manhattan case we have the following proposition (see Proposition 10):

Proposition 19. *S^h and S^v have corank* 1.

Now we are interested only in the rows and columns of S^h and S^v relative to the outflows and inflows, i.e. we compute $H^h = C^T S^h R^T$ and $H^v = C^T S^v R^T$.

By some computations we get the following:

Proposition 20. *If (l,k) is such that $min(s,t)+1 \le l+k \le max(s,t)+1$ then H^h and S^v have rank $max(s,t)-1$. Otherwise $rank(H^h) = rank(H^v) = max(s,t)$.*

The proof of Theorem 1.2, **O.M.N.** easily follows:

Proof. (**Theorem 1.2, O.M.N.**) The nodes lk described in Propositions 18 and 20 are those for which a breakage entails a loss of rank of the mappings H^h and H^v, hence of the mapping *inflow* $\mapsto$ *outflow*. Therefore the cited nodes are critical for the Oriented Manhattan Network. Moreover these nodes lie along the anti–diagonals of maximal length of the network as shown in Figure 2. □

7. Full Manhattan networks.

7.1. Qualitative solutions for full Manhattan networks.

Following rule **H** (see [20]) we get a totally similar result to that obtained for the circular manhattan case:

Theorem 7.1. *We have*

$$\begin{aligned} \mathcal{N} = \quad & \{\{J_{ij} = II0,\ i = 2\dots,s-1, j = 2,\dots,t-1\},\\ & \{J_{ij} = III0,\ i = 2\dots,s-1, j = 2,\dots,t-1\},\} \end{aligned}$$

that is the equilibria on the network are given by either type $II0$ or type $III0$ apart at most the first line and column and the last line and column.

For a damaged network we have the following:

Theorem 7.2. *If a failure occurs at any interior node of the network, then the new equilibrium is of the same type of original one.*

In the next section we consider networks where the nodes are all either of type $II0$ or of type $III0$.

7.2. Exact solutions for undamaged full Manhattan networks.

For a $s \times t$ network we have $8st$ flux variables: $incoming^k_{ij}$, $k = 1,2,3,4$, $outgoing^k_{ij}$, $k = 1,2,3,4$, with $i = 1,\dots,s$ and $j = 1,\dots,t$.

Next we describe the constraints. Equations (48) and (53) say that the flux is constant along lines connecting adjacent nodes: for nodes all of type $II0$ we write the incoming flows in terms of outgoing flows while for nodes all of type $III0$ we write outgoing flows in terms of incoming. These equations correspond to $2s(t-1)+2t(s-1)$ constraints. Moreover, equation (6) is written for each node in (49). Finally equations (50) and (54) describe the constraints of a node of being of type $II0$ or $III0$ and allow us to respectively write $outgoing^k$ and $incoming^k$ in terms of the total flux Γ incoming in (hence outgoing from) the node. Finally we get a total of $8st - 2s - 2t$ constraints in $8st$ variables.

For a network with all nodes of type $II0$ we have:

$$\begin{aligned} incoming^1_{ij} &= outgoing^3_{i-1,j},\\ incoming^3_{ij} &= outgoing^1_{i+1,j},\\ incoming^2_{ij} &= outgoing^4_{i,j+1},\\ incoming^4_{ij} &= outgoing^2_{i,j-1}. \end{aligned} \tag{48}$$

for all $i = 1, \ldots, s$, $j = 1, \ldots, t$, where

$out =$

$$\left[\begin{array}{c|ccc|c} \begin{matrix} outgoing^1_{11}+ \\ outgoing^4_{11} \end{matrix} & outgoing^1_{12} & \cdots & outgoing^1_{1\,t-1} & \begin{matrix} outgoing^1_{1t}+ \\ outgoing^2_{1t} \end{matrix} \\ \hline outgoing^4_{21} & & & & outgoing^2_{2t} \\ \vdots & & 0 & & \vdots \\ outgoing^4_{s-1\,1} & & & & outgoing^2_{s-1\,t} \\ \hline \begin{matrix} outgoing^4_{s1}+ \\ outgoing^3_{s1} \end{matrix} & outgoing^3_{s2} & \cdots & outgoing^3_{s\,t-1} & \begin{matrix} outgoing^2_{st}+ \\ outgoing^3_{st} \end{matrix} \end{array}\right],$$

is the outflow matrix and $vec(out)$ is the outflow vector. Moreover we have, for $i = 1, \ldots, s$ and $j = 1, \ldots, t$,

$$\begin{array}{c} incoming^1_{ij} + incoming^2_{ij} + incoming^3_{ij} + incoming^4_{ij} = \\ outgoing^1_{ij} + outgoing^2_{ij} + outgoing^3_{ij} + outgoing^4_{ij} \end{array} \tag{49}$$

and

$$outgoing^k_{ij} = \alpha_k \Gamma_{ij}, \;\; k = 1, 2, 3, 4, \tag{50}$$

where $\sum_{k=1}^4 \alpha_k = 1$ and $\Gamma_{ij} = \sum_{k=1}^4 incoming^k_{ij} = \sum_{k=1}^4 outgoing^k_{ij}$. Substituting equations (48) and (50) into (49) we get

$$\alpha_3 \Gamma_{i-1,j} + \alpha_4 \Gamma_{i,j+1} + \alpha_1 \Gamma_{i+1,j} + \alpha_2 \Gamma_{i,j-1} = \Gamma_{ij}. \tag{51}$$

We can write equation (51) as a matrix equation as follows.

$$\Gamma = (\alpha_3 L(s) + \alpha_1 L(s)^T)\Gamma + \Gamma(\alpha_4 L(t) + \alpha_2 L(t)^T) + in$$

where in is the inflows matrix:

$in =$

$$\left[\begin{array}{c|ccc|c} \begin{matrix} incoming^1_{11}+ \\ incoming^4_{11} \end{matrix} & incoming^1_{12} & \cdots & incoming^1_{1\,t-1} & \begin{matrix} incoming^1_{1t}+ \\ incoming^2_{1t} \end{matrix} \\ \hline incoming^4_{21} & & & & incoming^2_{2t} \\ \vdots & & 0 & & \vdots \\ incoming^4_{s-1\,1} & & & & incoming^2_{s-1\,t} \\ \hline \begin{matrix} incoming^4_{s1}+ \\ incoming^3_{s1} \end{matrix} & incoming^3_{s2} & \cdots & incoming^3_{s\,t-1} & \begin{matrix} incoming^2_{st}+ \\ incoming^3_{st} \end{matrix} \end{array}\right].$$

Hence, recalling that $A(n) = -I(n) + L(n)$, we get:

$$\left(\alpha_1 A(s)^T + \alpha_3 A(s)\right)\Gamma + \Gamma\left(\alpha_2 A(t)^T + \alpha_4 A(t)\right) + in = 0$$

and the following Sylvester equation

$$A\Gamma + \Gamma B + in = 0 \tag{52}$$

where $A = \alpha_1 A(s)^T + \alpha_3 A(s)$ and $B = \alpha_2 A(s)^T + \alpha_4 A(s)$.

By the first Gerschgorin theorem (see [14] for details), the eigenvalues of A are contained in the union of its Gerschgorin circles, which in this case is a unique circle centered in $-(\alpha_1 + \alpha_3)$ and radius $(\alpha_1 + \alpha_3)$. Then A is a stable matrix. The same holds for B. Then we compute Γ as

$$\Gamma = \frac{1}{2}\mathcal{G}_\infty,$$

where

$$\begin{array}{l}\mathcal{G}_0 = in,\\ \mathcal{G}_{k+1} = \frac{1}{2}(\mathcal{G}_k + A_k^{-1}\mathcal{G}_k B_k^{-1}).\end{array}$$

In the case where all nodes are of type $III0$ we have the following relations:

$$\begin{array}{l} outgoing_{ij}^1 = incoming_{i+1,j}^3,\\ outgoing_{ij}^3 = incoming_{i-1,j}^1,\\ outgoing_{ij}^2 = incoming_{i,j-1}^4,\\ outgoing_{ij}^4 = incoming_{i,j+1}^2.\end{array} \tag{53}$$

for all $i = 1, \dots, s$, $j = 1, \dots, t$.

Moreover we have for $i = 1, \dots, s$ and $j = 1, \dots, t$,

$$incoming_{ij}^k = p_k \Gamma_{ij},\ \ k = 1, 2, 3, 4, \tag{54}$$

where $\sum_{k=1}^4 p_k = 1$.

Substituting equations (53) and (54) into (49) we get

$$p_3\Gamma_{i+1,j} + p_4\Gamma_{i,j-1} + p_1\Gamma_{i-1,j} + p_2\Gamma_{i,j+1} = \Gamma_{ij}. \tag{55}$$

We can write equation (55) as a matrix equation as follows.

$$\Gamma = (p_3 L(s)^T + p_1 L(s))\Gamma + \Gamma(p_4 L(t)^T + p_2 L(t)) + out,$$

i.e.

$$\left(p_1 A(s) + p_3 A(s)^T\right)\Gamma + \Gamma\left(p_2 A(t) + p_4 A(t)^T\right) + out = 0.$$

Then we get the following Sylvester equation

$$A^T\Gamma + \Gamma B^T + out = 0 \tag{56}$$

where $A = p_1 A(s)^T + p_3 A(s)$ and $B = p_2 A(s)^T + p_4 A(s)$.

Then we compute Γ as

$$\Gamma = \frac{1}{2}\mathcal{G}_\infty,$$

where

$$\begin{array}{l}\mathcal{G}_0 = out,\\ \mathcal{G}_{k+1} = \frac{1}{2}(\mathcal{G}_k + A_k^{-T}\mathcal{G}_k B_k^{-T}).\end{array}$$

7.3. **Exact solutions for the full Manhattan network with damaged node at position lk.** Assume a breakage at an interior node lk, thus $2 \le l \le s-1$ and $2 \le k \le t-1$. After some time a new equilibrium is reached where the flux, along lines departing from and arriving to node lk, is annihilated. In order to describe this situation we set $outgoing_{ij}^k = incoming_{ij}^k = 0$ for $k = 1, 2, 3, 4$ and some appropriate indices ij, in the equations (48),(53),(49),(50),(54) in previous section.

Therefore the number of variables is now $8st - 8$ and the number of constraints is $8st - 2s - 2t - 4$ for a total of $2s + 2t + 4$ degrees of freedom.

Consider a network with all nodes of type $II0$. In place of constraint (50) and in order to describe the fact that some variables are zero we set the following relation among unknowns:

$$O^i = \alpha^i \circ (O^1 + O^2 + O^3 + O^4), \quad i = 1, 2, 3, 4, \tag{57}$$

where

$$\alpha^1_{ij} = \begin{cases} 0 & \text{if } (i,j) = (l+1,k) \\ \frac{\alpha_1}{(1-\alpha_3)} & \text{if } (i,j) = (l-1,k) \\ \frac{\alpha_1}{(1-\alpha_4)} & \text{if } (i,j) = (l,k+1) \\ \frac{\alpha_1}{(1-\alpha_2)} & \text{if } (i,j) = (l,k-1) \\ \alpha_1 & \text{otherwise,} \end{cases}$$

$$\alpha^2_{ij} = \begin{cases} \frac{\alpha_2}{(1-\alpha_1)} & \text{if } (i,j) = (l+1,k) \\ \frac{\alpha_2}{(1-\alpha_3)} & \text{if } (i,j) = (l-1,k) \\ \frac{\alpha_2}{(1-\alpha_4)} & \text{if } (i,j) = (l,k+1) \\ 0 & \text{if } (i,j) = (l,k-1) \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$\alpha^3_{ij} = \begin{cases} \frac{\alpha_3}{(1-\alpha_1)} & \text{if } (i,j) = (l+1,k) \\ 0 & \text{if } (i,j) = (l-1,k) \\ \frac{\alpha_3}{(1-\alpha_4)} & \text{if } (i,j) = (l,k+1) \\ \frac{\alpha_3}{(1-\alpha_2)} & \text{if } (i,j) = (l,k-1) \\ \alpha_3 & \text{otherwise,} \end{cases}$$

$$\alpha^4_{ij} = \begin{cases} \frac{\alpha_4}{(1-\alpha_1)} & \text{if } (i,j) = (l+1,k) \\ \frac{\alpha_4}{(1-\alpha_3)} & \text{if } (i,j) = (l-1,k) \\ 0 & \text{if } (i,j) = (l,k+1) \\ \frac{\alpha_4}{(1-\alpha_2)} & \text{if } (i,j) = (l,k-1) \\ \alpha_4 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} &O^1 + O^2 + O^3 + O^4 = \\ &\quad L(s)O^3 + L(s)^T O^1 + O^4 L(t) + O^2 L(t)^T + in. \end{aligned} \tag{58}$$

From equations (57) we get:

$$\begin{aligned} &vec(O^i) = \\ &\quad diag(vec(\alpha^i)) \left(vec(O^1) + vec(O^2) + vec(O^3) + vec(O^4) \right), \end{aligned} \tag{59}$$

for $i = 1, 2, 3, 4$, and, from equation (58), we get:

$$A(s)O^3 + A(s)^T O^1 + O^4 A(t) + O^2 A(t)^T + in = 0, \tag{60}$$

hence

$$\begin{aligned} &\left(I(t) \otimes A(s)^T\right) vec(O^1) + \left(I(t) \otimes A(s)\right) vec(O^3) + \\ &\left(A(t) \otimes I(s)\right) vec(O^2) + \left(A(t)^T \otimes I(s)\right) vec(O^4) + vec(in) = 0. \end{aligned} \tag{61}$$

Equations (59), give:

$$\begin{aligned} vec(O^1) &= B_1(vec(O^2) + vec(O^3) + vec(O^4)), \\ vec(O^2) &= B_2(vec(O^1) + vec(O^3) + vec(O^4)), \\ vec(O^3) &= B_3(vec(O^1) + vec(O^2) + vec(O^4)), \\ vec(O^4) &= B_4(vec(O^1) + vec(O^2) + vec(O^3)), \end{aligned}$$

where $B_i = (I(st) - \alpha^i)^{-1}\alpha^i$, $i = 1, 2, 3, 4$, hence

$$\begin{aligned} &vec(O^1) = C_1(vec(O^3) + vec(O^4)), \\ &vec(O^2) = C_2(vec(O^3) + vec(O^4)), \\ &C_3 vec(O^3) + C_4 vec(O^4) = 0, \end{aligned}$$

where

$$\begin{aligned}C_1 &= B_1(I(st) + C_2),\\ C_2 &= (I(st) - B_2B_1)^{-1}B_2(I(st) + B_1),\\ C_3 &= (I(st) - B_3(C_1 + C_2)),\\ C_4 &= -B_3(C_1 + C_2 + I(st)).\end{aligned}$$

Substituting the expressions for $vec(O^1)$ and $vec(O^2)$ into equation (61), we get:

$$\begin{aligned}&\left(\left(I(t)\otimes A(s)^T\right)C_1 + (A(t)\otimes I(s))\,C_2 + (I(t)\otimes A(s))\right)vec(O^3)+\\ &\left(\left(I(t)\otimes A(s)^T\right)C_1 + (A(t)\otimes I(s))\,C_2 + \left(A(t)^T\otimes I(s)\right)\right)vec(O^4)+\\ &\qquad\qquad vec(in) = 0.\end{aligned}$$

By writing

$$T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right]$$

where

$$\begin{aligned}T_1 &= \left(\left(I(t)\otimes A(s)^T\right)C_1 + (A(t)\otimes I(s))\,C_2 + (I(t)\otimes A(s))\right),\\ T_2 &= \left(\left(I(t)\otimes A(s)^T\right)C_1 + (A(t)\otimes I(s))\,C_2 + \left(A(t)^T\otimes I(s)\right)\right),\\ T_3 &= C_3,\\ T_4 &= C_4,\end{aligned}$$

$w = [vec(O^3), vec(O^4)]^T$ and $z = [vec(in), 0]^T$, we get the following compact form for equations (59) and (61):

$$Tw + z = 0.$$

Now T_1 is invertible hence

$$vec(O^3) = -T_1^{-1}vec(in) - T_1^{-1}T_2vec(O^4)$$

and

$$\begin{aligned}&0 = T_3vec(O^3) + T_4vec(O^4) =\\ &-T_3T_1^{-1}vec(in) - (T_3T_1^{-1}T_2 - T_4)vec(O^4) =\\ &-T_3T_1^{-1}vec(in) + (T|T_1)vec(O^4)\end{aligned}$$

where $(T|T_1) = T_4 - T_3T_1^{-1}T_2$ is the Schur complement of T_1 in T. Hence

$$\begin{aligned}vec(O^3) &= S^3vec(in),\\ vec(O^4) &= S^4vec(in),\end{aligned}$$

where

$$\begin{aligned}S^4 &= (T|T_1)^{-1}T_3T_1^{-1},\\ S^3 &= -T_1^{-1} - T_1^{-1}T_2S^4 = -(T|T_2)^{-1}T_4T_2^{-1}.\end{aligned}$$

Moreover

$$\begin{aligned}vec(O^1) &= S^1vec(in),\\ vec(O^2) &= S^2vec(in),\end{aligned}$$

where

$$\begin{aligned}S^1 &= C_1(S^3 + S^4),\\ S^2 &= C_2(S^3 + S^4).\end{aligned}$$

Similarly to the oriented and circular manhattan cases we have the following:

Proposition 21. $S^i,\ i = 1,2,3,4$ *have corank* 1.

Now we are interested only in the rows and columns of S^i, for $i = 1,2,3,4$, relative to the outflows and inflows, i.e. we compute $H^i = RS^iC$. We first observe that $R = C^T$, $RR^T = I(2s + 2(t-2))$ and $R^TR = I(st)$.

Proposition 22. *For all $i = 1, 2, 3, 4$, H^i is always full rank apart from the cases:*

$$(l, k) \in \{(2, j),\ j = 1, \dots, t\}$$

for which H^3 has corank 1;

$$(l, k) \in \{(s-1, j),\ j = 1, \dots, t\}$$

for which H^1 has corank 1;

$$(l, k) \in \{(i, 2),\ i = 1, \dots, s\}$$

for which H^2 has corank 1;

$$(l, k) \in \{(i, t-1),\ i = 1, \dots, s\}$$

for which H^4 has corank 1. Moreover the above nodes are not critical.

Proof. Before showing the first part of the proposition we explain why the cited node are not critical. Take for example H^1. Such a map looses rank if there is breakage at nodes on the row $s-1$. We modeled a breakage by setting to zero all the flows incoming to and outgoing from that node. In particular for a breakage at node lk with $l = s-1$, we have that $O^1_{s,k} = 0$, hence necessarily H^1 has corank 1. The same reasoning applies to H^2, H^3 and H^4.

Next we prove the first part of the proposition. Denote by v_i the generator of the null space of α^i. More precisely we have

$$\begin{aligned} v_1 &= e_{(k-1)s+l+1} \\ v_2 &= e_{(k-2)s+l} \\ v_3 &= e_{(k-1)s+l-1} \\ v_4 &= e_{ks+l}. \end{aligned}$$

Now v_i is also the annihilator of B_i. Also, we notice that B_i has no eigenvalues $\lambda = -1$ hence $I(st) + B_i$ is invertible. For $i = 2$, $C_2 + I(st)$ is invertible indeed

$$\begin{aligned} &det(I(st) + C_2) = \\ &det((I(st) - B_2B_1)^{-1})det(I(st) - B_2B_1 + B_2(I(st) + B_1)) = \\ &det((I(st) - B_2B_1)^{-1})det(I(st) + B_2)) \neq 0. \end{aligned}$$

Moreover, if $C_2v = 0$ then $B_2(I(st) + B_1)v = 0$, i.e. $(I(st) + B_1)v \parallel v_2$ and $v = (I(st) + B_1)^{-1}v_2$. Now, recalling that v_2 has only one non zero component and B_1 is diagonal, then $v \parallel v_2$. Therefore v_2 is also the annihilator of C_2. Similarly we get that v_1 is also the annihilator of C_1 (since C_i is diagonal for all i.)

Next we analyze C_3. We have

$$\begin{aligned} &C_3 = I(st) - B_3(B_1 + (B_1 + I(st))C_2) = \\ &I(st) - B_3(B_1 + (B_1 + I(st))(I(st) - B_2B_1)^{-1}B_2(I(st) + B_1)) = \\ &I(st) - B_3(I(st) - B_2B_1)^{-1}((I(st) - B_2B_1)B_1 + (B_1 + I(st))B_2(I(st) + B_1)) = \\ &I(st) - B_3(I(st) - B_2B_1)^{-1}(B_1 + 2B_1B_2 + B_2) = \\ &(I(st) - B_2B_1)^{-1}\left((I(st) - B_2B_1) - B_3(B_1 + 2B_1B_2 + B_2)\right) \end{aligned}$$

with

$$\begin{aligned} &(I(st) - B_2B_1 - B_3B_1 - B_3B_2 - 2B_1B_2B_3) = \\ &\textstyle\prod_{i=1}^3 (I(st) - \alpha_i)^{-1} \Big(\prod_{i=1}^3 (I(st) - \alpha^i) - \alpha^1\alpha^2(I(st) - \alpha^3) - \alpha^1\alpha^3(I(st) - \alpha^2) \\ &\qquad -\alpha^2\alpha^3(I(st) - \alpha^1) - 2\alpha^1\alpha^2\alpha^3\Big) = \\ &\textstyle\prod_{i=1}^3 (I(st) - \alpha_i)^{-1} \Big(\prod_{i=1}^3 (I(st) - \alpha^i) - \alpha^1\alpha^2 - \alpha^1\alpha^3 - \alpha^2\alpha^3 + \alpha^1\alpha^2\alpha^3\Big) = \\ &\textstyle\prod_{i=1}^3 (I(st) - \alpha_i)^{-1}\alpha^4 \end{aligned}$$

where, in the last equality, we have used the fact that $\sum_{i=1}^4 \alpha^i = I(st)$. Then $C_3 v_4 = 0$. Finally, for C_4 we have:

$$C_4 = -(I(st) + B_3) + C_3,$$

with $I(st) + B_3 = (I(st) - \alpha^3)^{-1}$. Now

$$C_3 = (I(st) - B_2 B_1)^{-1} \left(\prod_{i=1}^{3} (I(st) - \alpha_i)^{-1} \alpha^4 \right)$$

where

$$\begin{aligned}&(I(st) - B_2 B_1) = \\ &\big((I(st) - \alpha^1)^{-1} (I(st) - \alpha^2)^{-1} ((I(st) - \alpha^1)(I(st) - \alpha^2) - \alpha^1 \alpha^2)\big) = \\ &(I(st) - \alpha^1)^{-1} (I(st) - \alpha^2)^{-1} \left(I(st) - \alpha^1 - \alpha^2\right).\end{aligned}$$

Then

$$\begin{aligned}&C_4 = (I(st) - B_2 B_1)^{-1} \left(\textstyle\prod_{i=1}^{3} (I(st) - \alpha_i)^{-1} \alpha^4 - (I(st) - B_2 B_1)(I(st) + B_3) \right) = \\ &(I(st) - B_2 B_1)^{-1} \left(\textstyle\prod_{i=1}^{3} (I(st) - \alpha_i)^{-1} (\alpha^4 - (I(st) - \alpha^1 - \alpha^2) \right) = \\ &(I(st) - B_2 B_1)^{-1} \textstyle\prod_{i=1}^{3} (I(st) - \alpha_i)^{-1} \alpha^3\end{aligned}$$

and $C_4 v_3 = 0$.

Finally

$$H^4 = (R(T|T_1)^{-1} R^T)(R T_3 R^T)(R T_1^{-1} R^T)$$

and $det(H^4) = 0$ if and only if $det(R T_3 R^T) = 0$.
Indeed $(R T_1^{-1} R^T)$ and $(R(T|T_1)^{-1} R^T)$ are full rank principal submatrices of T_1^{-1} and of $(T|T_1)^{-1}$ respectively.
Now $det(R T_3 R^T) = 0$ if and only if v_4 is a row of R. Similarly

$$H^3 = (R(T|T_2)^{-1} R^T)(R T_4 R^T)(R T_2^{-1} R^T)$$

and $det(H^3) = 0$ if and only if $det(R T_4 R^T) = 0$ and $det(R T_4 R^T) = 0$ if and only if v_3 is a row of R.

For H^1 and H^2 we have that

$$H^i = (R C^i R^T)(H^3 + H^4), \ i = 1, 2$$

and $det(H^i) = 0$ if and only if $det(R C^i R^T) = 0$ that is if and only if v_i is a row of R.

Now $v_1 = e_{(k-1)s+l+1}$ row of R means that $l = s - 1$ and $k = 1, \dots, t$,
$v_2 = e_{(k-2)s+l}$ row of R means that $k = 2$ and $l = 1, \dots, t$,
$v_3 = e_{(k-1)s+l-1}$ row of R means that $l = 2$ and $k = 1, \dots, t$ and finally
$v_4 = e_{ks+l}$ row of R means that $k = t$ and $l = 1, \dots, s$. □

Assume now that the network has all nodes of type $III0$. In place of constraint (54) and in order to describe the fact that some variables are zero we set the following relation among unknowns:

$$E^i = p^i \circ (E^1 + E^2 + E^3 + E^4), \quad i = 1, 2, 3, 4, \tag{62}$$

where

$$p^1_{ij} = \begin{cases} 0 & \text{if } (i,j) = (l+1, k) \\ \frac{p_1}{(1-p_3)} & \text{if } (i,j) = (l-1, k) \\ \frac{p_1}{(1-p_4)} & \text{if } (i,j) = (l, k+1) \\ \frac{p_1}{(1-p_2)} & \text{if } (i,j) = (l, k-1) \\ p_1 & \text{otherwise,} \end{cases}$$

$$p^2_{ij} = \begin{cases} \frac{p_2}{(1-p_1)} & \text{if } (i,j) = (l+1,k) \\ \frac{p_2}{(1-p_3)} & \text{if } (i,j) = (l-1,k) \\ \frac{p_2}{(1-p_4)} & \text{if } (i,j) = (l,k+1) \\ 0 & \text{if } (i,j) = (l,k-1) \\ p_2 & \text{otherwise,} \end{cases}$$

$$p^3_{ij} = \begin{cases} \frac{p_3}{(1-p_1)} & \text{if } (i,j) = (l+1,k) \\ 0 & \text{if } (i,j) = (l-1,k) \\ \frac{p_3}{(1-p_4)} & \text{if } (i,j) = (l,k+1) \\ \frac{p_3}{(1-p_2)} & \text{if } (i,j) = (l,k-1) \\ p_3 & \text{otherwise,} \end{cases}$$

$$p^4_{ij} = \begin{cases} \frac{p_4}{(1-p_1)} & \text{if } (i,j) = (l+1,k) \\ \frac{p_4}{(1-p_3)} & \text{if } (i,j) = (l-1,k) \\ 0 & \text{if } (i,j) = (l,k+1) \\ \frac{p_4}{(1-p_2)} & \text{if } (i,j) = (l,k-1) \\ p_4 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} &E^1 + E^2 + E^3 + E^4 = \\ &\quad L(s)^T E^3 + L(s)E^1 + E^4 L(t)^T + E^2 L(t) + out. \end{aligned} \tag{63}$$

From equations (62) we get:

$$\begin{aligned} &vec(E^i) = \\ &\quad diag(vec(p^i))\left(vec(E^1) + vec(E^2) + vec(E^3) + vec(E^4)\right), \end{aligned} \tag{64}$$

for $i = 1, 2, 3, 4$, and, from equation (63), we get:

$$A(s)^T E^3 + A(s)E^1 + E^4 A(t)^T + E^2 A(t) + out = 0, \tag{65}$$

hence

$$\begin{aligned} &\left(I(t) \otimes A(s)\right) vec(E^1) + \left(I(t) \otimes A(s)^T\right) vec(E^3) + \\ &\left(A(t)^T \otimes I(s)\right) vec(E^2) + \left(A(t) \otimes I(s)\right) vec(E^4) + vec(out) = 0. \end{aligned} \tag{66}$$

Equations (64), give:

$$\begin{aligned} vec(E^1) &= B_1(vec(E^2) + vec(E^3) + vec(E^4)), \\ vec(E^2) &= B_2(vec(E^1) + vec(E^3) + vec(E^4)), \\ vec(E^3) &= B_3(vec(E^1) + vec(E^2) + vec(E^4)), \\ vec(E^4) &= B_4(vec(E^1) + vec(E^2) + vec(E^3)), \end{aligned}$$

where $B_i = (I(st) - p^i)^{-1} p^i$, $i = 1, 2, 3, 4$, hence

$$\begin{aligned} &vec(E^1) = C_1(vec(E^3) + vec(E^4)), \\ &vec(E^2) = C_2(vec(E^3) + vec(E^4)), \\ &C_3 vec(E^3) + C_4 vec(E^4) = 0, \end{aligned}$$

where

$$\begin{aligned} C_1 &= B_1(I(st) + C_2), \\ C_2 &= (I(st) - B_2 B_1)^{-1} B_2 (I(st) + B_1), \\ C_3 &= (I(st) - B_3(C_1 + C_2)), \\ C_4 &= -B_3(C_1 + C_2 + I(st)). \end{aligned}$$

Substituting the expressions for $vec(E^1)$ and $vec(E^2)$ into equation (66), we get:

$$\begin{aligned} &\left(\left(I(t) \otimes A(s)\right) C_1 + \left(A(t)^T \otimes I(s)\right) C_2 + \left(I(t) \otimes A(s)^T\right)\right) vec(E^3) + \\ &\left(\left(I(t) \otimes A(s)\right) C_1 + \left(A(t)^T \otimes I(s)\right) C_2 + \left(A(t) \otimes I(s)\right)\right) vec(E^4) + vec(out) = 0. \end{aligned}$$

By writing

$$T' = \left[\begin{array}{c|c} T_1' & T_2' \\ \hline T_3' & T_4' \end{array}\right]$$

where

$$\begin{array}{l} T_1' = \left((I(t)\otimes A(s))\,C_1 + \left(A(t)^T \otimes I(s)\right)C_2 + \left(I(t)\otimes A(s)^T\right)\right), \\ T_2' = \left((I(t)\otimes A(s))\,C_1 + \left(A(t)^T \otimes I(s)\right)C_2 + (A(t)\otimes I(s))\right), \\ T_3' = C_3, \\ T_4' = C_4, \end{array}$$

$w' = [vec(E^3), vec(E^4)]^T$ and $z' = [vec(out), 0]^T$, we get the following compact form for equations (64) and (66):

$$T'w + z = 0.$$

Now T_1' is invertible hence

$$vec(E^3) = -(T_1')^{-1}vec(out) - (T_1')^{-1}T_2'vec(E^4)$$

and

$$\begin{array}{l} 0 = T_3'vec(E^3) + T_4'vec(E^4) = \\ -T_3'(T_1')^{-1}vec(out) - (T_3'(T_1')^{-1}T_2' - T_4')vec(E^4) = \\ -T_3'(T_1')^{-1}vec(out) + (T'|T_1')vec(E^4), \end{array}$$

where $(T'|T_1') = T_4' - T_3'(T_1')^{-1}T_2'$ is the Schur complement of T_1' in T'. Hence

$$\begin{array}{l} vec(E^3) = S^3vec(out), \\ vec(E^4) = S^4vec(out), \end{array}$$

where

$$\begin{array}{l} S^4 = (T'|T_1')^{-1}T_3'(T_1')^{-1}, \\ S^3 = -(T_1')^{-1} - (T_1')^{-1}T_2'S^4 = -(T'|T_2')^{-1}T_4'(T_2')^{-1}. \end{array}$$

Moreover

$$\begin{array}{l} vec(E^1) = S^1vec(out), \\ vec(E^2) = S^2vec(out), \end{array}$$

where

$$\begin{array}{l} S^1 = C_1(S^3 + S^4), \\ S^2 = C_2(S^3 + S^4). \end{array}$$

Similarly to the full manhattan case for nodes of type $II0$, we have the followings:

Proposition 23. $S^i,\ i = 1,2,3,4$ *have corank* 1.

Proposition 24. *For all* $i = 1,2,3,4$, H^i *is always full rank apart from the cases:*

$$(l,k) \in \{(2,j),\ j = 1,\dots,t\}$$

for which H^3 *has corank* 1*;*

$$(l,k) \in \{(s-1,j),\ j = 1,\dots,t\}$$

for which H^1 *has corank* 1*;*

$$(l,k) \in \{(i,2),\ i = 1,\dots,s\}$$

for which H^2 *has corank* 1*;*

$$(l,k) \in \{(i,t-1),\ i = 1,\dots,s\}$$

for which H^4 *has corank* 1. *Moreover the above nodes are not critical.*

The proof of Theorem 1.2, **F.M.N.** easily follows:

Proof. (**Theorem 1.2, F.M.N.**) By Propositions 22 and 24 each node of the network is not critical hence the network is robust. □

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Received August 2008; revised May 2009.

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