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TRACE THEOREMS FOR TREES AND APPLICATION TO THE HUMAN LUNGS

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Abstract. The aim of this paper is to develop a model of the respiratory system. The real bronchial tree is embedded within the parenchyma, and ventilation is caused by negative pressures at the alveolar level. We aim to describe the series of pressures at alveolae in the form of a function, and to establish a sound mathematical framework for the instantaneous ventilation process. To that end, we treat the bronchial tree as an infinite resistive tree, we endow the space of pressures at bifurcating nodes with the natural energy norm (rate of dissipated energy), and we characterise the pressure field at its boundary (i.e. set of simple paths to infinity). In a second step, we embed the infinite collection of leafs in a bounded domain $\Omega \subset \mathbb{R}^d$, and we establish some regularity properties for the corresponding pressure field. In particular, for the infinite counterpart of a regular, healthy lung, we show that the pressure field lies in a Sobolev space $H^s(\Omega)$, with $s \approx 0.45$. This allows us to propose a model for the ventilation process that takes the form of a boundary problem, where the role of the boundary is played by a full domain in the physical space, and the elliptic operator is defined over an infinite dyadic tree.

1. **Introduction, modelling aspects.**

The present work addresses some theoretical issues raised by the modelling of the bronchial tree and its interactions with the elastic medium in which it is embedded. The actual bronchial tree can be represented as an assembly of connected pipes, structured in a dyadic way, through which air flows. According to Poiseuille's law (which we shall assume valid in all branches, see [16, 17] for more details on this assumption), the flow rate Q through a pipe is proportional to the drop in pressure between its ends, which can be expressed in the manner of a Ohmic law:

$$P_{in} - P_{out} = RQ,$$

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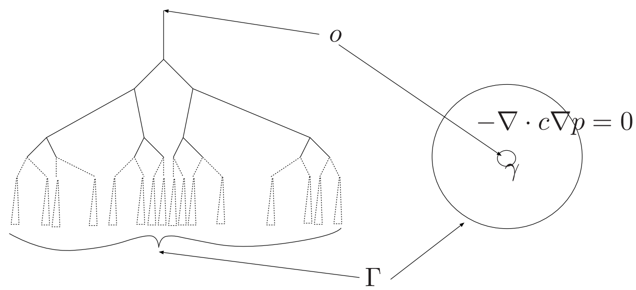

FIGURE 1. Analogy with Darcy problem.

where R is the resistance of the pipe, which varies with L/D^4 (where L is the length, and D the diameter). According to measurements detailed in Weibel [22], all pipes have a similar shape, and their size varies with λ^n, where n denotes the generation and λ is a parameter close to 0.85 for a healthy lung.

As far as fluid flow is concerned, the respiratory tract can therefore be treated as a resistive finite tree with 23 generations, with geometrically increasing resistances. The starting point of the present approach is an extrapolation of the actual tree to an infinite dyadic tree. Gradient and divergence operators can be defined for such a tree, so that modelling of the air flow takes the form of a Darcy problem over the tree. Figure 1 illustrates this analogy. The tree itself plays the role of a domain occupied by a porous medium through which a fluid flows according to Darcy's law, and the root of the tree may be compared to a place in the domain at which the pressure is set to 0. The typical situation is as follows: a negative pressure is applied on the boundary Γ of the porous domain, driving fluid through the domain from γ to Γ.

Our approach has two main objectives :

1. To model the instantaneous ventilation process in a proper mathematical framework (irrigation of the parenchyma driven by an applied alveolar pressure field) as a Dirichlet to Neuman operator defined for a Darcy-like problem over the infinite tree in the same way as in the standard Darcy problem. As a first step, we shall build this operator for functions defined in the abstract set of ends $\{0,1\}^{\mathbb{N}}$. The second step will consist in embedding the set of ends onto a domain of the physical space Ω, and to extend the definition of this operator to the embedded situation.
2. To explore the design of new types of constitutive models for the lung considered as a viscoelastic material. The parenchyma is a complex medium. In part, its complexity derives from the fact that inner dissipation is due to the flow of air expelled or driven in the alveolae by non-volume-preserving deformations throughout the dyadic bronchial tree. The special character of these dissipation effect calls for non-standard damping models. A first step towards accounting for this type of dissipation effect was proposed in [11] for the one-dimensional problem. For higher dimensions, the description of trace spaces for pressure and fluxes which we propose here makes it possible to propose a new class of constitutive models for these kinds of materials.

In the PDE case, the fact that derivatives of a function are bounded in some way (e.g. for functions in Sobolev spaces) makes it possible to define their trace on zero measure manifolds. In the present case of a tree, a similar approach can be taken to define a trace on the set of ends of the dyadic tree, which is represented as $\Gamma = \{0,1\}^{\mathbb{N}}$. This question is strongly related to the properties of Dirichlet functions and Harmonic functions with finite energy at the end of the tree, as discussed extensively in the literature. In the 1970's, Cartier in [8] studied Harmonic functions via the Green Kernel, introduced the notion of Martin boundary, and gave a characterisation of positive harmonic functions; subsequently many authors provided results on this subject (see Woess [24], Ancona [4], Mouton [18]). That functions with finite energy converge almost surely along random walks is also well known, as discussed in articles by Yamasaki [21], Ancona-Lyons-Peres [5].

In the present approach, which aims at embedding the set of ends Γ (geometrical boundary of the tree as defined in [18]) in the physical space, we are especially interested in describing intrinsic regularity of trace functions. To that end, we propose a new approach that involves the construction of an explicit basis of harmonic functions in the energy space, which allows to identify the trace space as a subset of $L^2(\Gamma)$. We will see that, in the case of geometric trees, trace functions can be described accurately in terms of A^r regularity (Cohen [9]), where the parameter r depends on the resistances of the tree. As the real tree is embedded in a three dimensional medium (the parenchyma), it is not clear *a priori* whether the regularity in the tree is likely to induce any kind of regularity for the corresponding field in the physical space. In particular, two ends of the tree can be very close to each other in the physical space, while being far apart in the tree. We prove here that, under some conditions on the way the tree irrigates the domain, the embedded pressure field presents indeed some correlation in space, which can be expressed in a precise way in terms of fractional Sobolev regularity.

As for similar approaches in the context of PDE's, let us also mention [2, 3], in which the authors propose a theoretical framework and associated numerical tools to handle elliptic problems set on bi-dimensional tree-like fractal domains. Their approach uses the auto-similar structure of the domain intensively. Our framework is different: we start with a simpler model (Poiseuille's law in a network of pipes), with a much lighter set of assumptions. As a consequence, our model can handle non-homogeneous situations, and all considerations related to the flowing of air through branches are completely independent from geometric considerations (only dimensions of the pipes are involved, through the value of local resistance): geometrical aspects are involved only in the embedding process of the boundary leafs onto a physical domain.

Let us add some remarks related to (apparently) similar questions in the context of Sobolev spaces to illustrate the difficulties pertaining to these objectives in the context of trees. As will become apparent, the natural norm (i.e. based on energy considerations) for the tree is some kind of H^1 semi-norm (whose square is the rate of dissipated energy), to which one adds a term involving the value at a vertex to handle constant fields. A natural counterpart in $\mathbb{R}^d$ of this norm is defined as follows

$$\|u\|^2 = \int_B |u|^2 + \int_{\mathbb{R}^d} |\nabla u|^2,$$

where B is the d-dimensional unit ball. Let us denote by H^1 the corresponding Hilbert space (set of all those functions in L^2_{loc} for which the previous quantity is bounded). The question of how such functions may behave at infinity can be formulated by introducing H^1_0 as the closure of $C^\infty_c(\mathbb{R}^d)$ in H^1, and the quotient space $\tilde{H}^{1/2} = H^1/H_0^1$. The first question we will address for the resistive tree is as follows: is $\tilde{H}^{1/2}$ trivial or not ? The situation in $\mathbb{R}^d$ is quite poor, as, when the space is not trivial, it is in fact almost trivial. More precisely, the situation is as follows:

for $d = 1, 2$: no trace can be defined at infinity, $\tilde{H}^{1/2} = \{0\}$;

for $d \geq 3$: the trace space $\tilde{H}^{1/2}$ is not trivial, but it is one-dimensional. The only non-trivial instance corresponds to constant functions.

The poorness of $\tilde{H}^{1/2}$ for $d \geq 3$ is due to the fact that the finite character of the H^1 semi-norm imposes some rigidity in the transverse direction so that only asymptotically constant functions qualify (see [10] for a detailed proof, in the case $d = 3$). A richer situation can be obtained by integrating some non-isotropic weight in the norm, in order to lower the correlation in the transverse direction. This transverse de-correlation corresponds to the native situation for trees: the very tree-structure allows fields to exhibit highly different values at vertices belonging to the same generation (as soon as those vertices are far away from each other with respect to the natural distance in the tree). As a consequence, the quasi-trivial situation (one dimensional trace space) will not be met for the trees we intend to explore: the trace space will be either trivial (case $H^1 = H_0^1$), or infinite dimensional.

The paper is structured as follows. In Section 2, we revisit standard results for functional spaces on general infinite networks: we give an abstract definition of the trace space we aim to identify, and we characterise dyadic trees for which the abstract trace space is not trivial. In Section 3, the abstract trace space introduced in the previous section is identified with a functional space on the boundary $\Gamma = \{0,1\}^{\mathbb{N}}$, and we define a counterpart to the trace of the normal derivative on Γ. Those considerations are applied to a geometric tree for a rigorous definition of the Dirichlet to Neuman and Neuman to Dirichlet operators associated with the Poisson problem on the tree. In section 4, the tree is embedded in a domain of $\mathbb{R}^N$ (as the bronchial tree is embedded in the parenchyma), and we investigate the regularity of the corresponding pressure fields, as well as whether it is possible to define N to D and D to N operators in the embedded situation. Finally, in Section 5, we discuss how these considerations can be applied in modelling the human lungs.

2. **Functional spaces in infinite trees.**

This section contains mainly alternative proofs to some standard properties for Sobolev spaces on trees (see e.g. [19]).

2.1. **General setting.**

In what follows we shall use notation T to define a general network, as those considerations are to be applied to special networks, namely dyadic trees. Yet, we do not suppose in this first section that T is actually a tree.

Let $T = (V, E, r)$ denote a resistive network: V is the set of vertices (possibly infinite), E, subset of $V \times V$, is the set of edges, and $r \in (0, +\infty)^E$ a resistance field. We will follow the convention that edges are counted only once in E, that is to say $(x, y) \in E \Longrightarrow (y, x) \notin E$. As for resistances, we will of course consider that $r(y, x)$ is defined and equal to $r(x, y)$ as soon as $(x, y) \in E$. We will also use of the field of conductances, defined by $c(e) = 1/r(e)$ for any $e \in E$.

We shall simply assume in this introductory section that the number of vertices is countable, and the number of neighbours is uniformly bounded:

$$\sup_{x \in V} \sharp\, \{y\,,\ (x, y) \in E\} < +\infty.$$

As the only functions defined over edges are fluxes, we whall design by $\mathbb{R}^E$ the set of skew symmetric functions over edges:

$$u(y, x) = -u(x, y) \text{ as soon as } (x, y) \in E.$$

Definition 2.1. For any $u \in \mathbb{R}^E$ (skew symmetric flux distribution as defined above), we define

$$\partial u \in \mathbb{R}^V, \ \partial u(x) = \sum_{y \sim x} u(x, y),$$

where $y \sim x$ stands for $(x, y) \in E$ or $(y, x) \in E$. Symmetrically, for any field $p \in \mathbb{R}^V$ (collection of pressures at nodes), we define

$$\partial^\star p \in \mathbb{R}^E, \ \partial^\star p(x, y) = -\partial^\star p(y, x) = p(y) - p(x).$$

Remark 1. The operator ∂ can be seen as the divergence operator on the tree. Notice that $\partial u(x) > 0$ means that some fluid enters the domain at x.

Consider now that some viscous fluid flows through the edges of the network according to Poiseuille's law, and that some fluid is injected at constant rate 1 through some vertex o. This model takes the form of a discrete Darcy problem:

$$\begin{cases} u + c\partial^\star p = 0, \\ \partial u = \delta_o, \end{cases}$$

where c is the conductance field $c(e) = 1/r(e)$, δ_o is 1 at o, and vanishes everywhere else.

Remark 2. Note that $\partial^\star$ is formally the *opposite* of the adjoint of ∂, the same way the divergence operator is the opposite of the adjoint of the gradient operator.

We are interested in solutions with finite energy, or more precisely finite instantaneous loss of energy over the network by viscous dissipation. It leads to energy spaces:

$$L^2(T) = \{u \in \mathbb{R}^E, \ \|u\|_2^2 = \sum_{(x,y)\in E} r(x, y) u(x, y)^2 < +\infty\},$$

and its pressure counterpart

$$H^1(T) = \{p \in \mathbb{R}^V, \ |p|_1^2 = \sum_{(x,y)\in E} c(x, y)[p(y) - p(x)]^2 < +\infty\}.$$

The flux space is a standard weighted ℓ^2 space, and $c\partial^\star$ is an isometry from $H^1(T)/\mathbb{R}$ onto $L^2(T)$. We shall endow $H^1(T)$ with the norm

$$\|p\|_1^2 = c(o)p(o)^2 + |p|_1^2.$$

Remark 3. The way to handle constant functions may affect significantly the result. In the spirit of [20], an alternative choice would consist in replacing $p(o)^2$ by an L^2-like quantity

$$\sum_{x \in V} \rho(x) p(x)^2,$$

where ρ is for example the average resistance (i.e. average length, see Remark 5 below) of edges that contain x. This choice would lead to different results, much more similar to the $\mathbb{R}^d$ context: in particular, $H_0^1(T)$ identifies to $H^1(T)$ as soon as the diameter of the tree (i.e. the maximal resistance of a single path to infinity) is infinite. The choice we made is therefore essential, and justified by modelling considerations: the L^2 norm of the pressure does not make sense in terms of energy, whereas the H^1 semi-norm is the rate of dissipated energy.

Definition 2.2. We denote by $D(T)$ the set of finitely supported functions in $\mathbb{R}^V$, and we define $H_0^1(T)$ as the closure in $H^1(T)$ of $D(T)$.

Let us now formulate a discrete counterpart of the Green formula (see [1]).

Proposition 1. *(Green formula on the whole network)*
Let $p \in H^1(T)$ and $q \in H_0^1(T)$ be given. For any sequence (q_n) in $D(T)$ which converges to q, the quantity

$$-\sum_{x\in V} \partial c \partial^\star p(x) q_n(x)$$

converges to a value which does not depend on the chosen sequence. This defines $-\partial c\partial^\star p$ as an element of $H^{-1}(T) = (H_0^1(T))'$, and it holds

$$\langle -\partial c\partial^\star p \, , q \rangle_{(H^{-1}(T), H_0^1(T))} = \sum_{e\in E} c(e)\partial^\star p(e)\partial^\star q(e). \tag{1}$$

Proof. For $q_n \in D(T)$, a summation by parts gives

$$-\sum_x \partial c\partial^\star p(x) q_n(x) = \sum_e c(e)\partial^\star p(e)\partial^\star q_n(e),$$

and the right-hand side converges to $\sum_e c(e)\partial^\star p(e)\partial^\star q(e)$. □

Remark 4. Note the absence of boundary terms in the Green formula. It is due to the fact that all vertices of T are considered as *inner* vertices, including those which are involved in a single connection (like the root of the dyadic tree which we introduce in the next section). A Green formula with boundary terms can be recovered by introducing a partition $V = \mathring{V} \cup \partial V$, where ∂V stands for any subset of V (arbitrarily considered as the boundary of V). For any $x \in \partial V$, the quantity $-\partial c\partial^\star p(x)$ is the flux entering the network. Denoting by $g(x) = \partial c\partial^\star p(x)$ the flux *getting out* of the network (g plays the role of $c\partial p/\partial n$ in the context of PDE's), we may write (1) as follows:

$$-\sum_{x\in \mathring{V}} \partial c\partial^\star p(x) q(x) = \sum_{e\in E} c(e)\partial^\star p(e)\partial^\star q(e) + \sum_{x\in\partial V} g(x)q(x).$$

This formula is particularly adapted to the case of a network with some fluid entering (or flowing out) at some vertices $x \in \partial V$, and conservative (i.e. with no leak) at vertices in $\mathring{V}$. Assuming $p \in H_0^1$ (the case of fluid entering the network at infinity is postponed to the next sections), and taking $q = p$, we get the instantaneous energy balance

$$\sum_{e\in E} c(e)|\partial^\star p(e)|^2 = \sum_{e\in E} r(e)|u(e)|^2 = -\sum_{x\in\partial V} g(x)p(x),$$

where the left-hand side is the rate of dissipated energy within the network, and the right-hand side is the power of external forces.

The abstract trace space of $H^1(T)$ can be defined in this general framework as follows:

Definition 2.3. (Abstract trace space)
Let T be a resistive network. The abstract trace space is defined as $H^1(T)/H_0^1(T)$. We denote by $\tilde{\gamma}_0$ the abstract trace operator (canonical surjection).

Notice that H^1/H_0^1 is trivial for any finite network.

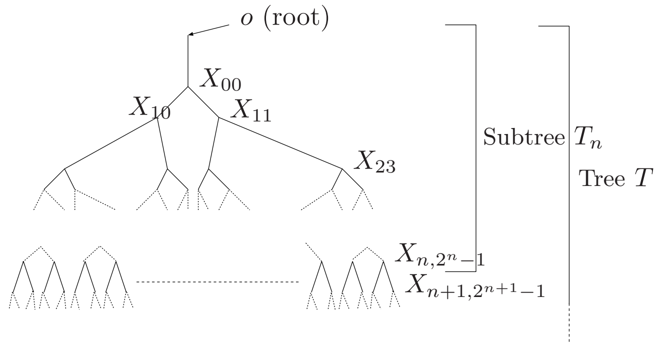

FIGURE 2. Tree T

2.2. **Abstract trace space on a dyadic tree.** We investigate here whether the abstract trace space is trivial or not, in the case of a dyadic tree. The main properties presented in this section are also established in [19], in a slightly different formalism. We propose here alternative proofs, some parts of which will be used in the next section to define trace operators.

From now on, T designs an infinite dyadic tree with root o (see Fig. 2),

$$V = \{o\} \cup \{X_{nk}\,,\ n \in \mathbb{N}\,,\ 0 \le k \le 2^n - 1\}\,,$$

$$E = \{(o, X_{00})\} \cup \{(X_{nk}, X_{n+1,2k})\ ,\ (X_{nk}, X_{n+1,2k+1})\ ,\ n \in \mathbb{N}\,,\ 0 \le k \le 2^n - 1\}\,.$$

Definition 2.4. A tree such that the resistance of edges is constant for each generation is called *regular*. We denote by r_n the common value at generation n:

$$r_n = r\left(X_{n-1,k}, X_{n,2k}\right) = r\left(X_{n-1,k}, X_{n,2k+1}\right)\ ,\ n \ge 1\,,\ 0 \le k \le 2^{n-1} - 1,$$

and $r_0 = r(o, X_{00})$.

Remark 5. We presented in the introduction some considerations in the context of $\mathbb{R}^d$. In the comparison with this standard context, it is clear that n plays the role of the radial component (distance to the origin), and k the role of what we called the transverse direction. To detail a bit this comparison, let us mention here that the set of vertices of the tree can be seen as a metric space for the distance canonically induced by $\text{dist}(x, y) = r(x, y)$ for any two connected vertices. Now considering T as a one-dimensional manyfold $\mathcal{T}$, and identifying a pressure field $p \in \mathbb{R}^V$ with a piecewise affine function over $\mathcal{T}$, we get (s is the curvilinear abscissa)

$$\int_{\mathcal{T}} \left|\frac{\partial p}{\partial s}\right|^2 ds = \sum_{(x,y)\in E} r(x, y)\frac{|p(y) - p(x)|^2}{r(x, y)^2},$$

which is exactly the quantity we defined as $|p|_1^2$. The transverse decorrelation we mentioned in the introduction is due to the fact that two vertices of the same generation may have very little common ancestors, so that their distance may be large.

Definition 2.5. Let $n \ge 1$. We define T_n as the subtree of T with same root o and height n (see Fig. 2), we denote by $V(T_n)$ (resp. $E(T_n)$) its set of vertices (resp. edges), and by R_n the equivalent resistance of T_n, defined as follows: Consider that a uniform zero pressure is applied at the leafs of the tree, whereas its root is maintained at pressure 1. The resistance R_n is defined as the reciprocal of the flux flowing out of o, so that Poiseuille's law

$$1 = p_{\text{root}} - p_{\text{leafs}} = R_n \times (\text{global flux})$$

is verified.

Note that (R_n) is an increasing sequence, which leads us to the following definition:

Definition 2.6. The global resistance (or simply resistance when no confusion is possible) of the infinite tree T is defined by

$$R = \lim_{n\to+\infty} R_n \ \in \ (0, +\infty].$$

Remark 6. The resistance of a regular tree is simply $R = \sum r_n/2^n$.

Before stating the main result of this section, which shows that trees with $R < +\infty$, and only those, exhibit a non-trivial trace space, we present some definitions and lemmas which will be used also in the following sections.

Definition 2.7. The capacity of o with respect to T_n is defined by

$$C(o, T_n) = \inf\{|q|_1^2\,,\ q(o) = 1 \text{ and } q(X_{nk}) = 0\,,\ 0 \le k \le 2^n - 1\}.$$

Lemma 2.8. *Let T_n be a finite resistive tree of height n. It holds*

$$C(o, T_n) = |p_n|_1^2$$

with p_n satisfying

$$\begin{cases} p_n(o) & = & 1 \\ p_n(X_{nk}) & = & 0 \quad \forall k\,,\ 0 \le k \le 2^n - 1 \\ -\partial c \partial^\star p_n(X_{jk}) & = & 0 \quad \forall j \le n-1\,,\ 0 \le k \le 2^j - 1. \end{cases} \tag{2}$$

Moreover $|p_n|_1^2 = 1/R_n$.

Proof. Let p_n be defined as above. For any q satisfying $q(o) = 1$, $q(X_{nk}) = 0$ for $0 \le k \le 2^n - 1$, we introduce $d = q - p_n$. It holds

$$|q|_1^2 = |p_n + d|_1^2 = |p_n|_1^2 + |d|_1^2 + 2(p_n\,, d)_1. \tag{3}$$

As d is 0 but at internal nodes, Green formula (see Prop. 1) gives

$$(p_n\,, d)_1 = -\langle \partial c \partial^\star p\,, d\rangle = 0,$$

which yields $|q|_1^2 \ge |p_n|_1^2$, so that $C(o, T_n) \ge |p_n|_1^2$, thus equality holds.

Now $|p_n|_1^2 = 1/R_n$ simply expresses the fact that the rate of dissipated energy is the square of the pressure jump divided by the resistance. More precisely, using again Prop. 1 and the fact that p_n vanishes at the leafs,

$$|p_n|_1^2 = (p_n\,, p_n)_1 = -\langle \partial c \partial^\star p_n\,, p_n\rangle = -p_n(o)\partial c \partial^\star p_n(o). \tag{4}$$

Noting that $-\partial c \partial^\star p_n(o)$ is just the global flux, and that $p_n(o) = 1$, we obtain $|p_n|_1^2 = 1/R_n$, by definition of R_n (see Definition 2.5). □

The previous considerations extend straightforwardly to infinite trees:

Definition 2.9. Let T be an infinite resistive tree. The capacity of o w.r.t. T is defined by

$$C(o, T) = \inf\left\{|p|_1^2\,,\ p \textit{ finitely supported and } p(o) = 1\right\}.$$

Lemma 2.10. *Let T be an infinite resistive tree. Then $C(o,T) = R^{-1} \in [0,+\infty)$.*

Proof. It is a consequence of $C(o,T) = \lim C(o,T_n) = \lim 1/R_n = 1/R$. □

Lemma 2.11. *Let $\mathbb{1}_T$ be the function defined on V identically equal to 1. It holds*

$$R = +\infty \Leftrightarrow \mathbb{1}_T \in H_0^1(T).$$

Proof. If $\mathbb{1}_T \in H_0^1(T)$, then $C(o,T) = 0$, and the resistance is infinite. If the resistance is $+\infty$, then the field $p_n \in D(T)$ (defined on T_n by (2) and extended by 0 outside T_n) takes value 1 at o, and its H^1 semi-norm goes to 0 as n goes to infinity, so that its distance to $\mathbb{1}_T$ goes to 0. □

Theorem 2.12. *Let R be the global resistance of the resistive tree (V,E,r). We have*

$$R = +\infty \Leftrightarrow H_0^1(T) = H^1(T).$$

Proof. The sufficient condition is straightforward: identity $H^1(T) = H_0^1(T)$ implies $\mathbb{1}_T \in H_0^1(T)$ so that, by Lemma 2.11, the resistance is infinite.

Suppose now $R = +\infty$. The purpose is to establish that any $f \in H^1(T)$ can be approximated by a sequence (f_ℓ) of finitely supported functions. Applying Lemma 2.11 gives $\mathbb{1}_T \in H_0^1(T)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of finitely supported functions converging to $\mathbb{1}_T$, so that

$$\lim_{n\to+\infty} |u_n|_1 = 0. \tag{5}$$

Let f be in $H^1(T)$ and let ε be nonnegative. There exists $n \in \mathbb{N}^\star$ such that

$$\sum_{E\setminus E(T_n)} c(x,y)(f(y) - f(x))^2 < \varepsilon, \tag{6}$$

where $E(T_n)$ is the set of edges of T_n. The construction of the sequence $(f_\ell)_{\ell\in\mathbb{N}}$ is then possible. If $x \in V(T_{n-1})$ (set of vertices of T_n except generation n), we set $f_\ell(x)$ equal to $f(x)$. Otherwise there exists k in $\{0,\ldots,2^n-1\}$ such that $x \in T_{nk}$, which is the infinite subtree rooted in X_{nk} (see Def. 3.1 and Fig. 3.1). We then set

$$f_\ell(x) = f(X_{nk})u_\ell(x)/\max(1/2, u_\ell(X_{nk})).$$

As u_ℓ converges pointwisely to 1, one has $\max(1/2, u_\ell(X_{nk})) = u_\ell(X_{nk})$ for ℓ sufficiently large, so that f_ℓ identifies with f on T_n (including the n^{th} generation). The function f_ℓ is finitely supported by construction. Let us now establish that $|f - f_\ell|_1$ converges to zero. The quantity $|f - f_\ell|_1^2$ reduces to a sum over $E \setminus E(T_n)$:

$$|f - f_\ell|_1 = |f - f_\ell|_{1,T\setminus T_n} \le |f|_{1,T\setminus T_n} + |f_\ell|_{1,T\setminus T_n}.$$

The first contribution is less than $\sqrt{\varepsilon}$ by definition of n. The second one can be estimated

$$\begin{aligned} \sum_{E\setminus E(T_n)} c(x,y)(f_\ell(x) - f_\ell(y))^2 \quad &\le \quad \sum_{k=0}^{2^n-1} \frac{f(X_{nk})^2}{u_\ell(X_{nk})^2} \sum_{E(T_{nk})} c(x,y)(u_\ell(x) - u_\ell(y))^2 \\ &\le \quad \max_{k\in\{0,\ldots,2^n-1\}} \frac{f(X_{nk})^2}{u_\ell(X_{nk})^2} |u_\ell|_1^2. \end{aligned}$$

Now as u_ℓ converges uniformly (for n fixed) to 1 on the set of leafs of T_n, the maximum in the previous expression is bounded. As $|u_\ell|_1^2$ goes to 0 as ℓ goes to infinity, the quantity can be controlled by ε for ℓ sufficiently large, which ends the proof. □

We finish this section by establishing a Poincaré-like inequality for functions in $H^1_0(T)$, and some general properties which will be useful in the following.

Proposition 2. *(Poincaré inequality)*
Let T be a tree with finite resistance. Then there exists $C > 0$ satisfying for each $f \in H^1_0(T)$

$$\|f\|_1 \leq C|f|_1. \tag{7}$$

Consequently, $|\cdot|_1$ is a norm on $H^1_0(T)$.

Proof. For any $f \in D(T)$, there exists n such that f is supported within T_n so that, by Lemma 2.8,

$$f(0)^2 \leq R_n|f|_1^2 \leq R|f|_1^2.$$

The density of $D(T)$ in $H^1_0(T)$ ends the proof (with a constant $C = \sqrt{1+R}$). ☐

Definition 2.13. The set of finite energy harmonic functions is defined by

$$H^1_\Delta(T) = \{p \in H^1(T) / \ \partial c \partial^\star p = 0\}.$$

Lemma 2.14. *We assume R finite. Then $H^1_0(T) \cap H^1_\Delta(T) = \{0\}$.*

Proof. Let f be in $H^1_0(T) \cap H^1_\Delta(T)$. Green formula (Prop. 1) gives $|f|_1 = 0$. As $|\cdot|_1$ is a norm on $H^1_0(T)$, it yields $f \equiv 0$. ☐

It is now possible to define properly an abstract non-homogeneous Dirichlet problem. Let T be an infinite dyadic tree, and let $\tilde{g} \in H^1/H^1_0$ be given. We consider the following problem:

$$\begin{cases} p \in H^1(T), \\ -\partial c \partial^\star p = \delta_o, \\ \tilde{\gamma}_0(p) = \tilde{g}. \end{cases} \tag{8}$$

As a first step, we establish well-posedness (in the finite resistance case) of the homogeneous Dirichlet problem

$$\begin{cases} p \in H^1_0(T), \\ -\partial c \partial^\star p = \delta_o. \end{cases} \tag{9}$$

Theorem 2.15. *Homogeneous Dirichlet problem (9) is well-posed if and only if the global resistance is finite.*

Proof. Let us assume $R < +\infty$. We consider the sequence of fields $q_n = R_n p_n$ (extended by 0 outside T_n), where p_n is defined by (2). The H^1 semi-norm of q_n is $\sqrt{R_n}$, so that it is bounded in $H^1(T)$ (the value at o is R_n), therefore one can extract a subsequence (still denoted by (q_n)) which converges weakly towards $q \in H^1_0(T)$. As weak convergence implies pointwise convergence, one has

$$-\partial c \partial^\star q = \delta_o.$$

Uniqueness is a direct consequence of Lemma 2.14 (the only harmonic function in $H^1_0(T)$ is 0).

Let us now assume that the resistance is $+\infty$, and that a solution p to Problem (9) exists. We denote by q_n the projection of p onto the affine subspace of all those fields q which take value $p(o)$ at o, and which vanishe outside T_n:

$$q(X_{mk}) = 0 \quad \forall m \geq n\,,\ 0 \leq k \leq 2^m - 1.$$

This projection is performed with respect to the H^1 norm, which amounts to minimize the H^1 semi-norm, as the value at o is prescribed. As p is harmonic (if one excludes the root o), it turns out that q_n identifies (up to a multiplicative constant) to the field p_n which was built in Lemma 2.8 (see Eqs. (2)). More precisely, $q_n = p(o)p_n$. As $|q_n|_1 = p(o)/\sqrt{R_n}$, it goes to 0 as n goes to infinity. On the other hand, as p is in $H_0^1(T)$, q_n converges strongly to p, so that $|p|_1 = 0$, and p is constant, which is in contradiction with any non-zero flux flowing through o. ☐

Proposition 3. *(Royden decomposition (Soardi [19]))*
We assume $R < +\infty$. For each $f \in H^1(T)$ there exists a unique $q \in H_0^1(T)$ and a unique $p \in H_\Delta^1(T)$ such that

$$\begin{cases} f = p + q, \\ |f|_1^2 = |p|_1^2 + |q|_1^2. \end{cases} \tag{10}$$

Proof. As $(\cdot\,,\cdot)_1$ is a scalar product on $H_0^1(T)$ (see Prop. 2), there exists a unique $q \in H_0^1(T)$ which minimizes the distance (for the H^1 semi-norm) over $H_0^1(T)$. Optimality conditions ensure harmonicity of $p = f - q$. ☐

Theorem 2.16. *We suppose $R < +\infty$. Let $\tilde{g} \in H^1/H_0^1$ be given. Then there exists a unique solution to the non-homogeneous Dirichlet problem: Find $p \in H^1(T)$ such that*

$$\begin{cases} -\partial c \partial^\star p &= \delta_o \\ \tilde{\gamma}_0(p) &= \tilde{g}. \end{cases} \tag{11}$$

Proof. We denote by g the harmonic instance of $\tilde{g}$ (given by Prop. 3), and by p_∞ the solution to the homogeneous Dirichlet Problem (9). Then $p_\infty + g$ is a solution to (11). Uniqueness is a consequence of the uniqueness for the homogeneous problem. ☐

In the context of modelling the respiratory process, we will consider the problem with natural boundary conditions at the root:

Corollary 1. *Let $\tilde{g} \in H^1/H_0^1$ be given. There exists a unique solution p to*

$$\begin{cases} -\partial c \partial^\star p &= 0 \quad \text{in } T \setminus \{o\} \\ p(o) &= 0 \\ \tilde{\gamma}_0(p) &= \tilde{g}. \end{cases}$$

Proof. Let p_1 denote the solution to Eq. (11), and p_0 the solution to homogeneous Dirichlet problem (9). The solution to this new problem is then defined in a unique way as $p_1 - (p_1(o)/p_0(o))p_0$. ☐

3. **Trace theorems.**

3.1. **Preliminaries.**

In the context of PDEs, trace theorems rely on an extension by density of the notion of restriction to a subset, for regular functions (typically in $\mathcal{D}(\overline{\Omega})$). In the case of a tree there is no natural counterpart for the space of regular functions defined beyond the tree. The strategy we propose is based on the construction of a Hilbert basis for the set of harmonic function with finite energy. The basis functions are in some sense asymptotically piecewise constant at infinity, so that a trace can be defined canonically. The trace operator is then defined by density.

Proposition 4. *We assume $R < +\infty$. We define $I : H^1/H_0^1 \longrightarrow H_\Delta^1(T)$ as the operator which maps any $\tilde{q} \in H^1/H_0^1$ onto its unique harmonic instance (see Prop. 3). Then I is a bicontinuous isomorphism.*

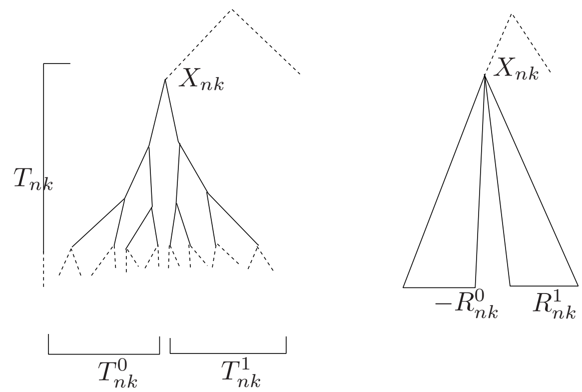

FIGURE 3. Subtrees T^0_{nk}, T^1_{nk}, and function ψ_{nk}.

Proof. The mapping I is a one-to-one and onto linear mapping by Prop. 3. As it is continuous between Hilbert spaces, it is bicontinuous by Banach-Steinhaus Theorem. □

Let us now construct a Hilbert basis of $H^1_\Delta(T)$.

Definition 3.1. Let T be an infinite resistive tree, $n \in \mathbb{N}$, $k \in \{0, \ldots, 2^n - 1\}$ and T_{nk} the associated infinite subtree with root X_{nk}. Subtree T_{nk} can be divided into two infinite subtrees denoted by T^0_{nk} and T^1_{nk} with $V(T^0_{nk})$ and $V(T^1_{nk})$ the corresponding set of vertices. Let R^0_{nk} and R^1_{nk} be defined as the corresponding global resistances.

Let $n \in \mathbb{N}$ and $k \in \{0, \ldots, 2^n - 1\}$ be fixed. We assume that R^0_{nk} and R^1_{nk} are finite. Theorem 2.15 ensures existence of a unique solution to the following problem

$$\begin{cases} \psi \in H^1_0(T^0_{nk}) \\ -\partial c \partial^\star \psi = \delta_{X_{nk}}. \end{cases} \tag{12}$$

Let $\tilde{\psi}^0_{nk}$ be this solution. Similarly, we define $\tilde{\psi}^1_{nk}$ as the unique solution to

$$\begin{cases} \psi \in H^1_0(T^1_{nk}) \\ -\partial c \partial^\star \psi = -\delta_{X_{nk}}. \end{cases} \tag{13}$$

The idea is to build out of those functions a non trivial function (i.e. non constant at infinity) on T. To that purpose, we add a constant to each of those functions in order to set a common pressure value at X_{nk} (the vertex at which both pressure fields are to be connected). Therefore we define ψ^0_{nk} and ψ^1_{nk} as

$$\begin{aligned} \psi^0_{nk} &= \tilde{\psi^0_{nk}} - \tilde{\psi^0_{nk}}(X_{nk}), \\ \psi^1_{nk} &= \tilde{\psi^1_{nk}} - \tilde{\psi^1_{nk}}(X_{nk}). \end{aligned} \tag{14}$$

We define now ψ_{nk} on the overall tree T:

$$\psi_{nk}(x) = \begin{cases} \psi^0_{nk}(x) & \textit{if} \quad x \in V(T^0_{nk}), \\ \psi^1_{nk}(x) & \textit{if} \quad x \in V(T^1_{nk}), \\ 0 & \textit{otherwise}. \end{cases} \tag{15}$$

Remark 7. Let us describe the behaviour of ψ_{nk} "at infinity" . As ψ^0_{nk} minimizes the H^1 semi-norm among all those fields on $H^1_0(T^0_{nk})$ which vanish at infinity, and which share the same value at X_{nk}, one has

$$\tilde{\psi^0_{nk}}(X_{nk}) = 1/C(X_{nk}, T^0_{nk}) = R^0_{nk},$$

and a similar result for the second subtree. As a consequence, one can say (this assertion will be made more precise in the following), that ψ_{nk} behaves assymptotically like $-R^0_{nk}$ on T^0_{nk}, like R^1_{nk} on T^1_{nk}, and like 0 everywhere else (see Fig. 3.1).

Definition 3.2. Let T be an infinite resistive tree. We will say that T is a uniformly bounded resistive tree if

$$R^0_{nk} < +\infty\,,\ R^1_{nk} < +\infty \quad \forall n \in \mathbb{N}\,,\ \forall k \in \{0, \ldots, 2^n - 1\}.$$

Notice that a uniformly bounded resistive tree is automatically finitely resistant, as R^0_{00} and R^1_{00} are finite.

Theorem 3.3. *Let T be a uniformly bounded resistive tree. Then $(\varphi_0, \varphi_{nk})_{n,k}$ defined as*

$$\begin{cases} \varphi_0 = \sqrt{r_0}\, \mathbb{1}_T, \\ \varphi_{nk} = \dfrac{\psi_{nk}}{\sqrt{R^0_{nk} + R^1_{nk}}}\,,\ n \in \mathbb{N}\,,\ 0 \le k \le 2^n - 1, \end{cases} \tag{16}$$

is a Hilbert basis of $H^1_\Delta(T)$ (r_0 is the resistance of the first edge, and $\mathbb{1}_T$ is the function which is identically equal to 1 on T).

Proof. The functions φ_{nk} are harmonic and normalized by construction.

Let us now show that $(\varphi_0, \varphi_{nk})$ is an orthogonal family. Firstly, as the H^1 semi-norm of φ_0 is 0, and $\varphi_{nk}(o) = 0$, $(\varphi_0, \varphi_{nk})_1$ vanishes for $k \in \{0, \ldots, 2^n - 1\}$. As for $(\varphi_{nk}\,, \varphi_{n'k'})_1$ products, it is sufficient to establish

$$(\varphi_{00}, \varphi_{nk})_1 = 0 \quad \forall n\,,\ k, \tag{17}$$

the other situations can be handled in the same manner. The scalar product can be written as a sum of contributions of the two subtrees of T_{00}. As φ_{nk} is 0 on one of those subtrees, only one contribution remains, say on T^0_{00}. As the function φ_{00} is in $H^1_0(T^0_{00})$ up to an additive constant (by construction), and as the constant does not affect the $(\cdot,\cdot)_1$ product, Green formula (see Prop 1) can be applied on T^0_{00}:

$$(\varphi_{nk}, \varphi_{00})_{1,T} = (\varphi_{nk}, \varphi_{00})_{1,T^0_{00}} = \langle -\partial c \partial^\star \varphi_{nk}\,, \varphi_{00}\rangle_{\left(H^{-1}(T^0_{00}), H^1_0(T^0_{00})\right)} = 0,$$

by harmonicity of φ_{nk}.

It remains to prove that the family is total. In order to do that let f be in $H^1_\Delta(T)$ satisfying

$$(f\,, \varphi_0)_1 = 0 \quad \text{and } (f\,, \varphi_{nk})_1 = 0 \quad \forall n\,,\ k.$$

Firstly, as f is harmonic at the root o, $\partial^\star f(o, X_{00}) = 0$. The purpose of this second part of the proof is to obtain that the jump of f on each edge is related up to a constant to $\partial^\star f(o, X_{00})$. Consider any vertex $X_{nk} \neq o$. Orthogonality conditions

$$(f, \varphi_{nk})_1 = 0 \text{ and } \partial c \partial^\star f(X_{nk}) = 0,$$

together with Green formula (using again the fact that the restriction of φ_{nk} to T^i_{nk} is in $H^1_0(T^i_{nk})$ up to a constant) lead to a Cramer system. A direct resolution of this Cramer system proves that each flux through downstream edges of X_{nk} is proportional to the upstream flux arriving at X_{nk}. As the flux through the root edge (o, X_{00}) is zero, direct induction over n proves that $\partial^\star f$ is identically 0 over T, so that f is constant. As its scalar product with $\mathbb{1}_T$ is 0, f vanishes over T. $\square$

3.2. **Trace operators γ_0 and γ_1.** In this section we present how the trace operator γ_0 can be defined over $H^1(T)$ and we establish that, under some regularity assumptions on the tree T, $\gamma_0(H^1(T))$ can be identified to a subset of $L^2(\Gamma)$ (where $\Gamma = \{0,1\}^{\mathbb{N}}$ is the set of infinite paths). We also define the normal derivative operator γ_1 over a subspace of H^1, and we identifiy the corresponding trace space to $\gamma_0(H^1(T))'$.

Definition 3.4. The cylinder $C_{nk} \subset \Gamma = \{0,1\}^{\mathbb{N}}$ is defined as

$$C_{nk} = \{(\nu_n)_{n\geq 1} \in \Gamma\,, \quad \nu_i = \beta_i \ \text{ for } i = 1, \dots, n \text{ with } k = \sum_{i=1}^{n} \beta_i 2^{n-i}\}.$$

We denote by C^0_{nk} (resp. C^1_{nk}) the left-hand (resp. right-hand) half of C_{nk}:

$$C^i_{nk} = \{\nu = (\nu_n)_{n\geq 1} \in \Gamma\,, \quad \nu \in C_{nk}\,,\ \nu_{n+1} = i\} \quad i = 0\,,\ 1.$$

Note that C_{nk} can be seen as the set of infinite paths through the tree whose tail is contained in T_{nk}.

Definition 3.5. Let $\Gamma = \{0,1\}^{\mathbb{N}}$ be the limit set of the infinite tree T. Let $\sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$, the set of cylinders C_{nk} defined above. We denote by $(\Gamma, \sigma(\mathcal{C}), \mu)$ the standard Bernoulli space, with $\mu(C_{nk}) = 2^{-n}$, and by $L^2(\Gamma)$ the space of square integrable functions over Γ.

Let us now define the set F which is to play the role of regular functions for PDE problems, for which a proper restriction to the boundary of a domain can be defined. In the present context, F is spanned by functions which vanish on the boundary of T (in the sense of Definition 2.2), except on one subtree T_{nk}, where they vanish up to an additive constant.

Definition 3.6. Let F be the linear space spanned by all those pressure fields p such that there exists $n \in \mathbb{N}$, $k \leq 2^n - 1$, and $\pi \in \mathbb{R}$ such that

$$p_{|T_{nk}} - \pi \in H^1_0(T_{nk}) \text{ and } p_{|T\setminus T_{nk}} \in H^1_0(T \setminus T_{nk}).$$

We define $\gamma_0 : F \to L^2(\Gamma)$ as follows: for all $p \in F$ satisfying $p_{|T_{nk}} - \pi \in H^1_0(T_{nk})$, and $p_{|T\setminus T_{nk}} \in H^1_0(T \setminus T_{nk})$, we set

$$\gamma_0(p) = \pi \mathbb{1}_{C_{nk}}.$$

γ_0 is defined over F by linearity.

In particular, it is now possible to define the trace of the Hilbert basis of $H^1_\Delta(T)$.

Proposition 5. *Let T be a uniformly bounded resistive tree. φ_0 and φ_{nk} belong to F, and it holds*

$$\gamma_0(\varphi_0) = \sqrt{r_0}\mathbb{1}_\Gamma$$

and

$$\gamma_0(\varphi_{nk})(x) = \begin{cases} \dfrac{-R^0_{nk}}{\sqrt{R^0_{nk} + R^1_{nk}}} & \text{if } x \in C^0_{nk}, \\ \dfrac{R^1_{nk}}{\sqrt{R^0_{nk} + R^1_{nk}}} & \text{if } x \in C^1_{nk}, \\ 0 & \text{otherwise}. \end{cases} \tag{18}$$

Moreover

$$\left\{\begin{array}{lcl} \|\gamma_0(\varphi_0)\|^2_{L^2(\Gamma)} & = & r_0, \\ \|\gamma_0(\varphi_{nk})\|^2_{L^2(\Gamma)} & = & \dfrac{1}{2^{n+1}}\dfrac{(R^0_{nk})^2+(R^1_{nk})^2}{R^0_{nk}+R^1_{nk}}. \end{array}\right. \tag{19}$$

Proof. Recalling the construction of φ_0 and φ_{nk}, (18) is immediate. Then a simple computation shows (19), as $\mu(C^0_{nk}) = \mu(C^1_{nk}) = 1/2^{n+1}$. ☐

The previous considerations lead to a first unformal definition of the trace space as the set of linear combinations of $\gamma_0(\varphi_{nk})$ with coefficients in ℓ^2 :

$$\left\{ f = f_0\gamma_0(\varphi_0) + \sum_{n\geq 0}\sum_{k=0}^{2^n-1} f_{nk}\gamma_0(\varphi_{nk})\,,\ (f_{nk})\in\ell^2 \right\}$$

In general, this set does not identify to any proper functional space over Γ. It can be established that this set can be identified to some distribution space over Γ, as soon as the tree is subgeometric (i.e. $|r_{nk}| \leq C\alpha^n$, with $\alpha < 2$). Yet, in order to define properly traces as functions on Γ, we must restrict ourselves to the case of regular trees. Notice that, for such trees, $R^0_{nk} = R^1_{nk} = 2R^\infty_n$, where R^∞_n is the global resistance of any subtree T_{nk}. As a consequence, one has

$$\gamma_0(\varphi_{nk})(x) = \left\{\begin{array}{rl} -\sqrt{R^\infty_n} & \text{if } x\in C^0_{nk}, \\ \sqrt{R^\infty_n} & \text{if } x\in C^1_{nk}, \\ 0 & \text{otherwise}. \end{array}\right. \tag{20}$$

Notice also that functions $\gamma_0(\varphi_{nk})$ are orthogonal in $L^2(\Gamma)$ in the case of a regular tree.

Proposition 6. *Let T be a regular resistive tree with $R < +\infty$. Then γ_0 is continuous from $(F,\ \|.\|_{H^1(T)})$ to $(L^2(\Gamma),\ \|.\|_{L^2(\Gamma)})$. As a consequence it can be extended by density to an operator in $\mathcal{L}(H^1(T), L^2(\Gamma))$.*

Proof. Combining $p\in F$ and Proposition 3 (R being finite), there exists $p_0 \in H^1_0(T)$ and $p_h\in H^1_\Delta(T)$, such that $(p_0,p_h)_1 = 0$, and

$$p = p(o)\mathbb{1}_T + p_0 + p_h.$$

Note that necessarily $p_0(o) = -p_h(o)$. As $p\in F$, the harmonic component p_h writes

$$p_h = d_0\varphi_0 + \sum_{n=0}^{N}\sum_{k=0}^{2^n-1} d_{nk}\varphi_{nk},$$

for some $N < +\infty$, with $\sum d^2_{nk} = |p_h|^2_1$. Now noticing that the constant mode φ_0 is excited by $p(o)\mathbb{1}_T$ and $d_0\varphi_0 = d_0\sqrt{r_0}\mathbb{1}_T$, and using orthogonality of the family $(\gamma_0(\varphi_{nk}))$ in $L^2(\Gamma)$, one gets

$$\|\gamma_0(p)\|^2_{L^2(\Gamma)} = (p(o)+d_0\sqrt{r_0})^2 + \sum_{n=0}^{N}\sum_{k=0}^{2^n-1} d^2_{nk}\,\|\gamma_0(\varphi_{nk})\|^2\,.$$

In the case of a regular tree, Eq. (19) identifies $2^{n+1}\|\gamma_0(\varphi_{nk})\|^2$ as the global resistance of the subtree T^0_{nk}, which is less than $2^{n+1}R$ (the 2^{n+1} resistive subtrees $T^i_{nk'}$, $k' = 0,\ldots,2^n-1$, $i = 0,1$ are in parallel).

As for the first term, one has $d_0^2 r_0 = p_h(o)^2 = p_0(o)^2$, and Poincaré inequality (see Lemma 2) implies $p_0(o) \leq C|p_0|_1$. Finally, one gets

$$\|\gamma_0(p)\|^2_{L^2(\Gamma)} \leq 2p(o)^2 + 2C^2|p_0|_1^2 + R|p_h|_1^2$$

which is controled by $\|p\|^2_{H^1(T)}$. □

Corollary 2. *Suppose that T is a regular resistive tree with $R < +\infty$, then the trace space $\tilde{H}^{1/2}(\Gamma) = \gamma_0(H^1(T))$ is exactly*

$$\left\{ f \in L^2(\Gamma)\,,\ \exists\ f_0 \in \mathbb{R}\,,\ (f_{nk}) \in \ell^2,\ f = f_0\gamma_0(\varphi_0) + \sum_{n\geq 0}\sum_{k=0}^{2^n-1} f_{nk}\gamma_0(\varphi_{nk}) \right\}.$$

We denote it $\tilde{H}^{1/2}(\Gamma)$. This is a Hilbert space for the norm $\|f\|^2_{\tilde{H}^{1/2}(\Gamma)} = |f_0|^2 + |f_{nk}|^2_{\ell^2}$

Remark 8. The upperscript 1/2 in $\tilde{H}^{1/2}(\Gamma)$ is purely formal. We shall see that the trace space, in the case of dyadic trees, can be identified in some sense with Sobolev spaces H^s, where s depends on the resistances (see Section 4).

Proof. This is a direct consequence of the above results. □

We describe now how a counterpart to the normal derivative on the boundary can be defined in the present context, at least for functions which are harmonic in the neighbourhood of Γ. To be more precise, we will define γ_1 as the oulet flux, which is the discrete counterpart of $-c\partial p/\partial n$ in the context of standard Darcy problem (see Fig. 1).

Such a quantity is defined in the PDE context for functions p in H^1 such that $c\nabla p$ is divergence free, or at least has a divergence which is controlled in L^2. We define here this counterpart to the trace of the normal derivative for functions in $H^1(T)$ which are harmonic in $T \setminus \{o\}$:

Definition 3.7. We define $\mathring{H}^1_\Delta(T) \subset H^1(T)$ as the subspaces of functions which are harmonic over $T \setminus \{o\}$. This space can be written

$$\mathbb{R}\varphi_1 + H^1_\Delta(T),$$

with $\varphi_1 = p - p(o)\mathbb{1}_T$, where p is the solution to the homogeneous Dirichlet problem (9).

The construction of γ_1 is based on the following decomposition :

$$p \in \mathring{H}^1_\Delta(T) \Longleftrightarrow p = p_0\varphi_0 + p_1\varphi_1 + \sum\sum p_{nk}\varphi_{nk}\,,\ (p_{nk}) \in \ell^2.$$

Notice that φ_0 corresponds to a uniform pressure, with no flux associated. Moreover, as the tree is regular, the flux associated to φ_1 is uniformly distributed over Γ. As its integral balances the inlet flux exactly, this flux identifies to the uniform density 1 over Γ. Similarly, the flux associated to the pressure field φ_{nk} is uniforlmy distributed on each generation of the subtrees T^0_{nk} and T^1_{nk}. The normal trace γ_1 of a basis function φ_{nk} is then simply defined over C^i_{nk} as the global flux through T^i_{nk} divided by the measure of C^i_{nk} :

Definition 3.8. Le T be a regular tree with finite resistance, and let Λ be defined as span$(\varphi_0, \varphi_1, \varphi_{nk})$ (see Th. 3.3 for the definition of φ_0, φ_{nk}, and Def. 3.7 for the definition of φ_1). The operator $\gamma_1 : \Lambda \to L^2(\Gamma)$ is defined as follows : $\gamma_1(\varphi_0) \equiv 0$, $\gamma_1(\varphi_1) \equiv 1$ and, for any $n \in \mathbb{N}$, $0 \le k \le 2^n - 1$,

$$\gamma_1(\varphi_{nk})(x) = \begin{cases} -2^n/\sqrt{R_n^\infty} & \textit{if } x \in C_{nk}^0, \\ 2^n/\sqrt{R_n^\infty} & \textit{if } x \in C_{nk}^1, \\ 0 & \textit{otherwise}. \end{cases} \tag{21}$$

Proposition 7. *Let T be a regular resistive tree with $R < +\infty$. Then γ_1 is continuous from $(\Lambda, \|.\|_{H^1(T)})$ to the dual space $\tilde{H}^{1/2}(\Gamma)'$, which we denote by $\tilde{H}^{-1/2}(\Gamma)$. As a consequence it can be extended by density to an operator in $\mathcal{L}(\mathring{H}^1_\Delta(T), \tilde{H}^{-1/2}(\Gamma))$.*

Proof. Let us consider $g \in \tilde{H}^{1/2}(\Gamma)$, and $q \in \mathring{H}^1_\Delta(T)$ its harmonic extension (see Corollary 1). It admits the decomposition (Proposition 3 and Theorem 3.3)

$$q = q_0\varphi_0 + \sum\sum q_{nk}\varphi_{nk}\,, \quad \text{with } \|g\|^2_{\tilde{H}^{1/2}(\Gamma)} = |q_0|^2 + \sum\sum |q_{nk}|^2$$

so that one can compute explicitly

$$|\langle\gamma_1(\varphi_{nk}), g\rangle| = \left|\int_\Gamma \gamma_1(\varphi_{nk})\gamma_0(q)\right| = |q_{nk}|\left|\int_\Gamma \gamma_1(\varphi_{nk})\gamma_0(\varphi_{nk})\right| = |q_{nk}|.$$

As the tree is regular, functions $\gamma_1(\varphi_{nk})$ are proportional to $\mathbb{1}_{C^1_{nk}} - \mathbb{1}_{C^0_{nk}}$, for any n, k, and therefore $(\gamma_1(\varphi_1), \gamma_1(\varphi_{nk}))$, is an orthogonal system in $L^2(\Gamma)$. As a consequence, one has, for any $g \in \tilde{H}^{1/2}(\Gamma)$, any $u \in \mathring{H}^1_\Delta$, (we omit the φ_1 component for convenience)

$$\begin{aligned} &\left|\left\langle \gamma_1\left(\sum_n\sum_k u_{nk}\varphi_{nk}\right), g\right\rangle\right|^2 \\ &\le \left(\sum_n\sum_k |u_{nk}||\langle\gamma_1(\varphi_{nk})\,, g\rangle|\right)^2 \le \|u\|^2_{H^1}\,\|g\|^2_{\tilde{H}^{1/2}(\Gamma)}\,, \end{aligned}$$

which ends the proof. □

3.3. **Geometric trees.** We finish this section by considering the case of geometric trees, i.e. trees whose resistances follow the geometric law $r_n = r_0\alpha^n$, where α is a positive parameter. We suppose $\alpha < 2$, so that the resistance is finite (see Remark 6).

Definition 3.9. The Haar basis (Φ_0, Φ_{nk}) of $L^2(\Gamma)$ is defined as $\Phi_0 = \mathbb{1}_\Gamma$ and

$$\Phi_{nk}(x) = \begin{cases} -2^{n/2} & \text{if } x \in C_{nk}^0, \\ 2^{n/2} & \text{if } x \in C_{nk}^1, \\ 0 & \text{otherwise}. \end{cases} \tag{22}$$

It is a Hilbert basis of $L^2(\Gamma)$.

Definition 3.10. For any $r > 0$, let $A^r(\Gamma)$ be defined by

$$A^r(\Gamma) = \{f \in L^2(\Gamma)\,, \sum_{n\ge 0}\sum_{k=0}^{2^n-1} 2^{2nr}c_{nk}(f)^2 < +\infty\},$$

with $c_{nk}(f) = (f, \Phi_{nk})_{L^2(\Gamma)}$. If we denote by V_n the space of functions spanned by the characteristics functions of the cylinders C_{nk}, for $k = 1, \ldots, 2^n - 1$, $A^r(\Gamma)$ can be defined equivalently (see Cohen [9]) as

$$A^r(\Gamma) = \{f \in L^2(\Gamma)\,,\ (\mathrm{dist}_{L^2}(f, V_n)2^{nr})_{n\in\mathbb{N}} \in \ell^2\}.$$

Proposition 8. *Let T be a geometric tree with $\alpha \in (0,2)$. Then*

$$\tilde{H}^{1/2}(\Gamma) = \gamma_0(H^1(T)) = A^r(\Gamma) \quad \textit{with } r = \frac{1}{2} - \frac{\ln\alpha}{2\ln 2}.$$

Proof. Let $f \in \tilde{H}^{1/2}(\Gamma)$ be given. By Corollary 2, it expresses

$$f = f_0\gamma_0(\varphi_0) + \sum_{n\geq 0}\sum_{k=0}^{2^n-1} f_{nk}\gamma_0(\varphi_{nk}).$$

A simple computation shows that

$$\Phi_{nk} = \frac{\gamma_0(\varphi_{nk})}{\|\gamma_0(\varphi_{nk})\|_{L^2(\Gamma)}},\ \textit{ with } \|\gamma_0(\varphi_{nk})\|_{L^2(\Gamma)} = C\left(\frac{\alpha}{2}\right)^{n/2}.$$

This, combined with the fact that $r = \dfrac{1}{2} - \dfrac{\ln\alpha}{2\ln 2}$, leads to $2^{nr}\left(\dfrac{\alpha}{2}\right)^{n/2} = 1$, which yields

$$2^{nr}c_{nk}(f) = Cf_{nk} \quad \forall n \in \mathbb{N}\,,\ \forall k \in \{0, \ldots, 2^n - 1\}\,,$$

which ends the proof. □

Proposition 9. *Let T be a geometric tree, with $\alpha \in (0,2)$. Then, for any function $g \in A^r(\Gamma)$, with $r = (1-\ln\alpha/\ln 2)/2$, there exists a unique pressure field $p \in \mathring{H}^1_\Delta(T)$ such that $\gamma_0(p) = g$. In other terms, the Dirichlet problem*

$$\left\{\begin{array}{rcl} -\partial c\partial^\star p &=& 0 \quad \textit{in } T\setminus\{o\}, \\ p(o) &=& 0, \\ \gamma_0(p) &=& g \end{array}\right. \tag{23}$$

admits a unique solution.

Proof. This is a direct consequence of Proposition 8 and Corollary 1. □

Definition 3.11. We define A^{-r} as the dual space of A^r. It can be identified to the complete closure of $L^2(\Gamma)$ for the norm

$$\|f\|^2 = c_0(f)^2 + \sum_{n\geq 0}\sum_{k=0}^{2^n-1} 2^{2nr}c_{nk}(f)^2 \text{ with } c_0(f)$$
$$=(f,\Phi_0)_{L^2(\Gamma)}\,,\ c_{nk}(f) = (f,\Phi_{nk})_{L^2(\Gamma)}.$$

Proposition 10. *Let T be a homogeneous geometric tree with $\alpha \in (0,2)$. Then*

$$\tilde{H}^{-1/2}(\Gamma) = \gamma_1(\mathring{H}^1_\Delta(T)) = A^{-r}(\Gamma).$$

Proof. This is a direct consequence of proposition 7. □

We may now define operators $\mathcal{C}$ and $\mathcal{R}$ (as *Conductance* and *Resistance* operators). The conductance operator $\mathcal{C}$ models the instantaneous ventilation process: to a pressure field it associates a flux field which corresponds to the air which flows through the set of ends Γ. We shall consider here the situation of free out/in-let condition at the root, to model the fact that the root is connected to the outside world, at atmospheric pressure (set to 0 here, see Corollary 1).

Definition 3.12. Let T be a geometric tree with $\alpha \in (0, 2)$. The conductance operator is defined as follows: For any $g \in A^r$, with $r = (1 - \ln\alpha/\ln 2)/2$, one denotes by p the unique solution to Problem (23) (see Proposition 9). The image of g is then defined as $\mathcal{C}g = \gamma_1 p$. Thanks to Proposition 10, it is well defined as an element of $\mathcal{L}(A^r, A^{-r})$. The resistance operator $\mathcal{R} \in \mathcal{L}(A^{-r}, A^r)$ is defined as its reciprocal.

Remark 9. The resistance operator can be seen as the Neuman to Dirichlet operator for the Laplace problem : for any $v \in A^{-r}$, if one denotes by p the unique solution to

$$\begin{aligned} -\partial c \partial^\star p &= 0 \quad \text{in } T \setminus \{o\} \\ p(o) &= 0 \\ \gamma_1 p &= v \end{aligned}$$

the image of v is then defined as $\mathcal{R}v = \gamma_0 p$.

We finish this section by some numerical analysis for the conductance operator $\mathcal{C}$.

Definition 3.13. Let T be a geometric tree, with $\alpha \in (0, 2)$. For any $N > 0$, we recall that T_N designs the subtree of T with same root and N generation (see Definition 2.5). We design by $\tilde{T}_N$ the tree obtained by condensation of the 2^N truncated subtrees. More precisely, $\tilde{T}_N$ has the same set of vertices and edges than T_N, but the resistances of the edges containing a leaf, which are $r_N = \alpha^N$, for T_N, are replaced by

$$r_N + R_N^\infty = \alpha^N + \frac{1}{2}\alpha^{N+1} + \frac{1}{2^2}\alpha^{N+2} + \dots.$$

We denote by $\tilde{c}_N$ the corresponding collection of conductances, and by by Γ_N the set of leafs of $\tilde{T}_N$ (whose cardinal is 2^N). For any $g \in L^2(\Gamma)$, we denote by $P_N g \in L^2(\Gamma)$ its projection onto V_N (see Definition 3.10), and by g_N the corresponding collection of 2^N values. We define now p_N as the solution to the truncated Dirichlet problem

$$\left\{ \begin{aligned} -\partial \tilde{c}_N \partial^\star p_N &= 0 \quad \text{in } \tilde{T}_N \setminus \{\{o\} \cup \Gamma_N\}, \\ p_N(o) &= 0, \\ p_N &= g_N \quad \text{on } \Gamma_N. \end{aligned} \right. \tag{24}$$

$\mathcal{C}_N g \in V_N$ is defined as the piecewise constant function over Γ which is equal to $u(X_{Nk})2^N$ (the correcting factor 2^N stands for the reciprocal of $\mu(C_{Nk})$) on cylinder C_{Nk}, where $u(X_{Nk})$ is the flux getting out of the tree through X_{Nk}.

As in the context of finite element methods, one may not expect more than pointwise convergence of $\mathcal{C}_N$ towards $\mathcal{C}$ if one considers both operators in $\mathcal{L}(A^r(\Gamma), A^{-r}(\Gamma))$. Yet, a controlled uniform convergence can be established if g is assumed more regular than A^r, as asserted by the following proposition:

Proposition 11. *Let T be a α-geometric tree with $\alpha \in (0, 2)$, $r = (1 - \ln\alpha/\ln 2)/2$, and consider r' such that $r < r'$. There exists $C > 0$ such that*

$$\|\mathcal{C}_N g - \mathcal{C}g\|_{A^{-r}} \le \frac{C}{2^{N(r'-r)}} \|g\|_{A^{r'}} \quad \forall g \in A^{r'}.$$

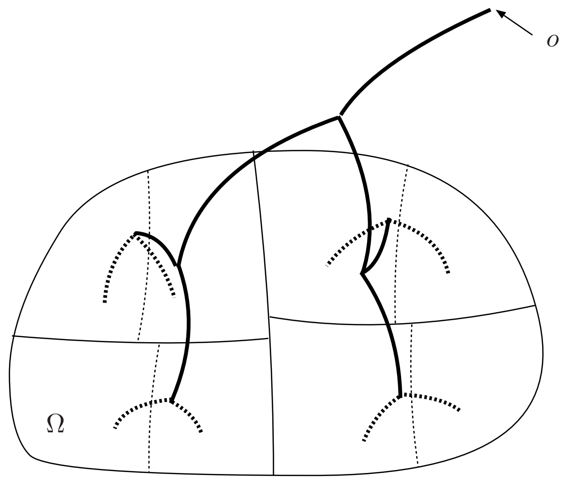

FIGURE 4. Embedding onto a domain.

Proof. As a straightforward consequence of the second part of Definition 3.10, there exists $C > 0$ such that

$$\|P_N g - g\|_{A^r} \leq C 2^{-N(r'-r)} \|g\|_{A^{r'}} \quad \forall g \in A^{r'}.$$

As Problem (24) is set in $\tilde{T}_N$ (with condensation of resistances), it is clear that $\mathcal{C}_N g = \mathcal{C} P_N g$, so that

$$\begin{aligned} \|\mathcal{C}_N g - \mathcal{C} g\|_{A^{-r}} = \|\mathcal{C}(P_N g - g)\|_{A^{-r}} &\leq \|\mathcal{C}\|_{\mathcal{L}(A^r, A^{-r})} \|P_N g - g\|_{A^r} \\ &\leq \frac{C}{2^{N(r'-r)}} \|g\|_{A^{r'}}. \end{aligned}$$

□

4. **Embedding onto a domain of $\mathbb{R}^d$.** The approach presented in this section is based on the following strategy: Firstly, we introduce a dyadic decomposition of a connected bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ (which aims at modelling the parenchyma in the case $d = 3$). With respect to this decomposition, we define a trace operator γ_0^Ω from $H^1(T)$ onto $L^2(\Omega)$. We then investigate the properties of the obtained pressure fields in terms of standard fractional Sobolev regularity.

The main result of this section states that if the tree irrigate the domain in a regular way, and if the tree is geometric ($r_n = \alpha^n$), the embedded trace space $\gamma_0^\Omega(H^1(T))$ can be identified to a standard Sobolev space H^s, under some conditions on α.

4.1. **Trace space as a subset of $L^2(\Omega)$.** We consider a connected, bounded domain $\Omega \subset \mathbb{R}^d$. Each vertex X_{nk} of the tree irrigates, through the subtree of its descendants, a portion of the parenchyma (see Fig. 4) which we denote by $\Omega_{nk} \subset \Omega$. This irrigation process exhibits a hierarchical structure. Thus, it is natural to introduce for Ω the following multi-scale decomposition:

Definition 4.1. (Multiscale decomposition)
Let $(\Omega_{nk})_{n\in\mathbb{N},k=0,\ldots,2^n-1}$ be a sequence of open nonempty connected subsets of Ω. We say that $\mathcal{O} = (\Omega_{nk})$ is a multi-scale decomposition of Ω if

(i) $\bigcup_{k=0}^{2^n-1} \overline{\Omega}_{nk} = \overline{\Omega} \quad \forall n,$

(ii) $\Omega_{nj} \cap \Omega_{nk} = \varnothing$ as soon as $j \neq k$,

(iii) $\overline{\Omega}_{n+1,2k} \cup \overline{\Omega}_{n+1,2k+1} = \overline{\Omega}_{nk} \quad \forall n \in \mathbb{N},\ \forall k = 0, \ldots, 2^n - 1.$

Definition 4.2. (Balanced multiscale decomposition)
The multiscale decomposition $\mathcal{O} = (\Omega_{nk})$ is said to be balanced if

$$|\Omega_{nk}| = 2^{-n}|\Omega| \quad \forall n\,,\ \forall k = 0, \ldots, 2^n - 1,$$

where $|A|$ denotes the Lebesgue measure of the measurable set A.

We aim here at defining the trace of a function in $H^1(T)$ as a function defined over Ω, according to a multiscale decomposition (Ω_{nk}). To that purpose, we consider again the subspace $F \subset H^1(T)$ spanned by cylindrical functions, i.e. functions which are constant on the boundary of a subtree, and vanish on the rest of the boundary (see Def. 3.6 for a proper definition of this space). Note that, thanks to the construction of the trace operator in Section 3, spanning functions of F can now be defined as functions whose trace is the characteristic function of a cylinder.

Definition 4.3. Let $\mathcal{O} = (\Omega_{nk})$ be a multiscale decomposition of Ω. We define the mapping $\gamma_0^\Omega : F \to L^2(\Omega)$ as follows: for all spanning function $p \in F$, $\gamma_0(p) = \pi \mathbb{1}_{C_{nk}}$, $\pi \in \mathbb{R}$, we set

$$\gamma_0^\Omega(p) = \pi \mathbb{1}_{\Omega_{nk}}.$$

γ_0^Ω is defined over F by linearity.

Note that γ_0^Ω is highly dependent on the decomposition $\mathcal{O} = (\Omega_{nk})$. We drop this explicit dependence to alleviate notations.

Proposition 12. *Let T be a regular resistive tree with $R < +\infty$, and (Ω_{nk}) a balanced multiscale decomposition (see Definition 4.2). Then γ_0^Ω is continuous from $(F, \|.\|_{H^1(T)})$ to $L^2(\Omega)$. As a consequence it can be extended by density to a mapping in $\mathcal{L}(H^1(T), L^2(\Omega))$. Its range will be simply denoted by $\gamma_0^\Omega(H^1)$.*

Proof. As no regularity is required, the proof of proposition 6 can be reproduced. Indeed, considering $(\varphi_0, \varphi_{nk})$ the Hilbert basis of $H^1(T)$ (which is included in F), the balanced character of the decomposition ensures orthogonality in $L^2(\Omega)$ of the family $(\gamma_0^\Omega(\varphi_0), \gamma_0^\Omega(\varphi_{nk}))$. Furthermore, as $|\Omega_{nk}| = 2^{-n}|\Omega|$, the proof of Proposition 6 can be reproduced here, up to a multiplicative factor $|\Omega|$ as soon as integrals over the domain Ω are involved. □

4.2. **Regularity results.** This section is devoted to a finer desciption of $\gamma_0^\Omega(H^1) \subset L^2(\Omega)$. We establish here that the trace space $\gamma_0^\Omega(H^1)$ lies in fractional Sobolev spaces, as soon as certain conditions are met by the mutliscale decomposition. Some properties presented here are proved in a different context in [9].

4.2.1. *Geometric issues.*

Definition 4.4. (Regular / quasi-regular multiscale decomposition)
We say that a multi-scale decomposition $\mathcal{O}$ of a bounded connected Lipschitz domain Ω is *regular* if the following properties hold

(i) $\mathcal{O}$ is balanced,

(ii) there exists a constant C such that for all $n \in \mathbb{N}$, $k \in \{0, \ldots, 2^n - 1\}$

$$\operatorname{diam}(\Omega_{nk}) \leq C 2^{-\frac{n}{d}}.$$

(iii) There exists $C > 0$ such that

$$\|\tau_h \mathbb{1}_{\Omega_{nk}} - \mathbb{1}_{\Omega_{nk}}\|_{L^1(\Omega)} \leq C|h| 2^{-\frac{n(d-1)}{d}}, \ \forall h \in \mathbb{R}^d, \ \forall n \in \mathbb{N}, \ k \in \{0, \ldots, 2^n - 1\},$$

where τ_h is the translation operator: $\tau_h \varphi(\cdot) = \varphi(\cdot + h)$.

We say that a multi-scale decomposition (Ω_{nk}) is *quasi-regular* if there exists a regular multi-scale decomposition $(\widetilde{\Omega}_{nk})$ and a bi-Lipschitz map $\phi : \Omega \to \widetilde{\Omega}$ such that for all $n \in \mathbb{N}$ and $k \in \{0, \ldots, 2^n - 1\}$

$$\phi(\Omega_{nk}) = \widetilde{\Omega}_{nk}.$$

Remark 10. The previous definition of a regular decomposition presents some similarities with the notion of a regular family of triangulations in the context of Finite Element discretization. Indeed, if we assume that all cells Ω_{nk} are piecewise smooth, a regular decomposition is a decomposition for which cells behave asymptotically like balls in the relations between volume, measure of the boundary, and diameter. For $d = 3$, it writes

$$\operatorname{diam}(\Omega_{nk}) \leq C|\Omega_{nk}|^{1/3}, \ \operatorname{area}(\partial\Omega_{nk}) \leq C|\Omega_{nk}|^{2/3}$$

The following lemma, which states essentially that the number of neighbors is controlled for regular and quasi-regular decompositions, will be essential in the proof of regularity results.

Lemma 4.5. *Let $\mathcal{O}$ be a quasi regular decomposition. There exists a constant C such that*

$$M_{nk} = \sharp\{j, \ \mathit{dist}(\Omega_{nj}, \Omega_{nk}) \leq 2^{-n/d}\} \leq C \quad \forall n, \ k.$$

Proof. As $\operatorname{diam}(\Omega_{nj}) \leq C_1 2^{-n/d}$, all cells which contribute to M_{nk} are entirely contained in

$$\{x \in \Omega, \ \operatorname{dist}(x, \Omega_{nk}) \leq (1 + C_1)2^{-n/d}\},$$

which is itself contained in some ball B of radius $C_2 2^{-n/d}$, whose measure $|B|$ behaves like $C_3 2^{-n}$. As $|\Omega_{nj}| \geq C_4 2^{-n}$, it yields $|B| \sim C_3 2^{-n} \geq M_{nk} C_4 2^{-n}$, which gives the expected estimate. □

4.2.2. *Sobolev spaces and A^s spaces.* This section contains some definitions, technical lemmas, mainly related to the spaces A^r which are somewhat similar to standard Sobolev spaces, but differ in the way oscillations are estimated: a Haar-like basis (which depends on the decomposition) is used instead of sine functions. Yet, as detailed in [9] for the case where Ω is a square, A^r can be identified to a space H^s for certain values of r, under suitable assumptions on the decomposition, This section is essential, as we aim at expressing the regularity of pressure fields in terms of standard criteria (Sobolev framework), whereas the requirement to have finite H^1 energy on the tree, the natural regularity exhibited by our problem, is expressed in terms of decreasing properties of spectral decompositions with respect to a Haar-like basis. Explicit references to the trace space that we aim at describing are put off until the next section.

Definition 4.6. Let $\mathcal{O}$ be a multiscale decomposition of Ω. We define

$$V_n = \operatorname{span}(\mathbb{1}_{\Omega_{nk}})_{n \in \mathbb{N}, k \in \{0, \ldots, 2^n - 1\}}.$$

We denote by $P_n f$ the projection of $f \in L^2(\Omega)$ onto V_n.

We denote by $m_{nk}(f)$ the average of f on the domain Ω_{nk}. The projection of f can be written explicitely:

$$P_n f = \sum_{k=0}^{2^n-1} m_{nk}(f)\mathbb{I}_{\Omega_{nk}}.$$

Proposition 13. *Let $\mathcal{O}$ be a decomposition of Ω such that (ii) of Definition 4.4 holds true. Then $\bigcup_{n\in\mathbb{N}} V_n$ is dense in $L^2(\Omega)$. As a consequence, density holds for quasi-regular decompositions.*

Proof. Because the diameter of the cells is assumed to go to 0 as n goes to infinity, any open subset of Ω contains a Ω_{nk}. As a consequence, any continuous function orthogonal to $\bigcup_{n\in\mathbb{N}} V_n$ is identically 0. □

The approximation properties of spaces V_n are linked to fractional Sobolev regularity:

Lemma 4.7. *(Jackson)*
Let $r < 1$ be given, $\mathcal{O}$ a decomposition which verifies (i) and (ii) of Definition 4.4, and (V_n) the associate family defined as above. There exists a constant C such that for all $n \in \mathbb{N}$ the following estimate holds

$$\|f - P_n f\|_{L^2(\Omega)} \le C 2^{-nr/d} \|f\|_{H^r(\Omega)} \quad \forall f \in H^r(\Omega).$$

Proof. The proof extends standard arguments (see e.g. [9]) to the present situation of general decompositions (no assumption is made on the cells, except on measure and diameter) and fractional Sobolev regularity.

We have

$$\|f - P_n f\|^2_{L^2(\Omega)} = \sum_{j=0}^{2^n-1} \|f - P_n f\|^2_{L^2(\Omega_{nj})} = \sum_{j=0}^{2^n-1} \|f - m_{nj}(f)\|^2_{L^2(\Omega_{nj})}.$$

Let $n \in \mathbb{N}$, $j \in \{0, \dots, 2^n - 1\}$ and $f \in H^r(\Omega_{nj})$. Then, we have

$$\int_{\Omega_{nj}} |f - m_{nj}(f)|^2 dx = \int_{\Omega_{nj}} |f|^2 - |\Omega_{nj}|\, m_{nj}(f)^2.$$

We deduce that

$$\frac{1}{2|\Omega_{nj}|} \int_{\Omega_{nj}} \int_{\Omega_{nj}} |f(x) - f(y)|^2 dxdy = \|f - m_{nj}(f)\|^2_{L^2(\Omega_{nj})}.$$

But

$$\int_{\Omega_{nj}} \int_{\Omega_{nj}} |f(x) - f(y)|^2 dxdy \le \int_{\Omega_{nj}} \int_{\Omega_{nj}} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2r}} dxdy \ \mathrm{diam}(\Omega_{nj})^{d+2r}.$$

Now (i) and (ii) of Definition 4.4 ensure existence of a constant C such that

$$\frac{1}{|\Omega_{nj}|} \mathrm{diam}(\Omega_{nj})^{d+2r} \le C 2^{-2rn/d} \qquad \forall n\,,\ 0 \le j \le 2^n - 1.$$

Putting everything together, we deduce that

$$\|f - P_n f\|^2_{L^2(\Omega)} \le C 2^{-2rn/d} \sum_{j=1}^{2^n-1} \int_{\Omega_{nj}} \int_{\Omega_{nj}} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2r}} dxdy \le C 2^{-2rn/d} \|f\|^2_{H^r(\Omega)},$$

which ends the proof. □

Remark 11. Notice that Condition (iii) of Definition 4.4 is not necessary to establish the result.

We may now define spaces A^r, following [9]:

Definition 4.8. Let $\mathcal{O}$ be a balanced multiscale distribution (Def. 4.2). Let $r \geq 0$. We define the space $A^r(\Omega)$ as follows

$$A^r(\Omega) = \{f \in L^2(\Omega)\,,\ (\mathrm{dist}_{L^2}(f, V_n)2^{nr})_{n\in\mathbb{N}} \in \ell^2\}$$

endowed with the norm

$$\|u\|_{A^r}^2 = \|P_0 u\|_{L^2(\Omega)}^2 + \|\mathrm{dist}_{L^2}(f, V_n)\, 2^{nr}\|_{\ell^2}^2.$$

Proposition 14. *We define $Q_n \in \mathcal{L}(L^2(\Omega), V_{n+1})$ as*

$$Q_n f = P_{n+1} f - P_n f.$$

For any $r > 0$, it holds

$$\|u\|_{A^r}^2 \sim \|P_0 u\|_{L^2}^2 + \sum_{n=0}^{+\infty} \|Q_n f\|_{L^2}^2\, 2^{2nr}.$$

Proof. See [9]. □

Remark 12. Although we do not make explicit reference to the decomposition $\mathcal{O}$ to alleviate notations, the space A^r defined above depends a priori on it. We will see that, under particular conditions on the geometry (quasi-regularity of the decomposition), and for $rd < 1/2$, it can be identified to the standard Sobolev space $H^{rd}(\Omega)$, which of course does not see the decomposition. Yet, in general, it is highly dependent on the way Ω is decomposed. It is particularly true for high values of r, for which belonging to A^r requires high regularity in a certain sense (high correlation of mean values on the two legs of the high frequency Haar functions), which for example does not imply continuity across interfaces between cells.

Definition 4.9. (Besov spaces)
Let $f \in L^2(\Omega)$. We set

$$w(1, t, f) = \sup_{|h|\leq t} \|\tau_h f - f\|_{L^2(\Omega_h)}$$

where $\Omega_h = \{x \in \Omega, x + h \in \Omega\}$, and τ_h is the translation operator. Let $0 < \gamma < 1$. We define the space $B_{2,2}^\gamma$ by

$$B_{2,2}^\gamma(\Omega) = \{f \in L^2(\Omega), \|f\|_{L^2}^2 + \sum_{j=0}^{+\infty} 2^{2j\gamma} w(1, 2^{-j}, f)^2 < +\infty\}.$$

In particular, $B_{2,2}^\gamma(\Omega) = H^\gamma(\Omega)$ for all $0 < \gamma < 1$, for any Lipschitz domain Ω.

In the next lemma, we prove that, because of the absence of high frequencies, the quantity $w(1, t, f)$ which conditions Besov regularity can be controlled with respect to t for functions of V_n.

Lemma 4.10. *Let $\mathcal{O}$ be a regular decomposition (see Def. 4.4), and (V_n) the associated family of functional spaces (see Def. 4.6). There exists a constant C such that for all $t \in \mathbb{R}^+$ the following estimate holds*

$$w(1, t, f) \leq C \min\left(1, 2^{n/d} t\right)^{1/2} \|f\|_{L^2} \quad \forall f \in V_n. \tag{25}$$

Proof. If $t \geq 2^{-n/d}$, then estimate holds trivially. Let us prove that (25) holds for $t = 2^{-\ell}$ where $\ell \geq n/d$. Any $f \in V_n$ writes

$$f = \sum_{k=0}^{2^n-1} f_{nk} \mathbb{1}_{\Omega_{nk}}.$$

Consider $|h| \leq 2^{-\ell}$, with $\ell \geq n/d$, and $x \in \Omega_{nk}$. As f is constant over Ω_{nk},

$$\begin{aligned} |f(x+h) - f(x)|^2 &= |\tau_h \mathbb{1}_{\Omega_{nk}} - \mathbb{1}_{\Omega_{nk}}|^2 |f(x+h) - f(x)|^2 \\ &\leq |\tau_h \mathbb{1}_{\Omega_{nk}} - \mathbb{1}_{\Omega_{nk}}|^2 \sup_{j \sim k} |f_{nj} - f_{nk}|^2, \end{aligned}$$

where $j \sim k$ stands here for $\mathrm{dist}(\Omega_{nj}, \Omega_{nk}) \leq 2^{-n/d}$, so that

$$\sup_{j \sim k} |f_{nj} - f_{nk}|^2 \leq C \sum_{j \sim k} |f_{nj}|^2.$$

By Condition (iii) of Definition 4.4,

$$\int_{\Omega_{nk}} |\tau_h \mathbb{1}_{\Omega_{nk}} - \mathbb{1}_{\Omega_{nk}}|^2 \leq |h| 2^{-n(d-1)/d}.$$

Now summing up over k and using Lemma 4.5 (which allows to control the number of j such that $j \sim k$), we get

$$\int_\Omega |f(x+h) - f(x)|^2 \leq C|h| 2^{-n(d-1)/d} \sum_j |f_{nj}|^2$$

The sum behaves like $2^n \|f\|^2_{L^2(\Omega)}$, so that

$$\|\tau_h f - f\|^2_{L^2(\Omega)} \leq C|h| 2^{n/d} \|f\|^2_{L^2(\Omega)}$$

which yields the estimate. □

The following proposition identifies the spaces A^r with a standard Sobolev space H^s, with $s = rd$, for some values of s. Note that such an identification cannot be expected to hold for large values of s, as straight discontinuities across hypersurfaces are ruled out as soon as s is greater than $1/2$, whereas highly "regular" functions in the A^r sense exhibits such discontinuities.

Remark 13. Such an identification between A^r spaces is proposed in [15], and it takes the form $A^r = H^r$, which is in apparent contradiction with the next result. This is due to the fact that definitions of A^r differ. More precisely, both are based on the same general definition (see [9]), but the approximation spaces are different. In [15], the regularity is expressed in terms of the behaviour of expansions with respect to a Haar-like basis, which is built in a tensor way from the one-dimensional Haar basis. Because of this construction, it is natural to make index n depend on the size of one-dimensional basis functions, and finite dimensional spaces are built in this spirit, so that functions at generation n explore oscillations at frequency 2^n. In our situation, as we aim at investigating unstructured decompositions connected with dyadic trees, n corresponds to the generation index. As a consequence, it takes us d (= the dimension) steps in the decomposition process to divide by 2 the average cell diameter, which amounts to double the explored frequency. This explains that A^r functions, according to our definition, are more regular.

Proposition 15. *Let $\mathcal{O}$ be a regular multiscale decomposition, and A^r the associated space (see Def. 4.8). Then, for all $r \geq 0$*

$$A^r(\Omega) = H^{rd}(\Omega) \quad \textit{if} \quad rd < \frac{1}{2},$$

$$A^r(\Omega) \hookrightarrow H^{\tilde{r}}(\Omega) \quad \textit{if} \quad rd \geq \frac{1}{2} \quad \forall\, \tilde{r} < \frac{1}{2},$$

$$H^{rd}(\Omega) \hookrightarrow A^{\tilde{r}}(\Omega) \quad \textit{if} \quad 0 \leq \tilde{r} \leq rd < 1.$$

Proof. Let us first prove $A^r(\Omega) \hookrightarrow H^{rd}(\Omega)$, under the condition $rd < \frac{1}{2}$. We have

$$f = P_0 f + \sum_{\ell=0}^{j-1} \big(P_{\ell+1} - P_\ell\big) f + f - P_j f = P_0 f + \sum_{\ell=0}^{j-1} Q_\ell f + f - P_j f.$$

Using the fact that $w(1,t,f+g) \leq w(1,t,f) + w(1,t,g)$, we obtain

$$w(1,2^{-j},f) \leq w(1,2^{-j},P_0 f) + \sum_{\ell=0}^{j-1} w(1,2^{-j},Q_\ell f) + w(1,2^{-j},f - P_j f).$$

By Lemma 4.10, we deduce that

$$w(1,2^{-j},f) \lesssim 2^{-j/2} \sum_{\ell=0}^{j-1} 2^{\ell/2d} \|Q_\ell f\|_{L^2} + \|f - P_j f\|_{L^2}$$

hence

$$w(1,2^{-j},f) \lesssim 2^{-j/2} \sum_{\ell=0}^{j} 2^{\ell/2d} \|f - P_\ell f\|_{L^2}. \tag{26}$$

Multiplying by 2^{jrd}, we find that

$$w(1,2^{-j},f) 2^{jrd} \lesssim 2^{(rd-1/2)j} \sum_{\ell=0}^{j} 2^{\ell/2d - r\ell} 2^{r\ell} \|f - P_\ell f\|_{L^2}.$$

Hence we obtain

$$w(1,2^{-j},f) 2^{jrd} \lesssim (a_n)_{n\in\mathbb{Z}} * (b_n)_{n\in\mathbb{Z}}$$

where

$$a_n = 2^{n(rd-1/2)} \mathbb{I}_{n\geq 0} \quad \text{and} \quad b_n = 2^{rn} \|f - P_n f\|_{L^2} \mathbb{I}_{n \geq 0}.$$

We have $b_n \in \ell^2$ because $f \in A^r(\Omega)$ and $a_n \in \ell^1$ because $rd < 1/2$. We deduce $A^r \hookrightarrow H^{rd}$ by using Young inequalities in the case $rd < 1/2$. In the case $rd \geq 1/2$, we follow the same reasoning except that we multiply equation (26) by $2^{j\gamma}$ where $0 \leq \gamma < 1/2$ to obtain

$$A^r(\Omega) \hookrightarrow H^\gamma(\Omega).$$

Let us prove that, for $rd < 1/2$,

$$\|f\|_{A^r} \leq C \|f\|_{B^{rd}_{2,2}} \sim \|f\|_{H^{rd}}. \tag{27}$$

Let $\beta = rd < 1/2$. We shall make use of the following characterization of H^β obtained by real interpolation between L^2 and H^β. Let

$$K(f,t) = \inf_{g \in H^\beta} \|f - g\|_{L^2} + t \|g\|_{H^\beta}.$$

Then (see for example [7]),

$$\|f\|_{H^\beta} = \|\rho^j K(f, \rho^{-j})\|_{\ell^2}$$

with $\rho > 1$. To prove estimate (27), we are reduced to showing that there exists a constant $C > 0$ such that

$$\|f - P_j f\|_{L^2} \leq CK(f, 2^{-j\beta/d}). \tag{28}$$

Let $f \in L^2$, $g \in H^\beta$. By Lemma 4.7, we obtain

$$\|f - P_j f\|_{L^2} \leq \|f - P_j g\|_{L^2} \leq \|f - g\|_{L^2} + \|g - P_j g\|_{L^2} \leq \|f - g\|_{L^2} + C2^{-j\beta/d}\|g\|_{H^\beta},$$

which ends the proof. □

4.3. **Sobolev regularity for embedded geometric trees.** In this section we establish that the embedded trace of $H^1(T)$ pressure fields (introduced by Proposition 12) possess some Sobolev regularity. We consider a geometric tree, as in the beginning of section 3.3, with

$$r_n = r_0 \alpha^n\,, \;\; \alpha \in (0, 2).$$

We choose $r_0 = 2(2 - \alpha)$, so that the values involved in (20) simplify down to $\alpha^{n/2}$. As a consequence, the trace of the Hilbert basis of $H^1(T)$ can be expressed

$$\gamma_0^\Omega(\varphi_{nk}) = \alpha^{n/2}\left(\mathbb{1}_{\Omega_{n+1,2k}} - \mathbb{1}_{\Omega_{n+1,2k+1}}\right).$$

Theorem 4.11. *Let T be a α geometric tree, Ω a bounded connected Lipschitz domain, $\mathcal{O} = (\Omega_{nk})$ a quasi-regular decomposition (see Def. 4.4), and γ_0^Ω the associated embedded trace operator (see Prop. 4.3). Then*

$$\gamma_0^\Omega(H^1(T)) = H^s(\Omega) \quad \textit{if} \quad s < \frac{1}{2},$$

$$\gamma_0^\Omega(H^1(T)) \hookrightarrow H^{s'}(\Omega) \quad \textit{if} \quad s \geq \frac{1}{2}\,,\ \textit{for all}\ s' < \frac{1}{2},$$

where $s = d(1 - \ln\alpha/\ln 2)/2$.

Proof. Assume that we are in the regular case. As T is regular and $\mathcal{O}$ is balanced, the family $(\gamma_0^\Omega(\varphi_0), \gamma_0^\Omega(\varphi_{nk}))$ is orthogonal in $L^2(\Omega)$. Let us now consider $p \in F$, with (the sum over n is actually finite)

$$p = p_0\varphi_0 + \sum_{n=0}^{+\infty}\sum_{k=0}^{2^k-1} p_{nk}\varphi_{nk}.$$

It holds

$$\begin{aligned}\|\gamma_0^\Omega(p)\|^2_{A^r(\Omega)} &= |p^0|^2 + \sum_{n=0}^{+\infty} 2^{2rn}\sum_{K=0}^{2^n-1}|p_{nk}|^2\|\gamma_0^\Omega(\varphi_{nk})\|^2_{L^2(\Omega)} \\ &\sim |p^0|^2 + \sum_{n=0}^{+\infty}\sum_{k=0}^{2^n-1}|p_{nk}|^2 = \|p\|^2_{H^1(T)}\end{aligned}$$

which, together with Proposition 15, proves Theorem 4.11 for a regular multi-scale decomposition.

As for the quasi-regular case, we recall that $H^r(\Omega)$, for $0 < r < 1$ is the set of all those $L^2(\Omega)$ functions such that the quantity

$$\int_{\Omega\times\Omega}\frac{|f(x) - f(y)|^2}{|x - y|^{d+2r}}dxdy$$

is finite. The stability of this criterium with respect to bi-lipschitz transformations of the domain ensures the conclusion in the quasi-regular case. □

4.4. **Resistance and conductance operators.** In this section we extend the approach proposed in Section 3.3 to the embedded situation. As a first step, we extend the definition of γ_1 to the embedded tree :

Definition 4.12. Let $\Lambda = \text{span}(\varphi_0, \varphi_1, \varphi_{nk})$ denote the set of finite linear combinations of the basis functions defined in Th. 3.3 and Def. 3.7. Operator $\gamma_1^\Omega : \Lambda \to L^2(\Omega)$ is defined as follows : $\gamma_1^\Omega(\varphi_0) \equiv 0$, $\gamma_1^\Omega(\varphi_1) \equiv 1/|\Omega|$ and, for any $n \in \mathbb{N}$, $0 \le k \le 2^N - 1$,

$$\gamma_1^\Omega(\varphi_{nk})(x) = \begin{cases} -\dfrac{1}{\sqrt{R_n^\infty}}\dfrac{1}{|\Omega_{n+1,2k}|} & \textit{if } x \in \Omega_{n+1,2k}, \\ \dfrac{1}{\sqrt{R_n^\infty}}\dfrac{1}{|\Omega_{n+1,2k+1}|} & \textit{if } x \in \Omega_{n+1,2k+1}, \\ 0 & \textit{otherwise}. \end{cases} \tag{29}$$

Proposition 16. *Let T be a regular resistive tree with $R < +\infty$, and let $\mathcal{O}$ be a regular multiscale decomposition. We assume that α is such that $s = d(1 - \ln\alpha/\ln 2)/2 < 1/2$. Then γ_1^Ω can be extended by density to an operator in $\mathcal{L}(\mathring{H}^1_\Delta(T), H^{-s}(\Omega))$.*

Proof. As the decomposition is regular, the extension of γ_1^Ω to an operator in $\mathcal{L}(\mathring{H}^1_\Delta(T), A^{-r}(\Omega))$, with $r = (1 - \ln\alpha/\ln 2)/2$ is straightforward (following the approach of Proposition 7), and the identification of the trace space with a dual Sobolev space follows from Theorem 4.11. □

Proposition 17. *Let T be a geometric tree, $\Omega \subset \mathbb{R}^d$, and $\mathcal{O}$ a regular multiscale decomposition. We assume that α is such that $s = d(1 - \ln\alpha/\ln 2)/2 < 1/2$. Then, for any $g \in H^s(\Omega)$, there exists a unique pressure field $p \in H^1(T)$ such that*

$$\begin{cases} -\partial c \partial^\star p &= 0 \quad \textit{in } T \setminus \{o\}, \\ p(o) &= 0, \\ \gamma_0^\Omega(p) &= g. \end{cases} \tag{30}$$

Proof. This is a direct consequence of Proposition 9 and Theorem 4.11. □

We may now define the conductance and resistance operators as follows:

Definition 4.13. Let T be a geometric tree with $s = d(1 - \ln\alpha/\ln 2)/2 < 1/2$, $\Omega \subset \mathbb{R}^d$, and $\mathcal{O}$ a regular multiscale decomposition. For any $g \in H^s(\Omega)$, we define $\mathcal{C}^\Omega g$ as $\gamma_1^\Omega p$, where p is the solution to (30). Operator $\mathcal{R}$ is defined as the reciprocal of $\mathcal{C}$.

Approximation of $\mathcal{C}^\Omega$. We end this section by extending Proposition 11 to the embedded operator $\mathcal{C}^\Omega$ by describing how the conductance operator $\mathcal{C}$ can be approximated by a truncated operator $\mathcal{C}_N$, for $N \in \mathbb{N}$.

Definition 4.14. For any $N > 0$, we recall that $\tilde{T}_N$ designs the subtree of T with same root, N generations, and condensated resistances (see Definition 3.13). We denote by Γ_N the set of leafs of $\tilde{T}_N$ (whose cardinal is 2^N). For any $g \in L^2(\Omega)$, we denote by $P_N g \in L^2(\Omega)$ its projection onto V_N (see Definition 4.6), and by g_N the corresponding collection of 2^N values. We define now p_N as the solution to the truncated Dirichlet problem

$$\begin{cases} -\partial c \partial^\star p_N &= 0 \quad \text{in } \tilde{T}_N \setminus \{\{o\} \cup \Gamma_N\}, \\ p_N(o) &= 0, \\ p_N &= g_N \quad \text{on } \Gamma_N. \end{cases} \tag{31}$$

$\mathcal{C}_N g$ is defined as the piecewise constant function over Ω which is equal to $u(X_{Nk})/|\Omega_{Nk}|$ on Ω_{Nk}, where $u(X_{Nk})$ is the flux getting out of the tree through X_{Nk}. Notice that the scaling factor is simply $2^N/|\Omega|$ in the case of a balanced decomposition.

Proposition 18. *Let T be a geometric tree, $\Omega \subset \mathbb{R}^d$, and $\mathcal{O}$ a regular multiscale decomposition of Ω. We assume that α is such that*

$$0 < s = d(1 - \ln\alpha/\ln 2)/2 < 1,$$

and we consider s' such that $s < s' < 1$. There exists $C > 0$ such that

$$\|\mathcal{C}_N g - \mathcal{C} g\|_{H^{-s}} \leq \frac{C}{2^{N(s'-s)/d}}.$$

Proof. Let us introduce $r = s/d$ and $r' = s'/d$. One has

$$\|P_N g - g\|_{A^r} \leq 2^{-N(r'-r)} \|g\|_{A^{r'}} \leq C 2^{-N(r'-r)} \|g\|_{H^{s'}},$$

by Proposition 15. The estimate is then a direct consequence of Propositions 11 and 15. □

5. **Application to the human lungs.** We present here how this theoretical framework can be applied to the modeling of the human lungs.

5.1. **Relevancy of the modelling assumptions.** First of all, Weibel's measurements ([23]) establish that the average healthy human respiratory tract presents some geometrical regularity which makes it possible to extrapolate it to an infinite tree. More precisely, all pipes have about the same shape, and a pipe at generation $n+1$ is 0.85 smaller than the pipes at generation n (see [23]). As Poiseuille's law gives a resistance of a pipe proportional to L/D^4 (L is the length, and D is the diameter), the homogeneity coefficient -3 with respect to the size (for a given shape) yields

$$\alpha = 0.85^{-3} \approx 1.63.$$

As it is smaller than 2 the tree has a finite resistance. Notice that this simple fact is quite sensitive to the 0.85 factor. Indeed, it converges because $0.85 > 2^{-1/3} \approx 0.79$. It is also noteworthy that $2^{-1/3}$ is a critical value for the volume also. As pointed out in [22], the volume of the extrapolated version of the actual tree is infinite, for the very same reason $0.85 > 2^{-1/3}$. We must admit that our infinite tree model does not make sense from the geometrical point of view, as far as the bronchial tree itself is concerned, because of this reason. It addresses only the functional nature of this system as a resistive network. Concerning the dimension in which the set of ends is to be embedded, Weibel [22] makes it clear that alveolae are quite uniformly distributed over the volume occupied by the lungs, except of course for the conducting tree itself. See Fig. 5 for a picture of the local structure of the lungs, in particular the uniform foamy zone (right-hand side of the picture) around the branches (essentialy on the left-hand side).

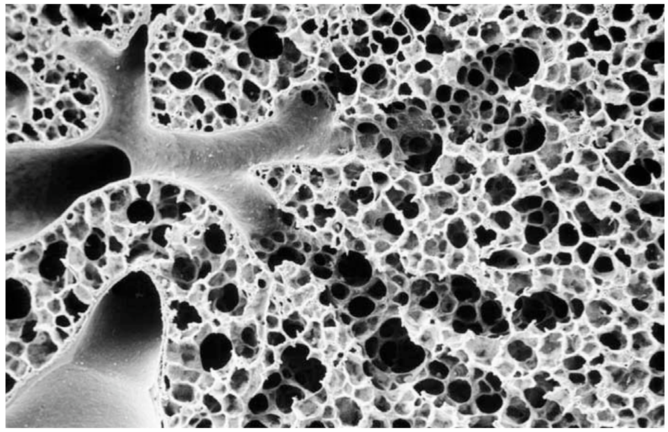

FIGURE 5. Scanning electron micrographs of alveolar ducts surrounded by alveoli arising from branched bronchioles in human lung, at two different scales (Courtesy E.R.Weibel, University of Berne).

As for the geometrical assumptions in Definition 4.4, although they correspond to a idealization of the actual tree, some facts support their relevancy in terms of modelling :

1. Sizes of the alveolae are quite uniformy distributed, at a given time of the ventilation cycle and, due to the very dyadic structure of the tree, the number of alveolae irrigated by a given bronchus does depend on its generation only, so that the balanced character of the decomposition is natural.
2. Assumptions (*ii*) and (*iii*) in Definition 4.4 assert that subdomain are not too far from balls, in the sense that their diameter and area vary asymptotically as those of a ball. This is more conjectural, as accurate measurements of irrigated subdomains for higher generations do not exist. Yet, the zones irrigated by the bronchi of the first generations are well documented, and they appear to have an aspect ratio close to 1, and a piecewise smooth surface.

5.2. **Ventilation process as a Dirichlet to Neumann operator.** We consider here a regular, geometric tree, which embeds onto the parenchyma which we identifiy to a Lipschitz domain $\Omega \subset \mathbb{R}^3$. Now assuming that the way the tree irrigates the parenchyma is quasi-regular, one can then define the operator γ_0^Ω onto $L^2(\Omega)$, whose range is $A^r(\Omega)$, with $r = (1-\ln\alpha/\ln 2)/2 \approx 0.15$. For the physical dimension $d = 3$, one gets $s = rd \approx 0.45$, which is (again, the inequality is tight) less than $1/2$, so that the embedded trace space is the standard Sobolev space $H^{0.45}$ (see Theorem 4.11).

We may now interpret Proposition 17 and Definition 4.13 as a model for the instantaneous ventilation process. The alveolar pressure field $g \in H^s$ drives some air through the tree, and the way this air irrigates the parenchyma is described by a flux field $u = \mathcal{C}^\Omega g$.

Remark 14. An interesting consequence of $0.45 \leq 1/2$ is that the set of feasible pressure fields over the parenchyma does not depend on the multiscale decomposition (as soon as it is quasi-regular). In the case of a smaller α, say $\alpha = 1.4$ for example, one would have $s = 0.25$, and consequently $\gamma_0^\Omega(H^1) = A^{0.75}$ in the three dimensional setting. In this latter situation, the trace space no longer identifies to a Sobolev spaces, and it strongly depends on the decomposition. For example, it admits functions with discontinuities accross the interface between the two lobes Ω_{10} and Ω_{11}, but discontinuities accross other interfaces might rule out the belonging to $A^{0.75}$

5.3. **Constitutive models for the parenchyma.** We end this paper by exploring how the presented approach can be used to design constitutive models for the lung. As already mentioned in the introduction, we proposed in [11] a first attempt to establish a constitutive equation for an elastic medium subject to dissipation phenomena due to the flow of incompressible fluid through a dyadic tree. The continuous model (only one-dimensional in [11]) is obtained as the limit of spring-mass systems intertwined with the leafs of a resistive tree. The equation takes the following form :

$$\partial_{tt}\eta - \partial_{xx}\eta - \partial_x \mathcal{R} \partial_x \partial_t \eta = 0.$$

The damping term can be interpreted the following way : $\partial_t \eta$ is the horizontal velocity (in the direction of the array of masses), so that $\partial_x \partial_t \eta$ is the defect in local conservation, which must be compensated by some flux through the tree. Operator $\mathcal{R}$ maps the flux field to the pressure field, which acts on the momentum equation by its gradient.

The present work allows to propose similar models in higher dimensions (in particular the dimension which is physically relevant, $d = 3$). Let us assume that the structure of the parenchyma (solid parts in Fig. 5) can be described by a constitutive relation $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{e})$, where $\boldsymbol{\sigma}$ is the stress tensor, and $\mathbf{e}$ the strain tensor. In the case of a Lam material (see [6] for more sophisticated models), one has for example

$$\boldsymbol{\sigma} = \mu \left(\nabla \boldsymbol{\eta} + (\nabla \boldsymbol{\eta})^T \right) + \lambda (\nabla \cdot \boldsymbol{\eta}) \mathrm{I}_d,$$

where μ and η are the Lam coefficients, and I_d is the identity tensor.

Considering now a geometric infinite tree, embedded in a regular way to a Lipschitz domain Ω assumed to be occupied by an elastic material, the previous considerations lead to the model

$$\partial_{tt}\boldsymbol{\eta} - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{\eta}) - \nabla \mathcal{R} \nabla \cdot \partial_t \boldsymbol{\eta} = 0.$$

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