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ISOSPECTRAL INFINITE GRAPHS AND NETWORKS AND INFINITE EIGENVALUE MULTIPLICITIES

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Abstract. We consider the continuous Laplacian on infinite locally finite networks under natural transition conditions as continuity at the ramification nodes and Kirchhoff flow conditions at all vertices. It is well known that one cannot reconstruct the shape of a finite network by means of the eigenvalues of the Laplacian on it. The same is shown to hold for infinite graphs in a L^∞–setting. Moreover, the occurrence of eigenvalue multiplicities with eigenspaces containing subspaces isomorphic to $\ell^\infty(\mathbb{Z})$ is investigated, in particular in trees and periodic graphs.

1. **Introduction.** It is well known that isospectral finite graphs with respect to the adjacency operator can be non isomorphic, see e.g. [12, 15], as well as isospectral finite networks, i.e. traces of finite topological graphs, with respect to the continuous Laplacian can be non isometric and, thereby, the underlying abstract graphs be non isomorphic, see [4, 6]. The node transition conditions for the Laplacian are the continuity at ramification nodes and a Kirchhoff incident flow condition at all nodes. The first aim of the present paper is to analyze the same phenomena on infinite uniformly locally finite graphs and networks. The second one is concerned with the existence of eigenvalues of infinite multiplicity, especially the occurrence of inseparable eigenspaces. An essential feature of our approach is the consideration of spaces of bounded functions and of bounded sequences and not a setting in possibly weighted Hilbert Sobolev spaces. It contains the L^2–eigenvalue approach, but seems to be more appropriate for spectral links between the network Laplacians and transition or adjacency operators. Often, parts of the continuous and residual spectrum in the L^2–setting belong to the point spectrum in spaces of bounded functions. Moreover, the canonical fundamental solutions built by sinus– and cosinus–functions and always belonging to the present setting, can only be treated in Sobolev spaces with sufficiently rapidly decreasing weights. In general,

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this approach cannot lead to an associated characteristic calculus for the transition operator, while keeping its symmetry properties.

The spectrum of the Laplacian on finite networks has been considered by many authors, see e.g. [1, 2, 3, 4, 6, 20, 19, 21, 24] and the references therein. For the infinite case we can refer to [9, 10, 11, 13, 22], for the finite algebraic graph theory to the monographs [12, 14, 15], while for the ℓ^2–setting in the infinite case we can refer to [23] and the monograph [27] and the references therein, and for the ℓ^∞–setting to [5, 8, 9, 10, 11].

The present paper is organized as follows. Some graph theoretical preliminaries and some results about the eigenvalues of the Laplacian under the aforementioned node transition conditions are summarized in Section 2. Throughout we shall assume that all edge lengths are equal to 1. Section 3 is devoted to the notion of isospectrality, to the black hole phenomenon, i.e. the occurrence of inseparable eigenspaces containing copies of $\ell^\infty(\mathbb{Z})$, and to the following result.

Theorem 3.2 *All infinite uniformly locally finite trees T with finitely many boundary vertices, but without ramification nodes of valency 2, are isospectral networks. More precisely, each $\lambda \in [0, \infty)$ is an eigenvalue of black hole type for $-\Delta_T^K$ in $\mathcal{C}_K^2(T) \cap L^\infty(T)$.*

In Section 4 two non isomorphic infinite graphs containing circuits are shown to be isospectral with finite geometric multiplicities, while the associated networks are isospectral and non isometric as well with finite and infinite multiplicities. Section 5 is devoted to the occurrence of black holes in periodic graphs or generalized lattices, in particular to the following results.

Corallary 5.1 *Any real number $\lambda > 0$ satisfying $\sin\sqrt{\lambda} = 0$ is a black hole eigenvalue for the Laplacian in each periodic graph of rank $m \geq 2$ and possesses eigenfunctions of compact support.*

Theorem 5.3 *Suppose that $\lambda > 0$ is an eigenvalue of the Laplacian on a band, i.e. a periodic graph of rank 1. Then λ has infinite geometric multiplicity in $\mathcal{C}_K^2(T) \cap L^\infty(T)$ iff λ is a black hole, or, iff λ possesses eigenfunctions belonging to $\mathcal{C}_K^2(T) \cap L^\infty(T)$ of finite support.*

In the final Section 6 examples of families of isospectral non isometric trees, as well as some examples of black hole eigenvalues in periodic graphs are presented.

2. **Graphs, networks and Laplacian.** For any graph $\Gamma = (V, E, \in)$, the vertex set is denoted by $V = V(\Gamma)$, the edge set by $E = E(\Gamma)$ and the incidence relation by $\in \subset V \times E$. The valency of each vertex v is denoted by $\gamma(v) = \text{card}\{e \in E \big| v \in e\}$. Unless otherwise stated, all graphs considered in this paper are assumed to be nonempty, simple, connected and *uniformly locally finite*, i.e.

$$\max_{v \in V(\Gamma)} \gamma(v) =: \gamma_{\max} < \infty. \tag{1}$$

The simplicity property means that Γ contains no loops, and at most one edge can join two vertices in Γ. Moreover, the conditions imply that Γ is countable. For a given numbering of the vertices v_i, $i \in \mathbb{N}$, set $\gamma_i = \gamma(v_i)$ and define the *adjacency* *matrix* or *adjacency operator* by

$$\mathcal{A}(\Gamma) = (a_{ih})_{i,h\in\mathbb{N}} : \mathbb{R}^{V(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)}, \tag{2}$$

where

$$a_{ih} = \begin{cases} 1 & \text{if } v_i \text{ and } v_h \text{ are adjacent in } \Gamma, \\ 0 & \text{else.} \end{cases}$$

Note that $\mathcal{A}(\Gamma)$ is indecomposable iff Γ is connected. Moreover, with the notation $\ell^p(\Gamma) := \ell^p(V(\Gamma))$, $\mathcal{A}(\Gamma)$ maps $\ell^p(\Gamma)$ into $\ell^p(\Gamma)$ for each $p \in [1, \infty]$. The same holds for the row–stochastic *transition matrix* or *transition operator* defined by

$$\mathcal{Z} = \mathrm{Diag}\left(\mathcal{A}(\Gamma)\ \mathbf{e}\right)^{-1} \mathcal{A}(\Gamma) = \mathrm{Diag}_i \left(\gamma_i^{-1}\right) \mathcal{A}(\Gamma), \tag{3}$$

where $\mathbf{e}$ denotes the sequence with entries equal to 1. For a subgraph Θ in Γ let $\bar{\Theta} = (V(\Theta), E(\bar{\Theta}), \in)$ denote the subgraph of Γ spanned by the vertices in Θ with

$$E(\bar{\Theta}) = \{e \big| e \in E(\Gamma),\ e \cap V(\Gamma) \subset V(\Theta)\}.$$

The subgraph Θ is called *induced* if $\bar{\Theta} = \Theta$. Two subgraphs are called *disjoint* if they have no vertex in common, and *essentially disjoint* if they have only a finite number of edges in common. The (combinatorial) *distance* between two vertices v_1 and v_2 is defined as the minimal number of edges of all paths joining v_1 and v_2. For further graph theoretical terminology we refer to [16, 25, 26].

Moreover, without loss of generality, we consider each graph as a connected *topological graph* in $\mathbb{R}^m$, i.e. $V(\Gamma) \subset \mathbb{R}^m$, and the edge set consists in a collection of Jordan curves $E(\Gamma) = \{\pi_j : [0,1] \to \mathbb{R}^m \big| j \in \mathbb{N}\}$ all of length 1 with the following properties: Each support $e_j := \pi_j([0,1])$ has its endpoints in the set $V(\Gamma)$, any two vertices in $V(\Gamma)$ can be connected by a path with arcs in $E(\Gamma)$, and any two edges $e_j \neq e_h$ satisfy $e_j \cap e_h \subset V(\Gamma)$ and $\mathrm{card}(e_j \cap e_h) \leq 1$. The arc length parameter of an edge e_j is denoted by t_j. The trace of the graph $\Gamma = (V, E, \in)$ defines its associated *network*

$$G = \bigcup_{j\in\mathbb{N}} \pi_j\left([0, l_j]\right),$$

that is called a $\mathcal{C}^\nu$-network, if all $\pi_j \in \mathcal{C}^\nu([0,1], \mathbb{R}^m)$. Throughout we shall assume that at least

$$\forall j \in \mathbb{N} :\ \pi_j \in \mathcal{C}^2([0,1], \mathbb{R}^m).$$

Thus, endowed with the induced topology of $\mathbb{R}^m$, G is a connected and locally compact space in $\mathbb{R}^m$. We shall distinguish the *boundary vertices* $V_b = \{v_i \in V \big| \gamma_i = 1\}$ from the *ramification nodes* $V_r = \{v_i \in V \big| \gamma_i \geq 2\}$, especially, we define the *essential ramification nodes* by $V_{\mathrm{ess}} = \{v_i \in V \big| \gamma_i \geq 3\}$.

Two networks G_1 and G_2 are *isometric* ($G_1 \cong G_2$) if there is an homeomorphism $H : G_1 \to G_2$ such that for each edge $e \subset G_1$, $H\big|_e$ is an isometric diffeomorphism onto some edge of G_2. In particular, H is length preserving. Moreover, the underlying abstract graphs Γ_1 and Γ_2 are called *isomorphic* as graphs ($\Gamma_1 \simeq \Gamma_2$) if there is a bijection $V(\Gamma_1) \longrightarrow V(\Gamma_2)$ that preserves the adjacency relation between vertices. If G_1 and G_2 are isometric networks, then the underlying abstract graphs Γ_1 and Γ_2 are isomorphic as graphs. In fact, in the present case of equal edge lengths, this is an equivalence:

$$G_1 \cong G_2 \iff \Gamma_1 \simeq \Gamma_2 \tag{4}$$

The orientation of the graph Γ is given by the *incidence matrix* or *incidence operator*

$$\mathcal{D}(\Gamma) = (d_{ik})_{i,k\in\mathbb{N}} : \mathbb{R}^{E(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)} \tag{5}$$

with

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(1) = v_i, \\ -1 & \text{if } \pi_j(0) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

By (1), $\mathcal{D}(\Gamma)$ is a bounded operator $\ell^\infty(E(\Gamma)) \longrightarrow \ell^\infty(V(\Gamma))$. The *corank* of the graph is defined by $\operatorname{corank}(\Gamma) = \dim\ker\mathcal{D}(\Gamma)$. By definition, a *circuit* ζ is a 2–regular connected graph. In the finite case ζ is a classical closed path, while in the infinite one ζ is the two–sided infinite path Γ_1, also known as double ray, with $V(\Gamma_1) = \mathbb{Z}$ and the adjacency relation

$$a_{ik} = 1 \iff |i-k| = 1. \tag{6}$$

Accordingly, the *circuit space* of the graph Γ is defined as the vector space spanned by elements of $\ker\mathcal{D}(\Gamma)$ having circuit like support:

$$\mathbf{\Pi}(\Gamma) = \big\langle c \in \ker\mathcal{D}(\Gamma) \big| \operatorname{supp}(c) \text{ is a circuit in } \Gamma \big\rangle.$$

In the infinite case, in general, the circuit space is a proper subspace of $\ker\mathcal{D}(\Gamma)$. But recall

Lemma 2.1. *([10]) The* $\operatorname{corank}(\Gamma)$ *is finite iff* $\dim\mathbf{\Pi}(\Gamma)$ *is finite. And in that case*

$$\mathbf{\Pi}(\Gamma) = \ker\mathcal{D}(\Gamma) \leq \ell^\infty(E(\Gamma)).$$

For a function $u : G \to \mathbb{R}$ we set $u_j := u \circ \pi_j : [0,1] \to \mathbb{R}$ and use the abbreviations

$$u_j(v_i) := u_j(\pi_j^{-1}(v_i)), \quad \partial_j u_j(v_i) := \frac{\partial}{\partial t_j} u_j(t_j)\Big|_{\pi_j^{-1}(v_i)} \quad \text{etc.}$$

As the basic geometric transition condition at ramification nodes we impose the *continuity condition*

$$\forall v_i \in V_r \,:\; e_j \cap e_s = \{v_i\} \implies u_j(v_i) = u_s(v_i), \tag{7}$$

that clearly is contained in the condition $u \in \mathcal{C}(G)$. Moreover, at all vertices we impose the Kirchhoff flow condition

$$\forall i \in \mathbb{N} \,:\; \sum_{j\in\mathbb{N}} d_{ij}\partial_j u_j(v_i) = 0. \tag{8}$$

Note that Condition (8) is the Neumann boundary condition at boundary vertices and that it does not depend on the orientation. The validity of (8) in a function space will be indicated by the subscript K. The *canonical* Laplacian Δ on a $\mathcal{C}^2$–network G is defined as the operator

$$\Delta = \Delta_G^K = \Big(u \mapsto \big(\partial_j^2 u_j\big)_{j\in\mathbb{N}} \Big)$$

with the domain $\mathcal{C}_K^2(G) = \{u \in \mathcal{C}(G) \big| \forall j \in \mathbb{N} : u_j \in \mathcal{C}^2([0,1]),\, u \text{ satisfies } (8)\}$ or a weighted Sobolev space $\mathcal{H}^2_{K,c}(G)$. The eigenvalues of $-\Delta_G^K$ in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ are real and nonnegative [10, 11]. Thus, we can write the corresponding eigenvalue problem in the form

$$0 \neq u \in \mathcal{C}_K^2(G) \cap L^\infty(G) \quad \text{and} \quad \partial_j^2 u_j = -\lambda u_j \quad \text{for} \quad j \in N. \tag{9}$$

For the sake of simplicity, we shall use the following notations for the point spectra and the geometric multiplicities under Kirchhoff conditions and for operators $\mathcal{T}$ on $\ell^\infty(\Gamma)$.

Definition 2.2.

$$\begin{aligned}\mathbf{S}(G) &= \sigma_p(-\Delta_G^K, \mathcal{C}_K^2(G)\cap L^\infty(G))\\ \mathbf{s}(\mathcal{T},\Gamma) &= \sigma_p(\mathcal{T}, \ell^\infty(\Gamma))\\ M(\lambda) &= M(\lambda, G) = m_g(\lambda, -\Delta_G^K, \mathcal{C}_K^2(G)\cap L^\infty(G))\\ m(\mu,\mathcal{T}) &= m_g(\mu, \mathcal{T}, \ell^\infty(\Gamma))\end{aligned}$$

Using the transition operator $\mathcal{Z}$ defined in (3), recall the

Theorem 2.3. *([10]) If λ is an eigenvalue of $-\Delta_G^K$ in $\mathcal{C}_K^2(G)\cap L^\infty(G)$ and $\varphi \in \ell^\infty(\Gamma)$ a node distribution of an eigenfunction belonging to λ, then*

$$\mathcal{Z}\varphi = \cos\sqrt{\lambda}\,\varphi. \tag{10}$$

Conversely, if $\cos\sqrt{\lambda}$ is an eigenvalue of $\mathcal{Z}$ admitting the eigenvector $\varphi \in \ell^\infty(\Gamma)$, then λ is an eigenvalue of $-\Delta_G^K$ in $\mathcal{C}_K^2(G)\cap L^\infty(G)$ and φ the node distribution of some eigenfunction belonging to λ. The geometric multiplicities are

$$M(\lambda, G) = \begin{cases} m(1,\mathcal{Z}) & \textit{if } \lambda = 0,\\ m(\cos\sqrt{\lambda},\mathcal{Z}) & \textit{if } \sin\sqrt{\lambda}\neq 0,\\ \operatorname{corank}(\Gamma)+1 & \textit{if } \lambda>0 \textit{ and } \cos\sqrt{\lambda}=1,\\ \operatorname{corank}(\Gamma)+1 & \textit{if } \lambda>0,\ \cos\sqrt{\lambda}=-1 \textit{ and } \Gamma \textit{ bipartite},\\ \operatorname{corank}(\Gamma)-1 & \textit{if } \lambda>0,\ \cos\sqrt{\lambda}=-1 \textit{ and } \Gamma \textit{ not bipartite}.\end{cases}$$

In the above formulae the multiplicity identities stem from the corresponding isomorphisms between the eigenspaces of $-\Delta_G^K$ and $\mathcal{Z}$. In particular for the corank, either the corresponding eigenspace contains a subspace of codimension one and isomorphic to the circuit space, or the circuit space contains a subspace of codimension one and isomorphic to the eigenspace. Especially in the case of k–regular graphs, the eigenvalues λ of the Laplacian and those μ of the adjacency operator $\mathcal{A}$ are related by the formula $\mu = \gamma\cos\sqrt{\lambda}$, and the non vanishing node distributions of eigenfunctions of $-\Delta_G^K$ in $\mathcal{C}_K^2(G)\cap L^\infty(G)$ are exactly the eigenvectors of $\mathcal{A}$ in $\ell^\infty(\Gamma)$.

3. **Isospectral infinite graphs and networks.** Two graphs Γ_1 and Γ_2 are called *isospectral* if $\mathbf{s}(\mathcal{A}(\Gamma_1),\Gamma_1) = \mathbf{s}(\mathcal{A}(\Gamma_2),\Gamma_2)$ and if the geometric multiplicities coincide

$$\forall\mu\in\mathbf{s}(\mathcal{A}(\Gamma_1),\Gamma_1):\ m_g(\mu,\mathcal{A}(\Gamma_1),\ell^\infty(\Gamma_1)) = m_g(\mu,\mathcal{A}(\Gamma_2),\ell^\infty(\Gamma_2)).$$

Two networks G_1 and G_2 are called *isospectral* if $\mathbf{S}(G_1) = \mathbf{S}(G_2)$ and if

$$\forall\lambda\in\mathbf{S}(G_1):\ m_g(\lambda,-\Delta_{G_1}^K,\mathcal{C}_K^2(G_1)\cap L^\infty(G_1)) = m_g(\lambda,-\Delta_{G_2}^K,\mathcal{C}_K^2(G_2)\cap L^\infty(G_2)).$$

Here the identities among the multiplicities are to be understood in the sense that they have both the same finite value or are both infinite. Of course, there cannot be an eigenspace having infinite, but countable basis. In the ℓ^2–setting all eigenspaces of $\mathcal{Z}$ are separable. But as an operator in ℓ^∞, $\mathcal{Z}$ can lead to inseparable eigenspaces. Of particular interest is the case of a black hole eigenvalue in which each bounded sequence can be considered as an eigenvector by means of a common automorphism of $\ell^\infty(\mathbb{Z})$:

Definition 3.1. An eigenvalue of an endomorphism of a Banach space is called a *black hole* if its eigenspace contains a subspace isomorphic to $\ell^\infty(\mathbb{Z})$.

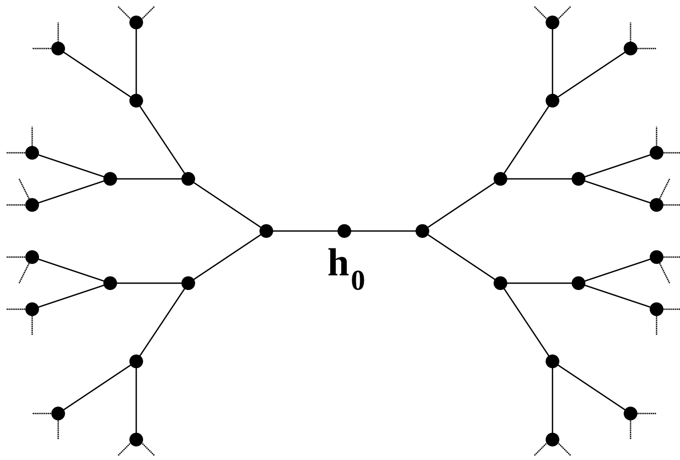

FIGURE 1. The tree H.

Examples of black holes for the adjacency operator can be found in [10] and in Section 6. It has been shown in [4, 6, 7] that finite isospectral networks are not necessarily isometric by using regular isospectral graphs that are not isomorphic. In the infinite case, Theorem 3.2 presents a family of infinite trees that are isospectral as networks, but not isospectral as graphs, since the regular (=homogeneous) trees $\mathcal{T}_\gamma$ belong to it and fulfill $\mathbf{s}\,(\mathcal{T}_\gamma) = [-\gamma, \gamma]$. In Section 4 we present a pair of two graphs that are isospectral in both senses and that have only finite multiplicities except for the eigenvalues belonging to vanishing node distributions, i.e. $\sin\sqrt{\lambda} = 0$. For the following result we have to bear in mind that all edge lengths are supposed to be equal.

Theorem 3.2. *All infinite uniformly locally finite trees T satisfying*

$$V_{\mathrm{ess}} = V_{\mathrm{r}} \quad \textit{and} \quad \operatorname{card} V_{\mathrm{b}}(T) < \infty$$

are isospectral networks, or more precisely: For such a tree, the point spectrum $\mathbf{S}(T)$ in $\mathcal{C}^2_K(T) \cap L^\infty(T)$ is equal to $[0, \infty)$, and each $\lambda \in [0, \infty)$ is a black hole eigenvalue of $-\Delta^K_T$ in $\mathcal{C}^2_K(T) \cap L^\infty(T)$.

Lemma 3.3. *Under the hypotheses of Theorem 3.2, the tree T contains a countable family $\{H_k \,|\, k \in \mathbb{Z}\}$ of disjoint subtrees each of them being isomorphic to the tree H having one distinguished node of valency 2 and all other valencies equal to 3, see Figure 1.*

Proof. By hypotheses, T contains a ramification node that is adjacent to two subtrees containing no boundary vertex of T. Thus, admitting one possible ramification node v_∞ of valency 2, we can assume that $V_{\mathrm{b}}(T) = \emptyset$.

If necessary, the following construction will be made in one of the trees adjacent to v_∞. Choose a ramification node v_0 and find a subtree H_0 isomorphic to H by identifying v_0 and h_0. Next, find a node v_1 with $\operatorname{dist}(H_0, v_1) \geq 3$ and find a subgraph $H_1 \simeq H$ with v_1 corresponding to h_0. Having found $n-1$ disjoint copies of H satisfying

$$\operatorname{dist}(H_i, v_{n-1}) \geq n - 1 + 2 - i \quad \text{for} \quad 0 \leq i \leq n-2,$$

find a vertex v_n with

$$\operatorname{dist}(H_i, v_n) \geq n + 2 - i \quad \text{for} \quad 0 \leq i \leq n-1$$

and a subgraph $H_n \simeq H$ with v_n corresponding to h_0. This defines inductively the desired family. □

Proof. (Theorem 3.2) By [9, Thm. 5.5] and [10, Thms. 8.4, 8.6],

$$\mathbf{S}(T) = [0, \infty) \quad \text{and} \quad \forall \lambda \in [0, \infty): \ M(\lambda, T) = \infty.$$

According to Lemma 3.3, we find a family $\{H_k \,|\, k \in \mathbb{Z},\, H_k \simeq H\}$ of disjoint subtrees in T. The constructions in the proofs of the cited theorems are such that for any $\lambda \in [0, \infty)$ and for each $k \in \mathbb{Z}$ there exists an eigenfunction u_k belonging to λ with support in H_k and vanishing at the node v_k corresponding to h_0. Then, to any $(x_k)_{k \in \mathbb{Z}} = x \in \ell^\infty(\mathbb{Z})$, we can associate the eigenfunction $u = \Phi(x)$ belonging to λ by defining

$$\forall k \in \mathbb{Z}: \quad u\Big|_{H_k} = x_k \, u_k$$

and by extending it by 0 to the remaining nodes of T. By disjointness, Φ defines a monomorphism from $\ell^\infty(\mathbb{Z})$ into the eigenspace. This permits to conclude. □

Remark 3.1. If the tree contains nodes of valency 2, then the assertion of Theorem 3.2 is no longer true since the eigenvalue 0 can be of finite multiplicity, see [9, Ex. 5.9]. Using the terminology and the results from [9] we obtain that the assertion of Theorem 3.2 remains true for all infinite uniformly locally finite trees T with constant edge lengths satisfying

$$\operatorname{card} V_{\mathrm{ess}} = \infty, \quad \operatorname{card} V_{\mathrm{b}}(T) < \infty \quad \text{and} \quad L(T) > \frac{1}{\min\{\gamma(v) \,|\, v \in V_{\mathrm{ess}}(T)\} - 1},$$

where $L(T)$ is the minimal ratio of incident viaducts in T. Recall that, by definition, a *viaduct* in a graph Γ is a path π of length at least 2 in Γ joining two distinct vertices u and v such that there is no other path in Γ joining u and v having a vertex in the set $V(\pi) \backslash \{u, v\}$.

Note further that in Γ_0 and Γ_1 each eigenvalue is of multiplicity at most 2, see e.g. [10].

Moreover, the condition of finitely many boundary vertices is essential. In the opposite case, the multiplicities can become finite, see the graphs Γ_1^n in 6.1, where two families of isospectral and non isomorphic trees for the Laplacian with finite multiplicities are presented that are also isospectral as graphs.

For the eigenvalues satisfying $\sin\sqrt{\lambda} = 0$, the finite and infinite circuits of the graph play a distinguished role for the multiplicities, especially when the circuit space possesses an uncountable basis.

Theorem 3.4. *Suppose that Γ contains infinitely many disjoint finite circuits or infinitely many essentially disjoint infinite circuits. Then any real number $\lambda > 0$ satisfying $\sin\sqrt{\lambda} = 0$ is a black hole eigenvalue for the network Laplacian, and possesses eigenfunctions of compact support in the first case.*

Proof. Consider first an infinite countable family of finite and mutually disjoint circuits $\{\zeta_k \,|\, k \in \mathbb{Z}\}$ in Γ. For $\sin\sqrt{\lambda} = 0$, each circuit ζ_k contains the support of a suitable eigenfunction f_k. These latter ones are all linearly independent and give rise to an injection of $\ell^\infty(\mathbb{Z})$ into the corresponding eigenspace by associating to $x = (x_k)_{k \in \mathbb{Z}}$ the eigenfunction defined by $x_k f_k$ on ζ_k and extended by 0 to the remaining edges. In the infinite circuit case we can proceed similarly for the black hole character. □

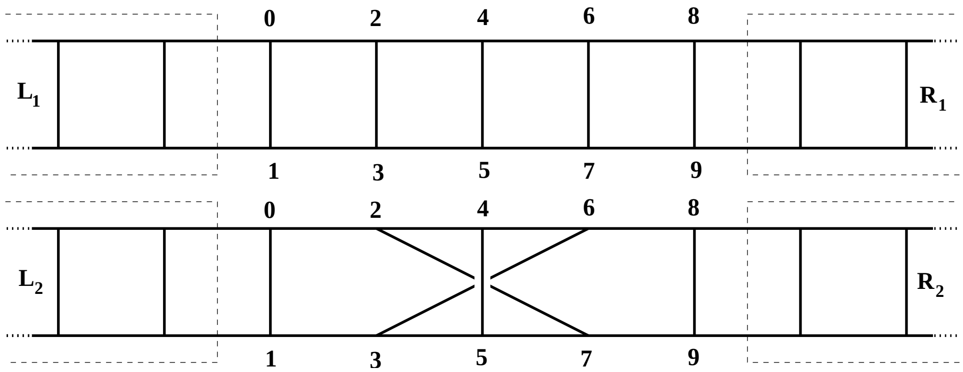

FIGURE 2. The graphs B_1 and B_2.

4. **Two isospectral infinite graphs with circuits.** Let B_1 denote the two–sided infinite ladder depicted in the upper part of Figure 2 and B_2 the graph depicted in the lower part of Figure 2. We shall show that both graphs are isospectral with only finite multiplicities, but not isomorphic, while the associated networks are isospectral with finite and infinite multiplicities, but not isometric.

Lemma 4.1. *The graphs B_1 and B_2 are not isomorphic.*

Proof. Numerate the vertices and denote by L_i and R_i the same infinite connected subgraphs of B_1 and B_2 not containing the vertices $v_0, \ldots, v_9$ as indicated in Figure 2. Suppose that $\varphi : V(B_1) \to V(B_2)$ is a graph isomorphism. Then φ maps circuits of length ℓ in B_1 into circuits of length ℓ in B_2, especially squares into squares, adjacent squares into adjacent squares and finite sequences of squares into finite sequences of squares preserving the adjacencies. Thus vertical edges in the graphs L_i and R_i have to be mapped into themselves. Thus, φ maps L_1 onto L_2 or R_2 and R_1 onto R_2 or L_2 respectively. In any case

$$\varphi\left(V(L_1 \cup R_1)\right) = V(L_2 \cup R_2).$$

This enforces that φ maps the vertices $v_0, \ldots, v_9$ into themselves and constitutes a permutation belonging to S_{10} compatible with the adjacencies in the finite graphs F_1 and F_2 depicted in Figure 3. But these two graphs are not isomorphic, since

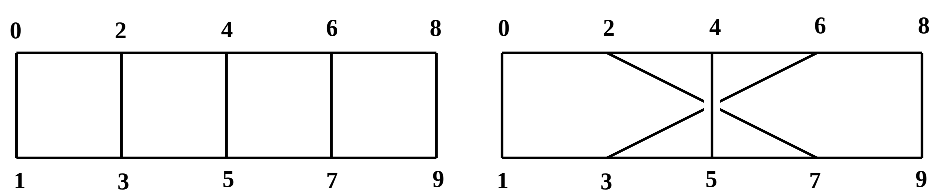

FIGURE 3. Two non isomorphic subgraphs F_1 and F_2.

F_1 contains four squares, while F_2 contains only two squares. This shows the assertion. □

We note in passing that the graph B_1 is planar, while B_2 is not. This yields another proof of Lemma 4.1, defining planarity in infinite graphs by planarity of all finite subgraphs according to [17].

Lemma 4.2. *The graphs B_1 and B_2 are isospectral: $\sigma_p(\mathcal{A}) = [-3, 3]$ and*

$$m(\mu, \mathcal{A}(B_i)) = \begin{cases} 1 & \text{for } \mu = \pm 3, \\ 2 & \text{for } -3 < \mu < -1 \text{ or } 1 < \mu < 3, \\ 3 & \text{for } \mu = \pm 1, \\ 4 & \text{for } -1 < \mu < 1. \end{cases}$$

Proof. We have to show that the point spectra of both adjacency operators coincide counting multiplicities. The proof consists in the following steps.

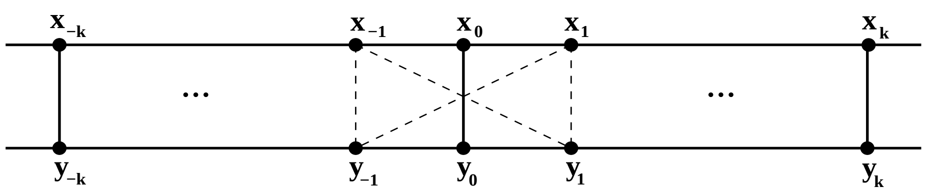

FIGURE 4. Eigenvector notation for both graphs.

(1) Throughout let $u \in \ell^\infty(B_i)$ be an eigenvector of the eigenvalue μ. Numerate the values of u at the vertices as indicated in Figure 4.
(2) First, note that any geometric multiplicity amounts at most to 4 since the values x_0, x_1, y_0, y_1 determine completely any eigenvector. Secondly, we can reduce the determination of the eigenvalues to the interval $[0, 3]$ since both graphs are bipartite.
(3) Let S denote the symmetry with respect to the horizontal middle line, i.e. the exchange of the values x_k and y_k for each $k \in \mathbb{Z}$. Each eigenvector u can be decomposed in a unique way into

$$u = \underbrace{\frac{1}{2}(u + Su)}_{=:a} \oplus \underbrace{\frac{1}{2}(u - Su)}_{=:b} \qquad \text{with} \quad Sa = a, \ Sb = -b.$$

(4) A symmetric vector $Su = u$ in B_1 leads to the Γ_1–recurrence

$$x_{k+1} + x_{k-1} = (\mu - 1)x_k, \qquad k \in \mathbb{Z}$$

while the antisymmetric case yields

$$x_{k+1} + x_{k-1} = (\mu + 1)x_k, \qquad k \in \mathbb{Z}.$$

(5) A symmetric vector $Su = u$ in B_2 leads to the conditions

$$x_1 + x_{-1} = (\mu - 1)x_0, \quad x_2 + x_0 + x_{-1} = \mu x_1, \quad x_{-2} + x_0 + x_1 = \mu x_{-1},$$

and the recurrence

$$x_{k+1} + x_{k-1} = (\mu - 1)x_k, \qquad k \in \mathbb{Z}, \ |k| \geq 2.$$

For $Su = -u$ we get

$$x_1 + x_{-1} = (\mu + 1)x_0, \quad x_2 + x_0 - x_{-1} = \mu x_1, \quad x_{-2} + x_0 - x_1 = \mu x_{-1},$$

and the recurrence

$$x_{k+1} + x_{k-1} = (\mu + 1)x_k, \qquad k \in \mathbb{Z},\ |k| \geq 2.$$

(6) For $\mu \in [0, 1)$ we find $m(\mu, \mathcal{A}(B_i)) = 4$ with the recurrences (4) and (5) that define on each graph four independent solutions since there are exactly two such solutions on Γ_1 in both cases:

$$\dim [\ker (\mathcal{A}(B_i) - \mu I) \cap \{u \in \ell^\infty(B_i) \,|\, Su = u\}] = \dim [\ker (\mathcal{A}(B_i) - \mu I) \cap \{u \in \ell^\infty(B_i) \,|\, Su = -u\}] = 2.$$

(7) For $\mu \in (1, 3)$, the antisymmetric part b of the eigenvector has to vanish since otherwise, it would lead to an unbounded sequence on Γ_1 according to the $\mu + 1$ cases in the recurrences (4) and (5). On the other hand, in the case $Su = u$, we obtain exactly two linearly independent solutions on both graphs. Thus, $m(\mu, \mathcal{A}(B_i)) = 2$.

(8) The eigenvalue $\mu = 3$ is simple for both graphs, since the constant vector is the only bounded solution of the recurrence $x_{k+1} + x_{k-1} = 2x_k$.

(9) For $\mu = 1$, the recurrences (4) and (5) corresponding to $\mu - 1 = 0$, define on each graph two linearly independent solutions, but only one such solution on Γ_1 for $\mu + 1 = 2$ for both graphs:

$$\dim [\ker (\mathcal{A}(B_i) - \mu I) \cap \{u \in \ell^\infty(B_i) \,|\, Su = u\}] = 2$$
$$\dim [\ker (\mathcal{A}(B_i) - \mu I) \cap \{u \in \ell^\infty(B_i) \,|\, Su = -u\}] = 1$$

Possible independent eigenvectors are given for B_1 by

$$x^1 = (\ldots, 1, 0, -1, 0, 1, 0, -1, x_0 = 0, 1, 0, -1, 0, 1, 0, -1, \ldots) = y^1,$$
$$x^2 = (\ldots, 0, -1, 0, 1, 0, -1, 0, x_0 = 1, 0, -1, 0, 1, 0, -1, 0 \ldots) = y^2,$$
$$x^3 = \mathbf{e} = -y^3,$$

and for B_2 by x^1, x^3 and

$$x^4 = (\ldots, -1, 2, 1, -2, -1, 2, 1, x_0 = 0, -1, -2, 1, 2, -1, -2, 1, \ldots) = y^4.$$

Thus, $m(\mu, \mathcal{A}(B_i)) = 3$.

□

Using Theorem 2.3 and Lemma 4.2, we obtain

Corollary 4.1. *The Laplacians of B_1 and B_2 in $\mathcal{C}^2_K \cap L^\infty$ have the same eigenvalues, their multiplicities coincide and satisfy*

$$M(\lambda, B_i) = \begin{cases} 1 & \text{if } \lambda = 0, \\ m\left(3\cos\sqrt{\lambda}, \mathcal{A}(B_i)\right) & \text{if } \lambda \in (0, \infty) \text{ and } \sin\sqrt{\lambda} \neq 0, \\ \infty & \text{if } \lambda \in (0, \infty) \text{ and } \sin\sqrt{\lambda} = 0. \end{cases}$$

5. **Infinite multiplicities in periodic graphs.** A *periodic graph* or *generalized lattice*, see [5, 27] is a uniformly locally finite graph whose automorphism group contains a transitive subgroup $\mathbf{G}$ isomorphic to some $\mathbb{Z}^m$. More precisely:

Definition 5.1. A uniformly locally finite graph Γ is called *periodic of rank* m with *translation group* $\mathbf{G} = \oplus_{i=1}^m \mathbb{Z} b_i \leq \mathrm{Aut}(\Gamma)$, with *kernel* N and with *cell* F, if the following conditions hold:

(a) Γ is connected.

(b) $\{b_i \mid 1 \leq i \leq m\}$ is a basis of the free abelian group $\mathbf{G}$.
(c) N and F are finite connected subgraphs of Γ such that $F = \overline{N \cup \bigcup_{i=1}^{m} N^{b_i}}$, $V(N)^{\mathbf{G}} = V(\Gamma)$ and $E(F)^{\mathbf{G}} = E(\Gamma)$.
(d) $\forall g, h \in \mathbf{G} : \quad g \neq h \quad \Longrightarrow \quad V(N^g) \cap V(N^h) = \emptyset$.

Here the group action on vertices and edges is indicated as exponent. The kernel defines the periodicity of the vertices, while the cell stands for the one of the edges. The group can always be thought of as a translation group of rank m in some $\mathbb{R}^n$. In the case of rank $m = 1$, periodic graphs are called *bands*. E.g. the two–sided infinite ladder graph B_1 from Section 4 is a band with each vertical edge as kernel.

Other classical examples of periodic graphs are given by the graphs of Kepler's plane tilings, as e.g. the tiling with regular triangles and dodecagons in Fig. 10, where a kernel is given by any pair of adjacent triangles. Examples of black holes in periodic graphs can be found in [7, 10] and in Section 6. Periodic networks are Liouville networks, and the eigenvalue 1 of their transition operators $\mathcal{Z}$ is simple, see [9].

If an eigenvalue of a periodic graph admits an eigenvector or an eigenfunction of finite support, then it must be a black hole. The simplest example is given by the graph in Figure 5 in which a pair of a big white and black dots in a square stands for

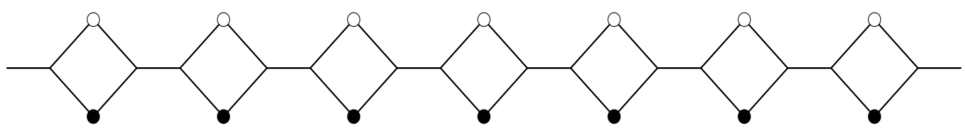

FIGURE 5. Eigenvectors of finite support leading to the black hole 0.

an arbitrary value and its negative, while no dot stands for the value 0. The values can change from one square to another. In this way eigenvectors of finite support are defined for the eigenvalue 0 for the transition operator $\mathcal{Z}$ and lead to a black hole. It corresponds to the black holes for the Laplacian defined by $\cos\sqrt{\lambda} = 0$.

According to [5, Lemma 8.10], a periodic graph Γ of rank $m \geq 2$ contains finite circuits. Then, by periodicity, there is an infinite countable family of finite and mutually disjoint circuits, and Theorem 3.4 yields the

Corollary 5.1. *Any real number $\lambda > 0$ satisfying $\sin\sqrt{\lambda} = 0$ is a black hole eigenvalue for the Laplacian in any periodic graph of rank $m \geq 2$ and possesses eigenfunctions of compact support.*

The assertion of Theorem 3.4 is false for bands, see e.g. Γ_1^0 in Section 6. Note that there are black holes without eigenvectors of finite support for $\mathcal{Z}$, and consequently black holes without eigenfunctions of compact support for $-\Delta_G^K$. An example is given by the eigenvalue -2 of Kepler's plane tiling with regular triangles, see [10], others by Examples 6.2 and 6.3. This phenomenon is not possible in the ℓ^2–setting where it is well–known that eigenvalues of infinite multiplicity have eigenfunctions of compact support, see e.g. [18]. Clearly, without periodicity, infinite graphs can have eigenfunctions or -vectors of compact or finite support with finite multiplicity. But in bands, eigenvalues of infinite multiplicity in the ℓ^∞–setting necessarily have eigenvectors of finite support. This is part of the following results.

Theorem 5.2. *An eigenvalue of the transition operator $\mathcal{Z}$ or of the adjacency operator $\mathcal{A}$ in a band B has infinite geometric multiplicity iff it admits eigenvectors of finite support.*

Proof. Let μ be an eigenvalue of $\mathcal{Z}$. If there is an eigenvector v of finite support S, then there is a sufficiently large induced kernel M containing S such that v vanishes at the nodes that are incident to nodes outside M. Under the one–dimensional group $\mathbb{Z}a_1$ and for the translated kernels M^{ka_1} with $k \in \mathbb{Z}$ and for any $x = (x_k)_{k\in\mathbb{Z}} \in \ell^\infty(B)$, define a corresponding eigenvector $u \in \ell^\infty(B)$ by

$$\forall k \in \mathbb{Z} : \; u\Big|_{M^{ka_1}} = x_k v\Big|_M .$$

Thus, μ is a black hole eigenvalue of $\mathcal{Z}$.

Next, suppose that μ has infinite multiplicity. Choose a kernel N in B and denote by N_k the translated kernel N^{kb_1} under the one–dimensional group $\mathbb{Z}b_1$. Without restriction we can assume that N is induced and sufficiently big such that the minimal distance between vertices in N being incident with N_1 and vertices in N_0 being incident with N_{-1} is at least 2. Let D_k denote the maximal subgraph of B containing N_k such that each value of an eigenvector belonging to μ is uniquely determined by its values in N_k. In fact, D_k is the maximal subgraph in B of all subgraphs Σ with property

$$x = 0 \quad \text{in} \quad N_k \quad \Longrightarrow \quad x = 0 \quad \text{in} \quad \Sigma,$$

since D_k clearly is one of those graphs Σ, while in each Σ the values of an eigenvector are uniquely determined by those in N_k due to linearity and injectivity of the determining operations. By periodicity it follows that $D_k = D_0^{kb_1}$ for all $k \in \mathbb{Z}$.

If $D_k = B$ for some $k \in \mathbb{Z}$, then μ is of finite multiplicity. Thus each D_k is a proper subgraph of B.

If there is a pair $k < h$ with an eigenvector x vanishing in N_k and N_h, and thereby vanishing also in D_k and D_h, but taking non zero values between both kernels, then there is another eigenvector, if not x, of finite support between and outside D_k and D_h.

If there is no such a pair, then each eigenvector of μ vanishing in N_k and N_h with $k < h$, vanishes also in the vertex set of the smallest connected induced subgraph containing $V(N_k) \cup V(N_h)$. But then D_k contains N_h, and D_h contains N_k. Thus $D_k = D_h$ for all $k, h \in \mathbb{Z}$, which is absurd since then $D_0 = B$. □

Corollary 5.2. *Each eigenvalue of infinite multiplicity of $\mathcal{Z}$ or $\mathcal{A}$ in a band is a black hole.*

For the Laplacian we can state the following equivalence.

Theorem 5.3. *Suppose that $\lambda > 0$ is an eigenvalue of the Laplacian on the band B. Then the following conditions are equivalent:*

1. *λ has infinite geometric multiplicity.*
2. *λ is a black hole.*
3. *λ possesses eigenfunctions of finite support.*

Proof. Clearly, the third condition implies the second one and the second one the first one. By Theorems 2.3 and 5.2 it remains to handle the case $\sin\sqrt{\lambda} = 0$ supposing that λ has infinite geometric multiplicity. But, then B cannot be a tree by Theorem 2.3, since a band like tree can only contain one infinite circuit up to translation. Thus, B contains finite circuits, and, again by periodicity, a countable family of those. Finally, Theorem 3.4 permits to conclude. □

For periodic graphs of rank $m \geq 2$, the finite support property has to be replaced by supports lying in periodic subgraphs of rank $m-1$ or less. The proofs of the corresponding results are quite technical and their details are omitted here. The simplest rank 2 example requiring band–like infinite supports is the eigenvalue $\mu = 0$ of the adjacency operator of the graph $\mathbf{K}_1$ of Kepler's plane tiling by squares, that corresponds to eigenvalues $\cos\sqrt{\lambda} = 0$ for the Laplacian. Other examples of finite support eigenvalues in higher rank periodic graphs are displayed in Example 6.4.

6. **Examples.**

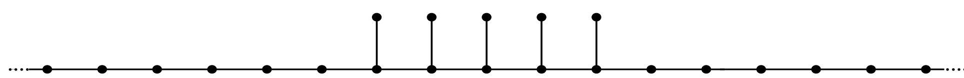

FIGURE 6. The graph $\Gamma_{1,n}$ for $n = 5$.

6.1. **Isospectral tree families with finite Laplacian multiplicities.** Two families of trees that are isospectral as graphs and as networks, but non isomorphic as graphs, and thereby not isomorphic as networks as well, can be found as follows. For $n \in \mathbb{N}^*$, let $\Gamma_{1,n}$ denote the tree obtained by adding in Γ_1 to n consecutive vertices one edge incident to a boundary vertex as displayed in Fig. 6. Then it is easily seen that $\mathbf{S}(\Gamma_{1,n}) = [0, \infty)$ and

$$M(\lambda, \Gamma_{1,n}) = \begin{cases} 1 & \text{if } \lambda = 0, \\ 2 & \text{if } \lambda > 0, \end{cases}$$

while $\mathbf{s}(\Gamma_{1,n}) = (-2, 2)$ and $m(\mu, \Gamma_{1,n}) = 2$ for $\mu \in (-2, 2)$. Note that the spectral radius 2 of the adjacency operator of $\Gamma_{1,n}$ is not an eigenvalue of $\mathcal{A}(\Gamma_{1,n})$ since a bounded eigenvector would have to be constant on the two one–sided infinite paths in $\Gamma_{1,n}$ that contain exactly one ramification nodes of valency 3 of $\Gamma_{1,n}$. By the bipartite character of $\Gamma_{1,n}$, the same holds for -2.

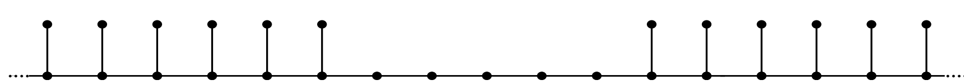

FIGURE 7. The graph Γ_1^n for $n = 5$.

Moreover, for $n \in \mathbb{N}$, let Γ_1^n be the infinite tree obtained by adding in Γ_1 to each vertex one edge incident to a boundary vertex except at n consecutive vertices as displayed in Fig. 7. Then $\mathbf{S}(\Gamma_1^n) = [0, \infty) \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)$ and

$$M(\lambda, \Gamma_1^n) = \begin{cases} 1 & \text{if } \lambda = 0, \\ 2 & \text{if } \lambda > 0 \text{ and } \cos\sqrt{\lambda} \neq 0. \end{cases}$$

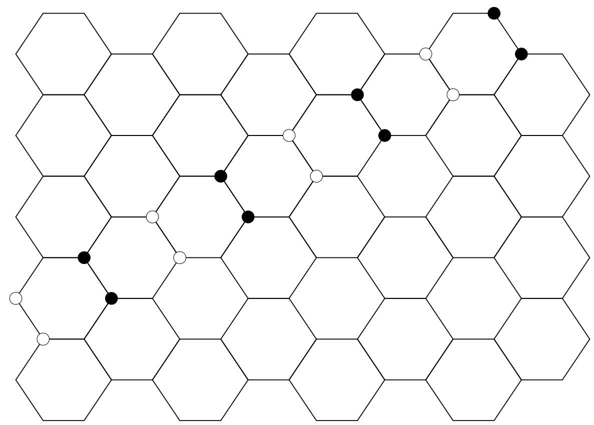

FIGURE 8. The black hole 1 for the adjacency operator of $\mathbf{K}_2$.

A reduction to the adjacency spectrum and to the eigenvectors of Γ_1 yields that $\mu \in \mathbf{s}\left(\Gamma_1^n\right) \iff \left(\mu - \frac{1}{\mu}\right) \in [-2,2]$ and, thereby,

$$\mathbf{s}\left(\Gamma_1^n\right) = \left[-1-\sqrt{2}, 1-\sqrt{2}\right] \cup \left[-1+\sqrt{2}, 1+\sqrt{2}\right]$$

and

$$m\left(\mu, \Gamma_1^n\right) = \begin{cases} 1 & \text{if } \mu = \pm(1+\sqrt{2}) \text{ or } \mu = \pm(1-\sqrt{2}), \\ 2 & \text{if } \mu \in (-1-\sqrt{2}, 1-\sqrt{2}) \cup (-1+\sqrt{2}, 1+\sqrt{2}). \end{cases}$$

6.2. **A black hole for $\mathbf{K}_2$.** The eigenvalue 1 (as well as -1) is a black hole for the adjacency operator of the graph $\mathbf{K}_2$ of Kepler's plane tiling by hexagons. In Figure 8 as well as in the following ones, a big white dot stands for the value 1 at the node, a big black dot for -1, and no dot for the value 0. In this way an eigenvector of band–like support is defined, that leads also to the node distribution of an eigenfunction for the Laplacian belonging to the black hole λ defined by $3\cos\sqrt{\lambda} = 1$.

6.3. **A black hole for $\mathbf{K}_4$.** The eigenvalue 1 (as well as -1) is a black hole for the adjacency operator of the graph $\mathbf{K}_4$ of Kepler's plane tiling by squares and octagons with an eigenvector of band–like support as defined in Figure 9. It leads to the node distribution of an eigenfunction for the Laplacian belonging to the black hole λ defined by $3\cos\sqrt{\lambda} = 1$.

6.4. **Black holes for $\mathbf{K}_5$.** Figure 10 displays band–like and finite supports of eigenvectors for the adjacency operator of Kepler's plane tiling by triangles and dodecagons, all to be read separately. The two upper ones belong to the black hole -2, while the two lower ones correspond to the black hole 0. They lead to node distributions of eigenfunctions for the Laplacian belonging to the black hole λ defined by $3\cos\sqrt{\lambda} = -2$ and $\cos\sqrt{\lambda} = 0$, respectively.

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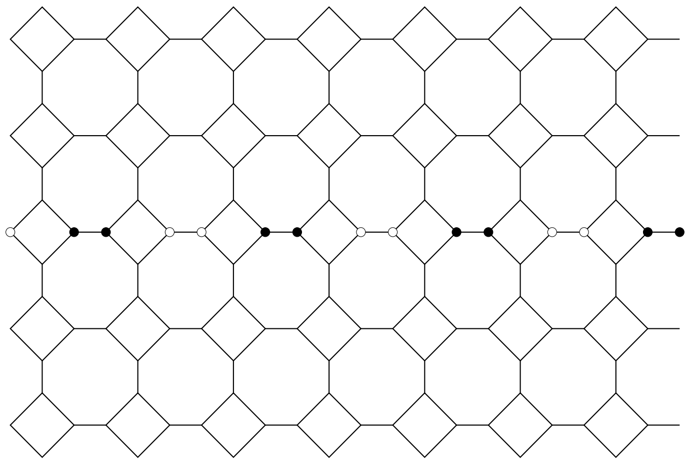

FIGURE 9. The black hole 1 for the adjacency operator of $\mathbf{K}_4$.

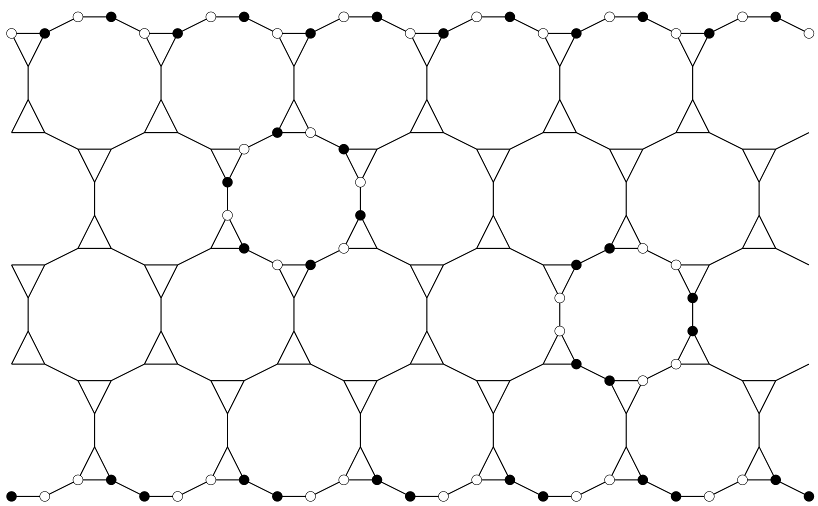

FIGURE 10. The black holes -2 and 0 for the adjacency operator of $\mathbf{K}_5$.

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