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CONTROL OF SYSTEMS OF CONSERVATION LAWS WITH BOUNDARY ERRORS

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Abstract. The general problem under consideration in this paper is the stability analysis of hyperbolic systems. Some sufficient criteria on the boundary conditions exist for the stability of a system of conservation laws. We investigate the problem of the stability of such a system in presence of boundary errors that have a small $\mathcal{C}^1$-norm. Two types of perturbations are considered in this work: the errors proportional to the solutions and those proportional to the integral of the solutions. We exhibit a sufficient criterion on the boundary conditions such that the system is locally exponentially stable with a robustness issue with respect to small boundary errors. We apply this general condition to control the dynamic behavior of a pipe filled with water. The control is defined as the position of a valve at one end of the pipe. The potential application is the study of hydropower installations to generate electricity. For this king of application it is important to avoid the waterhammer effect and thus to control the $\mathcal{C}^1$-norm of the solutions. Our damping condition allows us to design a controller so that the system in closed loop is locally exponential stable with a robustness issue with respect to small boundary errors. Since the boundary errors allow us to define the stabilizing controller, small errors in the actuator may be considered. Also a small integral action to avoid possible offset may also be added.

1. **Introduction.** The operation of many physical networks having an engineering relevance may be represented by hyperbolic partial differential equations (PDE). The main property of this class of PDE is the existence of the so-called Riemann coordinates which are a successful tool for the proof of classical solutions, the analysis and the control among other properties, see e.g. [3, 20].

The stability of homogeneous hyperbolic systems has been analyzed for a long time in the literature. A sufficient condition is that the Jacobian matrix of the boundary conditions has a spectral radius less than 1, see [12]. The result of [12] is based on the analysis of the Riemann coordinates. An other (and weaker) sufficient condition has been recently proven in [8]. In [21], using again a Riemann coordinates approach, it is stated a sufficient condition for the stability to be robust with respect to small non-homogeneous terms. In the present work, we consider the case of perturbations in the boundary conditions. These boundary errors are of two types: the errors proportional to the solutions and those proportional to the integral of the solutions. That part of our work is similar with [10] where, using a Lyapunov function, an integral action is considered for hyperbolic systems. See also [2, 7] (and [6]) for the use of Lyapunov functions for the stability of hyperbolic systems.

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Since many physical control problems are modelized by means of hyperbolic systems with boundary conditions defined by the actuator, our result may be understood as a sufficient criterion for the stability of some physical networks with a robustness issue with respect to small actuator errors. In our context the perturbations have to be small in $\mathcal{C}^1$-norm.

We apply our main result to the control of the dynamic behavior of a pipe filled with water. More precisely we consider a hydropower installation to generate electricity. Different equipments can be considered (see [15]) such as pipes, Francis turbines, surge tanks... Here we consider a pipe with a valve at one end. For such installations it is crucial to reduce the waterhammer effect, i.e. to control the $\mathcal{C}^1$-norm of the solutions. Our main result considers precisely this topology, and suggests a sufficient damping condition. Modelling the hydraulic system by a hyperbolic system, we succeed to design a controller so that the damping condition is satisfied. Roughly speaking we compute a suitable valve opening for the pipe. Moreover the exponential stability is shown to be robust with respect to small (proportional or integral) actions. On numerical simulations or on real applications, it may be necessary to add an integral action to the controller to cancel an offset (due to possible actuator imperfection e.g.). Our main result is thus a theoretical proof that adding this small integral doesnot destabilize the system and "control the waterhammer effect". Other techniques exist for the control of dynamic behavior of hydraulic installations. Let us consider the impedance method [22], the transfer matrix method [22, 18] and the method of characteristic curves [22, 23, 4] which use also the Riemann coordinates.

The number of other applications of hyperbolic systems is quite large. Let us consider e.g. the control of gas in pipeline networks [1], the study of road junction [13], or the regulation of open-channels [19]. The present work can be seen as a generalization of [19], since the Saint-Venant equations that are considered in that reference do not involve neither errors that are proportional to the fluid flow, neither errors that are integral of the fluid flow. These kind of errors are of great interest for practical applications since they can cancel some offset due to actuator imperfection as in [10, 11]. The main result can thus be seen as a theoretical proof of the second part of [11] where the interest of the integral action for the stability condition and the cancelation of the offset is illustrated on numerical simulations and on experiments.

The paper is organized as follows. First in Section 2, we state our main result, namely the necessary condition for the stability of two conservation laws. In Section 3, we apply our main result to the dynamic behavior of a pipe filled with water. In Section 4, we proof our main result. We conclude our paper in Section 5.

2. **Stability of systems of two conservation laws.** We consider the class of hyperbolic PDE obtained as a system of two conservation laws defined in Riemann coordinates as follows

$$\partial_t \boldsymbol{\xi} + \Lambda(\boldsymbol{\xi})\partial_x \boldsymbol{\xi} = 0 \tag{1}$$

with $\boldsymbol{\xi}$: $[0, L] \times [0, +\infty) \to \mathbb{R}^2$: $(x, t) \mapsto (\xi_1, \xi_2)(x, t)$, and $\Lambda(\boldsymbol{\xi}) = \operatorname{diag}(\lambda_1(\boldsymbol{\xi}), \lambda_2(\boldsymbol{\xi}))$ is a continuously differentiable function on a neighborhood of the origin (of $\mathbb{R}^2$); such that

$$\lambda_1(0) < 0 < \lambda_2(0) \ . \tag{2}$$

The generalization to hyperbolic system in $\mathbb{R}^n$, or to hyperbolic systems where all terms of the diagonal of the matrix Λ have the same sign is possible. However to ease the presentation we will consider only the case of a system of two conservation laws satisfying (2).

As usual, each component ξ_i of the PDE (1) may be called Riemann invariant. The reason is the following: each Riemann coordinate is constant along the corresponding characteristic curve, i.e.

$$d_t \xi_i(x(t), t)) = 0 ,$$

where x is the characteristic curve solution of the differential equation

$$\dot{x}(t) = \lambda_i(\xi(x(t), t)) .$$

In order to complete the problem statement, boundary conditions (BC) are needed. Here we consider the system (1) under the BC of the form

$$\begin{pmatrix} \xi_1(L,t) \\ \xi_2(0,t) \end{pmatrix} = \mathbf{g}\begin{pmatrix} \xi_1(0,t) \\ \xi_2(L,t) \end{pmatrix} + \mathbf{e}_p\begin{pmatrix} \xi_1(0,t) \\ \xi_2(L,t) \end{pmatrix} + \mathbf{e}_i\begin{pmatrix} \int_0^t \xi_1(0,s)\, ds \\ \int_0^t \xi_2(L,s)\, ds \end{pmatrix}, \tag{3}$$

where $\mathbf{g}$, $\mathbf{e}_p$, and $\mathbf{e}_i$ are three continuously differentiable functions defined on a neighborhood of the origin, and satisfying $\mathbf{g}(0) = \mathbf{e}_i(0) = \mathbf{e}_p(0) = 0$.

The characteristic solution ξ_1 (or ξ_2) that "leaves" the boundary at $x = L$ (or at $x = 0$) is a function of the characteristic solutions that "arrive" at the boundaries at the same instant. This form of the BC will be further motivated in the next section. In particular note that $\mathbf{e}_p$ and $\mathbf{e}_i$ will be interpreted as errors in the loop of the boundary control problem under consideration. The subscripts p and i will allow us to distinguish the error proportional to the value of the solution at the boundary with the error resulting as the integral of the boundary conditions (like a memory).

In order to state our main result, we need the following compatibility condition between the system (1) and the BC (3).

Definition 2.1. A function $\boldsymbol{\xi}^\# \in C^1(0, L; \mathbb{R}^2)$ satisfies the compatibility condition $\mathcal{C}$ if

$$\begin{pmatrix} \xi_1^\#(L) \\ \xi_2^\#(0) \end{pmatrix} = (\mathbf{g} + \mathbf{e}_p)\begin{pmatrix} \xi_1^\#(0) \\ \xi_2^\#(L) \end{pmatrix} + \mathbf{e}_i(0),$$

and

$$\begin{aligned} \begin{pmatrix} \lambda_1(\boldsymbol{\xi}^\#(L))\partial_x \xi_1^\#(L) \\ \lambda_2(\boldsymbol{\xi}^\#(0))\partial_x \xi_2^\#(0) \end{pmatrix} &= (\nabla \mathbf{g} + \nabla \mathbf{e}_p)\begin{pmatrix} \xi_1^\#(0) \\ \xi_2^\#(L) \end{pmatrix}\begin{pmatrix} \lambda_1(\boldsymbol{\xi}^\#(0))\partial_x \xi_1^\#(0) \\ \lambda_2(\boldsymbol{\xi}^\#(L))\partial_x \xi_2^\#(L) \end{pmatrix} \\ &\quad - \nabla \mathbf{e}_i(0)\begin{pmatrix} \xi_1^\#(0) \\ \xi_2^\#(L) \end{pmatrix} , \end{aligned}$$

where $\nabla \mathbf{g}(\boldsymbol{\xi})$, $\nabla \mathbf{e}_p(\boldsymbol{\xi})$, and $\nabla \mathbf{e}_i(\boldsymbol{\xi})$ denote the Jacobian matrices at $\boldsymbol{\xi} \in \mathbb{R}^2$ of $\mathbf{g}$, $\mathbf{e}_p$, and $\mathbf{e}_i$ respectively. □

Some additional notations and definitions are also needed:

- The norm $|\cdot|$ in $\mathbb{R}^2$ is defined, for all $\boldsymbol{\xi} \in \mathbb{R}^2$, by $|\boldsymbol{\xi}| = \max(|\xi_1|, |\xi_2|)$. $B(\varepsilon)$ denotes the ball centered in $0 \in \mathbb{R}^2$ with radius $\varepsilon > 0$.
- Given Φ continuous on $[0, L]$ and Ψ continuously differentiable on $[0, L]$, we denote
$$\begin{aligned} |\Phi|_{C^0(0,L)} &= \max_{x \in [0,L]} |\Phi(x)| , \\ |\Psi|_{C^1(0,L)} &= |\Psi|_{C^0(0,L)} + |\Psi'|_{C^0(0,L)} ; \end{aligned}$$
- $B_{\mathcal{C}}(\varepsilon)$ denotes the set of continuously differentiable functions $\boldsymbol{\xi}^\# \colon [0, L] \to \mathbb{R}^2$ satisfying the compatibility assumption $\mathcal{C}$ and $|\boldsymbol{\xi}^\#|_{C^1(0,L)} \le \varepsilon$;

- For a given matrix $A = (a_{ij})$, $\rho(A)$ denotes its spectral radius and $\mathrm{abs}(A)$ is the matrix defined by $\mathrm{abs}(A) = (|a_{ij}|)$.

The main result of this paper is the following

Theorem 2.2. *If*

$$\rho(abs(\nabla \mathbf{g}(0))) < 1, \tag{4}$$

then there exist $\varepsilon > 0$, $E > 0$, $\mu > 0$ and $C > 0$ such that, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon) \to \mathbb{R}^2$ such that

$$\mathbf{e}_p(0) = \mathbf{e}_i(0) = 0\ . \tag{5}$$

holds together with

$$\max\{|\nabla \mathbf{e}_p(0)|,\, |\nabla \mathbf{e}_i(0)|\} \leq E\ , \tag{6}$$

for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon)$, there exists an unique function $\boldsymbol{\xi} \in C^1([0,L] \times [0,+\infty)\ ;\mathbb{R}^2)$ satisfying the PDE (1), the boundary conditions (3) and the initial condition

$$\boldsymbol{\xi}(x,0) = \boldsymbol{\xi}^\#(x)\ ,\forall x \in [0,L]. \tag{7}$$

Moreover, this function satisfies

$$|\boldsymbol{\xi}(.,t)|_{C^1(0,L)} \quad \leq \quad Ce^{-\mu t}|\boldsymbol{\xi}^\#|_{C^1(0,L)}\ ,\forall t \geq 0. \tag{8}$$

The proof of this result will be based on an estimation of the influence of the boundary condition on the evolution of the Riemann coordinates. In particular, we have to prove that the damping condition (4) is strong enough to manage the unknown errors $\mathbf{e}_p$ and $\mathbf{e}_i$, whose derivative is assumed to be small at the origin due to (6). This result will be proved in Section 4 below.

For many physical networks, the boundary conditions of the hyperbolic system modelling the applications are defined by the controller. This is the case for the application of the dynamic behavior of a pipe filled with water as considered in Section 3 below. More precisely, for this application, we succeed to design a stabilizing controller by selecting a feedback so that the damping condition (4) for the hyperbolic model is satisfied. Also we succeed to ensure a robustness issue with respect to small proportional and integral actuator errors. When designing stabilizing controller, it may be also fruitful to add an integral action. The interest of the integral action is to cancel a possible offset (due to actuator imperfection e.g.) on real experiments or on numerical simulations. In this context Theorem 2.2 can be reinterpreted as a proof that adding a small integral action doesnot cancel the stability property. See also the example of the control of the flow in an open channel as considered in [11]. In particular note that in [11, Sections IV.D and V.C] an integral action is added to counteract the actuator imperfection. These numerical simulations and these experiments can be seen as other illustrations of Theorem 2.2.

Remark 1. Combining this main result and the main result of [21], we obtain that the damping condition (4) is strong enough to manage small non-homogeneous terms. More precisely instead of the PDE (1), we may consider

$$\partial_t \boldsymbol{\xi} + \Lambda(\boldsymbol{\xi})\partial_x \boldsymbol{\xi} = \mathbf{h}(\xi)$$

for a given continuously differentiable function $\mathbf{h}$ defined on a neighborhood of the origin. We may prove that if (4) holds, if $\mathbf{h}(0) = 0$ and if $|\nabla \mathbf{h}(0)|$ is sufficiently small then the conclusions of Theorem 2.2 holds. We illustrate this latter result in Section 3 below.

3. **Application to the dynamic behavior of a pipe filled with water.** In this section, we show how our main result may be applied for the control of the dynamic behavior of a pipe filled with water. Let us consider an exploitation of hydropower resources to generate electricity.

A mathematical model based on mass and momentum conservation can properly describe the dynamic behavior of a pipe filled with water. Hydraulic installations feature longitudinal dimensions greater than transversal dimensions, thus justifying a one-dimensional approach based on the following assumption:

- the flow is normal to the (constant) pipe cross-section A $[m^2]$;
- the pressure p $[Pa]$, the flow velocity C $[m/s]$ and the density ρ $[kg/m^3]$ are uniform in a cross-section A.

For a survey and the computation of the model see [15]. The momentum equation reads (see [15, page 26])

$$\partial_t C + \frac{1}{\rho}\partial_x p + C\partial_x C + g\sin(\alpha) + \frac{\lambda C|C|}{2D} = 0$$

where

- x is the abscissa along the pipe of length L $[m]$;
- t is the time $[s]$;
- λ is the local loss coefficient;
- g is the gravity constant $[m/s^2]$;
- α is the slope;
- D is the pipe diameter $[m]$.

The absolute value of the velocity ensures always dissipative term.

The continuity equation is given by

$$\partial_t p + \rho a^2 \partial_x C + C\partial_x p = 0$$

where a is the wave speed $[m/s]$.

Denoting the discharge by Q and the piezometric head by h, we have

$$Q = CA\,, \quad h = Z + \frac{p}{\rho g}$$

where Z is the elevation of the pipe $[m]$. Noticing that $\partial_x Z = \sin(\alpha)$, and assuming no vertical displacements of the pipe (i.e. $\partial_t Z = 0$), high wave speed (large a) and low flow velocity (small C and thus the convective terms $C\partial_x$ can be neglected with respect to the propagative term ∂_t), we obtain the simplified equations (see [15, page 29]):

$$\partial_t \begin{pmatrix} Q \\ h \end{pmatrix} + \begin{pmatrix} 0 & gA \\ \frac{a^2}{gA} & 0 \end{pmatrix} \partial_x \begin{pmatrix} Q \\ h \end{pmatrix} = \begin{pmatrix} -\frac{\lambda Q|Q|}{2DA} \\ 0 \end{pmatrix} . \tag{9}$$

The boundary conditions depend on the hydraulic components. A large variety of components may be considered in hydropower exploitations. Consider e.g. the pipes, the singular losses, the valves, the Francis turbines (see [16, 15]). For the boundary condition at the beginning of the pipe, we assume that the piezometric line is given and constant:

$$h(x = 0, t) = h_0\,, \tag{10}$$

for a given h_0.

For the boundary condition at the other end of the pipe, we assumed that the hydraulic installation is equipped with a valve. A valve induces head losses in hydraulic systems which are function of the valve obturator position s. The head losses through a valve are given by (see [15, page 85]):

$$h(x = L, t) - h_L = \frac{K_v(s)}{2gA_v^2} Q^2(x = L, t) \tag{11}$$

where $K_v(s)$ is the valve head loss coefficient, A_v is the reference of the valve area and h_L is given. The system under consideration in this application is depicted by Figure 1.

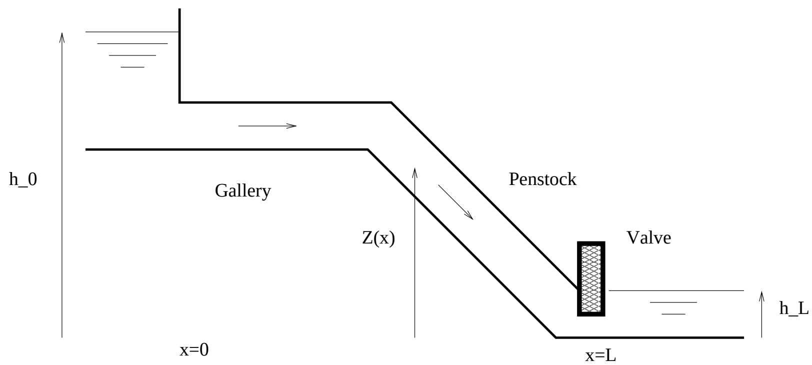

FIGURE 1. A simple hydraulic circuit

Hydraulic machines are increasingly subject to off-design operation, startup and shutdown sequences, quick set point changes, etc. The closure of the guide vanes induces a waterhammer effect in the penstock leading to head fluctuations (see [15, page 38 and following]. To overcome this waterhammer phenomenon, many hydraulic installations are equipped with surge tanks. The surge tank is a protection device against waterhammer effect behaving as a free surface for wave reflection but where the water level is function of the discharge time history. An hydraulic circuit with a surge tank is depicted in Figure 2. The surge tanks are often equipped with pumps which may be regulated. The literature about the study of systems equipped with controlled surge tanks is immense. Let us cite [9, 24]. See also [17, 25] where PI control strategies is applied. However as remarked in the introduction of [5], although PID controllers have been used to regulate the fluid, only few works develop modern design methods for nonlinear controllers. In particular, the controller should restrict peak values of the rate of the change of outflow. This problem may be easily interpreted as a constraint on the norm of the differentiable function of the flow, and thus a constraint on the $\mathcal{C}^1$-norm of the flow. This is exactly the framework of the present work, and thus our main result is particularly interesting in this context.

Let us show how we can apply our main result to the system[1] (9) with the boundary conditions (10) and (11). To apply our main result to the system depicted on Figure 3 we need to consider different boundary conditions. We will come back on this in the conclusion.

[1] Let us note that the right-hand side of (9) is not continuously differentiable due to the absolute value. However, we will see that around the steady-state, the Q variable has a constant sign and thus, locally around the equilibrium, we may remove the absolute value in (9).

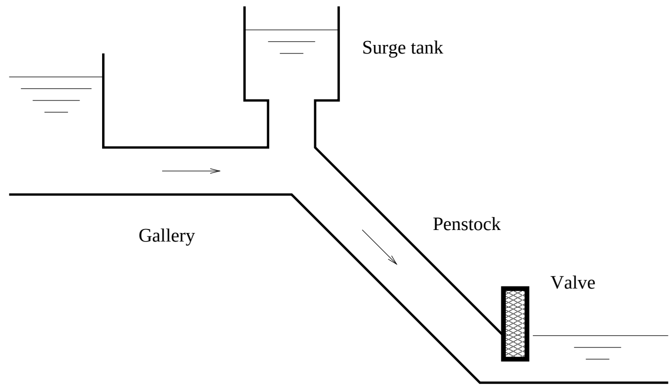

FIGURE 2. A hydraulic circuit with a surge tank

Considering the system (9) and the boundary conditions (10) and (11), let us first compute the steady-state solution of (9). A steady-state solution is such that $\partial_t \overline{h} = 0$ and $\partial_t \overline{Q} = 0$. We compute $d_x \overline{Q} = 0$ and, for all x in $[0, L]$,

$$d_x \overline{h}(x) = -\frac{\lambda \overline{Q}|\overline{Q}|}{2gDA^2} ,$$

in other words, $\overline{Q}$ is constant and $\overline{h}$ is an affine function. Assuming $\overline{Q} > 0$, we may remove the absolute value in (9) around the steady-state solution. Assuming moreover $h_0 > h_L$ (these latter assumptions are valid on hydropower installation) and using the boundary conditions (10) and (11) we get

$$\overline{Q} = \sqrt{\frac{2gDA^2 A_v^2 (h_0 - h_L)}{K_v(s) DA^2 + \lambda L A_v^2}} ,$$

and, for all $x \in [0, L]$,

$$\overline{h}(x) = -\frac{\lambda \overline{Q}^2}{2gDA^2} x + h_0 .$$

Now we note that the characteristic speeds are $-a$ and a with the corresponding Riemann coordinates $\xi_1 = Q - \overline{Q} - \frac{gA}{a}(h - \overline{h})$ and $\xi_2 = Q - \overline{Q} + \frac{gA}{a}(h - \overline{h})$. In this coordinates system (9) rewrites locally around the steady-state:

$$\partial_t \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \mathrm{diag}(-a, a)\partial_x \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \frac{\lambda g}{2Da}(\frac{\xi_1+\xi_2}{2} + \overline{Q})^2 + \frac{\lambda}{2DA}\overline{Q}^2 \\ -\frac{\lambda g}{2Da}(\frac{\xi_1+\xi_2}{2} + \overline{Q})^2 + \frac{\lambda}{2DA}\overline{Q}^2 \end{pmatrix} .$$

Let us now describe the boundary conditions in the Riemann coordinates. First let us note that the boundary condition (10) rewrites

$$\xi_2(x = 0, t) = \xi_1(x = 0, t) . \tag{12}$$

Now for any $k \in \mathbb{R}$, the boundary condition (11) holds as soon as the boundary condition

$$\xi_1(x = L, t) = k\xi_2(x = L, t) \tag{13}$$

is satisfied together with

$$\frac{a}{2gA}(1 - k)\xi_2(L, t) + \overline{h}(L) - h_L = \frac{K_v(s)}{2gA_v^2}(\frac{1 + k}{2}\xi_2(L, t) + \overline{Q}) .$$

This latter condition allows us to define (locally around the equilibrium) the controller

$$K_v(s) = \frac{2aA_v^2(1-k)\xi_2(L,t)}{(1+k)A\xi_2(L,t)+2A\overline{Q}} + \frac{4gA_v^2}{(1+k)\xi_2(L,t)+2\overline{Q}}(\overline{h}(L)-h_L) \tag{14}$$

where $\xi_2(L,t) = Q(L,t) - \overline{Q} + \frac{gA}{a}(h(L,t) - \overline{h}(L))$. The valve position s may be computed by inverting the function K_v (the graph of K_v is given in [15, Figure 5.6, page 86] and it can be checked that the function K_v is indeed invertible).

Note that this controller is a state-feedback. However to compute it we need only the values of the state at $x = L$, i.e. at the end where the control is implemented.

Now due to the boundary conditions (12) and (13), the damping condition (4) of Theorem 2.2 holds as soon as $-1 < k < 1$. Therefore by defining the controller with (14) for any $-1 < k < 1$, we get a stabilizing controller: the Q and h variables exponentially converge to the equilibrium $\overline{Q}$ and $\overline{h}$ in the topology of the $\mathcal{C}^1$-norm i.e. by avoiding too large value of the derivatives (roughly speaking it avoids the waterhammer effect). Moreover we obtain a robustness issue with respect to small actuator errors and non-homogeneous terms as proven in Theorem 2.2 (see also Remark 1), i.e. for the left-hand side of (9) sufficiently small.

Note moreover that for real implementations of the controller (14) it may interesting to add a small integral action to cancel a possible offset. This integral action can be seen as a small integral error in the boundary conditions of our model. If this integral action is sufficiently small then the closed-loop system is exponentially stable.

4. **Proof of Theorem 2.2.** This section is devoted to the proof of Theorem 2.2. In Section 4.1, we recall an existence result of a solution in finite time. In Sections 4.2 and 4.3, estimates of $|\boldsymbol{\xi}(\cdot,t)|_{C^0(0,L)}$ and $|\partial_x\boldsymbol{\xi}(\cdot,t)|_{C^0(0,L)}$ are derived, and we conclude the proof of Theorem 2.2 in Section 4.4.

Note that the outline of this proof is analogous to the one of the proof of the main result of [21] where it is studied the effect of small non-homogeneous terms on hyperbolic systems. Therefore to ease the presentation, we prefer to focus on the main difficulties and the differencies between the proof of Theorem 2.2 with the proof of the main result of [21] (consider in particular Claim 4.3).

4.1. **Existence result.** The following existence result on a finite time interval is a basic tool for the proof of Theorem 2.2. It combines the existence result of [20, Chap. 5, Theo. 1.1] together with the result of the continuity with respect to parameters as given in [3, Chap. 3] (see also the proof of [21, Lemma 1]):

Lemma 4.1. *([20, 3]) Let $T_2 > T_1 > 0$ and $T = T_2 - T_1$. Assume that the BC satisfy (4). Then there exist $\varepsilon_1(T) > 0$, $C_1(T) > 0$ and $E_1(T)$ such that, for all $0 < E < E_1(T)$, for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_1(T))$ and for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_1(T)) \to \mathbb{R}^2$ such that (5) holds and*

$$\max\{|\nabla\mathbf{e}_p(0)|, |\nabla\mathbf{e}_i(0)|\} \leq E , \tag{15}$$

there exists a unique function $\boldsymbol{\xi} \in C^1([0,L]\times[T_1,T_2];\mathbb{R}^2)$ satisfying the PDE (1) with boundary conditions (3) and initial condition (7). Moreover, this function $\boldsymbol{\xi}$ satisfies, for all $t \in [T_1,T_2]$,

$$|\boldsymbol{\xi}(.,t)|_{C^0(0,L)} \leq C_1(T)|\boldsymbol{\xi}^\#|_{C^0(0,L)} , \tag{16}$$

$$|\boldsymbol{\xi}(.,t)|_{C^1(0,L)} \leq C_1(T)|\boldsymbol{\xi}^\#|_{C^1(0,L)} . \tag{17}$$

In the following, Lemma 4.1 is applied several times on intervals which will be defined with the help of two decreasing sequences of positive numbers ε_2, ε_3, ... and E_2, E_3, ... We consider initial conditions $\boldsymbol{\xi}^{\#}$ successively in $B_{\mathcal{C}}(\varepsilon_2)$, $B_{\mathcal{C}}(\varepsilon_3)$, ...

Let, for $i \in \{1,2\}$,

$$s_i = \frac{L}{|\lambda_i(0)|}, \tag{18}$$

$$\tau_1 > \max\{s_1,\ s_2\}. \tag{19}$$

Let $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ and $a > 1$ such that

$$|(\nabla \mathbf{g})_{ij}(0)| \ < \ a_{ij} \ < \ a, \quad \forall (i,j) \in \{1,2\}^2, \tag{20}$$

$$\rho(A) \ < \ 1. \tag{21}$$

From (21), there exists a sufficiently larger integer $K \geq 2$ such that $c(2\tau_1) \sum_{k\geq K} |A^k| < 1$, where $c(2\tau_1)$ is given by Lemma 4.1 applied on $[0, 2\tau_1]$. Let

$$\tau_2 := (K+2)\tau_1\ , \tag{22}$$

and

$$\nu = c(2\tau_1) \sum_{k\geq K} |A^k| < 1\ . \tag{23}$$

4.2. **Estimation of $|\xi(.,t)|_{C^0(0,L)}$.** Let $\varepsilon_2 = \varepsilon(\tau_2)$ and $E_2 = E(\tau_2)$ given by Lemma 4.1 applied on $[0, \tau_2]$. For all $0 < E < E_2$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_2)$, the PDE (1), with the boundary conditions (3) and the initial condition (7), admits a unique solution $\boldsymbol{\xi} \in C^1([0,L]\times[0,\tau_2];\mathbb{R}^2)$.

In view of (2), and by using a continuity argument, we may assume without loss of generality (i.e. with ε_2 sufficiently small) that

$$\lambda_1(\boldsymbol{\xi}(x,t)) < 0 < \lambda_2(\boldsymbol{\xi}(x,t))\ . \tag{24}$$

The aim of this section is to prove the following

Lemma 4.2. *We have the existence of ε_5 in $(0,\varepsilon_2)$ and $E_5 > 0$, such that for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_5)$, we have*

$$|\boldsymbol{\xi}(.,\tau_2)|_{C^0(0,L)} \leq \nu |\boldsymbol{\xi}^{\#}|_{C^0(0,L)}. \tag{25}$$

Let $x \in [0,L]$. Hereafter the characteristic curves are defined backwards in time from (x,τ_2). In what follows, $\mathbb{N}$ denotes the set of nonnegative integers. For $k \in \mathbb{N}\setminus\{0\}$ and for $(i_1,...,i_k) \in \{1,2\}^k$, we define $t_{i_1\ldots i_k} \in [0,T]$ and $p_{i_1\ldots i_k} \in [0,L]\times\{t_{i_1\ldots i_k}\}$ by induction on k as follows (see also [21] for the construction and an illustration of these values).

Initial step $k=1$

- Let us consider the solution y_1 of the Cauchy problem

$$\dot{y}_1(t) = \lambda_1(\boldsymbol{\xi}(y_1(t),t)), \qquad y_1(\tau_2) = x.$$

  In view of (24), it allows us to define the time instant $t_1 \leq \tau_2$ by $y_1(t_1) = L$ and we set $p_1 = (L, t_1)$.

- Let us consider the solution y_2 of the Cauchy problem

$$\dot{y}_2(t) = \lambda_2(\boldsymbol{\xi}(y_2(t),t)), \qquad y_2(\tau_2) = x.$$

In view of (24), it allows us to define the time instant $t_2 \leq \tau_2$ by $y_2(t_2) = 0$. and we set $p_2 = (0, t_2)$.

By the invariance of the Riemann coordinates along the characteristic curves, we get

$$\xi_i(x, \tau_2) = \xi_i(p_i). \tag{26}$$

General induction step

Now let $k \in \mathbb{N} \setminus \{0\}$ be arbitrarily fixed and assume that $t_{i_1 \dots i_k} \in [0, \tau_2]$ and $p_{i_1 \dots i_k} \in [0, L] \times \{t_{i_1 \dots i_k}\}$ are defined. Then, for $i_{k+1} \in \{1, 2\}$, we define $t_{i_1 \dots i_{k+1}} \in [0, \tau_2]$ and $p_{i_1 \dots i_{k+1}} \in [0, L] \times \{t_{i_1 \dots i_{k+1}}\}$ by considering two cases (as done above for $k = 1$):

- Consider the Cauchy problem

$$d_t y_1(t) = \lambda_1(\boldsymbol{\xi}(y_1(t),t)), \qquad y_1(t_{i_1 \dots i_k}) = 0$$

and define $t_{i_1 \dots i_k i} \in [0, t_{i_1 \dots i_k})$ by $y_1(t_{i_1 \dots i_k i}) = L$. If such $t_{i_1 \dots i_k i}$ exists, it is unique and we define $p_{i_1 \dots i_k i}$ by $p_{i_1 \dots i_k i} = (L, t_{i_1 \dots i_k i})$. In contrast, if such $t_{i_1 \dots i_k i}$ does not exist, we do not define $t_{i_1 \dots i_k i}$, neither $p_{i_1 \dots i_k i}$, nor $t_{i_1 \dots i_k i \dots i_l}$ and $p_{i_1 \dots i_k i \dots i_l}$ for $l > k + 1$.

- Consider the Cauchy problem

$$d_t y_2(t) = \lambda_2(\boldsymbol{\xi}(y_2(t),t)), \qquad y_2(t_{i_1 \dots i_k}) = L,$$

and define $t_{i_1 \dots i_k j} \in [0, t_{i_1 \dots i_k})$ by $y_2(t_{i_1 \dots i_k j}) = 0$. Again, if such $t_{i_1 \dots i_k j}$ exists, it is unique and then we define $p_{i_1 \dots i_k j}$ by $p_{i_1 \dots i_k j} = (0, t_{i_1 \dots i_k j})$. However if such $t_{i_1 \dots i_k j}$ does not exist, we do not define $t_{i_1 \dots i_k j}$, neither $p_{i_1 \dots i_k j}$, nor $t_{i_1 \dots i_k j \dots i_l}$ and $p_{i_1 \dots i_k j \dots i_l}$ for $l > k + 1$.

Similarly to (26), by construction of $t_{i_1 \dots i_{k+1}}$ and $p_{i_1 \dots i_{k+1}}$, we have

$$\xi_1(0, t_{i_1 \dots i_k}) = \xi_1(p_{i_1 \dots i_k 1}) \ , \tag{27}$$

and

$$\xi_2(L, t_{i_1 \dots i_k}) = \xi_2(p_{i_1 \dots i_k 2}) \ . \tag{28}$$

Note that, in view of (18)-(19) and (22), there exists a finite number of $k \geq 1$ such that

$$s_{i_1} + \dots + s_{i_k} \leq \tau_2 - \tau_1 \ , \ \forall i_1, \dots, i_k \in \{1, 2\} \ .$$

Let us prove the following result which studies the effect of the boundary errors on the Riemann coordinates evaluated at the points $p_{i_1 \dots i_k}$.

Claim 4.3. *There exist $\varepsilon_4 \in (0, \varepsilon_2)$ and $E_4 \in (0, E_2)$, such that, for all $0 < E < E_4$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_4) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_4)$ and for all $x \in [0, L]$, for all integer $k \geq 1$, for all $(i_1, ..., i_k, i_{k+1}) \in \{1, 2\}^{k+1}$ such that $s_{i_1} + \dots + s_{i_k} \leq \tau_2 - \tau_1$, the time instant $t_{i_1 \dots i_k i_{k+1}}$ and the point $p_{i_1 \dots i_k i_{k+1}}$ exist.*

Moreover we have

$$|\xi_{i_k}(p_{i_1 \dots i_k})| \leq \sum_{j=1,2} a_{i_k j} |\xi_j(p_{i_1 \dots i_k j})|. \tag{29}$$

Proof. (Proof of Claim 4.3)

Let $(i_1, ..., i_k, i_{k+1}) \in \{1,2\}^{k+1}$ such that $s_{i_1} + \cdots + s_{i_k} \leq \tau_2 - \tau_1$. Due to (18), (19), the time instant $t_{i_1 \ldots i_k i_{k+1}}$ and the point $p_{i_1 \ldots i_k i_{k+1}}$ exist.

Now, by the Mean-Value Inequality, also called Finite-Increment Theorem (see [26, Prop. 2. p. 78]), and conditions (5) and (15), and using Lemma 4.1, there exists positive values ε_3 and E_3, and an increasing positive function w, defined for sufficiently small positive values and satisfying $w(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that, for all $0 < E < E_3$, for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_3)$, for all continuously differentiable functions $\mathbf{e}_p$: $B(\varepsilon_3) \to \mathbb{R}^2$, for all $s \in [t_{i_1}, \tau_2]$, we have

$$|\mathbf{e}_p(\boldsymbol{\xi}(y_{i_1}(s), s))| \leq (E + w(c(\tau_2)\varepsilon_3))|\boldsymbol{\xi}(y_{i_1}(s), s)|.$$

Similarly, up to reducing the values ε_3 and E_3 and the function w, we may assume that for all $0 < E < E_3$, for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_3)$, for all continuously differentiable functions $\mathbf{e}_i$: $B(\varepsilon_3) \to \mathbb{R}^2$, for all $s \in [t_{i_1}, \tau_2]$, we have

$$|\mathbf{e}_i(\boldsymbol{\xi}(y_{i_1}(s), s))| \leq \frac{(E + w(c(\tau_2)\varepsilon_3))}{\tau_2}|\boldsymbol{\xi}(y_{i_1}(s), s)|.$$

Therefore by using the boundary conditions (3) and inequalities (20) and by picking ε_4 and E_4 such that

$$|(\nabla \mathbf{g}(0))_{ij}| + \frac{1}{2}(E_4 w(c(\tau_2)\varepsilon_4)\varepsilon_4 < a_{ij}$$

we get for all $0 < E < E_4$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_4) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_4)$, the estimation (29). This concludes the proof of Claim 4.3. □

By a repetitive application of Claim 4.3, we obtain the following result.

Claim 4.4. *For all $0 < \varepsilon < \varepsilon_4$, for all $0 < E < E_4$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_4)$ and for all $x \in [0, L]$, for all integer $k \geq 1$, for all $(i_1, ..., i_k, i_{k+1}) \in \{1,2\}^{k+1}$ such that*

$$\tau_2 - 2\tau_1 \leq s_{i_1} + ... + s_{i_k} \leq \tau_2 - \tau_1 \ ,$$

the existence of $t_{i_1 \ldots i_k i_{k+1}}$ is guaranteed and the time instant $t_{i_1 \ldots i_k}$ is in the interval $[0, 2\tau_1]$. Moreover

$$|\boldsymbol{\xi}(p_{i_1 \ldots i_k})| \leq c(2\tau_1)|\boldsymbol{\xi}^{\#}|_{C^0(0,L)}. \tag{30}$$

Proof. (Proof of Claim 4.4) The existence of $t_{i_1 \ldots i_k i_{k+1}}$ follows from Claim 4.3. The estimation $t_{i_1 \ldots i_k} \leq 2\tau_1$ follows from $\tau_2 - 2\tau_1 \leq s_{i_1} + ... + s_{i_k}$ and the definition of the time instant $t_{i_1 \ldots i_k}$.

Estimation (30) is a consequence of Lemma 4.1 applied on $[0, 2\tau_1]$. □

By an decreasing induction on l, and following the same lines as the proof of [21, Claim 7], we may prove the following

Claim 4.5. *There exist $\varepsilon_5 > 0$ and $E_5 > 0$ such that for all l in $\mathbb{N}$, we have the property ($\mathcal{P}_l$):*

For all $0 < \varepsilon < \varepsilon_5$, for all $0 < E < E_5$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_5) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_5)$, for all $(i_1, \ldots, i_l) \in \{1,2\}^l$ such that $s_{i_1} + \cdots + s_{i_l} \leq \tau_2 - 2\tau_1$, we have

$$|\xi_{i_l}(p_{i_1 \ldots i_l})| \quad \leq \quad \sum_{k \geq l} \sum_{I_k} \sum_{j=1,2} a_{i_l i_{l+1}} a_{i_{l+1} i_{l+2}} \cdots a_{i_k j} |\xi_j(p_{i_1 \cdots i_k j})|$$

where I_k denotes the set of indices i_j, $j \in \{1,\dots,k\}$ such that

$$\tau_2 - 2\tau_1 \leq s_{i_1} + \dots + s_{i_l} + s_{i_{l+1}} + \dots + s_{i_k} \leq \tau_2 - \tau_1 \ .$$

We are now in a position to prove Lemma 4.2.

Proof. (Proof of Lemma 4.2) Due to Claim 4.5, $(\mathcal{P}_1)$ is true and thus with (26), for all $0 < \varepsilon < \varepsilon_5$, for all $0 < E < E_5$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_5) \to \mathbb{R}^2$ satisfying (5) and (15), for all $i_1 \in \{1,2\}$, for all $x \in [0,L]$ and $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_5)$,

$$|\xi_{i_1}(x,\tau_2)| \quad \leq \quad \sum_{k\geq 1} \ \sum_{\tau_2-2\tau_1\leq s_{i_1}+\dots+s_{i_k}\leq\tau_2-\tau_1} \ \sum_{j=1,2} a_{i_1i_2}a_{i_2i_3}\dots a_{i_kj}|\xi_j(p_{i_1\dots i_kj})|$$

which gives with (30)

$$\begin{aligned} |\xi_{i_1}(x,\tau_2)| \quad \leq \quad & c(2\tau_1)|\boldsymbol{\xi}^\#|_{C^0(0,L)} \sum_{k\geq 1} \ \sum_{\tau_2-2\tau_1\leq s_{i_1}+\dots+s_{i_k}\leq\tau_2-\tau_1} \\ & \sum_{j=1,2} a_{i_1i_2}a_{i_2i_3}\dots a_{i_kj} \end{aligned} \tag{31}$$

Note that the sums in (31) are finite. Moreover, due to (19) and (22)

$$(s_{i_1} + \dots + s_{i_k} \geq \tau_2 - 2\tau_1 = K\tau_1) \Rightarrow k \geq K. \tag{32}$$

Observe also that, by the definition of matrix product, we have, for all $N \in \mathbb{N}$,

$$\sum_{(i_2,\dots,i_k,j)\in\{1,\dots,N\}^k} a_{i_1i_2}a_{i_2i_3}\dots a_{i_kj} = \sum_{j=1}^{N}(A^k)_{i_1j} \leq |A^N|. \tag{33}$$

From (31), (32) and (33), we get, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_5) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\bar{\varepsilon})$, we have

$$|\xi_{i_1}(x,\tau_2)| \leq c(2\tau_1)|\boldsymbol{\xi}^\#|_{C^0(0,L)} \sum_{k\geq K}|A^k|$$

which gives with (23) $|\xi_{i_1}(x,\tau_2)| \leq \nu|\boldsymbol{\xi}^\#|_{C^0(0,L)}$, for all i_1 in $\{1,2\}$ and for all x in $[0,L]$. This is (25). This concludes the proof of Lemma 4.2. □

4.3. **Estimation of $|\partial_x\xi(.,t)|_{C^0(0,L)}$.** Let $\boldsymbol{\eta}$: $[0,L]\times[0,\tau_2] \to \mathbb{R}^2$ be defined by $\boldsymbol{\eta} = \bar{\Lambda}\partial_x\boldsymbol{\xi}$ where $\boldsymbol{\xi} \in C^1([0,L]\times[0,\tau_2];\mathbb{R}^2)$ is defined by $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\bar{\varepsilon})$, the PDE (1), the boundary conditions (3), and the initial condition (7).

Similarly let us define η_1: $[0,L]\times[0,\tau_2] \to \mathbb{R}$ and η_2: $[0,L]\times[0,\tau_2] \to \mathbb{R}$ defined respectively by $\boldsymbol{\eta} = (\eta_1,\ \eta_2)^T$.

Differentiating (1) with respect to x and denoting $\bar{\Lambda} = \Lambda(0)$, we get

$$\partial_t\boldsymbol{\eta} + \bar{\Lambda}\Lambda(\boldsymbol{\xi})\bar{\Lambda}^{-1}\partial_x\boldsymbol{\eta} = -\bar{\Lambda}(\nabla\Lambda(\boldsymbol{\xi})\partial_x\boldsymbol{\xi})\partial_x\boldsymbol{\xi} \ , \tag{34}$$

along the characteristic curves.

Moreover, differentiating (3) and using (1), it gives

$$\begin{aligned} & \begin{pmatrix} (-\Lambda(\boldsymbol{\xi})\bar{\Lambda}^{-1}\boldsymbol{\eta})_1(L,t) \\ (-\Lambda(\boldsymbol{\xi})\bar{\Lambda}^{-1}\boldsymbol{\eta})_2(0,t) \end{pmatrix} \\ = \; & (\nabla \mathbf{g} + \nabla \mathbf{e}_p) \begin{pmatrix} \xi_1(0,t) \\ \xi_2(L,t) \end{pmatrix} \begin{pmatrix} (-\Lambda(\boldsymbol{\xi})\bar{\Lambda}^{-1}\boldsymbol{\eta})_1(0,t) \\ (-\Lambda(\boldsymbol{\xi})\bar{\Lambda}^{-1}\boldsymbol{\eta})_2(L,t) \end{pmatrix} \\ & + \nabla \mathbf{e}_i \begin{pmatrix} \int_0^t \xi_1(0,s)\, \mathrm{d}s \\ \int_0^t \xi_2(L,s)\, \mathrm{d}s \end{pmatrix} \begin{pmatrix} \xi_1(0,t) \\ \xi_2(L,t) \end{pmatrix} . \end{aligned}$$

Note that (34) is a system of conservation laws perturbed by non-homogeneous terms (as those considered in [21]), whose boundary conditions are given by the previous equation. A development similar to $\boldsymbol{\xi}$ can be used as for ξ_i along the trajectories of (34). We obtain the

Lemma 4.6. *There exist $\varepsilon_6 > 0$, $E_6 > 0$, and $0 < \nu' < 1$ such that, for all $0 < E < E_6$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_6) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_6)$, we have*

$$|\boldsymbol{\eta}(.,\tau_2)|_{C^0(0,L)} \leq \nu' |\boldsymbol{\eta}(.,0)|_{C^1(0,L)}. \tag{35}$$

4.4. **Proof of Theorem 2.2.** In this section, we conclude the proof of Theorem 2.2. Let $\nu'' = \min\{\nu, \nu'\}$, $\varepsilon_7 = \min\{\varepsilon_5, \varepsilon_6\}$ and $E_7 = \min\{E_5, E_6\}$. We combine (25) and (35) to get for all $0 < E < E_7$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_7) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_7)$,

$$|\boldsymbol{\xi}(.,\tau_2)|_{C^1(0,L)} \leq \nu'' |\boldsymbol{\xi}^\#|_{C^1(0,L)} \ .$$

This estimate allows a repeated application of Lemma 4.1 on intervals of length τ_2 to give, for all $0 < E < E_7$, for all continuously differentiable functions $\mathbf{e}_p$ and $\mathbf{e}_i$: $B(\varepsilon_7) \to \mathbb{R}^2$ satisfying (5) and (15), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_7)$, the existence of a unique solution of (1), (3), and (7) over any interval $[0, N\tau_2]$ with $N \in \mathbb{N} \setminus \{0\}$ and

$$|\boldsymbol{\xi}(.,N\tau_2)|_{C^1(0,L)} \leq \nu''^N |\boldsymbol{\xi}^\#|_{C^1(0,L)} \ .$$

Thus, by letting $C = \max(c(\tau_2), 1)e^{-\ln \nu''}$ and $\mu = -\frac{\ln(\nu'')}{\tau_2}$, we get (8). This concludes the proof of Theorem 2.2.

5. **Conclusion.** The aim of this paper is to state a sufficient condition for the stability of systems of conservation laws. We consider the case of small (proportional or integral) perturbations in the boundary conditions. To prove this theoretical result we use the Riemann coordinates and we study the effect of the perturbations on the characteristic curves. We need to assume that the perturbations are small in $\mathcal{C}^1$-norm.

Then we apply this result to the dynamic behavior of a pipe filled with water. The potential application is the hydropower installations to generate electricity. For such applications it is important to avoid the waterhammer effect and thus to control the $\mathcal{C}^1$-norm of the solutions. Our damping condition allows us to design a controller so that the system in closed-loop is locally exponential stable with a robustness issue with respect to small boundary errors. Since the boundary conditions allow us to define the stabilizing controller it appears that small errors in the actuator may be considered. Also a small integral action to avoid possible offset may be also added. For this hydraulic application the boundary condition consist of a constant piezometric line at one end, and of a valve at the other end. The case of hydraulic equipment with a surge tank asks to consider more complicate boundary conditions and two coupled systems (one for the gallery, and an other one for the penstock). This study needs further investigation.

An other illustration of this theoretical result is [11]. Consider in particular Sections IV.D and V.C where a small integral action has been added on the controller. It is noted in [11] that a small integral action cancels the offset on experiments, and that this integral action doesnot destabilize the system. Our main result Theorem 2.2 can be seen as a proof that the controller of [11] is robust with respect to such integral actions.

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