NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2009.4.249
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Volume **4**, Number **2**, June **2009** pp. **249–266**

A HAMILTONIAN PERSPECTIVE TO THE STABILIZATION OF SYSTEMS OF TWO CONSERVATION LAWS

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Abstract. This paper aims at providing some synthesis between two alternative representations of systems of two conservation laws and interpret different conditions on stabilizing boundary control laws. The first one, based on the invariance of its coordinates, is the representation in Riemann coordinates which has been applied successfully for the stabilization of linear and non-linear hyperbolic systems of conservation laws. The second representation is based on physical modelling and leads to port Hamiltonian systems which are extensions of infinite-dimensional Hamiltonian systems defined on Dirac structure encompassing pairs of conjugated boundary variables. In a first instance the port Hamiltonian formulation is recalled with respect to a canonical Stokes-Dirac structure and then derived in Riemann coordinates. In a second instance the conditions on the boundary feedback relations derived with respect to the Riemann invariants are expressed in terms of the port boundary variable of the Hamiltonian formulation and interpreted in terms of the dissipation inequality of the Hamiltonian functional. The p-system and the Saint-Venant equations arising in models of irrigation channels are the illustrating examples developed through the paper.

1. **Introduction.** In this paper we shall be concerned with the stabilization via boundary control of hyperbolic systems of two conservation laws mainly motivated by the control of the shallow water equations used as models of irrigation channels, but also consider as paradigm the p-system [25]. The stabilization by boundary control of irrigation channels has been intensively studied for instance in [2, 10, 11] for both linear and non linear cases. Stability of hyperbolic partial differential equations on a one-dimensional spatial domain is widely studied in the literature [1, 3]. One of the most often suggested approaches, uses Riemann invariants to derive a stabilizing boundary control [12]. In recent publications, some extensions are suggested and based on the suitable choice of control Lyapunov function expressed in terms of the Riemann coordinates of the system [1, 3, 5, 6].

The use of physically motivated control Lyapunov function for the derivation of stabilizing control laws for non-linear finite-dimensional systems has proven to be very efficient and has lead to a great variety of results [4, 24, 26]. Very often, when the system stems from physical modelling, one may derive dissipation inequalities

2000 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Stabilization, irrigation channels, port-Hamiltonian systems.

related to energy balance equations and energy dissipating phenomena [31]. Using the dissipative port-Hamiltonian formulation for controlled physical systems [4, 21, 26], one may go one step further and assign in closed loop not only some dissipation inequality for some suitable control Lyapunov function but also assign the dynamic behavior by the structure matrices of the Hamiltonian system in closed loop [19, 22]. For infinite-dimensional systems very similar techniques, based on dissipation inequalities, which in terms of PDE's amounts to consider some entropy function [1, 6], have been used for the stabilization of boundary control systems [8, 16]. Recent works have used a boundary port-Hamiltonian formulation of systems of conservation laws [18, 27] in order to derive stabilizing boundary control for a class of linear systems defined on one-dimensional spatial domains [15, 17, 29, 30].

The aim of this paper is twofold. Firstly, after a brief recall of the formulation of a system of two conservation laws in terms of port-Hamiltonian systems defined with respect to the canonical Stokes-Dirac structure [18, 27], it derives the port-Hamiltonian formulation of these systems in terms of Riemannian coordinates. Secondly the conditions expressed on the boundary values of the Riemann coordinates used in [12] are related to some conditions expressed on the boundary port variables of the Hamiltonian formulation. Using the fact that the Hamiltonian functional satisfies a conservation equation, that makes it become a natural candidate Lyapunov function, these conditions are then interpreted in terms of dissipativity of the Hamiltonian function and allow an easy physical interpretation.

The sketch of the paper is the following. In the first part, we start with the motivating example of the p-system and recall briefly the results on boundary stabilization of systems of two conservation laws presented in [12]. In the second part we recall the definition of port-Hamiltonian systems defined with respect to Stokes-Dirac structure [18, 27] and then derive its expression using the Riemann invariants. The formulation is given for the example of the shallow water equation. In the third part, under some assumptions, we relate the conditions given in terms of the Riemann invariants with conditions given in terms of the port variables. Secondly we interpret them in terms of a dissipation equality of the Hamiltonian function. The last part gives some conclusions.

2. Motivation through the stabilization of the p-system using the Riemann invariants.

2.1. Reminder on the stabilization of system of two conservations laws.

In this section, we shall very briefly recall the main result on the stabilization of a hyperbolic system of two conservation laws suggested by Greenberg & Li [12]. Consider a spatial domain consisting of the finite interval $[0, L] \ni x$ with $L \in I\!R_+^*$ and time domain being the real interval $[0, +\infty) \ni t$. The state space is a non-empty connected open set in $I\!R^2$, denoted by Ω. Consider the system of two conservations laws:

$$\partial_t Y + \partial_x f(Y) = 0, \tag{1}$$

where

- $Y = (y_1 \quad y_2)^T : [0, +\infty) \times [0, L] \to \Omega$ is the vector of the two dependent variables;
- $f : \Omega \to I\!R^2$ is a C^1-function called *the flux vector*.

Note that the system (1) may also be written:

$$\partial_t Y + F(Y)\partial_x Y = 0 \tag{2}$$

where F is the Jacobian of the flux vector f. Assuming that the system is hyperbolic, implies that this system can be diagonalised using the Riemann invariants (see for instance [14, pages 34 - 35]). This means that there exists a change of coordinates $\xi(Y)$ whose Jacobian matrix is denoted $D(Y)$,

$$D(Y) = \frac{\partial \xi}{\partial Y}, \tag{3}$$

and diagonalises $F(Y)$ in Ω:

$$D(Y)F(Y) = \Lambda(Y)D(Y) \qquad Y \in \Omega.$$

In the coordinates ξ, the system (1) can then be rewritten in the following (diagonal) characteristic form:

$$\partial_t \xi + \Lambda(\xi)\partial_x \xi = 0 \tag{4}$$

with ξ: $[0, L] \times [0, +\infty) \to I\!R^2$, $(x, t) \mapsto \xi(x, t)$, and $\Lambda(\xi) = \mathrm{diag}(\lambda_1(\xi), \lambda_2(\xi))$, with $\lambda_1(\xi), \lambda_2(\xi)$ satisfying the conditions:

- the λ_i's are continuously differentiable functions on a neighborhood of the origin;
- $\lambda_2(0) < 0 < \lambda_1(0)$.

In this paper we shall consider the following result of Greenberg and Li [12] which may be recalled as follows.

Theorem 2.1. *Consider the hyperbolic system of conservation laws in Riemannian coordinates (4) with the following relations on the boundary variables:*

$$\xi_1(0) = \mathbf{K_1}(\xi_2(0)),\ \xi_2(L) = \mathbf{K_2}(\xi_1(L)) \tag{5}$$

with the functions K_1 and K_2 being C^1 and satisfying:

$$\mathbf{K_1}(0) = \mathbf{K_2}(0) = 0 \text{ and } |\mathbf{K_1}'(0)\mathbf{K_2}'(0)| < 1. \tag{6}$$

Consider initial values:

$$\lim_{t\to 0^+} (\xi_1, \xi_2)(x, t) = (\xi_{1,0}, \xi_{2,0})(x),\ 0 < x < L, \tag{7}$$

being C^1 and satisfying the assumption that to be small in the C^1 norm and the compatibility conditions:

$$\xi_{1,0}(0) = \mathbf{K_1}(\xi_{2,0}(0)) \tag{8}$$

$$\xi_{2,0}(L) = \mathbf{K_2}(\xi_{1,0}(L)) \tag{9}$$

$$\lambda_1(\xi_{1,0}, \xi_{2,0})(0)\, \partial_x \xi_{1,0}(0) = \lambda_2(\xi_{1,0}, \xi_{2,0})(0)\, \mathbf{K_1}'(0)\, \partial_x \xi_{2,0}(0) \tag{10}$$

$$\lambda_2(\xi_{1,0}, \xi_{2,0})(L)\, \partial_x \xi_{2,0}(L) = \lambda_1(\xi_{1,0}, \xi_{2,0})(L)\, \mathbf{K_2}'(L)\, \partial_x \xi_{1,0}(L) \tag{11}$$

Then the initial value problem, for this system, has a unique C^1 solution. Moreover, its solution decays to zero in the C^1 norm with an exponential rate.

2.2. **Application to the p-system.** In their paper [12], the authors consider the p-system defined by the following system of conservation laws:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} - \partial_x \begin{pmatrix} v \\ \sigma(u) \end{pmatrix} = 0 \tag{12}$$

where σ is a C^1 function satisfying:

$$\sigma(0) = 0,\ \sigma'(u) > 0. \tag{13}$$

They consider the following relations on the boundary values:

$$\sigma(u(0,t)) - rv(0,t) = 0,\ v(L,t) = 0 \tag{14}$$

with $r \in \mathbb{R}_+^*$

Using the Riemann invariants:

$$\xi_1 = \frac{1}{2}\left(v - \int_0^u \sqrt{\sigma'(\eta)}d\eta\right) \text{ and } \xi_2 = \frac{1}{2}\left(v + \int_0^u \sqrt{\sigma'(\eta)}d\eta\right) \tag{15}$$

the authors show that the theorem 2.1 applies under the condition that $r \in \mathbb{R}_+^*$. Let us define the functions $\gamma \to C(\gamma)$ and $\gamma \to U(\gamma)$ such that:

$$C(\gamma) = \sqrt{\sigma'(U(\gamma))},$$

and $U(\gamma)$ is defined implicitly by

$$\gamma = \int_0^{U(\gamma)} \sqrt{\sigma'(\eta)}d\eta.$$

Then the eigenvalues of this system are:

$$\lambda_1(\xi_1,\xi_2) = C(\xi_2 - \xi_1),\ \ \lambda_2(\xi_1,\xi_2) = -C(\xi_2 - \xi_1).$$

In [12], the authors construct a unique smooth solution using the properties of invariance of the Riemann coordinates along the Riemann forward characteristic $\dot{\xi} = \lambda_1$ and the backward one $\dot{\xi} = \lambda_2$.

This approach has been used too in [11] and others papers of the authors of [3]. Another approach is to construct a Lyapunov function in the Riemann coordinates [1, 2, 3, 5, 6, 10] which can but necessary be linked to the entropy of the system.

However, Greenberg and Li have made the remark that for the p-system example, the stability condition looks like the dissipativity condition of the energy of this system. Let's see how it can be interpreted in the following subsection in terms of dissipativity and Hamiltonian Structure.

2.3. **Link with Hamiltonian structure.** It is remarkable that, for this classical physical example, the authors use the physical variables (u being the volume, v the velocity of some isentropic gas for instance and σ denoting the opposite of the pressure) in order to express the boundary relations (14) and not the Riemann invariants. Furthermore they note that the condition on the parameter r in the boundary relations may be interpreted in terms of dissipativity properties. Indeed consider the energy of the system (sum of the kinetic and the internal energy of the gas):

$$H(u,v) = \int_0^L \left(\frac{v^2}{2} + \int_0^u \sigma(\eta)\,d\eta\right) dx \tag{16}$$

and its balance equation:

$$\frac{dH}{dt} = v(t,L)\,\sigma(u(t,L)) - v(t,0)\,\sigma(u(t,0)). \tag{17}$$

The boundary relations (14) and the positivity of r imply then energy dissipation:

$$d_t H = -rv^2(0,t) \leq 0. \tag{18}$$

Actually the relation between energy dissipation and the inequality in the conditions (6) on the boundary relation of the theorem 2.1, may be expressed as follows. Consider a slight generalization of the boundary relations (14) by assuming that the relation at $x = 0$ is given by some C^1 function G_0:

$$\sigma(u(0,t)) = G_0(v(0,t)) \tag{19}$$

Hence the energy balance equation becomes[1]:

$$d_t H = -G_0(v(0))^T v(0) \tag{20}$$

and the energy is dissipated if $G_0' > 0$, and $G(0) = 0$, this last point is obvious by definition of $G(x) = -rx$.

Using the definition of the Riemann invariants and that $\sigma(u)$ is a strictly increasing function, it follows that there exist a C^1 function $\bar{u}$ such that the inverse coordinate transformation may be written:

$$\left\{ \begin{array}{rcl} u & = & \bar{u}(\xi_2 - \xi_1) \\ v & = & \xi_2 + \xi_1 \end{array} \right. \tag{21}$$

Using the coordinate transformation and the implicit function theorem, there is a function K_1 such that the boundary relations on the physical variables translates as follows on the Riemann invariants:

$$\left\{ \begin{array}{rcl} \xi_1(0) & = & K_1(\xi_2(0)) \\ \xi_2(L) & = & K_2(\xi_1(L)) = -\xi_1(L) \end{array} \right. \tag{22}$$

The inequality conditions of the theorem 2.1 reduce in this case to:

$$|K_1'(0)| < 1. \tag{23}$$

Denoting the celerity of the system in Riemann coordinates by $\lambda(\xi_2 - \xi_1) = \lambda_1(\xi_1, \xi_2) = -\lambda_2(\xi_1, \xi_2) > 0$, a straightforward calculation on the boundary relations at $x = 0$, leads to the following relation between the function K_1 and G_0:

$$|K_1'(0)| = \left| \frac{\lambda(\xi_2 - \xi_1) + G_0'(\xi_2 + \xi_1)}{\lambda(\xi_2 - \xi_1) - G_0'(\xi_2 + \xi_1)} \right|. \tag{24}$$

This is clearly a Cayley transformation and hence the negativity of $-G'(0)$ which implies the energy dissipation (in equation (20)) is equivalent with the circle condition (23) given by the theorem 2.1.

This raised for us the question wether such an equivalence holds for some physical systems, in particular these systems stemming from physical systems models and their Hamiltonian formulation [18, 27].

[1] The transpose of a vector or matrix x is denoted as x^T.

3. **Port Hamiltonian formulation of a hyperbolic system of two conservation laws.**

3.1. **Boundary port Hamiltonian systems.** Boundary port Hamiltonian systems are extensions of infinite-dimensional systems defined with respect to some Hamiltonian differential operator [20] in the sense that they allow for functions whose support is not necessarily strictly included in the spatial domain. In such a way boundary port Hamiltonian systems allow the description of physical system which exchange a non-zero energy flow through its boundaries in the Hamiltonian frame [18, 27]. Numerous physical systems have been treated in this frame and the formulation has also been extended to dissipative systems [17], [29, 30]. For linear systems, the relation with boundary control systems, the well-posedness and the stabilization by positive real controllers on the boundary port variables have been carried out [15] [28, 29]. In this section we shall recall the definition of port Hamiltonian systems in the case of a system of two conservation laws on a one-dimensional domain. Let us first recall the definition of the *variational derivative* with respect to α, denoted by $\delta_\alpha H$, of a smooth functional H on smooth functions $\alpha(x)$ defined on the spatial domain, is defined as the function which satisfies:

$$H(\alpha + \epsilon\eta) = H(\alpha) + \epsilon \int_0^L \frac{\delta H}{\delta \alpha}(x, \alpha(x))\, \eta(x)\, dx + O(\epsilon^2) \tag{25}$$

for any $\epsilon \in \mathbb{R}$ and any smooth function $\eta(x)$ such that $\alpha + \epsilon\eta$ satisfies the same boundary conditions as α [20]. Let us also recall that a *Hamiltonian operator* is a differential operator which defines a Poisson bracket on the set of smooth functions with compact support strictly included in the interval $[0, L]$ by the relation:

$$\{e_1, e_2\} = \int_0^L e_1(x)\, \mathcal{J}\, e_2(x)\, dx. \tag{26}$$

The system of two conservation laws given in equation (1) may be written as a Hamiltonian system, if there exist a smooth functional $H(Y)$ and a Hamiltonian operator $\mathcal{J}$ such that the second term of the conservation laws may be written:

$$\partial_x\, e(Y) = \mathcal{J} \begin{pmatrix} \delta_{y_1} H \\ \delta_{y_2} H \end{pmatrix}. \tag{27}$$

Models of physical systems involving two conservation laws have a canonical structure corresponding to a dimensionless canonical coupling between two physical domains [18, 27]. In this case the Hamiltonian operator takes the elementary form:

$$\mathcal{J} = \epsilon \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{28}$$

where $\epsilon \in \{1, -1\}$.

Note that this is precisely the case of the p-system which may be written as a Hamiltonian system with respect to the Hamiltonian operator (28), generated by taking the Hamiltonian equal to the energy of the system given in (16).

If one wants to relax the condition on the considered functions and extend their domain to the whole spatial domain, one may define an extension of Hamiltonian systems to port boundary Hamiltonian systems. Although port-boundary systems may be defined for more general Hamiltonian operators [15] [17], we shall restrict ourselves to the operators (28) corresponding to the canonical physical models. The extension of the Hamiltonian system is based on the definition of a pair of external variables, in the sense of control theory [31], called *port boundary variables*.

Therefore, compute the symmetric pairing obtained from the bracket (26) and consider smooth functions which have a support is not necessarily strictly included in the spatial domain $[0, L]$:

$$\begin{aligned} \{e_1, e_2\} + \{e_2, e_1\} &= \int_0^L e_1(x)\, \mathcal{J}\, e_2(x)\, dx + \int_0^L e_2(x)\, \mathcal{J}\, e_1(x)\, dx \\ &= 2\epsilon \int_0^L \partial_x (e_1^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e_2) \\ &= \epsilon \left[(e_1^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e_2) \right]_0^L \end{aligned}$$

It is hence immediately seen that when the support of the functions is not strictly included in the spatial domain $[0, L]$, the bracket (26) is no more skew-symmetric. It has been shown however [15, 27] that the operator $\mathcal{J}$ may be extended in order to recover the property of skew-symmetry. Therefore one uses the concept of *Dirac structure* which has been suggested in [7, 9] and is a generalization of symplectic and Poisson tensors. It represents these skew-symmetric tensors in terms of vector subspaces (or distributions) which are isotropic and co-isotropic with respect to some non-degenerated symmetric pairing.

For the canonical Hamiltonian operator $\mathcal{J}$ defined in (28), the first step consist in considering additional functions defined on the boundary of the spatial domain, i.e. $\mathbb{R}^{\{0,\, L\}}$ and called boundary port variables. It has been shown in [27] that these port boundary variables may be chosen as a linear combination of the functions e_1 and e_2 restricted at the boundary:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \mathbf{diag}\,(1,\, 1,\, -1,\, 1) \begin{pmatrix} e_1(0) \\ e_1(L) \\ e_2(0) \\ e_2(L) \end{pmatrix} \tag{29}$$

where **diag** denotes a diagonal matrix with the coefficient of the diagonal being the arguments.

In the case of the p-system the boundary port variables are simply the velocity and the pressure at the boundaries of the domain.

Defining the space of flow variables is $\mathcal{F} = C^\infty([0, L]) \times C^\infty([0, L]) \times \mathbb{R}^{\{0,L\}}$ and the space of efforts $\mathcal{E} = C^\infty([0, L]) \times C^\infty([0, L]) \times \mathbb{R}^{\{0,L\}}$, the linear subset $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ defined by:

$$\mathcal{D} = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} \;/\; \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right. \\ \left. \text{and} \quad \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \Bigg|_{0,L} \right\} \tag{30}$$

is a Dirac structure with respect to the symmetric pairing:

$$\left\langle \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix} \right\rangle = \int_0^L (e_1\, f_1 + e_2\, f_2)\, dz + e_\partial(L)\, f_\partial(L) - e_\partial(0)\, f_\partial(0) \tag{31}$$

It is clear from the definition of the Dirac structure that it extends the definition of the Hamiltonian operator defined in (28).

The infinite Hamiltonian system defined with respect to the Hamiltonian operator (28) may then also been extended to an implicit Hamiltonian system defined with respect to the Stokes-Dirac structure as follows:

$$\left(\begin{pmatrix} \frac{\partial \alpha_1(t)}{\partial t} \\ \frac{\partial \alpha_2(t)}{\partial t} \\ f_\partial \end{pmatrix}, \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \\ e_\partial \end{pmatrix} \right) \in \mathcal{D} \tag{32}$$

As a consequence of the definition of the Stokes-Dirac structure, the Hamiltonian functional is no more conserved but is subject to the following balance equation:

$$\frac{dH}{dt} = e_\partial^T f_\partial \tag{33}$$

3.2. Boundary port Hamiltonian systems and Riemann coordinates.

3.2.1. *Hamiltonian operator expressed in the Riemann coordinates.*
In this section we shall consider a hyperbolic system of two conservation laws (1) which admits a Hamiltonian representation, that is such that vector of flux variables may be written following (27) and furthermore that it is defined with respect to the canonical Hamiltonian operator (28). Hence the system is written:

$$-\partial_t Y = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} (\delta_Y H). \tag{34}$$

In the sequel we express the Hamiltonian system in terms of the Riemannian invariants and give the expression of the Hamiltonian operator as well as of the boundary port variables. Denote by $\tilde{H}(\xi)$ the Hamiltonian expressed in the Riemann invariants: $\tilde{H}(\xi) = H \circ Y(\xi)$ where Y denotes, with an abuse of notation, the inverse change of coordinates to the Riemann coordinates.

Recall that D denotes the Jacobian of change of coordinates to the Riemann invariants defined in (3) and define $D_\xi = D \circ Y(\xi)$. One obtains by multiplying both terms of (34) by D:

$$\begin{aligned} & -D(Y)\partial_t Y = D(Y) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(\delta_Y H) \\ \Leftrightarrow \quad & -\partial_t \xi = D_\xi(\xi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D^T(\xi)\delta_\xi \tilde{H}(\xi)) \\ \Leftrightarrow \quad & -\partial_t \xi = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D_\xi^T)\delta_\xi \tilde{H}(\xi) + D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_\xi^T \partial_x(\delta_\xi \tilde{H}(\xi)). \end{aligned}$$

Hence in terms of the Riemann invariants the system is written:

$$-\partial_t \xi = (B\partial_x + C)\delta_\xi \tilde{H}(\xi), \tag{35}$$

where $B = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_\xi^T$ and $C = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D_\xi^T)$. The following properties may be noted: firstly the matrix B is symmetric and secondly it is related with the matrix C by:

$$\partial_x B = C^T + C. \tag{36}$$

3.2.2. *Boundary port variables.* In this section we shall check the formal skew-symmetry of the differential operator $(B\partial_x + C)$ and then define port boundary variables associated with it. Let us define the following bracket on smooth functions on the spatial domain $[0,\, L]$:

$$\{e_1, e_2\} = \int_0^L e_1^T (B\partial_x + C) e_2 \, dx \tag{37}$$

and consider the symmetric product:

$$\begin{aligned} \{e_1, e_2\} \quad + \quad \{e_2, e_1\} &= \int_\Omega e_1^T (B\partial_x + C)e_2 + e_2^T (B\partial_x + C) e_1 \, dx \\ &= \int_0^L \partial_x (e_1^T B e_2) - e_1^T \partial_x (B) e_2 + e_2^T \partial_x (B) e_2 + e_2^T C e_1 \\ &\quad + e_1^T \left(C + C^T - \partial_x B \right) e_2 \, dx. \end{aligned}$$

By using the fact that B is symmetric and the relation (36), the symmetric product reduces to:

$$\{e_1, e_2\} + \{e_2, e_1\} = \int_0^L \partial_x (e_1^T B e_2) = \left[(e_1^T B e_2) \right]_0^L . \tag{38}$$

The product (38) corresponds to Stokes theorem applied to the equation for the differential operator $(B\partial_x + C)$. Furthermore the second member of (38) vanishes for all functions e_1, e_2 with compact support strictly included in the domain $[0,\, L]$ and hence for these functions the bracket is skew-symmetric.

Remark 1. Actually in the original coordinates the skew-symmetry matrix differential operator is trivial and as it has constant coefficients, the Jacobi identities are satisfied. This property remains with the change of variables of course. Hence the system (35) is an infinite dimensional Hamiltonian system as long as one consider functions with compact support strictly included in the domain $[0,\, L]$.

However considering functions which do not vanish on the boundary of the domain, the bracket (37) is no more skew-symmetric. In this case the time variation of the Hamiltonian becomes:

$$\frac{d\tilde{H}(\xi)}{dt} = \left[(\delta_{\xi_1} \tilde{H}^T B \, \delta_{\xi_2} \tilde{H}) \right]_0^L . \tag{39}$$

Thus one may account for the energy flow through the boundary of the domain, however the structure of an infinite-dimensional Hamiltonian system is lost.

Using the notion of Dirac structure and following [27] one may still define a Hamiltonian system by extending the system with external variables associated with the boundary. Naturally the choice is not unique.

The most natural choice related to the balance equation (39) consist in choosing the following pair of variables:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \delta_{\xi_1} \tilde{H}(0) \\ \delta_{\xi_1} \tilde{H}(L) \\ -\delta_{\xi_2} \tilde{H}(0) \\ \delta_{\xi_2} \tilde{H}(L) \end{pmatrix} \tag{40}$$

then the balance equation (39) is written in form of a dissipation equality [31]:

$$\frac{d\tilde{H}(\xi)}{dt} = w_1^T \begin{pmatrix} B(0) & 0 \\ 0 & B(L) \end{pmatrix} w_2 \tag{41}$$

where $B(0) = B(\xi(0))$ and $B(L) = B(\xi(L))$.

Remark 2. Following the definition of Dirac structure [7] and adapting the proofs in [15] [27], one may prove that the differential operator $(B\partial_x + C)$combined with (40) generates a Dirac structure. This Dirac structure is defined with respect to the pairing (symmetric bilinear form) defined on $\left(C^\infty[0,L] \times C^\infty[0,L] \times I\!R^2\right) \times \left(C^\infty[0,L] \times C^\infty[0,L] \times I\!R^2\right) \ni ((f,\, w_1), (e,\, w_2))$:

$$\langle (f,\, w_1), (e,\, w_2) \rangle = \int_0^L e^T f\, dx - w_1^T B\, w_2.$$

The drawback of this choice of external variable is that they correspond to a non-canonical product which actually depends on the differential operator.

Another choice which corresponds rigourously to the definitions in [15] consists in choosing the so-called *boundary port variables*. Consider the matrix:

$$R_{ext} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Sigma_2 & -\Sigma_2 \\ I & I \end{pmatrix} \begin{pmatrix} D_\xi^T(L) & 0 \\ 0 & D_\xi^T(0) \end{pmatrix}. \tag{42}$$

which satisfies the condition:

$$\begin{aligned} R_{ext}^T \Sigma R_{ext} &= \begin{pmatrix} B(L) & 0 \\ 0 & -B(0) \end{pmatrix} \\ &= \begin{pmatrix} D_\xi(b) & 0 \\ 0 & D_\xi(a) \end{pmatrix} \begin{pmatrix} \Sigma_2 & 0 \\ 0 & -\Sigma_2 \end{pmatrix} \begin{pmatrix} D_\xi^T(L) & 0 \\ 0 & D_\xi^T(0) \end{pmatrix}. \end{aligned} \tag{43}$$

According to the port boundary variables may then be defined by:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{ext} \begin{pmatrix} e_1(0) \\ e_1(L) \\ e_2(0) \\ e_2(L) \end{pmatrix} \tag{44}$$

This definition of the boundary port variables follows strictly the construction suggested in [15]. Now the differential operator $B\partial_x + C$ completed with the definition defines the following vector space:

$$\tilde{\mathcal{D}} = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} \;/ \quad \begin{array}{l} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = (B\partial_x + C) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{ext} \begin{pmatrix} e_1(0) \\ e_1(L) \\ e_2(0) \\ e_2(L) \end{pmatrix} \end{array} \right\} \tag{45}$$

Adapting the proofs in [15], one may prove that the vector space $\tilde{\mathcal{D}}$ is a Dirac structure with respect to the pairing defined on $\left(C^\infty[0,L] \times C^\infty[0,L] \times I\!R^2\right) \times \left(C^\infty[0,L] \times C^\infty[0,L] \times I\!R^2\right) \ni ((f,\, f_\partial), (e,\, e_\partial))$:

$$\langle (f,\, w_1), (e,\, w_2) \rangle = \int_0^L e^T f\, dx - e_\partial^T\, \mathbf{diag}\,(1,\, 1\, -1,\, 1)\; f_\partial$$

which is canonical in the sense that it does not depend on the differential operator anymore.

3.3. **Application to the shallow water equations.** We consider the special case of a reach of an open channel delimited by two underflow gates as represented in Figure 1.

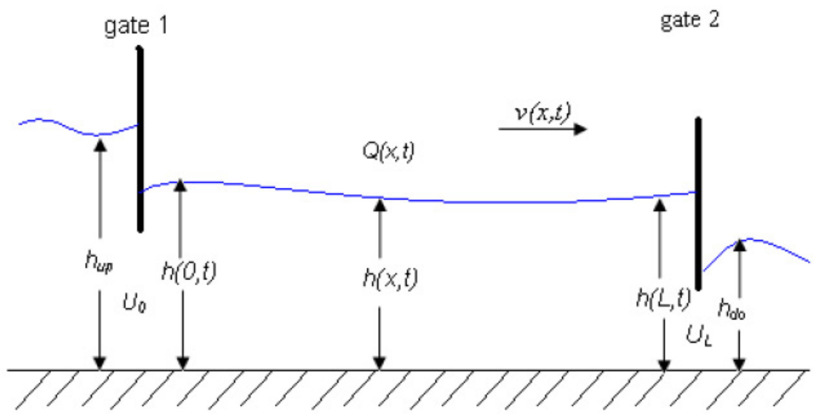

FIGURE 1. A reach of an open channel delimited by two adjustable underflow gates

We assume that the channel is horizontal, prismatic with a constant rectangular section and a width, and that the **friction effects are neglected**.

The flow is the canal may be described by the so-called *shallow water equations* or *Saint-Venant equations* which constitute a system of two conservation laws, actually the mass balance and the momentum balance equations:

$$\partial_t h + \partial_x (hv) = 0, \tag{46}$$

$$\partial_t v + \partial_x (\frac{1}{2} v^2 + gh) = 0, \tag{47}$$

with v denotes the velocity of the water flow and the water level h. Each underflow gates imposes a boundary condition of the form:

$$Wh(0,t)v(0,t) = U_0(t)\Psi_1(h(0,t)), \tag{48}$$

$$Wh(L,t)v(L,t) = U_L(t)\Psi_2(h(L,t)), \tag{49}$$

where W the channel width and
$\Psi_1(h(x,t)) = \alpha_0\sqrt{2g(h_{up} - h(x,t))}$, and $\Psi_2(h(x,t)) = \alpha_L\sqrt{2g(h(x,t) - h_{do})}$, $h_{do} < h(L)$ and $h(0) < h_{up}$ where h_{up} is the water level before the upstream gate, h_{do} is the water level after the downstream gate. α_0 and α_L are the product of the gate (or overflow) width and water-flow coefficient of the corresponding gate. U_0 and U_L are the control functions.

3.3.1. *Hamiltonian formulation of the shallow water equations.* In this section we shall present briefly the Hamiltonian formulation which is described in detail in the general case (with slope and friction) in [13], see also [23]. We shall consider the model of the channel on the spatial domain [0, L]. The energy of the water flow in the channel is defined in terms of two state variables:

- the momentum $p(x,t) = \rho v(x,t)$ where ρ the mass density of the water (constant as the water is assumed to be incompressible),
- the section area of the water $q(x,t) = W\,h(x,t)$ where W denotes the width of the channel.

In the sequel we use the state vector defined by $Y = \begin{pmatrix} q & p \end{pmatrix}^T$. The total energy of the systems consists in the sum of the kinetic and potential energy of the water and is given by:

$$H(Y) = \frac{1}{2}\int_0^L \frac{\rho g}{W} q^2 + \frac{1}{\rho} q p^2 dx, \tag{50}$$

where g denotes the constant of gravity. The variational derivatives of the Hamiltonian functional define the two co-energy variables: e_p the volumic flow and e_q the hydrodynamic pressure as follows:

$$e_q = \delta_q H(Y) = \frac{p^2}{2\rho} + \frac{\rho g}{W} q \left(= \frac{1}{2}\rho v^2 + \rho g h \right), \tag{51}$$

$$e_p = \delta_p H(Y) = \frac{qp}{\rho} \left(= W h v\right). \tag{52}$$

It may be shown [13] [23] that the shallow water equations may be expressed as a system of two conservation laws (1) in canonical interaction and admits a Hamiltonian formulation (34) by writing the flux vector:

$$f(Y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_q H \\ \delta_p H \end{pmatrix} = \begin{pmatrix} \frac{pq}{\rho} \\ \frac{\rho g}{W} q + \frac{p^2}{2\rho} \end{pmatrix}. \tag{53}$$

Finally the Hamiltonian system may be completed with port boundary variables according to the section 3.2.2:

$$\begin{pmatrix} e_\partial^0(t) \\ e_\partial^L(t) \\ f_\partial^0(t) \\ f_\partial^L(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_q H_{|x=0} \\ \delta_q H_{|x=L} \\ \delta_p H_{|x=0} \\ \delta_p H_{|x=L} \end{pmatrix} = \begin{pmatrix} \frac{g\rho}{W} q(0) + \frac{p^2(0)}{2\rho} \\ -\frac{g\rho}{W} q(L) - \frac{p^2(L)}{2\rho} \\ \frac{p(0)q(0)}{\rho} \\ \frac{p(L)q(L)}{\rho} \end{pmatrix}$$

The energy balance is then expressed by:

$$\begin{aligned} \frac{dH}{dt} &= e_\partial^T f_\partial \\ &= -e_q(L)e_p(L) + e_q(0)e_p(0) \\ &= -\left[\frac{\rho g}{W} q(L) + \frac{1}{2\rho} p^2(L)\right]\left[\frac{q(L)p(L)}{\rho}\right] + \left[\frac{\rho g}{W} q(0) + \frac{1}{2\rho} p^2(0)\right]\left[\frac{q(0)p(0)}{\rho}\right] \end{aligned}$$

3.3.2. *Expression in Riemann coordinates.* Using the Jacobian of the vector of flux variables

$$\nabla f(Y) = F(Y) = \begin{pmatrix} \frac{p}{\rho} & \frac{q}{\rho} \\ \frac{g\rho}{W} & \frac{p}{\rho} \end{pmatrix}$$

one obtains the following Riemann invariants:

$$\xi = \xi(Y) = \begin{pmatrix} \frac{p}{\rho} + 2\sqrt{\frac{gq}{W}} \\ \frac{p}{\rho} - 2\sqrt{\frac{gq}{W}} \end{pmatrix} \tag{54}$$

The Jacobian of this change of coordinates is:

$$D(Y)=\begin{pmatrix}\sqrt{\frac{g}{Wq}} & \frac{1}{\rho}\\ -\sqrt{\frac{g}{Wq}} & \frac{1}{\rho}\end{pmatrix} \tag{55}$$

and the celerities of the system are:

$$\Lambda(\xi)=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=\begin{pmatrix}\frac{p}{q}+\sqrt{\frac{gq}{W}} & 0\\ 0 & \frac{p}{q}-\sqrt{\frac{gq}{W}}\end{pmatrix}$$

According to the section 3.2.1, the Hamiltonian system is expressed in the Riemann invariants as the system (35) with matrices

$$\begin{aligned}B &= D\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}D^T\\ &= \begin{pmatrix}\sqrt{\frac{g}{Wq}} & \frac{1}{\rho}\\ -\sqrt{\frac{g}{Wq}} & \frac{1}{\rho}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{g}{Wq}} & -\sqrt{\frac{g}{Wq}}\\ \frac{1}{\rho} & \frac{1}{\rho}\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{\frac{g}{Wq}} & \frac{1}{\rho}\\ -\sqrt{\frac{g}{Wq}} & \frac{1}{\rho}\end{pmatrix}\begin{pmatrix}\frac{1}{\rho} & \frac{1}{\rho}\\ \sqrt{\frac{g}{Wq}} & -\sqrt{\frac{g}{Wq}}\end{pmatrix}=\begin{pmatrix}\frac{2}{\rho}\sqrt{\frac{g}{Wq}} & 0\\ 0 & -\frac{2}{\rho}\sqrt{\frac{g}{Wq}}\end{pmatrix}\end{aligned}$$

$$C=\begin{pmatrix}\frac{1}{\frac{g}{4W}(\xi_1-\xi_2)+\frac{W\bar{q}}{g}} & \frac{1}{\rho}\\ -\frac{1}{\frac{g}{4W}(\xi_1-\xi_2)+\frac{W\bar{q}}{g}} & \frac{1}{\rho}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-\frac{1}{2}\sqrt{\frac{g}{Wq}}\frac{q'}{q} & \frac{1}{2}\sqrt{\frac{g}{Wq}}\frac{q'}{q}\\ 0 & 0\end{pmatrix}=\begin{pmatrix}-\alpha & \alpha\\ -\alpha & \alpha\end{pmatrix}$$

with $\alpha=\frac{1}{2\rho}\sqrt{\frac{g}{Wq}}\frac{q'}{q}$.

4. **Stability and dissipation.**

In this section we shall elaborate on the question posed by Greenberg and Li [12] about the interpretation of the inequality conditions in (6) given in their stabilization theorem 2.1 in terms of dissipation of energy at the boundary. In a first section, we shall consider the port boundary variables associated with a Hamiltonian system and give some condition where the dissipation relations on these port boundary variables may be related to the inequality conditions (6). In a second part we shall treat the shallow water equations for which the conditions do not hold and still interpret the stabilizing boundary relations in terms of energy dissipation.

4.1. **Port boundary variables, Riemann coordinates and stabilizing boundary relations.**

4.1.1. *Stabilizing boundary relations with respect Riemann invariants and boundary port variables.* Consider a hyperbolic system of two conservation laws (1) which admits a Hamiltonian representation (34) with port variables (29). And let us assume the following.

Assumption 1. *There exist two C^1 real functions L_1, L_2 admitting an inverse and defining the following relation between the variational derivative of the Hamiltonian functional of system (1) and the Riemannian coordinates:*

$$\begin{array}{l}\delta_{y_1} H(x) = L_1(\xi_1 - \xi_2)(x) \\ \delta_{y_2} H(x) = L_2(\xi_1 + \xi_2)(x)\end{array} \tag{56}$$

and these functions satisfy:

$$L_1' < 0, \ \ \textit{and} \ \ L_2' > 0. \tag{57}$$

Remark 3.
These conditions on L_i are physical conditions as on can note that the sign of L_i' is linked with the sign of the eigenvalues λ_i.

Then consider relations on the boundary port variables defined by two C^1 functions G_0, G_L as follows:

$$\left\{\begin{array}{l} e_\partial^0 = G_0(f_\partial^0) \\ f_\partial^L = G_L(e_\partial^L) \end{array}\right.$$

As a consequence the energy balance equation depends on G_0, G_L and becomes:

$$\frac{dH}{dt} = e_\partial^L G_L(e_\partial^L) - f_\partial^0 G_0(f_\partial^0) \tag{58}$$

Using the implicit function theorem, the relations (58) on the port boundary variables, may be expressed in terms of relations (5) on the boundary values of the Riemann coordinates. Finally let us note that, using the derivative of the functions K_i of the boundary conditions (5) can be expressed in function of L_1, L_2, G_0, G_L:

$$K_1' = \frac{L_1' + G_0'(L_2)L_2'}{L_1' - G_0'(L_2)L_2'} \text{ and } K_2' = \frac{G_L'(L_1)L_1' - L_2'}{G_L'(L_1)L_1' + L_2'}.$$

Proposition 1. *Consider that the assumption 1 is satisfied. If the functions G_0, G_L are chosen such that the energy of the system is dissipated according to (58):*

$$G_L' < 0,\ G_L(0) = 0 \ \textit{and} \ G_0' > 0,\ G_0(0) = 0$$

then the functions K_1 and K_2 in (5) satisfy the inequalities:

$$|\mathbf{K_1}'(0)| < 1 \ \textit{and} \ |\mathbf{K_2}'(0)| < 1 \tag{59}$$

$$\Rightarrow |\mathbf{K_1}'(0)\mathbf{K_2}'(0)| < 1. \tag{60}$$

If furthermore the compatibility conditions (10)-(11) are satisfied, the conditions of theorem 2.1 are satisfied and the system is exponentially stable.

Remark 4.
Let consider again the example of the p-system treated in the section 2.3. Using the change of coordinates (21), the assumption 1 is indeed satisfied as one may the port boundary variables:

$$\begin{array}{l}\delta_u H(x) = \sigma \circ \bar{u}(\xi_2 - \xi_1)(x) \\ \delta_v H(x) = v = L_2(\xi_1 + \xi_2)(x)\end{array} \tag{61}$$

Hence one may apply the results of the proposition 1: assuming the boundary relation to be dissipative, the system is exponentially stable using Greenberg and Li theorem 2.1.

4.2. **Shallow water equations, dissipativity and stabilization.** The shallow water equations (described in section 3.3) constitute a counterexample to the preceding section. The main reason is that, contrary to the p-system, their Hamiltonian is not separated in the conserved quantities p and q and that the variational derivatives of the Hamiltonian are not expressed as functions of the sum and difference of the Riemann invariants and do not satisfy the assumption 1. However, we shall prove below that some similar results holds but using other variables than the port boundary variables in order to define boundary relations leading to energy dissipation.

Recall the energy balance equation (54):

$$\begin{aligned} d_t H &= e_\partial^T f_\partial \\ &= -e_q(L)e_p(L) + e_q(0)e_p(0) \\ &= -\left[\frac{\rho g}{W} q(L) + \frac{1}{2\rho} p^2(L)\right]\left[\frac{q(L)p(L)}{\rho}\right] + \left[\frac{\rho g}{W} q(0) + \frac{1}{2\rho} p^2(0)\right]\left[\frac{q(0)p(0)}{\rho}\right]. \end{aligned}$$

A particular property of this balance equation is that the section area q remains positive as well as the hydrodynamic pressure $e_q = \left(\frac{\rho g}{W} q + \frac{1}{2\rho} p^2\right)$. Furthermore considering the expression of the Riemann coordinates (54), one may notice that the *state variables* satisfy the first assumption 1:

$$p = L_1(\xi_1 + \xi_2) = \frac{\rho}{2}(\xi_1 + \xi_2) \; q = L_2(\xi_1 - \xi_2) = \left[\frac{1}{4}\sqrt{\frac{B}{g}}(\xi_1 - \xi_2)\right]^2.$$

The second assumption of assumption 1 which is expressed as $L_1' = \frac{\rho}{2} > 0$, $L_2' = A > 0$. L matrix is linked with the state variables by the following relation:

$$\begin{pmatrix} L_2' & 0 \\ 0 & L_1' \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}^{-1} D_Y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} d(e) = d(Y) \tag{62}$$

$$\begin{aligned} &\begin{pmatrix} L_2' & 0 \\ 0 & L_1' \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}^{-1} D_Y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} d(e) = F^{-1}(Y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} d(e) \\ \Rightarrow\; &\begin{pmatrix} L_2' & 0 \\ 0 & L_1' \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^{-1} D_Y = F^{-1}(Y) \\ \Rightarrow\; &\begin{pmatrix} L_2' & 0 \\ 0 & L_1' \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^{-1} D_Y F(Y) = Id \\ \Rightarrow\; &\begin{pmatrix} L_2' & -L_2' \\ L_1' & L_1' \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^{-1}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} D_Y = Id \\ \Rightarrow\; &\begin{pmatrix} L_2' & -L_2' \\ L_1' & L_1' \end{pmatrix}\begin{pmatrix} \sqrt{\frac{g}{Wq}} & \frac{1}{\rho} \\ -\sqrt{\frac{g}{Wq}} & \frac{1}{\rho} \end{pmatrix} = Id \\ \Rightarrow\; &\begin{pmatrix} 2L_2'\sqrt{\frac{g}{Wq}} & 0 \\ 0 & 2L_1'\frac{1}{\rho} \end{pmatrix} = Id \\ \Rightarrow\; &2L_2' = \sqrt{\frac{Wq}{g}} > 0,\; L_1' = \frac{\rho}{2} > 0 \end{aligned}$$

Using the same notations for the boundary relations applied to the states variables Y and not to $(e_q, e_p)^T$, and with $A = \frac{1}{2}\sqrt{\frac{Wq}{g}} > 0$, one gets

$$\begin{aligned} K_1' &= \frac{\frac{\rho}{2} + AG_0'}{-\frac{\rho}{2} + AG_0'} = \frac{G_0'(L_2)L_2' + L_1'}{G_0'(L_2)L_2' - L_1'}, \\ K_2' &= \frac{A - \frac{\rho}{2}(G_L')}{A + \frac{\rho}{2}(G_L')} = \frac{L_2' - G_L'(L_1)L_1'}{L_2' + G_L'(L_1)L_1'}, \end{aligned}$$

and so one can conclude that if the boundary relations dissipate the energy

$$\begin{aligned} d_t H &= < 0 \\ &\Rightarrow G_0' < 0 \text{ and } G_L' > 0 \\ &\Rightarrow |K_2'(0)| < 1 \text{ and } |K_1'(0)| < 1 \\ &\Rightarrow |K_1'(0)K_2'(0)| < 1 \end{aligned}$$

and under the compatibility conditions, then one can conclude on the stability of the shallow water equations.

Remark 5. A particular case is deal with here as the equilibrium state $(\bar{q}, \bar{p})$ has been taken equal to zero to simplify the equations. Just note that all the calculus have been done with an equilibrium $(\bar{q}, \bar{p}) \neq 0$ and under the conditions $G_L(\bar{x}) = 0$ and $G_0(\bar{x}) = 0$, where $\bar{x}$ is the equilibrium of the function x, the results previously presented are the same.

5. **Conclusions.** In the first part of this paper we have recalled the expression of the port Hamiltonian systems of two conservation laws and derived its expression in terms of Riemann invariant. In these coordinates we have shown that the matrix differential operator associated with the system is no more canonical and have given its expression. We have then derived the expression of the Dirac structure associated with this expression of the matrix differential operator, the Hamiltonian operator. The expression of the boundary port variables associated with the definition of the Dirac structure extending the matrix differential operator (the Hamiltonian operator) of the Hamiltonian formulation has also been derived. This construction has been detailed for the examples of of the p-system and the shallow water equations.

In the second part of the paper, under the assumption that the port boundary variables may be expressed as a function of the sum and difference of the Riemann invariants and one the monotonicity of these functions, we have expressed the stability conditions on the boundary values of the Riemann invariants with some conditions on the boundary constraints on the port variables. As a consequence we have given an interpretation of the stabilizing boundary relations in terms of the dissipation of the Hamiltonian function, in the case of physical systems identical with the total energy of the system, on the boundary of the system.

The first remark is that the latter are more restrictive that the conditions of Greenberg and Li as they imply that the energy is dissipated at each time instant which appears to be a stronger than the requirement on the characteristics. Secondly the assumptions on the relation between the boundary port variables and the Riemann invariants are true for the p-system (i.e. for nonlinear wave equations) but not for the shallow water equations. However in the latter case particular properties of the Hamiltonian allow to related again the energy dissipation to the conditions of Greenberg and Li's theorem using the remark that the state variables satisfy the assumption.

This paper aimed to be a first step towards a physically motivated and interpretable construction of stabilizing boundary control. A first extension of this paper could be to relax the requirement on the instantaneous dissipation of the energy (i.e. the Hamiltonian function in the physical model) and that eventually techniques on dissipation on time intervals (see for instance [28]) may lead to weaker conditions.

A second extension could be to use the Riemann coordinate for assigning a closed-loop Dirac structure and simultaneously the Hamiltonian functional in some analogy with the finite-dimensional case [19].

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Received September 2008; revised February 2009.

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