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KORN INEQUALITIES ON THIN PERIODIC STRUCTURES

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Abstract. We prove Korn-type inequalities for thin periodic structures of period ε and thickness $\varepsilon h(\varepsilon)$, where $h(\varepsilon)\to 0$ as $\varepsilon\to 0$, among which there are plane grids, spatial rod and box structures. These inequalities are important in homogenization of corresponding elasticity problems.

1. **Introduction.** As thin periodic structures F_ε^h, we will consider plane grids and trusses and also three-dimensional box and rod structures whose geometry depends on two parameters mutually related to one another, namely, ε defines the periodicity cell and εh is the thickness of constituting elements of the structure, where $h = h(\varepsilon) \to 0$. It was found in [1] (see also [2, 3, 4, 5, 6, 7]), that the elastic properties of periodic structures manifest a qualitative difference depending on the value of the limit $\lim\limits_{\varepsilon\to 0} h(\varepsilon)/\varepsilon$, and according to this, thin structures were classified as

(i) sufficiently thick structures if $h(\varepsilon)/\varepsilon \to \infty$;
(ii) structures of critical thickness if $h(\varepsilon)/\varepsilon \to \theta > 0$;
(iii) sufficiently thin structures if $h(\varepsilon)/\varepsilon \to 0$.

The following Korn inequalities clarify the scaling phenomenon mentioned above:

$$\int\limits_{F_\varepsilon^h} |u|^2 dx \le C_1 \left(1 + \left(\frac{\varepsilon}{h}\right)^2\right) \int\limits_{F_\varepsilon^h} |e(u)|^2 dx, \tag{1}$$

$$\varepsilon^2 \int\limits_{F_\varepsilon^h} |\nabla u|^2 dx \le C_2 \int\limits_{F_\varepsilon^h} \left[\left(\frac{\varepsilon}{h}\right)^2 |e(u)|^2 + |u|^2\right] dx, \quad u \in C_0^\infty(\mathbb{R}^N)^N, \tag{2}$$

where $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is a symmetric gradient of the vector-valued function u, the constants C_1 and C_2 do not depend on the geometric parameters ε and h but may depend on the diameter of the support of the function u.

These two inequalities are different in nature. Inequality (2) holds "piecewise" on the fragments of the structure F_ε^h of a certain type, while inequality (1) does not admit such a "decomposition" and requires special properties of connectedness or strength from the periodic structure.

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In comparison with (1), take note of quite different nature of the Poincaré-Friedrichs inequality

$$\int_{F_\varepsilon^h} |\varphi|^2 dx \le C_0 \int_{F_\varepsilon^h} |\nabla\varphi|^2 dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \tag{3}$$

where constant C_0 depends only upon the geometry of the singular structure F and the diameter of the support of φ. The inequality (3) is valid under the only supposition of connectedness of F, and this is its another noticeable feature. Since function $h(\varepsilon)$ may be arbitrary in (3), there is no scaling phenomenon in scalar problems.

2. **Thin periodic grids.**

2.1. **The simplest grids.** In Fig. 1, a thin periodic grid is presented. It is composed of infinite rods of thickness $h > 0$, and the periodicity cell $\square = [-\frac{1}{2}, \frac{1}{2})^2$ is indicated by the dotted line. In the same figure, the infinitely thin (or singular) grid is depicted. It corresponds to the thickness $h = 0$ and is a skeleton for the thin grid. This is a model example. It makes clear the relation between the singular structure and the corresponding to it thin structure, which remains the same in more complicated examples.

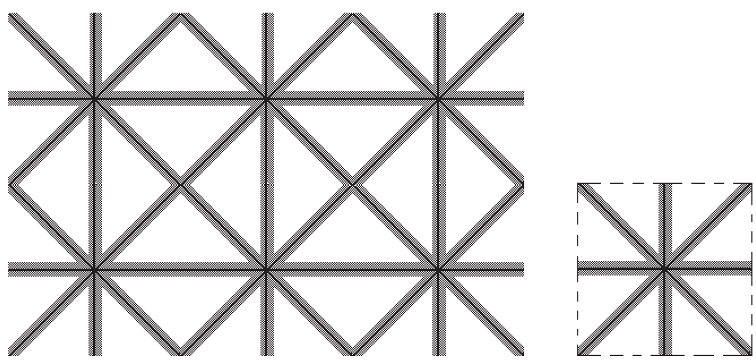

FIGURE 1.

Let F^h denote a 1-periodic grid of thickness $h > 0$ and F denote a singular grid corresponding to it; $F_\varepsilon^h = \varepsilon F^h$ is a homothetic contraction of the grid F^h. The thickness of rods of the grid F_ε^h is equal to εh.

The model singular (or thin) grid is composed of several periodically repeating infinite straight lines (or strips). Such grids are called *the simplest*. We exclude from consideration the degenerate case of a periodically repeating single straight line.

Theorem 2.1. *For the simplest grids F_ε^h, inequality (1) holds, where the constant C_1 depends on the geometry of the singular grid F and the diameter of supp u, and also the inequality*

$$h\varepsilon \int_{F_\varepsilon^h} |\nabla u|^2 dx \le C_2 \int_{F_\varepsilon^h} (|e(u)|^2 + |u|^2) dx, \quad u \in C_0^\infty(\mathbb{R}^2)^2, \tag{4}$$

holds, where the constant C_2 depends only on the geometry of the singular grid.

For sufficiently thin grids, the constant $C_1\left(1+\left(\frac{\varepsilon}{h}\right)^2\right)$ from (1) tends to infinity as $\varepsilon\to 0$. The following example shows that in this case, the exact constant in the Korn inequality indeed has the order $(\frac{\varepsilon}{h})^2$.

Example. Consider the square grid F^h. Let $a(t)$ be a smooth 1-periodic function equal to zero in a neighborhood of the points 0 and $\frac{1}{2}$, and let $a\not\equiv 0$. In the periodicity cell, we define the vector-valued function v by

$$v(y)=\begin{cases}(-y_2a'(y_1), \quad a(y_1)) \text{ for } y\in[-\frac{1}{2},\frac{1}{2}]\times[-\frac{h}{2},\frac{h}{2}],\\ (-a(y_2), a'(y_2)y_1) \text{ for } y\in[-\frac{h}{2},\frac{h}{2}]\times[-\frac{1}{2},\frac{1}{2}],\end{cases}$$

and extend to $\mathbb{R}^2$ periodically. It is easy to verify that

$$\int\limits_{\square\cap F^h}|v|^2dy=O(h),\qquad \int\limits_{\square\cap F^h}|e(v)|^2dy=O(h^3).$$

We set $u(x)=v(\frac{x}{\varepsilon})$ in $\square=(-\frac{1}{2},\frac{1}{2})^2$, where $\varepsilon=\frac{1}{n}$ and n is a positive integer. Then $u|_{\partial\square}=0$ owing to the properties of the function $a(t)$, and the relations

$$\int\limits_{\square\cap F_\varepsilon^h}|u|^2dx=\int\limits_{\square\cap F^h}|v|^2dy=O(h),\qquad \int\limits_{\square\cap F_\varepsilon^h}|e(u)|^2dx=\frac{1}{\varepsilon^2}\int\limits_{\square\cap F^h}|e(v)|^2dy=O(\frac{h^3}{\varepsilon^2}),$$

hold, which imply the desired property.

2.2. **Regular grids.** We now turn to grids that are not the simplest (see Fig. 2). We first make a convention on how to construct a thin grid F^h using a singular grid F. For this purpose, we elongate each link I in both sides by $\frac{h}{2}$ so that a segment I' of length $|I|+h$ will be obtained. Construct a band I^h of width h with middle line I'. The union of all these bands I^h defines the grid F^h. Note that each node of the singular grid F belongs to the grid F^h, together with the circle of radius $\frac{h}{2}$, by construction.

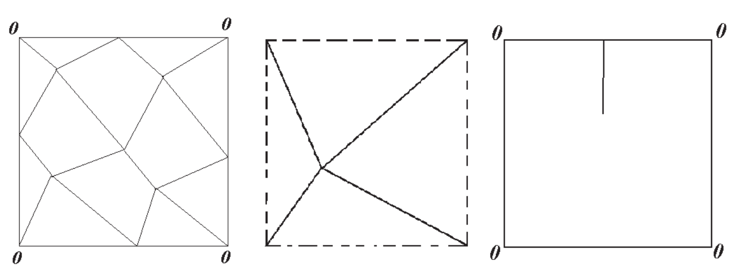

Figure 2.

Define one more class of grids for which we can prove inequality (1). A link of a grid is *piercing* if the whole straight line passing through this link belongs to the grid. We try to ascribe the mark (or label) $k=0,1,\ldots$ to each node of the grid F, proceeding by induction. The node gets the mark 0 if at least two noncollinear piercing links emanate from it. Let all the nodes to which one can ascribe the marks $k\le N$ be defined. Then we ascribe the mark $N+1$ to the yet unmarked node if one can go out from it along two noncollinear links into the neighboring nodes that already have marks. We say that the grid F is regular if all its nodes turn out to be marked by finitely many marks. In Fig. 3, regular grids are depicted; their nodes have marks from 0 to 3.

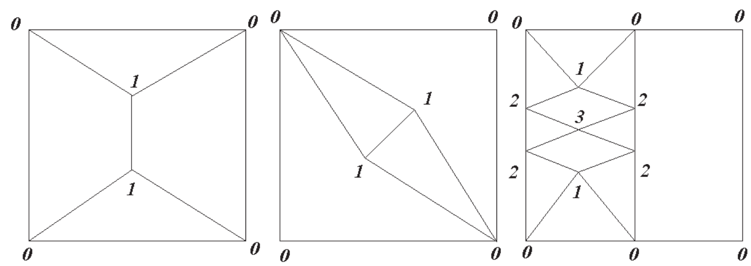

FIGURE 3.

By definition, the regular grid F contains piercing straight lines. They make up the simplest grid which forms the basis of the grid F, imparting to it the special connectedness and strength in two or more directions.

The notion of a regular grid can be extended in the following way by ascribing the marks not only to the nodes but also to the links. The node gets the mark 0 if at least two noncollinear piercing links issue from it. The zero mark get the links that connect nodes with the zero mark and also the piercing links. Let all the nodes and links to which we can ascribe the marks $k \leq N$ be defined. Then the mark $N+1$ is ascribed to the yet unmarked node if one can get out of it along two noncollinear links into nodes or to links that already have marks. Having defined all the nodes with the mark $N+1$, we ascribe the mark $N+1$ to the yet unmarked links that connect already marked nodes. In this extended sense, the first of the grids depicted in Fig. 2 is regular; the others are not regular.

Theorem 2.2. *Let F be a regular grid. Then the following inequality holds:*

$$\left(1+\left(\frac{\varepsilon}{h}\right)^2\right)^{-1}\int\limits_{F_\varepsilon^h}(|u|^2+\varepsilon^2|\nabla u|^2)dx \leq C\int\limits_{F_\varepsilon^h}|e(u)|^2dx, \quad u \in C_0^\infty(\mathbb{R}^2)^2,$$

where the constant C depends on the geometry of the singular grid F and the diameter of supp u.

Theorem 2.3. *For any periodic grid F_ε^h without exceptions, inequality (2) holds, where the constant C_2 depends only on the geometry of the singular structure of F.*

The Korn inequality (1) remains unproved for grids that are not regular, for example, for the second and the third in Fig. 2.

2.3. **Technical lemmas.** We start from certain inequalities on the unit square $\square = [0,1]^2$.

Lemma 2.4. *The following inequalities hold:*

$$\int\limits_\square |\nabla v|^2 dy \leq C_1 \int\limits_\square (|e(v)|^2 + |v_2|^2)dy, \tag{5}$$

$$\int_{\square} |\nabla v|^2 dy \leq C_2 \left(\int_{\square} |e(v)|^2 dy + \sum_{i=0}^{1} \frac{1}{h} \int_{K^i} v_2^2 dy \right), \quad v \in C^\infty(\square)^2, \tag{6}$$

where K^0, K^1 *are ellipses inscribed into the rectangles* $[0,h)\times[0,1]$, $[1-h,1)\times[0,1]$, *respectively.*

Proof. At first, we claim (5). If not, then there exist vectors v^k such that

$$\int_{\square} |\nabla v^k|^2 dy = 1, \quad \int_{\square} [|e(v^k)|^2 + |v_2^k|^2] dy \to 0, \quad \int_{\square} v_1^k dy = 0. \tag{7}$$

Sequence v^k is bounded in $H^1(\square)^2$, and we can suppose that $v^k \to v$, $\nabla v^k \rightharpoonup \nabla v$ in $L^2(\square)$. Then (7) implies the relations

$$e(v) = 0, \quad v_2 = 0, \quad \int_{\square} v_1 dy = 0,$$

which yield $v = 0$. On the other hand, the classical Korn inequality

$$\int_{\square} |\nabla w|^2 dy \leq C_1 \int_{\square} (|e(w)|^2 + |w|^2) dy, \quad w \in C^\infty(\square)^2, \tag{8}$$

applied for $w = v^k$ implies $C_1 \int_{\square} |v^k|^2 dy \geq \frac{1}{2}$ for a sufficiently large k by $(7)_1$, which contradicts the strong convergence $v^k \to 0$ in $L^2(\square)$. Inequality (5) is proved. $\square$

Inequality (6) is also proved by assuming the contrary. If it does not hold, we can find a sequence $h \to 0$ and vectors v^h such that

$$\int_{\square} |\nabla v^h|^2 dy = 1, \quad \int_{\square} |e(v^h)|^2 dy + \frac{1}{h} \int_{K^0} |v_2^h|^2 dy + \frac{1}{h} \int_{K^1} |v_2^h|^2 dy \to 0, \quad \int_{\square} v_1^h dy = 0. \tag{9}$$

Obviously, v_1^h is bounded in $H^1(\square)$. The same is true for v_2^h. Indeed, examine $\alpha^h = \int_{\square} v_2^h dy$.

The following inequality stems from the theorem on the trace of a function:

$$\frac{1}{h} \int_{T_0} w^2 dy \leq \delta \int_{\square} |\nabla w|^2 dy + C_\delta \int_{\square} w^2 dy, \quad \forall \delta > 0, \tag{10}$$

where $T_0 = [0,h] \times [0,1]$ and $w \in H^1(\square)$. From here, we derive

$$\begin{aligned} |\alpha^h|^2 &= \frac{1}{|K^0|} \int_{K^0} |\alpha^h|^2 dy \\ &= \frac{4}{\pi h} \int_{K^0} |\alpha^h|^2 dy \leq \frac{4}{\pi h} \left[\int_{T_0} |v_2^h - \alpha^h|^2 dy + \int_{K^0} |v_2^h|^2 dy \right] \leq C, \end{aligned} \tag{11}$$

by (9) and the boundedness of the sequence $v_2^h - \alpha^h$ in $H^1(\square)$. The property (11) implies, that v_2^h is bounded in $H^1(\square)$.

Thus, $\|v^h\|_{H^1(\square)^2} \leq C$. Without loss of generality, $v^h \rightharpoonup v$ in $H^1(\square)^2$, $v^h \to v$ in $L^2(\square)^2$. Then due to (10), (9),

$$\lim_{h\to 0} \frac{1}{|K^0|} \int\limits_{K^0} |v_2^h|^2 dy = \frac{4}{\pi} \int\limits_0^1 10|v_2(0,y_2)|^2 dy_2 = 0.$$

Similarly,

$$\lim_{h\to 0} \frac{1}{|K^1|} \int\limits_{K^1} |v_2^h|^2 dy = \frac{4}{\pi} \int\limits_0^1 |v_2(1,y_2)|^2 dy_2 = 0.$$

We thereby derive from (9) relations

$$e(v) = 0, \quad \int\limits_{\square} v_1 dy = 0, \quad v_2|_{y_1=0} = v_2|_{y_1=1} = 0,$$

which admit only $v = 0$. Therefore, $v^h \to 0$ in $L^2(\square)^2$.

On the other hand, from the Korn inequality (8) applied for $w = v^h$ it follows that $C_1 \int\limits_{\square} |v^h|^2 \geq \frac{1}{2}$, which contradicts the strong convergence $v^h \to 0$. Inequality (6) is proved, and Lemma 2.4 is proved together with it.

From Lemma 2.4, we derive

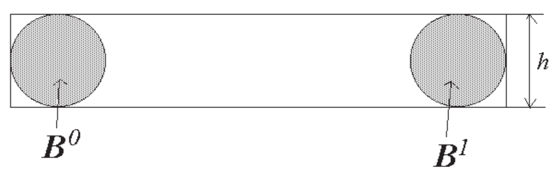

Figure 4.

Lemma 2.5. *Let* $I^h = [0,1] \times [0,h]$ *be a band and let* B^0 *and* B^1 *be the circles of radius* $\frac{h}{2}$ *adjacent to the end-walls of the band (see Fig. 4), Then the following inequalities hold:*

$$h^2 \int\limits_{I^h} |\nabla u|^2 dx \leq C_1 \int\limits_{I^h} (|e(u)|^2 + h^2 u_2^2) dx, \tag{12}$$

$$h^2 \int\limits_{I^h} |\nabla u|^2 dx \leq C_2 \left(\int\limits_{I^h} |e(u)|^2 dx + \sum_{i=0}^{1} h \int\limits_{B^i} u_2^2 dx \right), \quad u \in C^\infty(I^h)^2. \tag{13}$$

Proof. In inequalities (5) and (6), we make the following change of variables:

$$x_1 = y_1, \quad x_2 = hy_2, \quad v_1 = u_1, \quad v_2 = hu_2, \tag{14}$$

as a result of this change, for example, inequality (5) becomes

$$\int\limits_{I^h} \left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + h^2 \left(\frac{\partial u_1}{\partial x_2}\right)^2 + h^2 \left(\frac{\partial u_2}{\partial x_1}\right)^2 + h^4 \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right] dx$$

$$\leq C\int\limits_{I^h}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2+\frac{h^2}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)^2+h^4\left(\frac{\partial u_2}{\partial x_2}\right)^2+h^2u_2^2\right]dx.$$

From here,

$$h^2\int\limits_{I^h}\left[\left(\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2\right]dx\leq C\int\limits_{I^h}\left[|e(u)|^2+h^2u_2^2\right]dx,$$

and since

$$\int\limits_{I^h}\left(\left|\frac{\partial u_1}{\partial x_1}\right|^2+\left|\frac{\partial u_2}{\partial x_2}\right|^2\right)dx\leq\int\limits_{I^h}|e(u)|^2dx,$$

we obtain estimate (12).

In a similar way, inequality (13) is derived from (6). The change (14) transforms the ellipses K^i into circles B^i, $i=0,1$. Lemma 2.5 is proved. □

We now deduce consequences from Lemma 2.5.

For the rod $\varepsilon I^h=[0,\varepsilon]\times[0,\varepsilon h]$, from estimate (12), by homothety we obtain

$$\varepsilon^2\int\limits_{\varepsilon I^h}|\nabla u|^2dx\leq C_1\int\limits_{\varepsilon I^h}\left[\left(\frac{\varepsilon}{h}\right)^2|e(u)|^2+|u_2|^2\right]dx \tag{15}$$

and by summation come to (2). This completes the proof of the Th. 2.3.

Lemma 2.6. *Let* $P=x:|x_2|<\frac{t}{2}$. *Then*

$$t\int\limits_P|\nabla u|^2dx\leq C\int\limits_P(|e(u)|^2+|u_2|^2)dx,\quad u\in C_0^\infty(\mathbb{R}^2)^2, \tag{16}$$

where C *is an absolute constant.*

Proof. Indeed, cut the strip P of width t into equal parts P_i of length $\sqrt{t}$ and apply inequality (15) on each part P_i separately, assuming it to be the strip εI^h with $\varepsilon{=}h{=}\sqrt{t}$. Then

$$t\int\limits_{P_i}|\nabla u|^2dx\leq C_1\int\limits_{P_i}(|e(u)|^2+|u_2|^2)dx.$$

Summing these estimates, we arrive at inequality (16). □

2.4. **Proof of theorem 2.1.** Inequality (4) for the simplest grid F_ε^h follows from Lemma 2.6 if we take into account that the thickness of the strips that constitute the grid is equal to $t=\varepsilon h$.

We are to prove the estimate (1) for the simplest grid F_ε^h. To this end we will use the following inequality on the strip $\varepsilon I^h=[0,\varepsilon]\times[0,\varepsilon h]$:

$$\varepsilon^2\int\limits_{\varepsilon I^h}|\nabla u|^2dx\leq C\left[\left(\frac{\varepsilon}{h}\right)^2\int\limits_{\varepsilon I^h}|e(u)|^2dx+\frac{1}{h}\sum_{i=0}^{1}\int\limits_{\varepsilon B^i}u^2dx\right], \tag{17}$$

which is obtained by homothety from inequality (13).

The following variations of the Poincare-Friedrichs inequality for the scalar-valued function are useful.

(i) If $I^h = [0,1] \times [0,h]$ and B is a circle of radius $\frac{h}{2}$ belonging to the strip I^h, then

$$\int\limits_{I^h} w^2 dx \le C \left(\int\limits_{I^h} |\nabla w|^2 dx + \frac{1}{h} \int\limits_B w^2 dx \right), \quad w \in C^\infty(I^h).$$

From here, using the homothety, we obtain

$$\int\limits_{\varepsilon I^h} w^2 dx \le C \left(\varepsilon^2 \int\limits_{\varepsilon I^h} |\nabla w|^2 dx + \frac{1}{h} \int\limits_{\varepsilon B} w^2 dx \right), \quad w \in C^\infty(\varepsilon I^h). \tag{18}$$

(ii) If B and B_1 are circles of radius $\frac{h}{2}$ that belong to the strip $I^h = [0,1] \times [0,h]$, then

$$\frac{1}{h} \int\limits_{B^1} w^2 dx \le \frac{1}{h} \int\limits_B w^2 dx + C \int\limits_{I^h} |w \frac{\partial w}{\partial x_1}| dx.$$

From here, using the homothety, we obtain

$$\frac{1}{h} \int\limits_{\varepsilon B^1} w^2 dx \le \frac{1}{h} \int\limits_{\varepsilon B} w^2 dx + \varepsilon^2 C_\delta \int\limits_{\varepsilon I^h} |\frac{\partial w}{\partial x_1}|^2 dx + \delta \int\limits_{\varepsilon I^h} |w|^2 dx \quad \forall \delta > 0. \tag{19}$$

(iii) If P is a horizontal strip of width t and B is a circle of radius $\frac{t}{2}$ that belongs to P, then

$$\int\limits_P w^2 dx \le d^2 \int\limits_P \left| \frac{\partial w}{\partial x_1} \right|^2 dx, \tag{20}$$

$$\int\limits_B w^2 dx \le dt \int\limits_P \left| \frac{\partial w}{\partial x_1} \right|^2 dx, \quad w \in C_0^\infty(\mathbb{R}^2), \tag{21}$$

where d is the diameter of the support of the function u.

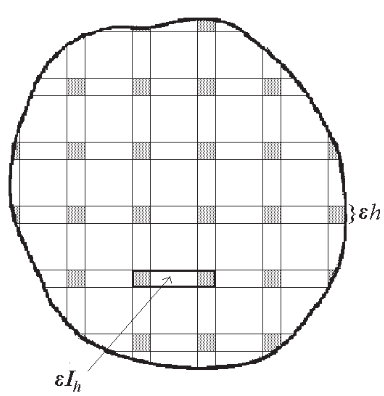

FIGURE 5.

Pass to the direct proof of inequality (1). For clarity, let F_ε^h be a square grid (Fig. 5) composed of horizontal strips $P = P_i$ and vertical strips $S = S_i$ of width

εh. On a separate horizontal strip P, by virtue of (20),

$$\int_P u_1^2 dx \leq C \int_P |e(u)|^2 dx. \tag{22}$$

To estimate u_2 on P, distinguish the fragment εI^h (see Fig.5), where by (18) and (17),

$$\int_{\varepsilon I^h} u_2^2 dx \leq C_1 \left\{ \left(\frac{\varepsilon}{h}\right)^2 \int_{\varepsilon I^h} |e(u)|^2 dx + \frac{2}{h} \sum_{i=0}^{1} \int_{\varepsilon B^i} u_2^2 dx \right\}.$$

The last integrals over circles are estimated by inequality (21) applied to the vertical strips S_0 and S_1 containing the circles εB^0 and εB^1, namely,

$$\frac{1}{h} \sum_{i=0}^{1} \int_{\varepsilon B^i} |u_2|^2 dx \leq C\varepsilon \sum_{i=0}^{1} \int_{S_i} |e(u)|^2 dx.$$

After summation over all fragments εI^h constituting the strip P, we obtain

$$\int_P u_2^2 dx \leq C \left\{ \left(\frac{\varepsilon}{h}\right)^2 \int_P |e(u)|^2 dx + \varepsilon \sum_j \int_{S_j} |e(u)|^2 dx \right\}. \tag{23}$$

The number of horizontal strips $P{=}P_i$ crossing $\operatorname{supp} \mathrm{u}$ is of order ε^{-1}. Therefore, summing estimates (23) over all these strips $P = P_i$, we derive

$$\sum_i \int_{P_i} u_2^2 dx \leq C \left(1 + \left(\frac{\varepsilon}{h}\right)^2\right) \int_{F_\varepsilon^h} |e(u)|^2 dx.$$

From here and from (22), and also from the similar inequalities for the vertical strips S_i, estimate (1) for the square grid follows. Theorem 2.1 is proved.

2.5. **Proof of theorem 2.2.** Since Theorem 2.3 is already established, it only remains to verify estimate (1) for the regular grid F_ε^h.

Consider an arbitrary short rod εI^h with circles εB^0 and εB^1 adjacent to its end-walls. The centers of these circles are located at the nodes of the singular grid F_ε. By inequality (18),

$$\int_{\varepsilon I^h} |u|^2 dx \leq C \left(\varepsilon^2 \int_{\varepsilon I^h} |\nabla u|^2 dx + \frac{1}{h} \int_{\varepsilon B^0} |u|^2 dx \right),$$

which, by (17), yields

$$\int_{\varepsilon I^h} |u|^2 dx \leq C_1 \left\{ \left(\frac{\varepsilon}{h}\right)^2 \int_{\varepsilon I^h} |e(u)|^2 dx + \frac{1}{h} \sum_{i=0}^{1} \int_{\varepsilon B^i} |u|^2 dx \right\}.$$

From here, by summation over all short rods, we have

$$\int_{F_\varepsilon^h} |u|^2 dx \leq C \left\{ \left(\frac{\varepsilon}{h}\right)^2 \int_{F_\varepsilon^h} |e(u)|^2 dx + \frac{1}{h} \sum \int_{B_{\varepsilon h}} |u|^2 dx \right\},$$

where the sum is taken over the totality of circles $B_{\varepsilon h}$ of radius εh centered at the nodes of the singular grid F_ε. It is required to give an appropriate estimate for the above sum.

If the grid F is the simplest, then the appropriate estimate follows from (21). For example, for the square grid, we have

$$\frac{1}{h}\int\limits_{B_{\varepsilon h}} |u_1|^2 dx \le C\varepsilon \int\limits_{P} |e(u)|^2 dx,$$

where P is the horizontal strip containing the circle $B_{\varepsilon h}$. Taking into account that the number of circles in this strip is of order ε^{-1} and summing over all horizontal strips, we get

$$\frac{1}{h}\sum \int\limits_{B_{\varepsilon h}} |u_1|^2 dx \le C_1 \int\limits_{F_\varepsilon^h} |e(u)|^2 .dx$$

We obtain an analogous estimate for the component u_2. Thus, we have proved inequality (1) once more for the simplest grid.

Now, consider the regular grid. Let $J_0, J_1, \dots, J_k$ be the sets of balls $B_{\varepsilon h}$ whose centers are located at the nodes with the marks $0, 1, \dots, k$.

If $B_{\varepsilon h} \subset J_0$, then to estimate the corresponding integral, we use two noncollinear piercing strips containing this circle. In essence, the situation here is the same as for the simplest grids, and

$$\sum_{B_{\varepsilon h} \subset J_0} \frac{1}{h}\int\limits_{B_{\varepsilon h}} |u|^2 dx \le C \int\limits_{F_\varepsilon^h} |e(u)|^2 dx. \tag{24}$$

Let $B_{\varepsilon h}$ be a circle centered at the arbitrary node O with the unit mark. Then the node O is connected with the nodes O_1 and O_2 whose marks are zero along two noncollinear links εI_1 and εI_2 of the singular grid F_ε. The directing vectors of these links are τ_1 and τ_2, and the rods εI_1^h and εI_2^h correspond to them in the grid F_ε. In the circle $B_{\varepsilon h}$ we use the expansion of the vector u over the basis τ_1, τ_2: $u = u_{\tau_1}\tau_1 + u_{\tau_2}\tau_2$; let $u_{\tau_j} = u_j$, $j = 1, 2$.

We will estimate the component u_j in the circle $B_{\varepsilon h}$ using the rod εI_j^h. For example, from estimate (19), we have

$$\frac{1}{h}\int\limits_{B_{\varepsilon h}} |u_1|^2 dx \le C_\delta \varepsilon^2 \int\limits_{\varepsilon I_1^h} |e(u)|^2 dx + \frac{1}{h}\int\limits_{B_{\varepsilon h}^1} |u_1|^2 dx + \delta \int\limits_{\varepsilon I_1^h} |u_1|^2 dx,$$

where $B_{\varepsilon h}^1$ is a circle centered at the node O_1 belonging to the rod εI_1^h. In sum, having considered all the circles centered at the nodes with unit marks, we obtain

$$\begin{aligned}\sum_{J_1} \frac{1}{h}\int\limits_{B_{\varepsilon h}} |u|^2 dx &\le C_\delta \varepsilon^2 \int\limits_{F_\varepsilon^h} |e(u)|^2 + \frac{1}{h}\sum_{J_0}\int\limits_{B_{\varepsilon h}} |u|^2 dx + \delta \int\limits_{F_\varepsilon^h} |u|^2 dx \\ &\le \tilde{C}_\delta \int\limits_{F_\varepsilon^h} |e(u)|^2 dx + \delta \int\limits_{F_\varepsilon^h} |u|^2 dx,\end{aligned}$$

by inequality (24).

Thus, we sequentially estimate all the sums $\sum\limits_{J_i} \frac{1}{h} \int\limits_{B_{\varepsilon h}} |u|^2 dx$ for all $k = 0, 1, \dots,$ possible for the given regular grid F^h. After that, choosing $\delta > 0$ sufficiently small, we come to estimate (1).

If F is regular in the extended sense (see Sect. 2.2), then the proof is based on the same ideas but is more cumbersome.

Remark 1. By construction of h-grid F^h approximating the single grid F (see the beginning of Sect. 2.2), we may omit initial elongation of links. In this case in our proofs we cannot use circles of radius $\frac{h}{2}$ centered at the nodes of F, and take instead of them circles of radius $c_0 h$, located in the intersection of h-rods joining the same node. Here, constant c_0 depends upon the geometry of F.

3. **Box structures.** In Fig. 6, we depict the fragment (within the unit cube) of 1-periodic box structure F^h of thickness h. In the same figure, we also present the singular structure F which is composed of three mutually orthogonal families of parallel planes. Each plane of the structure F turns out to be middle for the corresponding infinite plate from the structure F^h. Let $F^h_\varepsilon = \varepsilon F^h$.

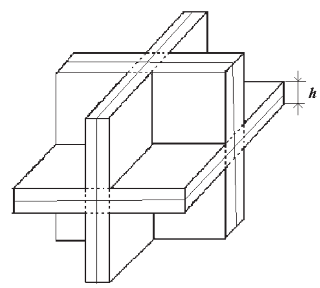

FIGURE 6.

Theorem 3.1. *Inequality (1) holds for the box structure F^h_ε, where the constant C_1 depends on the diameter of* supp u*, and also the inequality*

$$h\varepsilon \int\limits_{F^h_\varepsilon} |\nabla u|^2 dx \le C_2 \int\limits_{F^h_\varepsilon} (|e(u)|^2 + |u|^2)dx, \quad u \in C_0^\infty(\mathbb{R}^3)^3, \tag{25}$$

holds, where C_2 is an absolute constant.

Also, we note the following analog of Lemma 2.6 for an infinite plate.

Lemma 3.2. *Let $P = \{x \in \mathbb{R}^3 : |x_3| < \frac{t}{2}\}$ be an infinite plate of thickness $t > 0$. Then the following inequality holds:*

$$t \int\limits_{P} |\nabla u|^2 dx \le C \int\limits_{P} (|e(u)|^2 + |u|^2)dx, \quad u \in C_0^\infty(\mathbb{R}^3)^3, \tag{26}$$

where the constant C depends only on the diameter of supp u.

In the proof of this lemma, we use estimate (16) on the sections $x_1 = const$ and $x_2 = const$ and also the ordinary Korn inequalities for finite functions of two variables on the section $x_3 = const$.

From estimate (26), we immediately obtain inequality (25) if we take into account that the box structure consists of plates whose thickness is εh.

Now, our goal is to prove estimate (1). We start with auxiliary inequality on thin plate of thickness h.

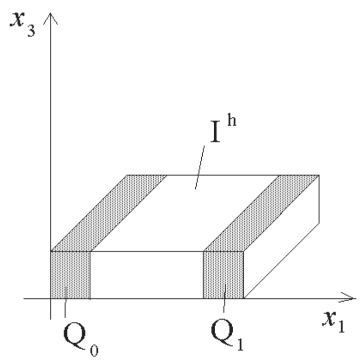

FIGURE 7.

Lemma 3.3. *Consider horizontal plate $I^h = [0,1]^2 \times [0,h]$ with rectangular beams Q_0 and Q_1 that adjoin to the end-walls $x_1 = 0$ and $x_1 = 1$ (see Fig. 7). Let C^0 and C^1 be cylinders of unit length inscribed in Q_0 and Q_1. Then the following inequality holds:*

$$h^2 \int\limits_{I^h} (|\frac{\partial}{\partial x_1} u_3|^2 + |\frac{\partial}{\partial x_3} u_3|^2) dx$$
$$\leq C \left(\int\limits_{I^h} |e(u)|^2 dx + h \sum_{i=0}^{1} \int\limits_{C^i} |u_3|^2 dx \right), \quad u \in C_0^\infty(I^h)^3. \tag{27}$$

For proof we make analysis in section $x_2 = t$ for two-dimensional vector (u_1, u_3), based on Lemma 2.5, and after that integrate over $t \in [0,1]$.

By homothety we deduce from (27) that

$$\varepsilon^2 \int\limits_{\varepsilon I^h} (|\frac{\partial}{\partial x_1} u_3|^2 + +|\frac{\partial}{\partial x_3} u_3|^2) dx \leq C \left(\left(\frac{\varepsilon}{h}\right)^2 \int\limits_{\varepsilon I^h} |e(u)|^2 dx + \frac{1}{h} \sum_{i=0}^{1} \int\limits_{\varepsilon C^i} |u_3|^2 dx \right). \tag{28}$$

Similarly, inequality (18) on the strip with circle yields the inequality on the plate εI^h with cylinder εC^0 considered in (28)

$$\int\limits_{\varepsilon I^h} \varphi^2 dx \leq C \left(\varepsilon^2 \int\limits_{\varepsilon I^h} (|\frac{\partial}{\partial x_1} \varphi|^2 + +|\frac{\partial}{\partial x_3} \varphi|^2) dx + \frac{1}{h} \int\limits_{\varepsilon C^0} \varphi^2 dx \right).$$

Hence, taking here $\varphi = u_3$, with the help of (28), we get

$$\int\limits_{\varepsilon I^h} |u_3|^2 dx \leq C \left(\left(\frac{\varepsilon}{h}\right)^2 \int\limits_{\varepsilon I^h} |e(u)|^2 dx + \frac{1}{h} \sum_{i=0}^{1} \int\limits_{\varepsilon C^i} |u_3|^2 dx \right). \tag{29}$$

We sum inequalities (29) over all horizontal fragments εI^h, then estimate the integrals over the cylinders εC^i by using an analogue of inequality (20), and finally

$$\sum_i \int\limits_{P_i} |u_3|^2 dx \leq C \left(\left(\frac{\varepsilon}{h}\right)^2 + 1 \right) \int\limits_{\Omega} |e(u)|^2 dx,$$

where the summation is performed over all horizontal plates P_i. The estimate of the longitudinal components u_1 and u_2 on the horizontal plates is trivial, since it follows from the Poincaré-Friedrichs inequality for a finite function of two variables x_1 and x_2. As a result, we have

$$\sum_i \int\limits_{P_i} |u|^2 dx \le C\left(\left(\frac{\varepsilon}{h}\right)^2 + 1\right) \int\limits_{\Omega} |e(u)|^2 dx.$$

A similar estimate for nonhorizontal plates leads to inequality (1).

Thus, Theorem 3.1 is proved.

Remark 2. Theorem 3.1 remains true for the box frame with only two families of parallel plates. Also, sloping plates may be included in the box frame. Further possible generalizations of its structure are omitted here.

4. **Spatial rod frames.**

4.1. Consider a rod frame F, 1-periodic in three directions, of the simplest form when F is composed of infinite straight lines. In Fig. 8, its image is presented. This frame can be complexified by adding to it some straight lines connecting the vertices of the unit cube. There are various possibilities for thickening of F to construct the frame F^h of thickness h. This can be done in the simplest way, first, from circular cylinders of radius $\frac{h}{2}$ whose axes coincide with the straight lines of the frame F, and, second, from beams that have a square with side h in the section. In Fig. 8, we present the simplest frame F^h composed of beams.

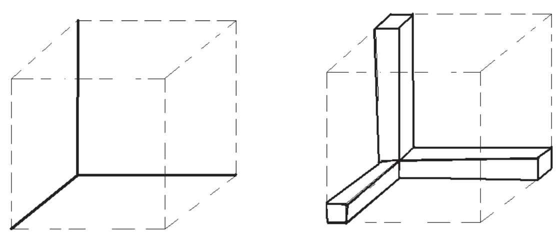

FIGURE 8.

Theorem 4.1. *For the rod frame $F^h_\varepsilon = \varepsilon F^h$, inequality (1), where the constant C_1 depends on the geometry of the singular structure F and the diameter of* supp u, *and inequality (2), where the constant C_2 depends only on the geometry of the singular structure F, hold.*

Next, we consider in details frames made from rectangular beams. Cylindrical frames with round (or even rather arbitrary) cross-section are studied similarly.

The proof of the estimate (1) starts with some auxiliary inequality on unit cube $\square = [0,1]^3$ which is verified similarly to (6).

Lemma 4.2. *The estimate*

$$\int\limits_{\square} |\nabla v|^2 dy \le C\left(\int\limits_{\square} |e(v)|^2 dy + \frac{1}{h}\sum_{i=0}^{1} \int\limits_{K^i} (v_2^2 + v_3^2) dy\right), \quad v \in C^\infty(\square)^3, \qquad (30)$$

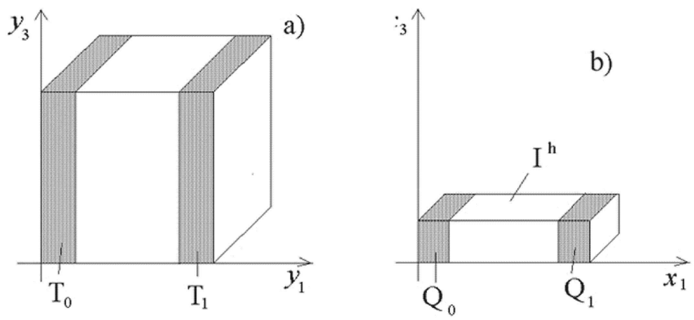

FIGURE 9.

holds. Here, K^0 and K^1 are ellipsoids inscribed into the parallelepipeds $T_0 = [0, h] \times [0, 1]^2$ and $T_1 = [1 - h, 1] \times [0, 1]^2$, respectively (see Fig. 9a).

As a corollary we obtain

Lemma 4.3. *Let $I^h = [0, 1] \times [0, h]^2$ be a horizontal beam, and let B^0 and B^1 be the balls of radius $\frac{h}{2}$ inscribed in it and, adjoining the wall-ends $x_1 = 0$, $x_1 = 1$. Then for the transverse components u_2 and u_3 of the vector-valued function u, the following inequality holds:*

$$h^2 \int_{I^h} |\nabla u_j|^2 dx$$

$$\leq C \left(\int_{I^h} |e(u)|^2 dx + h \sum_{i=0}^{1} \int_{B^i} (u_2^2 + u_3^2) dx \right), \quad j = 2, 3, \quad u \in C^\infty(I^h)^3. \tag{31}$$

Proof. In inequality (30), we make the following change of variables:

$$x_1 = y_1, \quad x_2 = hy_2, \quad x_3 = hy_3; \quad v_1 = u_1, \quad v_2 = hu_2, \quad v_3 = hu_3.$$

This transforms the cube $\square = [0, 1]^3$ into the beam I^h, the plates T_i into cubes Q_i with edge h, and the ellipsoids K^i inscribed in parallelepipeds T_i into balls B^i inscribed in Q_i (see Fig. 9a,b), $i = 0, 1$. For the function u, the inequality on the beam I^h appears from which estimate (31) follows. □

By homothety, we pass from (31) to the estimate on the beam $\varepsilon I^h = [0, \varepsilon] \times [0, \varepsilon h]^2$, i.e.,

$$\varepsilon^2 \int_{\varepsilon I^h} |\nabla u_j|^2 dx \leq C \left(\left(\frac{\varepsilon}{h}\right)^2 \int_{\varepsilon I^h} |e(u)|^2 dx + \frac{1}{h} \sum_{i=0}^{1} \int_{\varepsilon B^i} (u_2^2 + u_3^2) dx \right), \quad j = 2, 3. \tag{32}$$

From (32) and inequalities similar to it on the beams εI^h of general position, not necessarily horizontal, we derive estimate (1) for the rod frame F_ε^h. For this purpose, we partition F_ε^h into overlapping fragments, i.e., the beams containing a common ball in the intersection. Here, we use the same ideas as in Sec. 2 in the proof of Theorem 2.1.

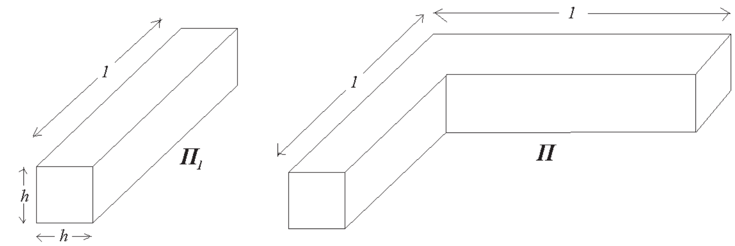

FIGURE 10.

In deriving estimate (2), a key role is played by an estimate on the fragment of the rod structure F^h, which is a pair of rods joint under the angle. For example, consider the "angle piece" Π (see Fig. 10) composed of two perpendicular beams Π_1 and Π_2 having unit length and the square with side h in the section: $\Pi_1 = [0,1] \times [-\frac{h}{2}, \frac{h}{2}]^2$ and $\Pi_2 = [-\frac{h}{2}, \frac{h}{2}] \times [0,1] \times [-\frac{h}{2}, \frac{h}{2}]$.

Lemma 4.4. *The following inequality holds:*

$$h^2 \int_{\Pi} |\nabla u|^2 dx \leq C \int_{\Pi} (|e(u)|^2 + h^2|u|^2) dx, \quad u \in C^\infty(\Pi)^3. \tag{33}$$

The proof of this lemma is given in Section 4.2.

We pass to the direct proof of inequality (2) for the rod frame F^h_ε. We cut the frame F^h_ε into fragments of the type $\varepsilon\Pi$. From estimate (33), using homothety we obtain the estimate on one such fragment, i.e.,

$$\varepsilon^2 \int_{\varepsilon\Pi} |\nabla u|^2 dx \leq C \int_{\varepsilon\Pi} \left(\left(\frac{\varepsilon}{h}\right)^2 |e(u)|^2 + |u|^2 \right) dx.$$

Summing this estimate over all fragments intersecting $\mathsf{supp}\ \mathsf{u}$, we come to inequality (2).

4.2. In this subsection, we prove Lemma 4.4. Let x_i be a longitudinal variable on the beam Π_i, $i = 1, 2$, so that, for example, $\Pi_1 = [0,1] \times [-h/2, h/2]^2$. Denote

$$|\nabla_{i,j} u|^2 \equiv \left|\frac{\partial u_i}{\partial x_j}\right|^2 + \left|\frac{\partial u_j}{\partial x_i}\right|^2.$$

Applying estimate (12) in the longitudinal sections $x_i = a$, $i = 2, 3$, $|a| < \frac{h}{2}$, of the beam Π_1 (see Fig. 10), we find that

$$h^2 \int_{\Pi_1} (|\nabla_{1,2} u|^2 + |\nabla_{1,3} u|^2) dx \leq C \int_{\Pi_1} [|e(u)|^2 + h^2(u_2^2 + u_3^2)] dx. \tag{34}$$

In a similar way, considering the longitudinal sections of the beam Π_2, we come to the relation

$$h^2 \int_{\Pi_2} (|\nabla_{1,2} u|^2 + |\nabla_{2,3} u|^2) dx \leq C \int_{\Pi_2} [|e(u)|^2 + h^2(u_1^2 + u_3^2)] dx. \tag{35}$$

We need the following inequality

$$\int_\Pi |\nabla u|^2 dx \le C\left(\int_\Pi |e(u)|^2 dx + \int_{\Pi'} |\nabla u|^2 dx\right), \quad u \in C^\infty(\Pi)^3, \tag{36}$$

where Π' is the construction of the "piece angle" type similar to Π composed of the beams Π_1' and Π_2', which are similar to Π_1 and Π_2 that are coaxial to them and have the length $1-h/2$ and the squares with side $h/2$ in the cross section. For example, one of the beams is $\Pi_1' = [h/4, 1-h/4] \times [-h/4, h/4]^2$. Note that the points of the set Π' are distant from the boundary $\partial\Pi$ no less than by $h/4$. To prove inequality (36) we partition Π into cubes $\Box_h$ with edge h, then take in each cube ball B_h of radius $h/4$ centered at the center of $\Box_h$ and write the uniform in h inequality

$$\|\nabla u\|_{L^2(\Box_h)} \le C(\|e(u)\|_{L^2(\Box_h)} + \|\nabla u\|_{L^2(B_h)}).$$

By summation we come to (36), see also [8] - [10].

As for (33), it remains (see (34)-(36)) to deduce

$$h^2 \int_{\Pi_1'} |\nabla_{2,3} u|^2 dx + h^2 \int_{\Pi_2'} |\nabla_{1,3} u|^2 dx \le C \int_\Pi [|e(u)|^2 + h^2|u|^2] dx, \quad u \in C^\infty(\Pi)^3. \tag{37}$$

For this purpose, we use the ideas of [8] (see also [10, Chapter 3, Sec. 1], [11, Chapter I, Sec. 2]).

At first we remark that the identity

$$\frac{\partial^2 w_i}{\partial x_j \partial x_p} = \frac{\partial}{\partial x_j} e_{ip}(w) + \frac{\partial}{\partial x_p} e_{ij}(w) - \frac{\partial}{\partial x_i} e_{pj}(w) \tag{38}$$

implies

$$\frac{\partial^2 u_i}{\partial x_j^2} = 2\frac{\partial}{\partial x_j} e_{ij}(u) - \frac{\partial}{\partial x_i} e_{jj}(u) \equiv F_j^i(u).$$

Decompose u into the sum $u = v + w$, where each component of the vector v is a variational solution of the following Dirichlet problem:

$$v_i \in H_0^1(\Pi), \quad \Delta v_i = \sum_{j=1}^3 F_j^i(u).$$

From integral identity for v_i, we derive the estimate

$$\|v_i\|_{H^1(\Pi)} \le C \tag{39}$$

where C is an absolute constant. Clearly, the vector $w = u - v$ has harmonic components $w_i \in C^\infty(\Pi) \cap H^1(\Pi)$:

$$\Delta w_i = 0 \quad \Longrightarrow \Delta e_{ij}(w) = 0 \text{ in } \Pi, \quad i, j = 1, 2, 3,$$

and due to (4.10),

$$\|e(w)\|_{L^2(\Pi)} \le C\|e(u)\|_{L^2(\Pi)}.$$

Recall some auxiliary result.

Let Π be an arbitrary bounded domain with Lipschitz boundary $\partial\Pi$ and let $\rho(x)$ be the distance function for $\partial\Pi$.

Lemma 4.5. ***(see** [8], [11]**)*** *Suppose that $v \in C^\infty(\Pi) \cap L^2(\Pi)$ is harmonic in Π, then*

$$\|\rho\nabla v\|_{L^2(\Pi)} \le C\|v\|_{L^2(\Pi)}, \quad C = const(\Pi).$$

Thus by lemma 4.5 we conclude in our situation that

$$\int_{\Pi} \rho^2 |\nabla e(w)|^2 dx \le C_1 \int_{\Pi} |e(w)|^2 dx \le C_2 \int_{\Pi} |e(u)|^2 dx,$$

where the constants are independent of h. Consequently (see also (38)),

$$\sum_{i,p} \int_{\Pi} \left| \rho \frac{\partial^2 w}{\partial x_i \partial x_p} \right|^2 dx \le C \int_{\Pi} |e(u)|^2 dx,$$

whence, in view of the choice of Π', it follows that

$$\sum_{i,p} h^2 \int_{\Pi'} |\frac{\partial^2 w}{\partial x_i \partial x_p}|^2 dx \le C \int_{\Pi} |e(u)|^2 dx. \tag{40}$$

By virtue of expansion $u = v + w$ with account for estimate (39), we can assume that the integrals with the function w stand in the left-hand side of the estimate (37) being proved.

Similarly, estimates (34) and (35) imply

$$\begin{aligned} h^2 \int_{\Pi_1'} |\nabla_{1,3} w|^2 dx \le C \int_{\Pi} (|e(u)|^2 + h^2 |u|^2) dx, \\ h^2 \int_{\Pi_2'} |\nabla_{2,3} w|^2 dx \le C \int_{\Pi} (|e(u)|^2 + h^2 |u|^2) dx. \end{aligned} \tag{41}$$

We are to extend each of the estimates (41) to the complementary part of Π_2' or Π_1' from Π'. To this end, the following inequalities are useful:

$$\int_{\Pi_i'} \varphi^2 dx \le C_1 \left(\int_{\Pi_i'} |\nabla \varphi|^2 dx + \frac{1}{h} \int_{Q} \varphi^2 dx \right), \tag{42}$$

$$\frac{1}{h} \int_{Q} \varphi^2 dx \le C_2 \int_{\Pi_i'} \left(\varphi^2 + \left| \frac{\partial \varphi}{\partial t} \right|^2 \right) dx, \quad i = 1, 2. \tag{43}$$

Here t is the longitudinal coordinate on Π_i, $i = 1, 2$, and Q is either a cube with edge $h/2$ or a ball of radius $h/2$, which lie in $\Pi_1' \cap \Pi_2'$. For example, we arrange the "transmission" of estimate (41_2) for $|\nabla_{2,3} w|^2$ to the part Π_1'. As φ, we take the derivatives $\frac{\partial w_2}{x_3}$ and $\frac{\partial w_3}{x_2}$.

Apply (42) first with $i = 1$ and then (43) with $i = 2$ to φ to find

$$h^2 \int_{\Pi_1'} |\nabla_{2,3} w|^2 dx \le C h^2 \left[\int_{\Pi_1'} \left(\left| \nabla \frac{\partial w_2}{\partial x_3} \right|^2 + \left| \nabla \frac{\partial w_3}{\partial x_2} \right|^2 \right) dx \right.$$

$$\left. + \frac{1}{h} \int_{Q} |\nabla_{2,3} w|^2 dx \right] \le C_1 h^2 \left(\int_{\Pi'} \left(\left| \nabla \frac{\partial w_2}{\partial x_3} \right|^2 + \left| \nabla \frac{\partial w_3}{\partial x_2} \right|^2 \right) dx + \int_{\Pi_2'} |\nabla_{2,3} w|^2 dx \right)$$

$$\le C \int_{\Pi} (|e(u)|^2 + h^2 |u|^2) dx,$$

where, at the final stage, estimates (40) and (41_2) are used.

Since $u = v + w$, the latter estimate and (39) imply

$$h^2 \int\limits_{\Pi_1'} |\nabla_{2,3} u|^2 dx \leq C \int\limits_{\Pi} (e(u)|^2 + h^2|u|^2) dx.$$

Therefore, a part of estimate (37) is checked. We verify its other part in the same way. Thus, Lemma 4.2 is proved.

4.3. Theorem 4.1 is true for a class of frames, which is wider than the class introduced in Sec. 4.1. It is seen from the proof that the estimate (2) holds for the frame F_ε^h if the corresponding singular frame F can be cut into "angular pieces" from two noncollinear segments. This requirement is fulfilled for the frame depicted in Fig. 11. Through each its node, two piercing straight lines pass, but not three as previously. Nevertheless, for this frame, estimate (1) can also be proved.

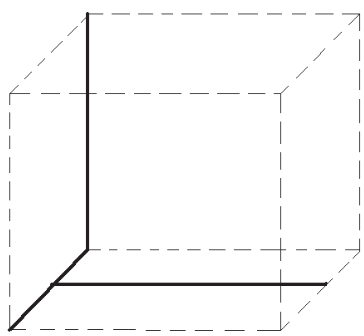

FIGURE 11.

We can introduce regular three-dimensional rod frames in analogy with the regular planar grids through some labelling procedure, assuming that the label 0 is ascribed to the nodes where three piercing straight lines meet. Then the frame depicted in Fig. 11, obviously, will not be regular in this sense. So the concept of regular rod structure should be revised in 3-D case. But here we omit suitable extensions of this notion, and demonstrate only the above example.

The questions pertaining to the Korn inequality on three-dimensional rod frames were also considered in [12, 13, 1].

5. **Trusses and constructions similar to them.**

5.1. By a planar 1-periodic singular truss F, we mean the set on the plane, 1-periodic in the variable x_1, composed, first, of several horizontal straight lines defined by the equation $x_2 = a$, $a \in [0, 1]$, among which there necessarily are boundary straight lines $x_2 = 0$, $x_2 = 1$, and, second, of the infinitely many nonhorizontal segments whose endpoints lie on the boundary straight lines (see Fig. 12).

In a natural way, starting from F, we define first the 1-periodic truss F^h of thickness h (see Sec. 2 devoted to grids) and then the ε-periodic truss $F_\varepsilon^h = \varepsilon F^h$ with the thickness of constituents equal to εh.

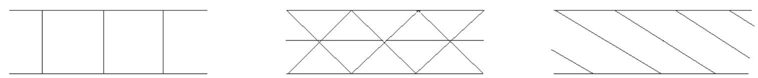

FIGURE 12.

Theorem 5.1. *The following weight inequality holds:*

$$\left(1+\left(\frac{\varepsilon}{h}\right)^2\right)^{-1}\int\limits_{F_\varepsilon^h}(u_1^2+\varepsilon^2u_2^2+\varepsilon^2|\nabla u|^2)dx \le C\int\limits_{F_\varepsilon^h}|e(u)|^2dx, \quad u\in C_0^\infty(\mathbb{R}^2)^2, \tag{44}$$

where the constant C depends only on the geometry of the singular structure F and the diameter of supp *u*.

Corollary 1. *If a truss F_ε^h is sufficiently thick or has a critical thickness, then*

$$\int\limits_{F_\varepsilon^h}(u_1^2+\varepsilon^2u_2^2+\varepsilon^2|\nabla u|^2)dx \le C\int\limits_{F_\varepsilon^h}|e(u)|^2dx; \tag{45}$$

if the truss F_ε^h is sufficiently thin, then

$$\int\limits_{F_\varepsilon^h}\left[\left(\frac{h}{\varepsilon}\right)^2u_1^2+h^2u_2^2+h^2|\nabla u|^2\right]dx \le C\int\limits_{F_\varepsilon^h}|e(u)|^2dx. \tag{46}$$

We see that inequality (45) (in contrast to inequality (46)) coincides in its form with the Korn inequality for the strip of width ε.

In the proof of Theorem 5.1, an important role is played by the following lemma.

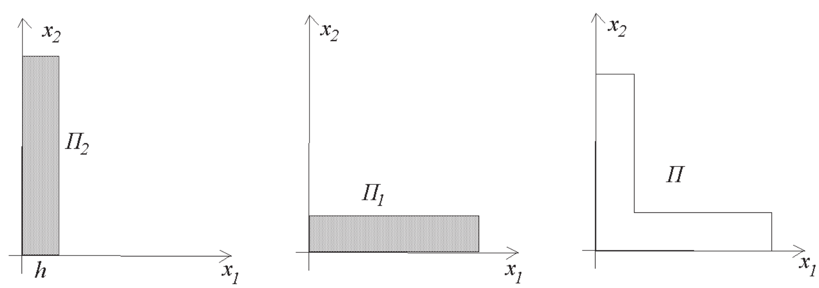

FIGURE 13.

Lemma 5.2. *Let* Π *be an "angle piece" composed of two rectangular strips* Π_1 *and* Π_2 *(see Fig. 13). Then the following estimate holds:*

$$h^2\int\limits_\Pi|\nabla u|^2dx \le C\left(\int\limits_\Pi|e(u)|^2dx+h^2\int\limits_{\Pi_2}u_1^2dx\right), \quad u\in C^\infty(\Pi)^2. \tag{47}$$

In a similar way, we can obtain anisotropic weighted Korn inequality for different thin spatial truss-type structures. We mean the rod truss, which is periodic in the variable x_1, and the structures, periodic in two variables x_1 and x_2, composed of rods or plates.

5.2. Here Lemma 5.2 is proved. Let $\Pi_1=[0,1]\times[0,h]$, $\Pi_2=[0,h]\times[0,1]$ and use the method for proving Lemma 4.2. In the domain Π, we find the expansion $u=v+w$ with the terms v and w such that

$$\|v\|^2_{H^1(\Pi)^3} \le C_1 \int\limits_{\Pi} |e(u)|^2 dx, \tag{48}$$

$$h^2 \int\limits_{\Pi'} |\nabla^2 w|^2 dx \le C_2 \int\limits_{\Pi} |e(u)|^2 dx. \tag{49}$$

The “angle piece” $\Pi' = \Pi'_1 \cup \Pi'_2$ is composed of the strips that are smaller than Π_1 and Π_2:

$$\Pi'_1 = [\frac{h}{4}, 1 - \frac{h}{4}] \times [\frac{h}{4}, \frac{3}{4}h], \quad \Pi'_2 = [\frac{h}{4}, \frac{3}{4}h] \times [\frac{h}{4}, 1 - \frac{h}{4}].$$

For the strip Π_2, we write the estimate (see (12))

$$h^2 \int\limits_{\Pi_2} |\nabla u|^2 dx \le C \int\limits_{\Pi_2} (|e(u)|^2 + h^2 u_1^2) dx,$$

which, see also (48), implies

$$h^2 \int\limits_{\Pi'_2} |\nabla w|^2 dx \le C \left(\int\limits_{\Pi} |e(u)|^2 dx + \int\limits_{\Pi_2} h^2 u_1^2 dx \right). \tag{50}$$

Add to this the inequality (its analog is (36))

$$\int\limits_{\Pi} |\nabla u|^2 dx \le C \left(\int\limits_{\Pi} |e(u)|^2 dx + \int\limits_{\Pi'} |\nabla u|^2 dx \right).$$

Thus, it remains to show

$$h^2 \int\limits_{\Pi'_1} |\nabla w|^2 dx \le C \left(\int\limits_{\Pi} |e(u)|^2 dx + \int\limits_{\Pi_2} h^2 u_1^2 dx \right).$$

For this purpose, we use the procedure of “transmission” of estimates from the set Π'_2 on the set Π'_1 through the circle Q of radius $\frac{h}{4}$, which is common for these sets (see the proof of Lemma 4.4), i.e.,

$$h^2 \int\limits_{\Pi'_1} |\nabla w|^2 dx \le h^2 C \left(\frac{1}{h} \int\limits_{Q} |\nabla w|^2 dx + \int\limits_{\Pi'_1} |\nabla^2 w|^2 dx \right)$$

$$\le h^2 C \left(\int\limits_{\Pi'_2} |\nabla w|^2 dx + \int\limits_{\Pi'} |\nabla^2 w|^2 dx \right) \le C \left(\int\limits_{\Pi} |e(u)|^2 dx + \int\limits_{\Pi_2} h^2 u_1^2 dx \right),$$

where the estimates (49) and (50) are applied at the final stage.

5.3. Here we prove Lemma 5.1 and, for the sake of clarity, consider the square truss F_ε^h (the corresponding singular frame F is the first in Fig. 12). In this case (see Fig. 14), the truss F_ε^h is composed of two infinite horizontal strips P_0 and P_1 of width εh and also of short vertical strips S_i whose length is equal to ε and width to εh. To the end-walls of each strip $S_i = S$, the circles B^j of radius $\frac{\varepsilon h}{2}$ adjoin.

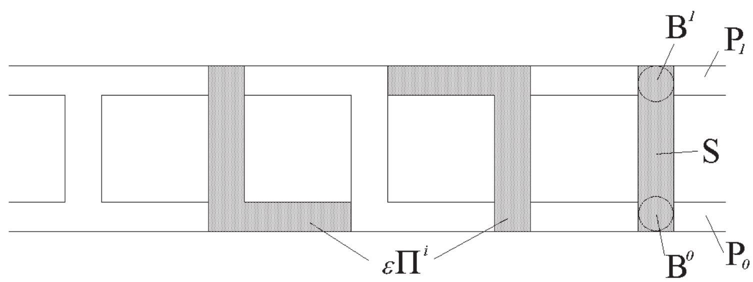

Figure 14.

First we estimate the longitudinal (relative to the whole truss) component u_1 of the vector-valued function u. From Sec. 2 we take the following inequalities for the finite vector function u:

$$\int\limits_{P_i} u_1^2 dx \le C \int\limits_{P_i} |e(u)|^2 dx, \quad i = 0, 1; \qquad \int\limits_{B^i} u_1^2 dx \le C\varepsilon h \int\limits_{P_i} |e(u)|^2 dx, \quad i = 0, 1;$$

$$\int\limits_{S} u_1^2 dx \le C \left(\varepsilon^2 \int\limits_{S} |\nabla u_1|^2 dx + \frac{1}{h} \int\limits_{B^0} u_1^2 dx \right);$$

$$\varepsilon^2 \int\limits_{S} |\nabla u_1|^2 dx \le C \left(\left(\frac{\varepsilon}{h}\right)^2 \int\limits_{S} |e(u)|^2 dx + \frac{1}{h} \sum_{j=0}^{1} \int\limits_{B^j} u_1^2 dx \right).$$

By summation, in accordance with the representation $F_\varepsilon^h = P_0 \cup P_1 \cup (\bigcup\limits_i S_i)$, we obtain

$$\int\limits_{F_\varepsilon^h} u_1^2 dx \le C \left(1 + \left(\frac{\varepsilon}{h}\right)^2 \right) \int\limits_{F_\varepsilon^h} |e(u)|^2 dx. \tag{51}$$

Next try to estimate the transverse (relative to the whole truss) component u_2 of the vector u. The following inequalities hold (see Sec. 2):

$$\int\limits_{P_i} u_2^2 dx \le C \int\limits_{P_i} \left| \frac{\partial u_2}{\partial x_1} \right|^2 dx, \quad i = 0, 1;$$

$$\int\limits_{S} u_2^2 dx \le C(\varepsilon^2 \int\limits_{S} |\nabla u_2|^2 dx + \frac{1}{h} \int\limits_{B^0} u_2^2 dx); \quad \frac{1}{h} \int\limits_{B^0} u_2^2 dx \le C\varepsilon \int\limits_{P_0 \cup P_1} \left| \frac{\partial u_2}{\partial x_1} \right|^2 dx.$$

By summation, we obtain

$$\int\limits_{F_\varepsilon^h} u_2^2 dx \le C \int\limits_{F_\varepsilon^h} |\nabla u_2|^2 dx. \tag{52}$$

The estimate of the component u_2 thereby reduces to the estimate of its gradient.

Now estimate the total gradient ∇u. For this purpose, partition the truss F_ε^h into fragments $\varepsilon\Pi^i$ of the "angle piece" type Π from Lemma 5.2 (see Fig. 14), i.e., $F_\varepsilon^h = \bigcup_i \varepsilon\Pi^i$. For a separate contracted "angle piece" $\varepsilon\Pi = \varepsilon\Pi^i$ containing the vertical strip S we have (see (47)):

$$\varepsilon^2 \int\limits_{\varepsilon\Pi} |\nabla u|^2 dx \leq C \left(\left(\frac{\varepsilon}{h}\right)^2 \int\limits_{\varepsilon\Pi} |e(u)|^2 dx + \int\limits_S u_1^2 dx \right),$$

whence, by summation over all fragments $\varepsilon\Pi^i$ and taking into account the estimate for the component u_1 (see (51)), we get

$$\varepsilon^2 \int\limits_{F_\varepsilon^h} |\nabla u|^2 dx \leq C \left(1 + \left(\frac{\varepsilon}{h}\right)^2 \right) \int\limits_{F_\varepsilon^h} |e(u)|^2 dx. \tag{53}$$

From (51)-(53), estimate (44) follows. Theorem 5.1 is proved.

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