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HOMOGENIZATION OF INTERFACIAL ENERGIES AND CONSTRUCTION OF PLANE-LIKE MINIMIZERS IN PERIODIC MEDIA THROUGH A CELL PROBLEM

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Abstract. We consider the homogenization of a periodic interfacial energy, such as considered in recents papers by Caffarelli and De La Llave [14], or Dirr, Lucia and Novaga [16]. In particular, we include the case where an external forcing field (which is unbounded in the limit) is present, and suggest two different ways to take care of this additional perturbation. We provide a proof of a Γ-limit, however, we also observe that thanks to the coarea formula, in many cases such a result is already known in the framework of BV homogenization. This leads to an interesting new construction for the plane-like minimizers in periodic media of Caffarelli and De La Llave, through a cell problem.

1. **Introduction.** In this paper, we consider the homogenization of a periodic interfacial energy, with an external forcing term, such as studied in a recent paper of L. Caffarelli and R. De La Llave [14]. We will show that after appropriate rescaling into ε-periodic energies, and sending ε to zero, we get convergence to an anisotropic perimeter (in the sense of Γ-convergence), with an interfacial energy simply characterized by the energies of plane-like minimizers in balls of large volume. In [16], a similar study has been performed, however there the perimeter itself is replaced with a two-phase singular perturbation problem (as in the seminal papers of Modica and Mortola [20] [21]), with some parameter $\delta > 0$ representing the width of the interface. Then, δ and ε are sent simultaneously to zero, however, also the ratio $\delta/\varepsilon \to 0$ so that in spirit the problem is the same as ours, and the limit is of course the same. See also [13], and the recent paper [22] which treats the (probably more complicated) case of interfacial energies which degenerate in the limit.

We provide here a direct proof of this homogenization result. It is quite standard (see [3] for a first result of this kind), and based on the standard blow-up approach which has recently be nicely reviewed in [12]. It turns out, also, that in most cases this proof is "useless" (and probably in all cases), in the sense that thanks to the

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coarea formula for BV functions, our problem can be cast into a more standard homogenization problem in the space of functions with bounded variation [6] [1] [11]. An interesting point, though, is the fact that the interfacial energy in both point of views is not given by the same formula: so that we deduce an equality between two problems, which is at first glance not completely obvious (however this identity is already observed, in some cases and up to minor changes, by Braides and Chiadò Piat in [11]).

Another interesting consequence is that we can use the cell problem in [6], [1] and [11] in order to derive a new proof of Caffarelli and De La Llave's result, with a quite different construction. This approach can probably be generalized to other, similar problems where quasi-periodic sets satisfying some kind of maximum principle are to be built (such as minimizers for nonlocal or degenerate perimeters, which are restrictions to sets of a functional satisfying some co-area formula, see for instance [15]), although, in some sense, the situation here is among the simplest.

In what follows, $Q = [0,1)^d$, and by $Q^\sharp$ we denote the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$. Functions or measures over $Q^\sharp$ will implicitly be identified with Q-periodic functions or measures in $\mathbb{R}^d$ (some care though has to be taken with periodic measures which weigh ∂Q). We consider here $g \in L^d(Q^\sharp)$ with $\int_Q g = 0$, and $F(x,p) : Q^\sharp \times \mathbb{R}^d \to [0,+\infty)$, continuous (periodic) in x, convex and one-homogeneous in p, with

$$c_*|p| \;\le\; F(x,p) \;\le c^*|p| \tag{1}$$

for any p, for some positive constants c_*, c^*.

We assume the existence of $\delta > 0$ such that for any $E \subset Q$ with finite perimeter,[1]

$$\mathcal{J}_Q(E) := \int_{Q\cap\partial^*E} F(x,\nu_E(x))\,d\mathcal{H}^{d-1}(x) + \int_{Q\cap E} g(x)\,dx \;\ge\; \delta Per(E,Q)\,, \tag{2}$$

where here and in the whole paper, ν_E is the inner normal to ∂^*E. Here $Per(E,Q) = \mathcal{H}^{d-1}(\partial^*E \cap Q)$ is the part of the perimeter of E inside Q, including the possible trace on $\partial Q \cap Q$. The inequality (2) holds (as observed in [16]) for instance if $\|g\|_d = \|g\|_{L^d(Q)}$ is small enough, indeed, we have in this case

$$\begin{aligned}\int_{Q\cap E} g(x)\,dx \;=\; -\int_{Q\setminus E} g(x)\,dx \\ \le\; \|g\|_d \min\{|Q\cap E|, |Q\setminus E|\}^{\frac{d-1}{d}} \;\le\; C\|g\|_d Per(E,Q)\end{aligned} \tag{3}$$

for some constant C depending only on the dimension (see for instance [4]), hence as soon as $\|g\|_d < c_*/C$ we can find $\delta > 0$ such that (2) holds.

Let us observe that a quite deep result of Bourgain and Brézis [9] [10] shows that if $g \in L^d(Q)$, there is a vector field $\sigma \in C^0(Q^\sharp, \mathbb{R}^d)$ (we can assume moreover that $\sigma = 0$ on ∂Q) with $\operatorname{div}\sigma = g$, hence

$$\begin{aligned}\int_{Q\cap\partial^*E} F(x,\nu_E(x))\,d\mathcal{H}^{d-1}(x) + \int_{Q\cap E} g(x)\,dx \\ = \int_{Q\cap\partial^*E} F(x,\nu_E(x)) - \sigma(x)\cdot\nu_E(x)\,d\mathcal{H}^{d-1}(x).\end{aligned}$$

We see that letting $F'(x,p) := F(x,p) - \sigma(x)\cdot p$, we can get rid of the external field g (and F' will satisfy (1) if $\|g\|_d$ is small enough). We discuss this in detail in

[1] We refer for instance to [19] and [18] for the definition and properties of sets of finite perimeter (a.k.a. Caccioppoli sets), and of their reduced boundary ∂^*E.

Section 4: in fact, we actually show that (2) yields the existence of such a σ. We also show that (2) can be a bit weakened, thanks to the results in [9] and [10].

We consider in this paper a first problem, quite standard, which regards the Γ-limit of the energies

$$\mathcal{E}_\varepsilon(E) \;=\; \int_{\partial^* E\cap\Omega} F\left(\frac{x}{\varepsilon},\nu_E(x)\right)\,d\mathcal{H}^{d-1}(x)+\frac{1}{\varepsilon}\int_{\Omega_\varepsilon\cap E} g\left(\frac{x}{\varepsilon}\right)\,dx\,, \tag{4}$$

defined on finite perimeter subsets $E\subset\Omega$ where Ω is a bounded open subset of $\mathbb{R}^d$, with Lipschitz boundary. Here Ω_ε is the union of all cubes $\varepsilon(k+Q)$, $k\in\mathbb{Z}^d$, which are contained in Ω. Considering also the integral of g over $\Omega\setminus\Omega_\varepsilon$ would produce annoying boundary effects.

The result we show (which is not new, see [16] where a similar issue is addressed in the framework of a singular perturbation problem, and the discussion below, but we give a direct proof for the reader's convenience) relies on a theorem of L. Caffarelli and R. De La Llave [14], that we now quote. Consider the functional

$$\mathcal{J}(E) \;=\; \int_{\partial^* E} F(x,\nu_E)\,d\mathcal{H}^{d-1}(x) \;+\; \int_E g(x)\,dx \tag{5}$$

(which is a priori finite only for sets E with compact boundary). Following [14] we introduce the following definition of a global minimizer in $\mathbb{R}^d$:

Definition 1.1. We say that $E\subset\mathbb{R}^d$ with locally finite perimeter is a class A minimizer for $\mathcal{J}$ if for any bounded set $B\subset\mathbb{R}^d$ and any $E'\subset\mathbb{R}^d$ with $E\triangle E'=(E\setminus E')\cup(E'\setminus E)\Subset B$, we have

$$\int_{B\cap\partial^* E} F(x,\nu_E)\,d\mathcal{H}^{d-1}(x) \;+\; \int_{B\cap E} g(x)\,dx \;\le\; \int_{B\cap\partial^* E'} F(x,\nu_{E'})\,d\mathcal{H}^{d-1}(x) \;+\; \int_{B\cap E'} g(x)\,dx\,.$$

The theorem of Caffarelli and De La Llave [14, Thm 4.1] is as follows.

Theorem 1.2. *For any $\nu\in\mathbb{R}^d\setminus\{0\}$, we can find a connected set E_ν (depending only on $\nu/|\nu|$) such that*

(i) For some M independent of ν, depending only on c_*, c^* and g, we have

$$\begin{aligned}\partial E_\nu &\subset \left\{x\in\mathbb{R}^d\,:\,|x\cdot\nu|\le M|\nu|\right\}\,,\\ E_\nu &\supset \left\{x\in\mathbb{R}^d\,:\,x\cdot\nu\ge M|\nu|\right\}\,,\\ E_\nu &\subset \left\{x\in\mathbb{R}^d\,:\,x\cdot\nu\ge -M|\nu|\right\}\,.\end{aligned}$$

(ii) E_ν is a class A minimizer for $\mathcal{J}$.

(iii) ∂E_ν is "quasi-periodic".

(For practical reasons we choose here to have ν pointing towards the interior of the set E_ν rather than the exterior.) The point (*iv*) of Theorem 4.1 in [14], which claims that the projection of ∂E_ν onto $Q^\sharp$ laminates the torus, also follows from the new proof (quite different from Caffarelli and De La Llave's—though relying essentially on the same properties) which we will give in Section 3, however we did not investigate further this point. In general, we do not expect this lamination to be a foliation (i.e., to be dense in the torus), see Remark 3.6 below. This is clearly not the case, for instance, when the direction is "rational", that is, if $\nu=p/|p|$ for some $p\in\mathbb{Z}^d$, since in that case the set E_ν can be shown to be periodic, which improves statement (*iii*). When the direction is not rational, the set is "quasi-periodic" in the following sense: for any integer p with $p \cdot \nu > 0$, then $E_\nu + p \subset E_\nu$, whereas if $p \cdot \nu < 0$, $E_\nu + p \supset E_\nu$. If p_n is a sequence of integer vectors with $p_n \cdot \nu \to 0$, then $E_\nu + p_n$ converges (locally in L^1) to E_ν. These statements are true provided E_ν is in minimal or maximal in some sense, we will not discuss this issue in this paper anymore since the proofs would be the same as in [14].

A fundamental point in this result is the fact that M is independent on the direction: letting $I_\nu = \{x \in \mathbb{R}^d : x \cdot \nu > 0\}$, the theorem provides given any direction $\nu \in \mathbb{S}^{d-1}$ a minimizer E_ν such that the Hausdorff distance between the surfaces ∂E_ν and $\partial I_\nu = \{x \cdot \nu = 0\}$ is bounded by the uniform bound M.

Another important result in [14] is Proposition 10.1 (and Equation (10.2)) which states that for any $\nu \in \mathbb{S}^{d-1}$, the limit

$$\phi(\nu) \;=\; \lim_{L\to\infty} \frac{1}{\omega_{d-1}L^{d-1}} \left(\int_{B(0,L)\cap\partial E_\nu} F(x,\nu_{E_\nu})\, d\mathcal{H}^{d-1} + \int_{B(0,L)_1\cap E_\nu} g(x)\, dx \right) \tag{6}$$

exists and defines, after one-homogeneous extension, a convex function in $\mathbb{R}^d$. Here ω_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$, and $B(0,L)_1 = \bigcup\{z + Q : z \in \mathbb{Z}^N, z + Q \subset B(0,L)\}$ so that g is integrated only on "complete" cells. The result, in our case, needs be a bit more precise, see Appendix A.

Using these results, we show in Section 2 the Γ-convergence of the energies $\mathcal{E}_\varepsilon$ of (4), as $\varepsilon \to 0$, to the anisotropic perimeter

$$\mathcal{E}(E) \;=\; \int_{\partial^* E} \phi(\nu_E(x))\, d\mathcal{H}^{d-1}(x)\,. \tag{7}$$

defined for any finite-perimeter set $E \subset \Omega$.

When $g = 0$, this boils down to an old, pioneering homogenization result of Ambrosio and Braides [3] (see also [2]), for "almost periodic" interfacial energies. The representation formula there (Eq. (4.2) and Thm. 4.2 in [3]) slightly differs from (6), since these results also apply in (nonconvex or almost periodic) situations where the plane-like minimizers need not exist (besides, their existence was not known at that time), but it can easily shown to be equivalent.

Using the coarea formula for functions with bounded variation [18] [4], it is easy to relate this Γ-convergence to more classical results on the homogenization of functionals with growth 1 (see [1] and [11]), for which the limit density ϕ is known to be given by a cell problem. This provides two independent proofs for the same result, but the interesting point is that it also gives two different representations of the limit (which of course must be equal). A second interesting point is that the cell problem for the functional $\mathcal{J}$ is solved by a function whose level sets are the same as the minimizers provided by Theorem 1.2. We give, in Section 3, a detailed proof of this fact, as a consequence we obtain a new proof of Theorem 1.2. This proof is quite different from the original one in [14] even if it shares some common steps. It might be not simpler, but we believe it has its own interest.

Eventually, in Section 4, we discuss the possibility of integrating out the external field g in the surface tension F, which yields another possible approach to reduce the problem to already well-known situations, and show that the results in this paper still hold under coercivity assumptions that are slightly milder than (2).

2. **Homogenization of the interfacial energy.**

Our goal in this section is to show the following. We assume here that the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{E}$ are extended to all Borel sets in Ω by letting $\mathcal{E}_\varepsilon(E) = \mathcal{E}(E) = +\infty$ if E does not have finite perimeter. It will be also convenient to introduce the "localized" version of $\mathcal{E}_\varepsilon$, denoted by $\mathcal{E}_\varepsilon(E, A)$ for A an open set, which is given by (4) with Ω replaced with A. In this localized version the second integral is also, by convention, on the set A_ε which is the union of the cubes $z + \varepsilon Q$, $z \in \varepsilon\mathbb{Z}^d$, such that $z + \varepsilon Q \subset A$. Then, we have (assuming, still, that $\partial\Omega$ is Lipschitz):

Theorem 2.1. *$\mathcal{E}_\varepsilon$ Γ-converges to $\mathcal{E}$ as $\varepsilon \to 0$, where the convergence is in the space of Borel sets endowed with the topology of the L^1-convergence of their characteristic functions.*

This means that given $\varepsilon_n \downarrow 0$, for any Borel set $E \subset \Omega$ we have:

- for any $(E_n)_{n\ge1}$ sequence of Borel sets with $|E_n \triangle E| \to 0$,
$$\liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(E_n) \ \ge\ \mathcal{E}(E)\,; \tag{8}$$
- there exists $(E_n)_{n\ge1}$, with $|E_n \triangle E| \to 0$ as $n \to \infty$, and
$$\limsup_{n\to\infty} \mathcal{E}_{\varepsilon_n}(E_n) \ \le\ \mathcal{E}(E)\,. \tag{9}$$

Here, $E_n \triangle E = (E_n \setminus E) \cup (E \setminus E_n)$ (the symmetric difference).

Remark 2.2. We also prove the following compactness property: if $(E_n)_{n\ge1}$ is a sequence of sets such that $\sup_n \mathcal{E}_{\varepsilon_n}(E_n) < +\infty$, then, up to a subsequence, E_n converges to a finite-perimeter set $E \subset \Omega$, in the sense that $|E_n \triangle E| \to 0$.

2.1. **Proof of (8).** We will use the standard blow-up approach (see [12] for a recent general description of this technique), but still detail a little the proofs in order to make the paper self-contained, and also because the presence of the (unsigned) external field g requires some special care. Consider $(E_n)_{n\ge1}$ a sequence of sets. Up to the extraction of a subsequence we may assume that $\liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(E_n) = \lim_{n\to\infty} \mathcal{E}_{\varepsilon_n}(E_n)$, and without loss of generality we assume it is finite (otherwise, there is nothing to prove). We show both (8) and the statement in Remark 2.2, that is, that E_n converges (up to a subsequence) to some set E if the energies $\mathcal{E}_{\varepsilon_n}(E_n)$ are uniformly bounded.

Let us define the measures μ_n by

$$\mu_n \ = \sum_{\substack{k\in\mathbb{Z}^d\\ \varepsilon_n(k+Q)\subset\Omega}} \left(\int_{\partial^* E_n \cap \varepsilon_n(k+Q)} F\left(\frac{x}{\varepsilon_n}, \nu_{E_n}(x)\right) d\mathcal{H}^{d-1}(x) + \frac{1}{\varepsilon_n} \int_{E_n\cap\varepsilon_n(k+Q)} g\left(\frac{x}{\varepsilon_n}\right) dx \right) \delta_{\varepsilon_n k}\,. \tag{10}$$

It is actually defined as a sum of Dirac masses on the points $\varepsilon_n k$ of $\varepsilon_n\mathbb{Z}^d \cap \Omega$, such that $\varepsilon_n(k+Q) \subset \Omega$. It is important here that Q is defined as $[0,1)^d$ (containing 0 and not 1), as the first integral is on a singular measures that might weigh $(d-1)$-dimensional surfaces, and we do not want some to be counted twice (or never) in the sum.

We have $\mathcal{E}_{\varepsilon_n}(E_n) \ge \mu_n(\Omega)$. By (2) and the definition (10) of μ_n, we have that $\mu_n(\Omega) \ge \delta Per(E_n, \Omega_{\varepsilon_n})$. By (1), we also get a bound on $Per(E_n, \Omega \setminus \Omega_{\varepsilon_n})$: hence we have that $(\chi_{E_n})_{n\ge1}$ is equibounded in $BV(\Omega)$, and that the μ_n are nonnegative measures, which are uniformly bounded. Hence, up to a subsequence we may assume there exists some measure μ and some finite-perimeter set E such that $\mu_n \stackrel{*}{\rightharpoonup} \mu$ as measures, and that $\chi_{E_n} \to \chi_E$ (in $L^1(\Omega)$: hence the compactness property in Remark 2.2 is shown). We have

$$\mu(\Omega) \;\le\; \liminf_{n\to\infty} \mu(\Omega_n) \;\le\; \liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(E_n)$$

so that (8) follows if we show that $\mu \ge \phi(\nu_E)\mathcal{H}^{d-1} \llcorner \partial^* E$.

A way to show such an inequality is to estimate the Radon-Nikodým derivative of the measure μ with respect to $\mathcal{H}^{d-1} \llcorner \partial^* E$, and to show that it is actually larger than $\phi(\nu_E)$. By the Besicovitch derivation theorem (see for instance [4, Thm. 5.52]), this derivative is given for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E$ by

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\mathcal{H}^{d-1}(B(x,r)\cap \partial^* E)}$$

In particular, at a regular point x_0 (where $\partial^* E$ has $(d-1)$-density 1, a normal vector $\nu_E(x_0)$, and the blow-up sequences of E converge to $\{(x - x_0)\cdot \nu_E(x_0) > 0\}$) the limit becomes

$$\ell \;=\; \lim_{r\to 0} \frac{\mu(B(x_0,r))}{\omega_{d-1} r^{d-1}}. \tag{11}$$

where ω_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$.

Let us now show that $\ell \ge \phi(\nu)$, where $\nu = \nu_E(x_0)$ is the inner normal to $\partial^* E$ at x_0. Notice that since x_0 is regular, we also have

$$\lim_{r\to 0} \frac{\int_{B(x_0,2r)} |\chi_{\{(x-x_0)\cdot \nu_E(x_0)>0\}} - \chi_E(x)|\,dx}{r^N} \;=\; 0.$$

For a.e. $r > 0$ (small), we have

$$\mu(B(x_0,r)) \;=\; \lim_{n\to\infty} \mu_n(B(x_0,r))\,,$$

and

$$\int_{B(x_0,2r)} |\chi_{\{(x-x_0)\cdot \nu_E(x_0)>0\}} - \chi_E(x)|\,dx$$
$$= \lim_{n\to\infty} \int_{B(x_0,2r)} |\chi_{\{(x-x_0)\cdot \nu_E(x_0)>0\}}(x-x_0) - \chi_{E_n}(x)|\,dx\,.$$

Hence, using a diagonal argument, there exist subsequences n_m and r_m such that $\varepsilon'_m = \varepsilon_{n_m}/r_m \to 0$,

$$\ell \;=\; \lim_{m\to\infty} \frac{\mu_{n_m}(B(x_0,r_m))}{\omega_{d-1} r_m^{d-1}} \tag{12}$$

and

$$\lim_{m\to\infty} \frac{\int_{B(x_0,2r_m)} |\chi_{\{(x-x_0)\cdot \nu_E(x_0)>0\}} - \chi_{E_{n_m}}(x)|\,dx}{r_m^d} \;=\; 0. \tag{13}$$

We let as before $I_\nu = I_{\nu_E(x_0)} = \{x \in \mathbb{R}^N\,, x\cdot \nu_E(x_0) > 0\}$. We now make for each m the change of variable $x = x_0 + r_m y$, and we define $E'_m = (E_{n_m} - x_0)/r_m \subset (\Omega - x_0)/r_m$. It follows from (13) that

$$\lim_{m\to\infty} \int_{B(0,2)} |\chi_{E'_m}(y) - \chi_{I_\nu}(y)|\,dy \;=\; 0. \tag{14}$$

Letting $B_m = \bigcup\{\varepsilon_{n_m}(k+Q) : \varepsilon_{n_m} k \in \varepsilon_{n_m}\mathbb{Z}^d \cap B(x_0,r_m)\}$ and $B'_m = (B_m - x_0)/r_m$, we have, on the other hand:

$$\frac{\mu_{n_m}(B(x_0,r_m))}{r_m^{d-1}} = \frac{1}{r_m^{d-1}}\left(\int_{\partial^* E_{n_m}\cap B_m} F\left(\frac{x}{\varepsilon_{n_m}}, \nu_{E_{n_m}}(x)\right) d\mathcal{H}^{d-1}(x) + \frac{1}{\varepsilon_{n_m}}\int_{E_{n_m}\cap B_m} g\left(\frac{x}{\varepsilon_{n_m}}\right) dx\right)$$
$$= \int_{\partial^* E'_m \cap B'_m} F\left(\frac{x_0}{\varepsilon_{n_m}} + \frac{y}{\varepsilon'_m}, \nu_{E'_m}(y)\right) d\mathcal{H}^{d-1}(y) + \frac{1}{\varepsilon'_m}\int_{E'_m\cap B'_m} g\left(\frac{x_0}{\varepsilon_{n_m}} + \frac{y}{\varepsilon'_m}\right) dy.$$

Let now $\theta_m \in [0,1)^d$ be the fractionary part of x_0/ε_{n_m}, that is, the vector $((\theta_m)_i)_{i=1}^d$ whose ith component is $(\theta_m)_i = (x_0)_i/\varepsilon_{n_m} - [(x_0)_i/\varepsilon_{n_m}]$ (where $[\,\cdot\,]$ is the integer part, and $(x_0)_i$ is the ith component of x_0). By periodicity, we may clearly replace the argument $x_0/\varepsilon_{n_m} + y/\varepsilon'_m$ in the two last integrals above with $(\varepsilon'_m\theta_m + y)/\varepsilon'_m$. Alternatively, we can change again variables and define $E''_m = E'_m + \varepsilon'_m\theta_m$ and $B''_m = B'_m + \varepsilon'_m\theta_m$: we find

$$\frac{\mu_{n_m}(B(x_0,r_m))}{r_m^{d-1}} = \int_{\partial^* E''_m\cap B''_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{E''_m}(y)\right) d\mathcal{H}^{d-1}(y) + \frac{1}{\varepsilon'_m}\int_{E''_m\cap B''_m} g\left(\frac{y}{\varepsilon'_m}\right) dy.$$

and it follows from (12) that

$$\omega_{d-1}\ell \;=\; \lim_{m\to\infty}\int_{\partial^* E''_m\cap B''_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{E''_m}(y)\right) d\mathcal{H}^{d-1}(y) \;+\; \frac{1}{\varepsilon'_m}\int_{E''_m\cap B''_m} g\left(\frac{y}{\varepsilon'_m}\right) dy\,, \tag{15}$$

moreover, it also follows from (14) that

$$\lim_{m\to\infty}\int_{B(0,3/2)} |\chi_{E''_m}(y) - \chi_{I_\nu}(y)|\, dy \;=\; 0. \tag{16}$$

Since $B_m = (B(x_0,r_m)\cap\varepsilon_{n_m}\mathbb{Z}^d) + \varepsilon_{n_m}Q$, the sets B'_m, B''_m above are given exactly by,

$$B'_m \;=\; \left(B(0,1)\cap\left\{\varepsilon'_m k - \frac{x_0}{r_n} \,:\, k\in\mathbb{Z}^d\right\}\right) \,+\, \varepsilon'_m Q$$

and

$$B''_m \;=\; \left((B(0,1)+\varepsilon'_m\theta_m)\cap\varepsilon'_m\mathbb{Z}^d\right) \,+\, \varepsilon'_m Q\,, \tag{17}$$

Observe that for any $s<1$, $B(0,s)\subset B''_m$ for m large enough.

Let $\eta>0$. Let E_ν be the set provided by Theorem 1.2, and for $s\in(1-2\eta,1-\eta)$ which will be choosen later on, define

$$\hat{E}_m \;=\; (\varepsilon'_m E_\nu \setminus B(0,s)) \cup (E''_m \cap B(0,s))\;.$$

Then, by the minimality of E_ν, we have

$$\int_{\partial^*\hat{E}_m\cap B''_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{\hat{E}_m}(y)\right) d\mathcal{H}^{d-1}(y) \,+\, \frac{1}{\varepsilon'_m}\int_{\hat{E}_m\cap B''_m} g\left(\frac{y}{\varepsilon'_m}\right) dy\,,$$
$$\geq\; \int_{\partial^*(\varepsilon'_m E_\nu)\cap B''_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{(\varepsilon'_m E_\nu)}(y)\right) d\mathcal{H}^{d-1}(y) \,+\, \frac{1}{\varepsilon'_m}\int_{(\varepsilon'_m E_\nu)\cap B''_m} g\left(\frac{y}{\varepsilon'_m}\right) dy\,,$$

which converges (see (6), and details in the appendix) to $\mathcal{H}^{d-1}(\partial I_\nu\cap B_1)\phi(\nu) = \omega_{d-1}\phi(\nu)$ as $m\to\infty$. Hence the inequality $\ell\geq\phi(\nu)$ will follow from (15) if we show that (for a suitable choice of s) the difference

$$\begin{aligned}\mathcal{E}_{\varepsilon'_m}(E''_m,&B''_m) - \mathcal{E}_{\varepsilon'_m}(\hat{E}_m, B''_m)\\ &= \int_{\partial^* E''_m\cap B''_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{E''_m}(y)\right) d\mathcal{H}^{d-1}(y) \,+\, \frac{1}{\varepsilon'_m}\int_{E''_m\cap B''_m} g\left(\frac{y}{\varepsilon'_m}\right) dy\\ &- \int_{\partial^*\hat{E}_m\cap B''_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{\hat{E}_m}(y)\right) d\mathcal{H}^{d-1}(y) \,+\, \frac{1}{\varepsilon'_m}\int_{\hat{E}_m\cap B''_m} g\left(\frac{y}{\varepsilon'_m}\right) dy\end{aligned} \tag{18}$$

is bounded from below, as $m \to \infty$, by some quantity which modulus is arbitrarily small.

Call R_m the region made of all cubes $z + \varepsilon'_m Q$, $z \in \varepsilon'_m\mathbb{Z}^d$, which intersect $\partial B(0,s)$ (we denote by $\mathcal{N}_m$ the number of such cubes), $S_m = (B''_m \setminus B(0,s)) \cup R_m$, $R'_m = S_m \setminus R_m$ (so that $S_m = R_m \cup R'_m$). In $B''_m \setminus S_m$, the sets E''_m and $\hat{E}_m$ coincide, so that the difference in (18) is also given by

$$\mathcal{E}_{\varepsilon'_m}(E''_m, B''_m) - \mathcal{E}_{\varepsilon'_m}(\hat{E}_m, B''_m) \;=\; \mathcal{E}_{\varepsilon'_m}(E''_m, S_m) - \mathcal{E}_{\varepsilon'_m}(\hat{E}_m, S_m). \tag{19}$$

On one hand, using (2), we have that

$$\begin{aligned}\mathcal{E}_{\varepsilon'_m}&(E''_m, S_m)\\ &= \int_{\partial^* E''_m\cap(R'_m\cup R_m)} F\left(\frac{y}{\varepsilon'_m}, \nu_{E''_m}(y)\right) d\mathcal{H}^{d-1}(y) + \int_{E''_m\cap(R'_m\cup R_m)} g\left(\frac{y}{\varepsilon'_m}\right) dy\\ &\geq \int_{\partial^* E''_m\cap R_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{E''_m}(y)\right) d\mathcal{H}^{d-1}(y) + \int_{E''_m\cap R_m} g\left(\frac{y}{\varepsilon'_m}\right) dy\,.\end{aligned} \tag{20}$$

On the other hand,

$$\begin{aligned}-\mathcal{E}_{\varepsilon'_m}(\hat{E}_m, S_m) \;=\; &-\mathcal{E}_{\varepsilon'_m}(\varepsilon'_m E_\nu, R'_m)\\ &- \int_{\partial^*(\varepsilon'_m E_\nu)\cap(R_m\setminus B(0,s))} F\left(\frac{y}{\varepsilon'_m}, \nu_{(\varepsilon'_m E_\nu)}(y)\right) d\mathcal{H}^{d-1}(y)\\ &- \int_{\partial B(0,s)\cap\partial^*\hat{E}_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{\hat{E}_m}(y)\right) d\mathcal{H}^{d-1}(y)\\ &- \int_{\partial^* E''_m\cap(R_m\cap B(0,s))} F\left(\frac{y}{\varepsilon'_m}, \nu_{E''_m}(y)\right) d\mathcal{H}^{d-1}(y) - \int_{\hat{E}_m\cap R_m} g\left(\frac{y}{\varepsilon'_m}\right) dy\,.\end{aligned} \tag{21}$$

Using (1), it follows

$$\begin{aligned}-\mathcal{E}_{\varepsilon'_m}(\hat{E}_m, S_m) \;\geq\; &\\ &- \;\mathcal{E}_{\varepsilon'_m}(\varepsilon'_m E_\nu, R'_m)\\ &- \;c^* Per(\varepsilon'_m E_\nu, R_m \setminus B(0,s)) \;-\; c^*\mathcal{H}^{d-1}(\partial B(0,s)\cap(E''_m \triangle(\varepsilon'_m E_\nu)))\\ &- \int_{\partial^* E''_m\cap R_m} F\left(\frac{y}{\varepsilon'_m}, \nu_{E''_m}(y)\right) d\mathcal{H}^{d-1}(y) - \int_{\hat{E}_m\cap R_m} g\left(\frac{y}{\varepsilon'_m}\right) dy\,.\end{aligned} \tag{22}$$

Adding (20) and (22), we deduce from (19) that

$$\begin{aligned}\mathcal{E}_{\varepsilon'_m}(E''_m,B''_m) - \mathcal{E}_{\varepsilon'_m}(\hat{E}_m,B''_m) \ \geq \\ &- \mathcal{E}_{\varepsilon'_m}(\varepsilon'_m E_\nu, R'_m) \\ &- c^* Per(\varepsilon'_m E_\nu, R_m \setminus B(0,s)) \ - \ c^*\mathcal{H}^{d-1}(\partial B(0,s)\cap(E''_m \triangle(\varepsilon'_m E_\nu))) \\ &+ \frac{1}{\varepsilon'_m}\int_{R_m} g\left(\frac{x}{\varepsilon}\right)(\chi_{E''_m} - \chi_{\hat{E}_m})(x)\,dx\,.\end{aligned} \tag{23}$$

We denote respectively by $-A^i_m$, $i = 1,2,3,4$ the four terms in the right-hand side of this expression. By (56),

$$\limsup_{m\to 0} A^1_m \ \leq \ \phi(\nu)\mathcal{H}^{d-1}(\partial I_\nu \cap (B(0,1)\setminus B(0,s))) \leq C(1-s) \leq 2C\eta\,. \tag{24}$$

Observe that the number $\mathcal{N}_m$ of cubes $z+\varepsilon'_m Q$, $z\in\varepsilon'_m\mathbb{Z}^d$, which compose the set R_m is (at most) of order $(1/\varepsilon'_m)^{d-1}$. (Indeed, $R_m \subset B(O,s+\sqrt{d}\varepsilon'_m)\setminus B(O,s-\sqrt{d}\varepsilon'_m)$ so that $\varepsilon'^d_m\mathcal{N}_m \leq C\varepsilon'_m$.) Moreover, the number of such cubes which intersect $\partial(\varepsilon'_m E_\nu)$ (which is at distance M from ∂I_ν by Theorem 1.2) is at most of order $(1/\varepsilon'_m)^{d-2}$ (using the same argument). Since the perimeter of $\varepsilon'_m E_\nu$ in each such cube is of order ε'^{d-1}_m, A^2_m is of order ε'_m hence

$$\lim_{m\to 0} A^2_m \ = \ 0\,. \tag{25}$$

Since both sets E''_m and $\varepsilon'_m E_\nu$ converge to I_ν as $m\to\infty$, up to a subsequence we know that for a.e. choice of $s\in(1-2\eta,1-\eta)$, $\mathcal{H}^{d-1}(\partial B(0,s)\cap(E''_m\triangle(\varepsilon'_\nu E_\nu)))\to 0$. Hence, if we choose well s,

$$\lim_{m\to 0} A^3_m \ = \ 0\,. \tag{26}$$

It remains to bound A^4_m. We have, for any cube $z+\varepsilon'_m Q$ which intersects $\partial B(0,s)$ $(z\in\varepsilon'_m\mathbb{Z}^d)$,

$$\frac{1}{\varepsilon'_m}\int_{z+\varepsilon'_m Q} g\left(\frac{x}{\varepsilon}\right)(\chi_{E''_m}-\chi_{\hat{E}_m})(x)\,dx \ \leq \ \|g\|_d\left(\int_{z+\varepsilon'_m Q}|\chi_{E''_m}-\chi_{\hat{E}_m}|\,dx\right)^{1-1/d}$$

so that (summing on all such cubes and recalling $\mathcal{N}_m$ is the number of cubes which constitute R_m)

$$\begin{aligned}A^4_m \ &\leq \ \mathcal{N}^{1/d}_m\|g\|_d\left(\int_{R_m}|\chi_{E''_m}-\chi_{\hat{E}_m}|\,dx\right)^{1-1/d} \\ &\leq \ C\left(\frac{1}{\varepsilon'_m}\int_{B(0,s+\sqrt{d}\varepsilon'_m)\setminus B(0,s-\sqrt{d}\varepsilon'_m)}|\chi_{E''_m}-\chi_{\hat{E}_m}|\,dx\right)^{1-1/d}\end{aligned} \tag{27}$$

where we have used the fact that $\mathcal{N}_m \leq C\varepsilon'^{1-d}_m$ and $R_m\subset B(0,s+\sqrt{d}\varepsilon'_m)\setminus B(0,s-\sqrt{d}\varepsilon'_m)$. Since (by Fubini's theorem)

$$\begin{aligned}\int_{1-2\eta}^{1-\eta}\left(\frac{1}{\varepsilon'_m}\int_{B(0,t+\sqrt{d}\varepsilon'_m)\setminus B(0,t-\sqrt{d}\varepsilon'_m)}|\chi_{E''_m}-\chi_{\hat{E}_m}|\,dx\right)dt \\ \leq \ 2\sqrt{d}\int_{B(O,1-\eta+\sqrt{d}\varepsilon'_m)\setminus B(0,1-2\eta-\sqrt{d}\varepsilon'_m)}|\chi_{E''_m}-\chi_{\hat{E}_m}|\,dx \ \to \ 0,\end{aligned}$$

as $m \to \infty$, up to a subsequence we find that for almost any choice of $s \in (1 - 2\eta, 1-\eta)$, the right-hand side of (27) goes to zero, hence:

$$\lim_{m\to\infty} A_m^4 \;=\; 0. \tag{28}$$

Collecting (24), (25), (26) and (28) we deduce from (23) that

$$\liminf_{m\to\infty} \mathcal{E}_{\varepsilon'_m}(E''_m, B''_m) - \mathcal{E}_{\varepsilon'_m}(\hat{E}_m, B''_m) \;\geq\; -2C\eta$$

for some constant C. It follows (from (6) and (15)) that $\ell \geq \phi(\nu) - 2C\eta/\omega_{d-1}$, and since η is arbitrary we get $\ell \geq \phi(\nu)$, which was our claim. Hence (8) holds.

2.2. **Proof of the inequality (9).** The proof of (9) in the particular case of polyhedral limit set is given in the Appendix A (Corollary A.3), where several "simple" limits of $\mathcal{E}_\varepsilon$ are investigated. We deduce here (9) in the general case.

Let $E \subset \Omega$ is an arbitrary set with finite perimeter. Here we need to assume that $\partial\Omega$ is Lipschitz. In this case, it is standard that it is possible to approximate E with sets E_n which are the intersection of Ω with a polyhedron, and such that $\lim_{n\to\infty} \mathcal{H}^{d-1}(\partial E_n \cap \Omega) = Per(E,\Omega)$. The Reshetnyak continuity Theorem (see [4, Theorem 2.39]), together with the continuity of ϕ (Corollay A.4) show that $\lim_{n\to\infty}\mathcal{E}(E_n) = \mathcal{E}(E)$. By corollary A.3 and a diagonal argument, we therefore can find sets $(E_\varepsilon)_{\varepsilon>0}$ such that $|E_\varepsilon \triangle E| \to 0$ as $\varepsilon \to 0$ and $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(E_\varepsilon) \leq \mathcal{E}(E)$. We deduce (9).

3. **A new construction for the plane-like minimizers.** The coarea formula for BV functions shows that if $u \in BV(\Omega)$ (the space of functions with bounded variation in Ω [18] [4]), then

$$\mathcal{F}_\varepsilon(u) \;:=\; \int_\Omega F\left(\frac{x}{\varepsilon}, Du\right) \;+\; \int_{\Omega_\varepsilon} g\left(\frac{x}{\varepsilon}\right) u(x)\,dx \;=\; \int_{-\infty}^{+\infty} \mathcal{E}_\varepsilon(\{u > s\})\,ds$$

and it is not difficult to deduce from Theorem 2.1 that $\mathcal{F}_\varepsilon$ (extended by the value $+\infty$ to functions $u \in L^1(\Omega) \setminus BV(\Omega)$) Γ-converges to

$$\mathcal{F}(u) \;:=\; \begin{cases} \int_\Omega \phi(Du) & \text{if } u \in BV(\Omega)\,, \\ +\infty & \text{if } u \in L^1(\Omega)\setminus BV(\Omega)\,. \end{cases}$$

See for instance [15, Prop. 3.5].

On the other hand, it is well-known (at least when $g = 0$, see [1] and [11]) that $\mathcal{F}_\varepsilon$ Γ-converges, as $\varepsilon \to 0$, to $\mathcal{F}$ if the convex one-homogeneous function ϕ is replaced with the solution ψ of the following cell problem: for each $p \in \mathbb{R}^d$,

$$\psi(p) \;=\; \min_{u\in BV(Q^\sharp)} \int_{Q^\sharp} F(x, p + Du) \;+\; \int_Q g(x)(p\cdot x + u(x))\,dx \tag{29}$$

where $BV(Q^\sharp)$ denotes the space of BV functions which are integer-periodic in $\mathbb{R}^d$. The classical proofs for this result are similar in spirit to our proof of Theorem 2.1, but do not rely on the minimizers introduced in Theorem 1.2. Of course, it will follow that $\phi = \psi$ (Cor. 3.7 below).

It is a priori quite important in the first integral here to consider the variation of the (periodic) measure $F(x, p + Du)$ on $Q^\sharp$ (rather than just Q, since it may be positive on ∂Q), however, for a given p and a minimizer u for (29), if $|Du|(\partial Q) > 0$, we might translate slightly u and g ($u \to u(\cdot - \tau)$, $g \to u(\cdot - \tau)$, or equivalently $\tau \to \tau + Q$, $\tau \in \mathbb{R}^d$) to get a new problem with the same value and such that $|Du|(\partial Q) = 0$. Hence in what follows we will not bother about this issue and implicitly consider that the derivatives of our functions do not charge ∂Q (and by periodicity, $k + \partial Q$, $k \in \mathbb{Z}^d$). Observe also that by standard regularization arguments [19], the min in (29) is also the infimum over smooth, periodic functions u — for which integrating over Q or $Q^\sharp$ does not make any difference.

It is clear that (29) defines a convex, one-homogeneous function ψ. Letting $u = 0$ in the problem yields

$$\psi(p) \;\leq\; (c^* + \|g\|_d)|p|\,. \tag{30}$$

On the other hand, provided as before that assumption (2) holds (for instance, if $\|g\|_d$ is small enough), we have that the functional which is minimized in (29) is coercive in BV, so that the problem is well-posed and admits actually a minimizer. Indeed, given a function u and letting $v(x) = p\cdot x + u(x)$, we have (using $\int_Q g = 0$ and (2))

$$\int_{Q^\sharp} F(x, p{+}Du) + \int_Q g(x)(p{\cdot}x{+}u(x)) \;\geq\; \int_{-\infty}^{+\infty} \mathcal{J}_Q(\{v > s\})\,ds \;\geq\; \delta|Dv|(Q)\,, \tag{31}$$

in particular we deduce that

$$\psi(p) \;\geq\; \delta|p|\,. \tag{32}$$

Fix now $p \in \mathbb{R}^d$, and let u be a minimizer in (29). Let $v(x) = p\cdot x + u(x)$ (which is in $BV_{loc}(\mathbb{R}^d)$). For any $s > 0$, let $E_s = \{v > s\}$. Then we show the following:

Proposition 3.1. *The set E_s is a class A minimizer for $\mathcal{J}$.*

Proof. The proof relies on convex duality and a calibration argument.
Step 1. *Existence of a "calibrating field".* First of all, we have that for any $p \in \mathbb{R}^d$ and $u \in BV(Q^\sharp)$,

$$\begin{aligned} H_p(u) \;:=&\; \int_{Q^\sharp} F(x, p + Du) \\ =&\; \sup\Big\{ p\cdot\int_Q \sigma(x)\,dx - \int_Q u(x)\operatorname{div}\sigma(x)\,dx \;: \\ &\qquad\qquad \sigma \in C^\infty(Q^\sharp;\mathbb{R}^d)\,, \sigma(x) \in C(x)\ \forall x \in Q^\sharp \Big\} \end{aligned} \tag{33}$$

where for each x, $C(x)$ is the convex set

$$C(x) \;=\; \left\{ q \in \mathbb{R}^d \,:\, q\cdot p \leq F(x,p)\ \forall p \in \mathbb{R}^d \right\}\,,$$

such that $\sup_{q\in C(x)} q\cdot p = F(x,p)$. This representation is found for instance in [7] and [8], and is not too difficult to show. The key point is the fact that — thanks to the continuity of F — for any $\theta < 1$, there exists $\eta > 0$ such that $|x - y| \leq \eta$ yields $\theta C(y) \subseteq C(x)$, so that building fields satisfying the constraint at each point, or regularizing these fields, is relatively easy. Given $u \in BV(Q^\sharp)$, a Besicovitch covering argument allows to build a measurable field σ, constant in balls, and such that $\sigma(x) \in C(x)$ a.e. and $\int_{Q^\sharp} \sigma\cdot(p + Du) \approx \int_{Q^\sharp} F(x, p + Du)$. Then for any $\theta < 1$, a mollification of $\theta\sigma$ will provide a C^∞ field with the same properties.

On the other hand, if $u \in L^{d/(d-1)}(Q^\sharp)\setminus BV(Q^\sharp)$, then the right-hand side of (33) is $+\infty$, and we also set $H_p(u) = +\infty$ in this case.

Let K_0 be the convex subset of $L^d(Q^\sharp)$:

$$K_0 \;=\; \left\{ -\operatorname{div}\sigma \,:\, \sigma \in C^\infty(Q^\sharp;\mathbb{R}^d)\,, \sigma(x) \in C(x)\ \forall x \in Q^\sharp \right\}.$$

and $K = \overline{K}^{L^d(Q^\sharp)}$ its closure in L^d. For $h \in K_0$, let

$$G_p(h) \;:=\; \inf\Big\{ -p\cdot\int_Q \sigma(x)\,dx \;:\; \sigma\in C^\infty(Q^\sharp;\mathbb{R}^d)\,,\\ \sigma(x)\in C(x)\ \forall x\in Q^\sharp\,, h=-\mathrm{div}\,\sigma\Big\}\,,$$

and let $G_p(h) = +\infty$ if $h \in L^d(Q^\sharp)\setminus K_0$. One checks that this defines a convex function of h, so that, in particular, its l.s.c. envelope (in L^d) is a convex function with domain K, which coincides with its convex l.s.c. envelope G_p^{**}. Then (33) expresses that

$$H_p(u) \;=\; G_p^*(u) \;=\; \sup_{h\in L^d(Q^\sharp)} \langle h,u\rangle_{L^d,L^{d/(d-1)}} \,-\, G_p(h)$$

is the Legendre-Fenchel conjugate of G_p (in the duality $(L^d, L^{d/(d-1)})$, see [17]) so that and $H_p^* = G_p^{**}$. Now, u is a minimizer for (29) if and only if

$$-g \;\in\; \partial H_p(u)$$

(this is obvious from the definition of the subdifferential $\partial H_p(u)$, which is the set of h such that $H_p(v) \ge H_p(u) + \int_Q h(v-u)\,dx$). The Legendre-Fenchel's identity shows that it is equivalent to

$$-\int_Q g(x)u(x)\,dx \;=\; H_p(u) \,+\, G_p^{**}(-g)\,.$$

Since there must exist $h_n \in K_0$ such that $h_n \to -g$ and $G_p^{**}(-g) = \lim_n G_p(h_n)$, it shows the existence of a sequence $\sigma_n \in C^\infty(Q^\sharp)$, such that $\mathrm{div}\,\sigma_n \to g$ in $L^d(Q)$, $-p\cdot\int_Q \sigma_n\,dx \to G_p^{**}(-g)$ and

$$-\int_Q \mathrm{div}\,\sigma_n(x)u(x)\,dx \,+\, p\cdot\int_Q \sigma_n(x)\,dx \;\to\; H_p(u) \tag{34}$$

as $n\to\infty$. Observe that since u has bounded variation (and is periodic), and σ_n is smooth and periodic, the integrals can be written $\int_{Q^\sharp} \sigma_n(x)\cdot(p+Du)$.

Step 2. *Proof of the minimality of E_s.* The sequence σ_n built in the previous step, seen as a periodic field over $\mathbb{R}^d$, is now used to show the minimality of the level sets E_s. Consider a large ball B and denote $B' = \cup_{k+Q\cap B\neq\emptyset} k+Q$ where $k\in\mathbb{Z}^d$. Let $v(x) = u(x) + p\cdot x$, where u is as before. The co-area formula for BV functions yields

$$\int_{-\infty}^{+\infty}\int_{B'\cap\partial^* E_s} F(x,\nu_{E_s}(x))\,d\mathcal{H}^{d-1}(x)\,ds \\ =\; \lim_{n\to\infty}\int_{-\infty}^{+\infty}\int_{B'\cap\partial^* E_s} \sigma_n(x)\cdot\nu_{E_s}(x)\,d\mathcal{H}^{d-1}(x)\,ds$$

and since $\sigma_n(x)\cdot\nu_{E_s}(x) \le F(x,\nu_{E_s}(x))$ we deduce that up to a subsequence, we have for a.e. $s\in\mathbb{R}$

$$\lim_{n\to\infty}\int_{B'\cap\partial^* E_s} \sigma_n(x)\cdot\nu_{E_s}(x)\,d\mathcal{H}^{d-1}(x) \;=\; \int_{B'\cap\partial^* E_s} F(x,\nu_{E_s}(x))\,d\mathcal{H}^{d-1}(x)\,.$$

Fix s such that this is true, and let now E' be a set with $E'\triangle E_s \Subset B$. We have

$$\begin{aligned}
&\int_{B'\cap\partial^*E'} F(x,\nu_{E'})\,d\mathcal{H}^{d-1} + \int_{B'\cap E'} g\,dx \\
&\geq \int_{B'} \sigma_n\cdot D\chi_{E'} + \int_{B'\cap E'} g\,dx = \int_{B'} \sigma_n\cdot D\chi_{E_s} + \int_{B'} \sigma_n\cdot D(\chi_{E'}-\chi_{E_s}) + \int_{B'\cap E'} g\,dx \\
&\qquad = \int_{B'} \sigma_n\cdot D\chi_{E_s} - \int_{B'} \operatorname{div}\sigma_n(\chi_{E'}-\chi_{E_s}) + \int_{B'\cap E'} g\,dx \\
&\qquad\qquad \to \int_{B'\cap\partial^*E_s} F(x,\nu_{E_s})\,d\mathcal{H}^{d-1}\,ds + \int_{B'\cap E_s} g\,dx
\end{aligned}$$

as $n\to\infty$, where we have used $\operatorname{div}\sigma_n \to g$. This shows the minimality of E_s. We deduce easily that for a.e. s, E_s is a class A minimizer for $\mathcal{J}$. The proof that E_s is a minimizer for all s follows from the fact that E_s is the limit of any sequence E_{s_j} with $s_j \downarrow s$ ($s_j > s$), s_j such that E_{s_j} is a class A minimizer, and the stability of class A minimizer, see [14, Sec. 9].[2] □

The next lemma is classical, and shown for instance in [14]. For the reader's convenience we provide a quick proof.

Lemma 3.2. *For any $s\in\mathbb{R}$, let E_s be defined as above, hence from Proposition 3.1 a class A minimizer for $\mathcal{J}$. There exists $r_0>0$ and $\gamma>0$ (independent on s) such that for any $x\in\mathbb{R}^d$:*

- *if $|B(x,r)\cap E_s|>0$ for any $r>0$ then for $r\leq r_0$, $|B(x,r)\cap E_s|\geq \gamma r^d$,*
- *if $|B(x,r)\setminus E_s|>0$ for any $r>0$ then for $r\leq r_0$, $|B(x,r)\setminus E_s|\geq \gamma r^d$.*

Proof. We just prove the first inequality (the proof of the second being identical). Letting $B_r = B(x,r)$, the idea is to compare the energy of E_s and the energy of $E_s\setminus B_r$ for $r>0$, small. The minimality of E_s yields for a.e. $r>0$:

$$\int_{B_r\cap\partial^*E_s} F(x,\nu_E)\,d\mathcal{H}^{d-1} + \int_{E_s\cap B_r} g(x)\,dx \leq \int_{\partial B_r\cap E_s} F(x,-\nu_{B_r})\,d\mathcal{H}^{d-1}$$

hence, using (1) and Hölder's inequality,

$$c_*\mathcal{H}^{d-1}(B_r\cap\partial^*E_s) \leq c^*\mathcal{H}^{d-1}(\partial B_r\cap E_s) + \|g\|_{L^d(B_r)}|E_s\cap B_r|^{\frac{d-1}{d}}.$$

Letting $f(r) = |E_s\cap B_r|>0$ for all $r>0$, and using the isoperimetric inequality in $\mathbb{R}^d$, we find

$$\begin{aligned}
c_d f(r)^{\frac{d-1}{d}} \leq Per(E_s\cap B_r) &= \mathcal{H}^{d-1}(B_r\cap\partial^*E_s) + \mathcal{H}^{d-1}(\partial B_r\cap E_s) \\
&\leq \frac{c_*+c^*}{c_*}\mathcal{H}^{d-1}(\partial B_r\cap E_s) + \frac{1}{c_*}\|g\|_{L^d(B_r)} f(r)^{\frac{d-1}{d}}.
\end{aligned}$$

Since $\mathcal{H}^{d-1}(\partial B_r\cap E_s) = f'(r)$ for all r but a finite or countable number, and choosing r_0 such that if $r<r_0$, $\|g\|_{L^d(B_r)}/c_* \leq c_d/2$ (which is possible since g is periodic and $|g|^d\in L^1(Q^\sharp)$ is equi-integrable), we deduce that if $r<r_0$,

$$\frac{c_d}{2} f(r)^{1-\frac{1}{d}} \leq \frac{c_*+c^*}{c_*} f'(r).$$

[2] Although the proof there is only sketched, but taking any competitor E' with $E_s\triangle E'\Subset B$, for B a big ball, one easily shows that one finds competitors $E'_j \to E'$ (of the form $(E'\cap(1+t)B)\cup(E_{s_j}\setminus(1+t)B)$ for a well-chosen $t\in(0,1/2)$, such that $\mathcal{H}^{d-1}(\partial(1+t)B\cap(E_{s_j}\triangle E_s))\to 0$) with $E_{s_j}\triangle E'_j\Subset 2B$ and $Per(E'_j,2B)\to Per(E',2B)$ as $j\to\infty$, from which the minimality of E_s is easily deduced.

Then, the conclusion follows from Gronwall's lemma, and the constant γ depends only on c_*, c^*, and the dimension d — while r_0 depends on c_* and g. To prove the second inequality, we follow the same lines, except that the energy of E_s is now compared with the energy of the sets $E_s \cup B_r$, $r > 0$. □

It follows that E_s (which a priori is "just" a Caccioppoli set) is a closed set with rectifiable boundary.

Corollary 3.3. *The sets of points of Lebesgue density, respectively,* 1 *and* 0 *of* E_s *are both open, hence we may consider* E_s *as a closed set (the complement of points of density* 0*), whose topological boundary coincides with the measure-theoretical boundary (which is the set of points of density neither* 0 *nor* 1*), hence, up to a* $\mathcal{H}^{d-1}$*-negligible set, to the reduced boundary* $\partial^* E$ [4] [18] [19].

The density estimates, together with the coarea formula, yield an estimate on the oscillation of v on a cube Q, defined by $\text{osc}_Q(v) = \text{ess sup}_Q\, v - \text{ess inf}_Q\, v$. Here as before, we recall that $p \in \mathbb{R}^d$ is a given vector, and v is defined by $v(x) = u(x) + p \cdot x$ where u is a minimizer of (29).

Corollary 3.4. *There exists* $C > 0$ *(depending on* c^*, c_*, g*, but not on* p*) such that* $\text{osc}_Q(v) \le C|p|$*. (Equivalently,* $\text{osc}_Q(u) \le C|p|$*.)*

Proof. If $x \in \partial E_s = \{v > s\}$, it follows from Lemma 3.2 that $|B(x, r_0) \cap E_s| \ge \gamma r_0^d$ and $|B(x, r_0) \setminus E_s| \ge \gamma r_0^d$. In particular, if $x \in \partial E_s \cap Q$, we have (assuming $r_0 < 1$) $\min\{|(-1,2)^d \cap E_s|, |(-1,2)^d \setminus E_s|\} \ge \gamma r_0^d$. We deduce that $Per(E_s, (-1,2)^d) \ge C\gamma^{\frac{d-1}{d}} r_0^{d-1}$ for a constant C depending only on the dimension. Hence,

$$\int_{(-1,2)^d} |Dv| \;\ge\; C r_0^{d-1} |\{s \in \mathbb{R} \,:\, \partial E_s \cap Q \neq \emptyset\}|\,,$$

and we observe that $|\{s \in \mathbb{R} \,:\, \partial E_s \cap Q \neq \emptyset\}| = \text{ess sup}_Q\, v - \text{ess inf}_Q\, v$. On the other hand, using (30) and (31) (and the fact that Dv is a periodic measure and that $[-1, d)^d$ is, up to integer translations, 3^d copies of $Q = [0,1)^d$),

$$\int_{[-1,2)^d} |Dv| \;=\; 3^d \int_Q |Dv| \;\le\; C|p|$$

where C depends on d, c^* and $\|g\|_d$ (and δ, which depends on the properties of g). We deduce that there exists $C > 0$, depending on c_*, c^* and g such that $|\text{ess sup}_Q\, v - \text{ess inf}_Q\, v| \le C|p|$, which shows the corollary. Of course the oscillation of $u = v - p \cdot x$ on Q is bounded by $(C + \sqrt{d})|p|$. □

Corollary 3.5. *There exists* M *which does not depend on* p *such that, if* s *is such that* $\partial E_s \cap Q \neq \emptyset$*: then* $\partial E_s \subset \{x : |x \cdot p| \le M|p|\}$*, more precisely* $\{x \,:\, x \cdot p \ge M|p|\} \subset E_s \subset \{x \,:\, x \cdot p \ge -M|p|\}$*.*

Proof. Just let $M = C + 2\sqrt{d}$ where C is the constant in the previous proof. Indeed, if $x \in E_s$, that is, $v(x) = u(x) + p \cdot x > s$, we have $p \cdot x > s - u(x)$. But since $\partial E_s \cap Q \neq \emptyset$, there is x' with $|x'| \le \sqrt{d}$ and $u(x') + p \cdot x' \le s$, hence $s \ge u(x') - |p|\sqrt{d}$. We deduce $p \cdot x > -\text{osc}_Q u - \sqrt{d}|p| \ge -(C - 2\sqrt{d})|p|$. □

To get a full proof of Theorem 1.2, it remains to show that the sets E_s are connected. In fact, we would just repeat here arguments similar to what is found in [14] (see in particular Prop. 7.3), which show that not only E_s, but also $\mathbb{R}^d \setminus E_s$, must be connected if E_s is a class A minimizer. Hence we admit this point, and this achieves our new proof of Theorem 1.2.

Remark 3.6. If ν is a rational direction, that is, if $\nu = p/|p|$ with $p \in \mathbb{Z}^d$, then the corresponding set E_s is clearly periodic: indeed, assuming for instance $p_d \neq 0$ and denoting by $(e_i)_{i=1}^d$ the canonical basis, there exist $d-1$ independent integer vectors $q_i = p_d e_i - p_i e_d$ such that $q_i \cdot p = 0$ so that $E_s + q_i = \{v(\cdot - q_i) > s\} = E_s$. In particular, it is expected that v is, in general, flat with a concentrated gradient. On the other hand, if ν is irrational, one could expect that Dv is not singular and ∂E_s might foliate the torus (i.e., its nonintersecting leaves might be dense in the torus), but this is not always true: for instance, if $g = 0$, $F(x,p) = a(x)|p|$ with a continuous, $a = 1$ outside of a ball in Q and $a >> 1$ in the ball half smaller, then the region where a is large will be avoided by ∂E_s for any direction ν, including irrational. See for instance [5] for a detailed analysis of this phenomenon in the classical context of KAM theory.

A consequence of this analysis is the following identity, which is already proved in [11, Thm. 5.1] (at least for $g = 0$ but if $g \neq 0$, we refer to the discussion in the next section where it is shown how to "eliminate" g: see in particular the proof of Proposition 4.4 below).

Corollary 3.7. *$\phi = \psi$: the limits in (6) and (29) coincide on $\mathbb{S}^{d-1}$.*

4. Elimination of the external field and weaker coercivity.

We show in this section that, thanks to a recent result of Bourgain and Brézis [10], the external field g can be removed in our formulation, in the sense that it can be integrated by part into the surface tension as soon as the global energy is coercive. Pushing further this remark (Sec. 4.2) allows then to weaken a little the coerciveness assumption which is necessary for Theorems 1.2 and 2.1. A simple two-dimensional example illustrates the differences between these various hypotheses, see Section 4.3.

4.1. The coercive case is equivalent to the case $g = 0$.

Proposition 4.1. *Assume (2) holds: then there exists $F'(x,p)$, continuous and periodic in x, convex and one-homogeneous in p, with*

$$c'_*|p| \ \le \ F'(x,p) \ \le {c^*}'|p| \tag{35}$$

(${c^*}' > c'_* > 0$) for any $p \in \mathbb{R}^d$ and such that for any $E \subset Q$ with finite perimeter,

$$\mathcal{J}_Q(E) \ = \ \int_{Q\cap\partial^* E} F'(x,\nu_E(x))\, d\mathcal{H}^{d-1}(x)\,. \tag{36}$$

Proof. Since (2) holds and $g \in L^d(Q)$ with $\int_Q g\,dx = 0$, we have (3), and we can find $\epsilon \in (0,1)$ small such that for any finite-perimeter set $E \subset Q$ (using Hölder's inequality and the relative isoperimetric inequality in Q),

$$-\epsilon \int_E g(x)dx \ = \ \epsilon \int_{Q\setminus E} g(x)\, dx \ \le \ \frac{\delta}{2} Per(E,Q)$$

so that

$$\int_Q F(x,Du) \ + \ \int_Q (1+\epsilon) g(x) u(x)\, dx \ \ge \ \frac{\delta}{2}|Du|(Q)\,, \tag{37}$$

for any $u \in BV(Q)$.

Thanks to (37), the problem

$$\min_{u\in BV(Q)} \int_Q F(x,Du) \ + \ \int_Q (1+\epsilon) g(x) u(x)\, dx$$

has a unique solution ($u = 0$). As in the previous section (but now we consider a functional defined for functions $u \in BV(Q)$, and *not* as in (29) for periodic functions defined on the torus $Q^\sharp$), there is the representation

$$\int_Q F(x, Du) = \sup\Big\{ -\int_Q u(x) \mathrm{div}\, \sigma(x)\, dx : \sigma \in C_c^\infty(Q; \mathbb{R}^d)\, , \sigma(x) \in C(x)\ \forall x \in Q \Big\}.$$

Hence, using similar convex analysis arguments, we deduce the existence of a sequence of compactly supported vector fields $\sigma_n \in C_c^\infty(Q; \mathbb{R}^d)$ such that as $n \to \infty$,

$$\mathrm{div}\, \sigma_n \ \to\ (1 + \epsilon)g$$

in $L^d(Q)$, while $\sigma_n(x) \in C(x)$ for any $x \in Q$. Letting $\sigma'_n = \sigma_n/(1 + \epsilon)$, we find smooth, compactly supported vector fields with $\mathrm{div}\, \sigma'_n \to g$ as $n \to \infty$, while $\sigma'_n \in C(x)/(1 + \epsilon)$ for all x.

Now, thanks to [10, Thm 3] and the fact that $\int_Q g - \mathrm{div}\, \sigma'_n\, dx = 0$, there exist $\sigma''_n \in C^0 \cap W_0^{1,d}(Q)$ with $\mathrm{div}\, \sigma''_n = g - \mathrm{div}\, \sigma'_n$, and

$$\|\sigma''_n\|_\infty \ \le\ C\|g - \mathrm{div}\, \sigma'_n\|_d \ \to\ 0$$

as $n \to \infty$.

Choose n large enough, in order to have $\|\sigma''_n\|_\infty \le c_*\epsilon/2$, and let $\sigma = \sigma'_n + \sigma''_n$. We have $\mathrm{div}\, \sigma = g$, and $\sigma = 0$ on ∂Q, so that

$$\int_Q F(x, Du) + \int_Q g(x)u(x)\, dx \ =\ \int_Q F(x, Du) - \sigma(x)\cdot Du \ =\ \int_Q F'(x, Du)\, , \quad (38)$$

where we have let $F'(x, p) = F(x, p) - \sigma(x) \cdot p$ for any $x \in Q$ and $p \in \mathbb{R}^d$. The function F', extended by periodicity to $\mathbb{R}^d \times \mathbb{R}^d$, is still continuous in x (since σ vanishes on ∂Q), 1-homogeneous and convex in p. Moreover we have for any x

$$\sigma(x) \cdot p \ =\ \sigma'_n(x) \cdot p + \sigma''_n(x) \cdot p \ \le\ \frac{1}{1 + \epsilon} F(x, p) + \frac{c_*\epsilon}{2}|p|$$

so that

$$\begin{aligned} F'(x, p) \ &=\ F(x, p) + \sigma(x) \cdot (-p) \ \le\ F(x, p) + \frac{1}{1 + \epsilon} F(x, -p)\ \frac{c_*\epsilon}{2}|p| \\ &\le\ \left(\frac{2 + \epsilon}{1 + \epsilon} + 1\right) c^*|p|\, , \end{aligned}$$

and

$$F'(x, p) \ =\ F(x, p) - \sigma(x) \cdot p \ \ge\ \frac{\epsilon}{1 + \epsilon} F(x, p) - \frac{c_*\epsilon}{2}|p| \ \ge\ \left(\frac{1 - \epsilon}{1 + \epsilon}\right) \frac{c_*\epsilon}{2}|p|\, ,$$

hence the new F' satisfies (35) with new constants $c'_* \le c_*$ and $c^{*\prime}$ (for which a bound can be easily estimated just as we did for c'_*). Observe that (38) is equivalent to (36). □

Returning to the functional $\mathcal{E}_\varepsilon$ in (4), we see that it is expressed as

$$\mathcal{E}_\varepsilon(E) \ =\ \int_{\partial^* E \cap (\Omega \setminus \Omega_\varepsilon)} F\left(\frac{x}{\varepsilon}, \nu_E(x)\right) d\mathcal{H}^{d-1}(x) + \int_{\partial^* E \cap \Omega_\varepsilon} F'\left(\frac{x}{\varepsilon}, \nu_E(x)\right) d\mathcal{H}^{d-1}(x)$$

and its Γ-limit can be deduced from classical results.

4.2. **Weaker coercivity.** Let us now assume that, instead of (2), F, g are such that for any finite-perimeter set E in the *torus* $Q^\sharp = \mathbb{R}^d/\mathbb{Z}^d$,

$$\int_{Q^\sharp\cap\partial^*E} F(x,\nu_E(x))\,d\mathcal{H}^{d-1}(x) \;+\; \int_{Q^\sharp\cap E} g(x)\,dx \;\ge\; \delta Per(E,Q^\sharp). \tag{39}$$

This assumption is weaker than (2) — it is simple to see that it is implied by (2), see Section 4.3 for an example where it is not equivalent. On the other hand, it is much more natural, since it does not depend on the "origin" of the periodicity cell. Now, the same proof as above (still using convex duality and the result of Bourgain and Brézis, this time in the torus [10, Thm 1']) shows the existence of a periodic field $\sigma \in C^0\cap W^{1,d}(Q^\sharp)$ such that $\operatorname{div}\sigma = g$, and $F'(x,p) = F(x,p) - \sigma(x)\cdot p \ge c'_*|p|$ for any $(x,p)\in Q^\sharp\times\mathbb{R}^d$, for some constant $c'_*>0$. In particular, for any $p\in\mathbb{R}^d$ and $u\in BV(Q^\sharp)$,

$$\begin{aligned}
&\int_{Q^\sharp} F(x,p+Du) \;+\; \int_Q g(x)(p\cdot x+u(x))\,dx\\
&\qquad = \int_{\partial Q}(p\cdot x)\sigma(x)\cdot n_Q(x)\,d\mathcal{H}^{d-1}(x) \;+\; \int_{Q^\sharp} F'(x,p+Du)\\
&\qquad\qquad = \hat\sigma\cdot p \;+\; \int_{Q^\sharp} F'(x,p+Du)\,,
\end{aligned}$$

where the vector $\hat\sigma\in\mathbb{R}^d$ is defined by

$$\hat\sigma_i \;=\; \int_{\partial Q\cap\{x_i=1\}} \sigma_i(x)\,d\mathcal{H}^{d-1}(x)$$

for $i=1,\dots,d$. Here, $n_Q = -\nu_Q$ denotes the outer normal to Q. Hence the cell problem (29) can be restated as

$$\psi(p) \;=\; \hat\sigma\cdot p + \min_{u\in BV(Q^\sharp)} \int_{Q^\sharp} F'(x,p+Du)\,, \tag{40}$$

and, again, it admits a solution. Clearly, again, one can construct the plane-like minimizers as before: it is enough to build them considering only the surface energy F', then, if E_ν is such a minimizer and $E\subset\mathbb{R}^N$ is such that $E_\nu\triangle E\Subset B$,

$$\begin{aligned}
&\int_{B\cap\partial E_\nu} F(x,\nu_{E_\nu}(x))\,d\mathcal{H}^{d-1}(x) \;+\; \int_{B\cap E_\nu} g(x)\,dx\\
&\quad = \int_{B\cap\partial E_\nu} F'(x,\nu_E(x))\,d\mathcal{H}^{d-1}(x) \;+\; \int_{\partial B\cap E_\nu}\sigma(x)\cdot n_B(x)\,d\mathcal{H}^{d-1}(x)\\
&\quad \le \int_{B\cap\partial^*E} F'(x,\nu_E(x))\,d\mathcal{H}^{d-1}(x) \;+\; \int_{\partial B\cap E}\sigma(x)\cdot n_B(x)\,d\mathcal{H}^{d-1}(x)\\
&\qquad\qquad = \int_{B\cap\partial^*E} F(x,\nu_E(x))\,d\mathcal{H}^{d-1}(x) \;+\; \int_{B\cap E} g(x)\,dx\,.
\end{aligned}$$

so that E_ν is also a class A minimizer for $\mathcal{J}$. We have shown the following:

Proposition 4.2. *Theorem 1.2 still holds under assumption* (39)*. Moreover, the limit* (6) *also exists (and the more precise results in Section A).*

Hence, one could expect again the Γ-convergence of the energies $\mathcal{E}_\varepsilon$, defined in (4), to $\int_\Omega \phi(D\chi_E) = \int_\Omega \psi(D\chi_E)$. The situation is slightly more complicated. In the limit case $\delta = 0$ in (2), we can still conclude:

Proposition 4.3. *Assume (39) holds. Assume moreover that for any $E \subset Q$,*

$$\mathcal{J}_Q(E) \;=\; \int_{Q\cap\partial^*E} F(x,\nu_E(x))\, d\mathcal{H}^{d-1}(x) + \int_{Q\cap E} g(x)\, dx \;\geq\; 0\,. \tag{41}$$

Then the thesis of Theorem 2.1 still holds: $\mathcal{E}_\varepsilon$ Γ-converges to $\mathcal{E}$.

We will discuss in the end what happens whenever (41) is not satisfied.

Proof. In fact, there is almost nothing to prove. The proof of Theorem 2.1 only uses (2) for essentially two purposes: (i) to show that the measures μ_n defined in (10) are nonnegative, or, similarly, when one needs to know that the energy decreases if computed on "less cubes", and (ii) to show that if (E_n) are sets with $\sup_n \mathcal{E}_{\varepsilon_n}(E_n) < +\infty$, then they are uniformly bounded in $BV(\Omega)$. In cases (i), assumption (41) is enough. To show (ii), that is, that the (E_n) converge up to a subsequence to a finite-perimeter set E, one just notices that, after integrating by part $(1/\varepsilon_n)g(x/\varepsilon_n) = \operatorname{div}(\sigma(x/\varepsilon_n))$, we have

$$\mathcal{E}_{\varepsilon_n}(E_n) \;\geq\; c_* Per(E_n,\Omega\setminus\Omega_{\varepsilon_n}) + c'_* Per(E_n,\Omega_{\varepsilon_n}) + \int_{\partial\Omega_{\varepsilon_n}} \chi_{E_n}\sigma\left(\frac{x}{\varepsilon_n}\right)\cdot n_{\Omega_{\varepsilon_n}} d\mathcal{H}^{d-1},$$

however, the last boundary integral is uniformly bounded as $n\to\infty$ (by some constant times $\mathcal{H}^{d-1}(\partial\Omega)$), so that still, the perimeters $Per(E_n,\Omega)$ are uniformly bounded. □

Now, what happens if (39) still holds but not (41)? The example in Section 4.3 shows that the Γ-limit of $\mathcal{E}_\varepsilon$ could be strictly lower than $\mathcal{E}$. However, it is not a very natural counterexample. In fact, it still holds:

Proposition 4.4. *Assume (39) holds. Let E_n, E be finite perimeter sets such that*

$$E_n \to E, \text{ that is, } |E_n \triangle E| \to 0 \quad \text{as } n\to\infty\,,$$

$$\limsup_{\delta\to 0}\left(\limsup_{n\to\infty} Per\,(E_n, \{x\in\Omega \,:\, \operatorname{dist}(x,\partial\Omega)<\delta\})\right) \;=\; 0\,, \tag{42}$$

that is, the measures $\mathcal{H}^{d-1}\llcorner\partial^*E_n$ do not accumulate on the boundary as $n\to\infty$. Then, there holds (8).

On the other hand, (9) holds under the weaker assumption (39), see Appendix A. A consequence (which in fact is simpler to prove that Proposition (4.4)) is that if $B \Subset \Omega$ is a subdomain of Ω, then $\mathcal{E}_\varepsilon$ still Γ-converges to $\mathcal{E}$ on the restricted class of finite-perimeter sets with support in B. A more general result is a Γ-convergence of $\mathcal{E}_\varepsilon$ to $\mathcal{E}$ with "well prepared" Dirichlet boundary conditions:

Corollary 4.5. *Let $E^0 \subset \Omega$ be a finite-perimeter set, and $B \Subset \Omega$ an open set. Let E^0_ε be a recovery sequence for E^0, as provided by (9). Let $\mathcal{E}^0_\varepsilon(E) := \mathcal{E}_\varepsilon(E)$ if E is a finite-perimeter set in Ω with $E\triangle E^0_\varepsilon \subset B$, and $+\infty$ else, and let $\mathcal{E}^0(E) := \mathcal{E}(E)$ if $E\triangle E^0 \subset B$ and $+\infty$ else. Assume (39) holds. Then $\mathcal{E}^0_\varepsilon$ Γ-converges to $\mathcal{E}^0$.*

Of course, the "most natural" convergence result in this paper is this one, since both Theorem 2.1 and Proposition 4.3 treat the boundary of the integral on g in a quite arbitrary way, which in particular depends on the "origin" of the cell of periodicity, see the discussion in Section 4.3. All these results should coincide for compactly supported sets.

Proof of Proposition 4.4. We first show that the identity $\phi = \psi$ (Cor. 3.7) still holds under (39). Denote respectively ϕ' and ψ' the interfacial energies corresponding to $F'(x,p) = F(x,p) - \sigma(x)\cdot p$, given by equations (6) and (29). By Corollary 3.7, $\phi' = \psi'$, and by (40), $\psi(p) = \psi'(p) + \hat\sigma\cdot p$. Hence we must just show that for any $\nu \in \mathbb{S}^{d-1}$,

$$\phi(\nu) \;=\; \phi'(\nu) \,+\, \hat\sigma\cdot\nu\,. \tag{43}$$

Let E_ν be a class A minimizer (for $\mathcal{J}$ or the surface tension F', it is of course equivalent) as provided by Theorem 1.2. We have

$$\begin{aligned}\int_{B(0,L)\cap\partial E_\nu} F(x,\nu_{E_\nu})\,d\mathcal{H}^{d-1} + \int_{B(0,L)_1\cap E_\nu} g(x)\,dx \\ = \int_{B(0,L)\cap\partial E_\nu} F'(x,\nu_{E_\nu})\,d\mathcal{H}^{d-1} + \int_{\partial B(0,L)_1} \chi_{E_\nu}(x)\sigma(x)\cdot n_{B(0,L)_1}(x)\,dx\end{aligned} \tag{44}$$

but since, by definition, $B(0,L)_1 = \bigcup\{z+Q \,:\, z\in\mathbb{Z}^d, z+Q\subset B(0,L)\}$, the last integral is an integral on a finite union of facets of translated unit cubes, and in particular the unit normal $n_{B(0,L)_1}$ is at each point an element of the canonical basis $(e_i)_{i=1}^d$ of $\mathbb{R}^d$ (or its opposite). We denote by $\langle\chi_{E_\nu}\rangle$ the function on $\partial B(0,L)_1$ which is equal, on each facet of a cube $z+Q$, $z\in\mathbb{Z}^d$ to the average of χ_{E_ν} on the same facet (and, more precisely, of the trace of χ_{E_ν} on the boundary of $B(0,L)_1$). Then, we observe that since this new function is constant on each facet, we have

$$\begin{aligned}\int_{\partial B(0,L)_1} \langle\chi_{E_\nu}\rangle(x)\sigma(x)\cdot n_{B(0,L)_1}(x)\,dx \\ = \int_{\partial B(0,L)_1} \langle\chi_{E_\nu}\rangle(x)\hat\sigma\cdot n_{B(0,L)_1}(x)\,dx \\ = \int_{\partial B(0,L)_1} \chi_{E_\nu}(x)\hat\sigma\cdot n_{B(0,L)_1}(x)\,dx \;=\; \int_{B(0,L)_1} \hat\sigma\cdot D\chi_{E_\nu}\,.\end{aligned} \tag{45}$$

Combining (44) and (45), we find

$$\begin{aligned}\int_{B(0,L)\cap\partial E_\nu} F(x,\nu_{E_\nu})\,d\mathcal{H}^{d-1} + \int_{B(0,L)_1\cap E_\nu} g(x)\,dx \\ = \int_{B(0,L)\cap\partial E_\nu} F'(x,\nu_{E_\nu}) + \hat\sigma\cdot\nu_{E_\nu}\,d\mathcal{H}^{d-1} \\ + \int_{\partial B(0,L)_1} (\chi_{E_\nu}(x) - \langle\chi_{E_\nu}\rangle(x))\sigma(x)\cdot n_{B(0,L)_1}(x)\,dx\,.\end{aligned} \tag{46}$$

The last integral in (46) is zero except in a M-neighborhood of ∂I_ν on the boundary $\partial B(0,L)_1$, hence on a set of measure of order $\sim CML^{d-2}$. Hence, dividing (46) by $\omega_{d-1}L^{d-1}$ and sending L to infinity, we find (43), which shows that $\phi = \psi$.

Now, let E_n, E be as in the thesis of Proposition 4.4. We have

$$\begin{aligned}\mathcal{E}_{\varepsilon_n}(E_n) \;\geq\; \int_{\Omega_{\varepsilon_n}\cap\partial^* E_n} F'\left(\frac{x}{\varepsilon_n}, \nu_{E_n}(x)\right)\,d\mathcal{H}^{d-1}(x) \\ + \int_{\partial\Omega_{\varepsilon_n}} \chi_{E_n}(x)\sigma\left(\frac{x}{\varepsilon_n}\right)\cdot n_{\Omega_{\varepsilon_n}}(x)\,d\mathcal{H}^{d-1}(x).\end{aligned} \tag{47}$$

By Theorem 2.1 (or standard results [6] [1] [11]),

$$\liminf_{n\to\infty} \int_{\Omega_{\varepsilon_n}\cap\partial^* E_n} F'\left(\frac{x}{\varepsilon_n}, \nu_{E_n}(x)\right)\,d\mathcal{H}^{d-1}(x) \;\geq\; \int_{\partial^* E} \phi'(\nu_E(x))\,d\mathcal{H}^{d-1}(x)\,. \tag{48}$$

On the other hand, introducing as before the functions $\langle \chi_{E_n} \rangle$, average of χ_{E_n} on the faces of the cubes $\varepsilon_n(z+Q)$, $z \in \mathbb{Z}^d$ which constitute $\partial\Omega_{\varepsilon_n}$ (while $\pm\hat{\sigma}_i$ is still the average of $\sigma(x/\varepsilon_n)\cdot n_{\Omega_{\varepsilon_n}}$ on the facets with $n_{\Omega_{\varepsilon_n}} = \pm e_i$), we find

$$\int_{\partial\Omega_{\varepsilon_n}} \chi_{E_n}(x)\sigma\left(\frac{x}{\varepsilon_n}\right)\cdot n_{\Omega_{\varepsilon_n}}(x)\, d\mathcal{H}^{d-1}(x)$$
$$= \int_{\partial\Omega_{\varepsilon_n}} (\chi_{E_n}(x) - \langle \chi_{E_n}\rangle(x))\sigma\left(\frac{x}{\varepsilon_n}\right)\cdot n_{\Omega_{\varepsilon_n}}(x)\, d\mathcal{H}^{d-1}(x) \;+\; \int_{\Omega_{\varepsilon_n}} \hat{\sigma}\cdot D\chi_{E_n}$$

We claim that assumption (42) yields

$$\lim_{n\to\infty} \int_{\partial\Omega_{\varepsilon_n}} (\chi_{E_n}(x) - \langle \chi_{E_n}\rangle(x))\sigma\left(\frac{x}{\varepsilon_n}\right)\cdot n_{\Omega_{\varepsilon_n}}(x)\, d\mathcal{H}^{d-1}(x) \;=\; 0\,, \tag{49}$$

so that we deduce from (47) and (48) that

$$\liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(E_n) \;\geq\; \int_{\partial^* E} \phi'(\nu_E(x))\, d\mathcal{H}^{d-1}(x) \;+\; \int_\Omega \hat{\sigma}\cdot D\chi_E\,,$$

which reduces to (8) by (43). Hence the proposition holds if we show (49). In fact, let $z \in \mathbb{Z}^d$ such that $\varepsilon_n(z+Q) \subset \Omega_{\varepsilon_n}$, and assume $\varepsilon_n(z+\partial Q)\cap\partial\Omega_{\varepsilon_n} \neq \emptyset$. Standard estimates show that there exists $C > 0$ (depending only on d) with

$$\int_{\varepsilon_n(z+\partial Q)\cap\partial\Omega_{\varepsilon_n}} |\chi_{E_n}(x) - \langle \chi_{E_n}\rangle(x)|\, d\mathcal{H}^{d-1}(x) \;\leq\; C|D\chi_{E_n}|(\varepsilon_n(z+Q))\,,$$

so that

$$\int_{\partial\Omega_{\varepsilon_n}} |\chi_{E_n}(x) - \langle \chi_{E_n}\rangle(x)|\, d\mathcal{H}^{d-1}(x) \;\leq\; C\,Per(E_n, \{x \in \Omega\,:\, \mathrm{dist}(x,\partial\Omega) \leq 2\sqrt{d}\varepsilon_n\}).$$

Hence we deduce (49) from (42). □

4.3. **A simple example.** We now discuss a very basic example (with a $g(x)$ taking only two values) to illustrate the differences between our various coercivity assumptions. It will show how the choice of the origin of the periodicity cell has an influence on the results (which is mostly artificial and due to the particular form of our external field term in (4)). We consider the two-dimensional case ($d = 2$). We choose $F(x,\nu) = 1$ and define $g \in L^d(Q^\sharp)$ as follows: for $a > 0$ we let $g(x) = -a$ if $0 < x_1 < 1/2$ and $g(x) = a$ if $1/2 < x_1 < 1$. Observe that $g = \operatorname{div}\sigma$, where for any $x = (x_1, x_2) \in Q$,

$$\sigma(x) \;=\; \begin{cases} (-ax_1, 0)^T & \text{if } 0 < x_1 < \frac{1}{2}\,,\\ (a(x_1-1), 0)^T & \text{if } \frac{1}{2} < x_1 < 1\,.\end{cases}$$

Hence if $E \subset Q$,

$$Per(E,Q) \;+\; \int_E g(x)\,dx \;=\; \int_{\partial^* E} (1 - \sigma\cdot\nu_E(x))\, d\mathcal{H}^1(x) \;\geq\; (1-\frac{a}{2})Per(E,Q).$$

Hence we see that if $a < 2$, (2) holds, while if $a = 2$, (41) holds. On the other hand, as soon as $a > 2$, neither (2) nor (41) do hold, as shown by the set $E = \{x \in Q : x_1 < 1/2\}$: we have $Per(E,Q) + \int_E g(x)\,dx = 1 - a/2 < 0$.

Now, what about (39)? If we show that it holds for some $a > 2$, then, for instance, Proposition (4.2) applies and $\mathcal{E}_\varepsilon$ Γ-converges to $\mathcal{E}$ also when $a = 2$. Notice, in this case, that $\phi(-1,0) = 0$, the class A minimizer corresponding to this direction being given by $E_{(-1,0)} = \{x_1 < 1/2\} \subset \mathbb{R}^2$.

We have the following relative isoperimetric inequality in the torus $Q^\sharp = \mathbb{R}^2/\mathbb{Z}^2$:

Lemma 4.6. *For any $E \subset Q^\sharp$ with $|E| \le 1/2$,*

$$|E| \;\le\; \frac{1}{8} Per(E, Q^\sharp)^2 \tag{50}$$

and the constant $1/8$ is optimal.

Hence: one has for any $E \subset Q^\sharp$

$$\begin{aligned} &Per(E, Q^\sharp) + \int_E g(x)\,dx \\ &\qquad \ge\; Per(E, Q^\sharp) - a\min\{|E|, |Q^\sharp \setminus E|\} \;\ge\; \left(1 - \frac{a}{8} Per(E, Q^\sharp)\right) Per(E, Q^\sharp) \end{aligned}$$

If $a < 4$, choosing a' with $a < a' < 4$, we can find $\delta > 0$ such that $Per(E, Q^\sharp) - a\min\{|E|, |Q^\sharp \setminus E|\} \ge Per(E, Q^\sharp) - a/2 \ge \delta Per(E, Q^\sharp)$ whenever $Per(E, Q^\sharp) \ge a'/2$. On the other hand, if $Per(E, Q^\sharp) < a'/2$, we have $(1 - aPer(E, Q^\sharp)/8) > 1 - aa'/16 > 0$, hence possibly choosing a smaller δ we get that (39) holds. If $a = 4$, it clearly does not hold since the set $E = \{0 < x_1 < 1/2\}$ has zero energy, while if $a > 4$, its energy is $2 - a/2 < 0$. Hence the bound 4 is optimal. In particular, we can conclude that actually for $a = 2$, Proposition (4.2) is true.

Proof of Lemma 4.6. Let E_n be a minimizing sequence for $Per(E, Q^\sharp)/\sqrt{|E|}$ under the constraint $|E| \le 1/2$. If $|E_n| \to 0$, also $Per(E_n, Q^\sharp) \to 0$, however in this case one can check for instance after an appropriate blow-up that the limit set should satisfy the isoperimetric equality in $\mathbb{R}^2$, hence it is a disc, and the ratio goes to $2\sqrt{\pi}$. If $|E_n|$ does not go to zero, we may assume E_n converges to some set E (in L^1) and we find that $Per(E, Q^\sharp)/\sqrt{|E|}$ is optimal (in particular, standard regularity results show that ∂E is analytic). Assume there exists $s, t \in (0, 1)$ such that ∂E does not cross neither $\{x_1 = s\}$ nor $\{x_2 = t\}$. Then, $(E - (s, t)) \cap Q$ is a subset of $\mathbb{R}^2$ which is optimal for the isoperimetric ratio, hence a disc. In the other case, we have for instance that $\{x_1 = s\} \cap \partial E$ for any s, and for a.e. s, this contains at least two points. Hence, integrating over $s \in (0, 1)$ we get $Per(E, Q^\sharp) \ge 2$. But in this case, the optimal set is a strip of width $1/2$ (for instance $E = \{0 < x_1 < 1/2\}$), and the ratio is $2\sqrt{2}$ (which is less than 2π). This proves the Lemma. ☐

Now, we consider the case where $2 < a < 4$, so that (39) holds and not (2). Let us explain why the Γ-limit of $\mathcal{E}_\varepsilon$ might be strictly below $\mathcal{E}$ in this case. In fact, this is very simple: let $\Omega = (0, 1)^2$ and $E \subset \Omega$ a finite perimeter set with smooth boundary. Let $\varepsilon_n = 1/n$ and E_n be the recovery sequence in (9). In this case, we can choose $\Omega_{\varepsilon_n} = \Omega$ for each n (although strictly speaking, with our definition, it should be $[1/n, 1) \times [1/n, 1)$). Set now $\hat{E}_n = E_n \cup ((0, 1/(2n)) \times (0, 1))$: we add to E_n a little strip where $g = -a$. Then, each time a cube $(0, 1/n) \times (k/n, (k + 1)/n)$ does not meet E_n, the additional energy is $1/n - n \times (a/(2n^2))$. Hence, if we let $\Sigma = \{s \in (0, 1) : (0, s) \in \overline{E}\}$, we get for n large enough

$$\mathcal{E}_{\varepsilon_n}(\hat{E}_n) \;\approx\; \mathcal{E}_{\varepsilon_n}(E_n) \,+\, (1 - |\Sigma|)\left(1 - \frac{a}{2}\right)$$

So that $\limsup_{n\to\infty} \mathcal{E}_{\varepsilon_n}(\hat{E}_n) < \mathcal{E}(E)$ as soon as $|\Sigma| < 1$. Of course, in some sense our sets $\hat{E}_n$ now converge to $E \cup \{0\} \times (0, 1)$ rather than E: this shows that in order to get still the convergence of $\mathcal{E}_\varepsilon$ to $\mathcal{E}$, one actually needs to impose some kind of Dirichlet boundary conditions on the sets (Cor. 4.5).

Of course, all this is a bit artificial: if we translate now g by $(1/4, 0)$: $g(x) = a$ if $0 < x_1 < 1/4$ or $3/4 < x_1 < 1$, and $-a$ if $1/4 < x_1 < 3/4$, and let now $\sigma = (ax_1, 0)^T$ if $0 < x_1 < 1/4$, $(-a(x_1-1/2), 0)^T$ if $1/4 < x_1 < 3/4$, $(a(x_1-1), 0)^T$ if $3/4 < x_1 < 1$, then again $g = \operatorname{div}\sigma$, but now if $E \subset Q$

$$Per(Q,E) + \int_E g(x)\,dx \;=\; \int_{Q\cap\partial^* E} (1-\sigma(x)\cdot\nu_E(x))\,d\mathcal{H}^1(x) \;\geq\; \left(1-\frac{a}{4}\right) Per(Q,E)$$

so that now the optimal a is the same for (2) and (39) (the latter is of course more natural, since independent on the (arbitrary) origin of the period which is chosen for defining $\mathcal{E}_\varepsilon$).

A question which is natural, is whether there exists (still for $F(x,p) = |p|$) a periodic $g \in L^d(Q^\sharp)$ such that (39) holds, while (2) never holds for any translation $g(\cdot - y)$, $y \in Q$: that is, whether there is really a difference between conditions (2) and (39). We do not know the answer, although it is likely to be true.

Appendix A. **Proof of** (6) **and some more general statements.** In this appendix, we prove (6), under the assumption that the set E_ν (which in fact may vary with L) is a class A minimizers whose boundary is contained in a strip of width $2M$. In fact, neither (1) nor (2) are really necessary for this section: as long as the minimizers exist and satisfy $\partial E_\nu \subset \{|x\cdot\nu| \leq M\}$, we just use the fact that $F(x,p) \leq c^*|p|$ for any x and p, and $g \in L^d(Q)$ with $\int_Q g\,dx = 0$. (Hence (6) also holds with the milder assumption (39), see Section 4.2.) The techniques in this section are quite classical and we merely adapt them to our cases, see for instance [14] [3].

For each $\nu \in \mathbb{S}^{d-1}$, we let Q^ν be the unit open cube $(-1/2, 1/2)^d$ rotated in a way such that ν is orthogonal to one face, and Q^ν_ε, as before, is the union of all cubes $z + \varepsilon Q \subset Q^\nu$ with $z \in \varepsilon\mathbb{Z}^d$. As before, $I_\nu = \{x \in \mathbb{R}^d : x\cdot\nu > 0\}$.

Let us first show the following lemma, which is quite standard (a variant is proven in [14]):

Lemma A.1. *We consider $g \in L^d(Q)$ with $\int_Q g\,dx = 0$, and $F(x,p)$ an interfacial energy (continuous, periodic in x and convex, one-homogeneous in p) with $F(x,p) \leq c^*|p|$ for any $x, p \in \mathbb{R}^d$. We assume that for each $\nu \in \mathbb{S}^{d-1}$, there exists a class A minimizer E_ν for $\mathcal{J}$ which satisfies point (i) of Theorem 1.2.*

Then, there exists $\phi(\nu)$ a bounded function, such that for any $\varepsilon_k \downarrow 0$ and any sequence of class A minimizers E^k_ν for $\mathcal{J}$ with $\partial E^k_\nu \subset \{|x\cdot\nu| \leq M\}$ for each k (so that, in particular, $\varepsilon_k E^k_\nu \to I_\nu$ as $k \to \infty$),

$$\phi(\nu) \;=\; \lim_{k\to\infty} \int_{\partial^*(\varepsilon_k E^k_\nu)\cap Q^\nu} F\left(\frac{x}{\varepsilon_k}, \nu_{(\varepsilon_k E^k_\nu)}(x)\right) d\mathcal{H}^{d-1}(x) \;+\; \frac{1}{\varepsilon_k}\int_{Q^\nu_{\varepsilon_k}\cap(\varepsilon_k E^k_\nu)} g\left(\frac{x}{\varepsilon_k}\right) dx\,. \tag{51}$$

Proof. We follow [14] and a similar proof in [15]. Observe first that for any $E \subset \mathbb{R}^d$ which is a class A minimizer of $\mathcal{J}$, by definition if Q' is any translate of $Q = [0,1)^d$ we have, comparing E with $E \setminus Q'$,

$$\mathcal{E}_1(E, \overline{Q'}) \;\leq\; \int_{\partial Q'\cap E} F(x, n_{Q'}(x))\,d\mathcal{H}^{d-1}(x) \;\leq\; c^*\,Per(Q),$$

(where here, $n_{Q'} = -\nu_{Q'}$ is the outer normal to $\partial Q'$) so that

$$\int_{\overline{Q'}\cap\partial^* E} F(x,\nu_E(x))\,d\mathcal{H}^{d-1}(x) \;\le\; 2dc^* \,+\, \|g\|_d \tag{52}$$

is bounded by a universal constant which depends only on g and the dimension.

We first prove that the limit (if it exists) must be bounded. The integrals in (51) can be written as a sum of integrals on small cubes $\varepsilon_k(z+Q)$, $z\in\mathbb{Z}^d$ of volume ε_k^d. Most of these contributions are zero, the only which may have a positive or negative contribution lie in the strip $\mathcal{S}_k = \{x\in\mathbb{R}^d : \operatorname{dist}(x, Q_\nu\cap\partial I_\nu)\le \varepsilon_k(M+\sqrt{d})\}$. When non zero, the contribution is (by (52)) between $-\varepsilon_k^{d-1}\|g\|_d$ and $\varepsilon_k^{d-1}(2dc^*+\|g\|_d)$. Hence,

$$-\lim_{k\to\infty}\varepsilon_k^{-1}\|g\|_d|\mathcal{S}_k| \;\le\; \phi(\nu) \;\le\; \lim_{k\to\infty}\varepsilon_k^{-1}(2dc^*+\|g\|_d)|\mathcal{S}_k|$$

and since $|\mathcal{S}_k| = 2\varepsilon_k(M+\sqrt{d}) + o(\varepsilon_k)$ as $\varepsilon_k\to 0$, we deduce

$$-2\|g\|_d(M+\sqrt{d}) \;\le\; \phi(\nu) \;\le\; 2(2dc^*+\|g\|_d)(M+\sqrt{d}). \tag{53}$$

Now, consider $\varepsilon > \varepsilon' > 0$ such that $\varepsilon' << \varepsilon$, and let E_ν, E'_ν be two class A minimizers of $\mathcal{E}$ with $\partial E_\nu\cup\partial E'_\nu \subset \{|x\cdot\nu|\le M\}$. We make the following construction. First of all, we can cover $\partial I_\nu\cap(1/\varepsilon')Q^\nu$ with $N = [(\varepsilon/\varepsilon')/(1+2\varepsilon\sqrt{d})]^{d-1}$ disjoint translates of $(2\sqrt{d}+1/\varepsilon)Q^\nu$, each centered on ∂I_ν (here, $[\,\cdot\,]$ denotes the integer part). Strictly inside each of these cubes (meaning at positive distance from the boundary), there is at least a translate of $(1/\varepsilon)Q^\nu$ which is centered on an point of $\mathbb{Z}^d$. We denote by $(Q_i)_{i=1}^N$ these translates. We also denote by $E_i\subset Q_i$ the corresponding translate of $E_\nu\cap(1/\varepsilon)Q^\nu$, and let

$$E' \;=\; \left(E'_\nu\setminus\bigcup_{i=1}^N Q_i\right)\cup\left(\bigcup_{i=1}^N E_i\right).$$

Then (observing that E'_ν and E' are identical on all cubes $z+Q$, $z\in\mathbb{Z}^d$, which are not contained in $(1/\varepsilon')Q^\nu$), we have by class A minimality of $\mathcal{E}'_\nu$:

$$\mathcal{E}_1(E'_\nu,\frac{1}{\varepsilon'}Q^\nu) \;\le\; \mathcal{E}_1(E',\frac{1}{\varepsilon'}Q^\nu)\,.$$

That is, denoting $R=\bigcup_{i=1}^N Q_i$ and S the union of the cubes $z+Q$, $z\in\mathbb{Z}^d$, with $z+Q\subset(1/\varepsilon')Q^\nu$ and $z+Q\not\subset R$,

$$\mathcal{E}_1(E'_\nu,\frac{1}{\varepsilon'}Q^\nu) \;\le\; N\mathcal{E}_1(E_\nu,\frac{1}{\varepsilon}Q^\nu) + \int_{\partial E'\cap(1/\varepsilon')Q^\nu\setminus R} F(x,\nu_{E'}(x))\,d\mathcal{H}^{d-1}(x) \,+\, \int_{S\cap E'} g(x)\,dx\,. \tag{54}$$

Let us decompose the "rest" in the previous estimate as follows:

$$\int_{\partial E'\cap(1/\varepsilon')Q^\nu\setminus R} F(x,\nu_{E'}(x))\,d\mathcal{H}^{d-1}(x) \,+\, \int_{S\cap E'} g(x)\,dx \le \int_{\partial E'_\nu\cap(1/\varepsilon')Q^\nu\setminus\overline{R}} F(x,\nu_{E'_\nu}(x))\,d\mathcal{H}^{d-1}(x) \,+\, c^*\mathcal{H}^{d-1}(\partial R\cap\partial E') + \int_{S\cap E'} g(x)\,dx \;=\; (I)+(II)+(III)\,. \tag{55}$$

By (52), (I) is bounded by a constant (C) times the number of cubes $z+Q$, $z\in\mathbb{Z}^d$, which intersect $\partial E'_\nu \subset \{|x\cdot\nu|\le M\}$. Hence,

$$\begin{aligned}(I) \;&\le\; C(M+\sqrt{d})\times\left\{\left(\frac{1}{\varepsilon'}+2\sqrt{d}\right)^{d-1} - N\left(\frac{1}{\varepsilon}-2\sqrt{d}\right)^{d-1}\right\}\\ &= \frac{C(M+\sqrt{d})}{\varepsilon'^{d-1}}\left\{\left(1+2\varepsilon'\sqrt{d}\right)^{d-1} - \left(\frac{\varepsilon'}{\varepsilon}\left[\frac{\varepsilon}{\varepsilon'}\frac{1}{1+2\varepsilon\sqrt{d}}\right](1-2\varepsilon\sqrt{d})\right)^{d-1}\right\}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \frac{A_I(\varepsilon',\varepsilon)}{\varepsilon'^{d-1}}\end{aligned}$$

where $A_I(\varepsilon',\varepsilon)\to 0$ if $\varepsilon\to 0$, $\varepsilon'\to 0$ and $\varepsilon'/\varepsilon\to 0$.

On the other hand, (II) is bounded by the total surface of $\partial R\cap\{|x\cdot\nu|\le M+2\sqrt{d}\}$:

$$(II)\;\le\; c^*N\frac{M+2\sqrt{d}}{\varepsilon^{d-2}}\;\le\; c^*\varepsilon\frac{M+2\sqrt{d}}{\varepsilon'^{d-1}}\left(\frac{\varepsilon'}{\varepsilon}\left[\frac{\varepsilon}{\varepsilon'}\frac{1}{1+2\varepsilon\sqrt{d}}\right]\right)^{d-1} \;=\; \frac{A_{II}(\varepsilon',\varepsilon)}{\varepsilon'^{d-1}}$$

where again, $A_{II}(\varepsilon',\varepsilon)\to 0$ if $\varepsilon\to 0$, $\varepsilon'\to 0$ and $\varepsilon'/\varepsilon\to 0$.

Then, $(III)=\int_{S\cap E'} g(x)\,dx$ is bounded by $\|g\|_d$ times the number of cubes $z+Q$ ($z\in\mathbb{Z}^d$) in S which meet $\partial E'$: again, since by construction $\partial E'\subset\{|x\cdot\nu|\le M+\sqrt{d}\}$, all these cubes lie in the strip $\{|x\cdot\nu|\le M+2\sqrt{d}\}$ and since they must not meet $R=\bigcup_{i=1}^N Q_i$ we find

$$\begin{aligned}(III)\;&\le\; \|g\|_d(M+2\sqrt{d})\left\{\frac{1}{\varepsilon'^{d-1}} - N\frac{1}{\varepsilon^{d-1}}\right\}\\ &= \frac{\|g\|_d(M+2\sqrt{d})}{\varepsilon'^{d-1}}\left\{1-\left(\frac{\varepsilon'}{\varepsilon}\left[\frac{\varepsilon}{\varepsilon'}\frac{1}{1+2\varepsilon\sqrt{d}}\right]\right)^{d-1}\right\} = \frac{A_{III}(\varepsilon',\varepsilon)}{\varepsilon'^{d-1}},\end{aligned}$$

where as before, $A_{III}(\varepsilon',\varepsilon)$ goes to zero if $\varepsilon,\varepsilon',\varepsilon'/\varepsilon$ go to zero.

Hence, letting $A(\varepsilon',\varepsilon)=A_I(\varepsilon',\varepsilon)+A_{II}(\varepsilon',\varepsilon)+A_{III}(\varepsilon',\varepsilon)$, we find that the "rest" in (54), that is, (55), is less than $A(\varepsilon',\varepsilon)/\varepsilon'^{d-1}$ where $A(\varepsilon',\varepsilon)$ goes to zero if $\varepsilon,\varepsilon',\varepsilon'/\varepsilon$ go to zero.

Consider now two possible limits a and a' of (51), along two different sequences ε_k and ε'_k (and E^k_ν, E'^k_ν the corresponding sequences of minimizers). Upon extracting a subsequence (and relabelling appropriately), we may assume that $\varepsilon'_k/\varepsilon_k\to 0$ as $k\to\infty$. Then, after an appropriate rescaling, (54) shows that

$$\mathcal{E}_{\varepsilon'_k}(\varepsilon'_k E'^k_\nu, Q^\nu)\;\le\;\left(\frac{\varepsilon'_k}{\varepsilon_k}\right)^{d-1} N_k\mathcal{E}_{\varepsilon_k}(\varepsilon_k E^k_\nu, Q^\nu)\;+\;A(\varepsilon'_k,\varepsilon_k)$$

where $N_k=[(\varepsilon_k/\varepsilon'_k)/(1+2\varepsilon_k\sqrt{d})]^{d-1}$. As $k\to\infty$, we deduce $a'\le a$. This shows the lemma. □

It then follows:

Corollary A.2. *Let $A\subset\mathbb{R}^d$ be an open set and let $I_\nu=\lim_{\varepsilon\to 0}(\varepsilon E_\nu)=\{x : x\cdot\nu\ge 0\}$. Then,*

$$\begin{aligned}&\mathcal{H}^{d-1}(\partial I_\nu\cap A)\phi(\nu)\\ &\quad= \lim_{\varepsilon\to 0}\int_{\partial^*(\varepsilon E_\nu)\cap A} F\left(\frac{x}{\varepsilon},\nu_{(\varepsilon E_\nu)}(x)\right)\,d\mathcal{H}^{d-1}(x) + \frac{1}{\varepsilon}\int_{A_\varepsilon\cap(\varepsilon E_\nu)} g\left(\frac{x}{\varepsilon}\right)\,dx \qquad (56)\end{aligned}$$

Observe that after a suitable rescaling, (6) follows from (56) taking $A = B(0,1)$ and $\varepsilon = 1/L$.

Proof. For any $n \geq 1$, we simply cover $\partial I_\nu \cap A$ by finitely many disjoint translates of $(1/n)Q^\nu$, centered on ∂I_ν, so that (denoting by R_n the union of all these cubes), $\mathcal{H}^{d-1}(\partial I_\nu \cap (A \setminus R_n)) \to 0$ as $n \to \infty$. Then, we estimate the error as in the proof of boundedness of ϕ in the previous lemma, to deduce from (51) show that both

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\varepsilon E_\nu, A) \;\geq\; \phi(\nu)\mathcal{H}^{d-1}(\partial I_\nu \cap R_n) \,-\, C\mathcal{H}^{d-1}(\partial I_\nu \cap (A \setminus R_n))$$

and

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\varepsilon E_\nu, A) \;\leq\; \phi(\nu)\mathcal{H}^{d-1}(\partial I_\nu \cap R_n) \,+\, C\mathcal{H}^{d-1}(\partial I_\nu \cap (A \setminus R_n))$$

for any n, where C is some constant. Sending $n \to \infty$, we deduce (56). □

Corollary A.3. *Let $E \subset \Omega$ be a polyhedral set, that is, such that $\partial E \cap \Omega$ is made of finitely many subsets of $x_i + \partial I_{\nu_i}$, for $x_i \in \mathbb{R}^d$, $\nu_i \in \mathbb{S}^{d-1}$, $i = 1, \ldots, N$ (and where ν_i coincides with ν_E). Then, there exist sets $E_\varepsilon \to E$ such that*

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(E_\varepsilon) \;\leq\; \mathcal{E}(E) \tag{57}$$

Proof. We sketch the proof. First, for any $\eta > 0$, we can cover ∂E with disjoint cylinders $A_i = \omega_i + (-\eta', \eta')\nu_i \subset \Omega$, $\eta' > 0$ small, where $\omega_i \subset (x_i + \partial I_{\nu_i}) \cap \partial E$, $\nu_i = \nu_E$ on ω_i, and $\mathcal{H}^{d-1}(\Omega \cap (\partial E \setminus \bigcup_{i=1}^N \omega_i)) \leq \eta$.

Then, we let for $\varepsilon > 0$ small enough (in particular, than η'/M)

$$E_\varepsilon \;=\; \left(E \setminus \bigcup_{i=1}^N A_i\right) \cup \left(\bigcup_{i=1}^N (x_i + \varepsilon E_{\nu_i}) \cap A_i)\right)$$

where each E_{ν_i} is a class A minimizer of $\mathcal{J}$ which satisfies *(ii)* in Theorem 1.2. Then, an accurate estimate of the error as in the previous proofs will show that

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(E_\varepsilon) \;\leq\; \sum_{i=1}^N \phi(\nu_i)\mathcal{H}^{d-1}(\omega_i) \,+\, C\eta$$

so that Corollary A.3 follows from a diagonal argument. □

If we assume that (2) holds, it now follows from Corollary A.3 and the estimate (8) that $\mathcal{E}_\varepsilon$ Γ-converge to $\mathcal{E}$ in the class of polyhedral sets (in particular, the lim sup in (57) is a limit). We deduce in particular (using quite standard semicontinuity arguments) the following:

Corollary A.4. *The function ϕ, extended to $\mathbb{R}^d$ by one-homogeneity, that is letting $\phi(p) = |p|\phi(p/|p|)$ if $p \neq 0$ and $\phi(0) = 0$, is convex (hence Lipschitz-continuous).*

In fact, still assuming "only" the same assumptions as in Lemma A.1, Corollary A.4 still holds. The proof is identical to the proof of [14, Lem. 10.2] whose idea is as follows: we choose ν_1, ν_2, $\nu = (\nu_1 + \nu_2)/|\nu_1 + \nu_2|$, and for any $\delta > 0$ we compare the energy in Q_ν of the "plane" $\varepsilon\partial E_\nu$, with the energy of the approximation E_ε provided by Corollary A.3 of a polyhedron E such that $\partial E \subset \{|x \cdot \nu| \leq \delta\}$ and $\nu_E = \nu_1$ on half of $\partial E \cap Q_\nu$, and ν_2 on the other half. Letting $\varepsilon \to 0$ we find $\phi(\nu) \leq (\phi(\nu_1) + \phi(\nu_2))/|\nu_1 + \nu_2| + C\delta$, and letting $\delta \to 0$ we deduce the convexity of the one-homogeneous extension of ϕ.

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