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ON FLUIDO-DYNAMIC MODELS FOR URBAN TRAFFIC

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Abstract. The aim of this paper is to address the following questions: which models, among fluido-dynamic ones, are more appropriate to describe urban traffic? While a rich debate was developed for the complicate dynamics of highway traffic, some basic problems of urban traffic are not always appropriately discussed. We analyze many recent, and less recent, models focusing on three basic properties. The latter are necessary to reproduce correctly queue formation at lights and junctions, and their backward propagation on an urban network.

1. **Introduction.** To study nontrivial phenomena in highways traffic, researchers from various areas (engineering, mathematics, physics) proposed a cornucopia of models, among which fluido-dynamic ones. The main aim of such modeling effort is to reproduce some behavior as: synchronized flows, wide jams, relaxation times to equilibrium velocities, etc. Of great impact in the scientific community it was the book [29] and prior work by the same author.

Fluido-dynamic models treat traffic from a macroscopic point of view: just the evolution of macroscopic variables, such as density and average velocity of cars, is considered. Since the basic model of Lighthill-Whitham-Richards, introduced in the 50s in [33, 36] and based on a single partial differential equation in conservation form, does not reproduce the observed rich dynamics (see e.g. [3, 27, 28]), many alternatives were searched for.

Some second order models, i.e. systems with two equations, were proposed since 70s by Payne [35] and by Whitham [38]. However, as showed by Daganzo in 1995 [12], all these second order models suffered of a main drawback: cars may travel backwards on unidirectional roads! A resurrection of such models happened with the work of Aw and Rascle [2] in 2000 and Zhang [40] in 2002. These papers became a starting point for a lot of other traffic models and derivations. Independently a third order model was proposed by Helbing in 1995, see [22]. We refer the reader to [4, 17, 22] for an account of the various approaches and also to [27, 28] for a

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derivation of the continuum Helbing's model from a mesoscopic gas-kinetic flow model.

The typical features of highways are long and straight streets, high speeds (or big differences in velocities), no flux interruption as traffic light or yielding signs, etc. The situation of urban traffic is quite different: short road segments, many flux interruptions, reduced speeds, intricate road networks, and so on. Classically, engineers used static models to capture some features, but the latter were not able to capture the main issue of urban traffic: queue formation and backward propagation. Thus, various *within day* models were considered (see [1]) and, only recently, some fluido-dynamics models, as the LWR equation, were completed with junction traffic assignments so to deal with real networks, see [6, 9, 15, 16, 17, 21, 23, 24, 25, 26, 30, 31, 32].

While a significant debate occurred on how the various models reproduce the highway traffic phenomena, see [4, 10, 11, 13, 14, 22, 29], less papers focused on typical issues of urban traffic. Inspired also by the basic problem considered by Daganzo to criticize second order models [12], the aim of the present paper is to analyze some basic properties, which we consider necessary to model appropriately urban road networks. Namely, we focus on the following three properties:

- **P1:** Cars may have only nonnegative speed.
- **P2:** Vehicles stop only at maximum density, i.e. the velocity v is 0 if and only if the density ρ is equal to the maximum density possible ρ_{max}.
- **P3:** The density at a red traffic light is the maximum possible, i.e. ρ_{max}.

Property **P1** says that cars can not go backward in unidirectional roads. It seems an obvious property for a traffic model, even if, as mentioned above, in the past some proposed second order models did not satisfy **P1**. Property **P2** says that cars have zero velocity only when the road is saturated and a queue is formed. Clearly property **P2** implies **P3** when a generic state can be connected with the zero-velocity state through waves with negative speed. The converse in general is not true. Thus it is reasonable to ask if systems, which do not satisfy **P2**, do satisfy at least **P3**.
Resuming, properties **P1**, **P2** and **P3** are necessary to correctly describe the formation of queues at lights and junctions, together with their backward propagation.

Along the paper the models are divided in scalar, second order and third order models. In Section 2 we consider the first order model introduced by Lighthill, Whitham and Richards (see [33] and [36]) and a multipopulation model; see [5] and [39]. Section 3 deals with second order models. In particular we deal with the Aw-Rascle-Zhang model (see [2, 40]), with the Greenberg-Klar-Rascle multilane model (see [18, 19]), with the Siebel-Mauser balanced vehicular model (see [37]) and with some phase transition models (see [8, 20]). Finally Section 4 deals with the third order model, introduced by Helbing [22] in 1995. In Section 5, we consider a property **P2'**, which is a relaxation with respect to **P2**, since it requires that cars may stop only if the density is sufficiently big. We show that the phase transition model, introduced by Goatin [20], is the only model satisfying **P2'**, but not **P2**.

The main results of the paper analysis are summarized in a table in Section 6. Resuming, the basic Lighthill-Whitham-Richards model, the multipopulation and the Colombo's phase transition hyperbolic models seem the more appropriate to address urban traffic modeling. However, also Siebel-Mauser BVT model and the Greenberg-Klar-Rascle multilane model satisfy some of the required properties. The discussion about the Helbing model is more involved and depends on the choice of functions appearing in the model description.

2. **First order models.**

This sections contains the analysis of the Lighthill-Whitham-Richards (LWR) model and an extension to the case of a multipopulation model, i.e. a model, which considers drivers with different characteristics. Throughout this section, we assume that the road is unidimensional and unidirectional; hence we model it with the real line $\mathbb{R}$.

2.1. **Lighthill-Whitham-Richards model.**

We denote with $\rho(t,x)$ and $v(t,x)$ respectively the density and the average speed of cars at the position $x \in \mathbb{R}$ and at time $t \geq 0$. The Lighthill-Whitham-Richards (LWR) model is simply described by the continuum equation

$$\rho_t + (f(\rho))_x = 0, \tag{1}$$

where $f(\rho) = \rho v(\rho)$ is the flux function. The basic assumptions for the model are the following ones:

(LWR1): the density ρ is positive and lower than or equal to ρ_{max};
(LWR2): f is a strictly concave function on $[0, \rho_{max}]$ such that $f(\rho_{max}) = 0$.

The following results hold.

Proposition 1. *Assume (LWR1)–(LWR2). Equation (1) is a strictly hyperbolic equation in conservation form and the characteristic field is genuinely nonlinear.*

This is a well known result; for a proof see for example [17]. Let us consider now properties **P1**, **P2** and **P3**.

Proposition 2. *Assume (LWR1)–(LWR2). The LWR model (1) satisfies properties **P1**, **P2** and hence **P3**.*

Proof. These properties are clearly satisfied since the fundamental hypothesis (LWR2) holds and $v = \frac{f(\rho)}{\rho}$. □

2.2. **A multipopulation model.**

Multipopulation models are extensions of the LWR model and their aim is to predict the behavior of different heterogeneous drivers. In this subsection we consider a model introduced by Benzoni-Gavage and Colombo [5] and independently by Wong and Wong [39].

Consider an unidimensional road of infinite length, modeled by $\mathbb{R}$ and the following system of n equations

$$\partial_t \rho_i + \partial_x(\rho_i v_i) = 0, \quad i = 1, \ldots, n, \tag{2}$$

where $\rho_i(t,x)$ is the density of cars belonging to the i-th class (or population) of drivers and v_i is the average speed of the i-th family and it is a function depending on $(\rho_1, \ldots, \rho_n)$. For example, different populations can be different typologies of vehicles, such as cars and trucks, or vehicles with different trips.

We consider the following simplifying assumptions:

(MP1): for every $i \in \{1, \ldots, n\}$, the function v_i depends only on the variable $\rho_1 + \cdots + \rho_n$.
(MP2): there exists a scalar decreasing function $\psi : [0, \rho_{max}] \to [0, 1]$ such that $\psi(0) = 1$, $\psi(\rho_{max}) = 0$ and, for every $i \in \{1, \ldots, n\}$,

$$v_i(r) = \psi(r) V_i, \tag{3}$$

where $V_i > 0$ is the maximal speed for the i-th population and ρ_{max} is the maximum possible density for the road.

For notational simplicity we rescale the system so that $\rho_{max} = 1$. In this case the model (2) is defined in the domain

$$\mathcal{D} = \{(\rho_1, \ldots, \rho_n) \in \mathbb{R}^n : \rho_i \geq 0 \text{ and } \rho_1 + \cdots + \rho_n \leq 1\}. \tag{4}$$

Proposition 3. *Assume (MP1) and (MP2). In the domain $\mathcal{D}$, if $\rho_i > 0$ for every $i = 1, \ldots, n$, then the system (2) is hyperbolic.*

For a proof see [5].

Proposition 4. *Assume (MP1) and (MP2) with $\psi(r) = 1 - r$. Consider $(\bar\rho_1, \ldots, \bar\rho_n) \in \mathcal{D}$ such that $\bar\rho_1 + \cdots + \bar\rho_n < 1$. There exists a point $(\tilde\rho_1, \ldots, \tilde\rho_n) \in \mathcal{D}$ with $\tilde\rho_1 + \cdots + \tilde\rho_n = 1$ such that the Riemann problem with initial data $(\bar\rho_1, \ldots, \bar\rho_n)$ and $(\tilde\rho_1, \ldots, \tilde\rho_n)$ is solved with an entropy admissible shock wave of the first family with negative speed.*

Proof. Define the function $\Phi : \mathcal{D} \to \mathbb{R}^n$ in the following way. For every $i \in \{1, \ldots, n\}$, the i-th component of Φ is

$$\Phi_i(\rho_1, \ldots, \rho_n) := \rho_i \left[1 + \frac{V_i(1 - (\rho_1 + \cdots + \rho_n))}{V_1\rho_1 + \cdots + V_n\rho_n}\right].$$

Therefore, the image of Φ is contained in the set $\mathcal{D} \cap \{\rho_1 + \cdots + \rho_n = 1\}$. Moreover, the function Φ has the property that the points $(\rho_1, \ldots, \rho_n)$ and $\Phi(\rho_1, \ldots, \rho_n)$ can be connected through a shock curve. More precisely, if we denote $(\tilde\rho_1, \ldots, \tilde\rho_n) = \Phi(\bar\rho_1, \ldots, \bar\rho_n)$, then the Riemann problem

$$\begin{cases} \partial_t \rho_1 + \partial_x(V_1 \psi(\rho_1 + \cdots + \rho_n)\rho_1) = 0, \\ \qquad \vdots \\ \partial_t \rho_n + \partial_x(V_n \psi(\rho_1 + \cdots + \rho_n)\rho_n) = 0, \\ (\rho_1(0,x), \ldots, \rho_n(0,x)) = \begin{cases} (\bar\rho_1, \ldots, \bar\rho_n), & \text{if } x < 0, \\ (\tilde\rho_1, \ldots, \tilde\rho_n), & \text{if } x > 0, \end{cases} \end{cases}$$

is solved by a shock wave with speed

$$\Lambda = -\frac{\psi(\bar\rho_1, \ldots \bar\rho_n)(V_1\bar\rho_1 + \cdots + V_n\bar\rho_n)}{1 - (\bar\rho_1 + \cdots + \bar\rho_n)} < 0.$$

The entropy admissibility of this wave is guaranteed by [5, Proposition 2.4].

The Jacobian matrix of the flux for (2) at a point $\rho_1 + \cdots + \rho_n = 1$ is

$$\begin{pmatrix} -V_1\rho_1 & \cdots & -V_1\rho_1 \\ \ldots & \ddots & \ldots \\ -V_n\rho_n & \cdots & -V_n\rho_n, \end{pmatrix}$$

and hence its eigenvalues are

$$\lambda_1 = -(V_1\rho_1 + \cdots + V_n\rho_n), \quad \lambda_2 = 0, \ldots, \lambda_n = 0.$$

Therefore the wave is of the first family. ∎

We deduce the following corollary.

Corollary 1. *Assume (MP1) and (MP2) with $\psi(r) = 1 - r$. The model (2) satisfies properties **P1**, **P2** and **P3**.*

Remark 1. In [15] a source destination model was introduced as a generalization of the LWR model. This model is based on the assumption that each vehicle has a preassigned path in the network. Thus the LWR model is supplemented with some transport equations of the form

$$\pi_t^i + v(\rho)\pi_x^i = 0$$

where π^i denotes the percentage of car traffic in a road with the same path.

In [24], the authors proved that the source destination model is a special case of the multipopulation model. In fact, it is sufficient to consider the variables $\rho_i = \rho\pi^i$.

3. **Second order models.**

This part deals with second order models. More precisely, we consider the Aw-Rascle-Zhang model, two phase-transition models, the Siebel-Mauser balanced vehicular model and the Greenberg-Klar-Rascle multilane model. As in the previous section, for each of these models, we consider a one-dimensional road, modeled by $\mathbb{R}$.

3.1. **AW-Rascle-Zhang model.**

The Aw-Rascle-Zhang model in conservation form is given by the following hyperbolic system

$$\begin{cases} \rho_t + (y - \rho^{\gamma+1})_x = 0, \\ y_t + \left(\frac{y^2}{\rho} - y\rho^\gamma\right)_x = 0, \end{cases} \tag{5}$$

where $\rho(t, x)$ denotes the density of cars at time $t \geq 0$ and at position $x \in \mathbb{R}$, $y = \rho(v + \rho^\gamma)$ is a generalized momentum, v is the average velocity of cars and $\gamma > 1$ is a constant.

Proposition 5. *If $\rho > 0$, then system (5) is strictly hyperbolic. The eigenvalues of the Jacobian matrix of the flux are*

$$\lambda_1(\rho, y) = \frac{y}{\rho} - (\gamma + 1)\rho^\gamma, \qquad \lambda_2(\rho, y) = \frac{y}{\rho} - \rho^\gamma. \tag{6}$$

Moreover the first characteristic field is genuinely nonlinear, while the second characteristic field is linearly degenerate.

For a proof, see [17]. Notice that the second eigenvalue λ_2 is equal to the velocity v of the cars. It is easy to see that both rarefaction and shock curves of the first family are lines passing through the origin.

We assume that (ρ, y) takes value in the domain

$$\mathcal{D} = \left\{(\rho, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \rho^{\gamma+1} \leq y \leq \rho\right\}; \tag{7}$$

see Figure 1. Thus, we are implicitly assuming that $\rho_{max} = 1$.

The next proposition analyzes the solution when a traffic light is red. The proof is contained in [16] or in [17, Proposition 6.2.1].

Proposition 6. *Let $(\rho_0, y_0) \in \mathcal{D}$, $(\rho_0, y_0) \neq (0, 0)$. Consider system (5) with the initial condition (ρ_0, y_0) for $x < 0$ and with the boundary condition $v = 0$ at $x = 0$. This problem is solved with a shock wave of the first family with negative speed. Moreover the trace $(\hat{\rho}, \hat{y})$ at $x = 0$ of the solution is given by the intersection between the curve of the first family through (ρ_0, y_0) and the curve $y = \rho^{\gamma+1}$ and satisfies $\hat{\rho} > 0$; see Figure 2.*

Proposition 7. *The Aw-Rascle-Zhang model satisfies property **P1**, but not properties **P2** and **P3**.*

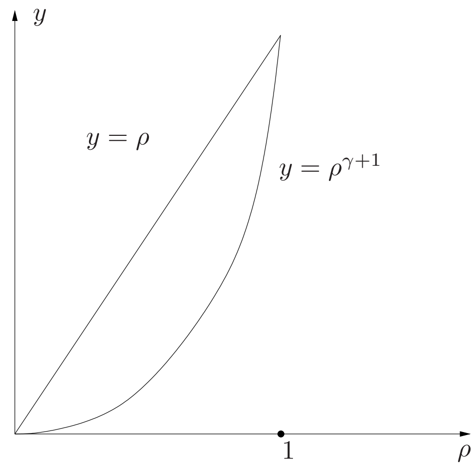

FIGURE 1. The domain $\mathcal{D}$ of equation (7).

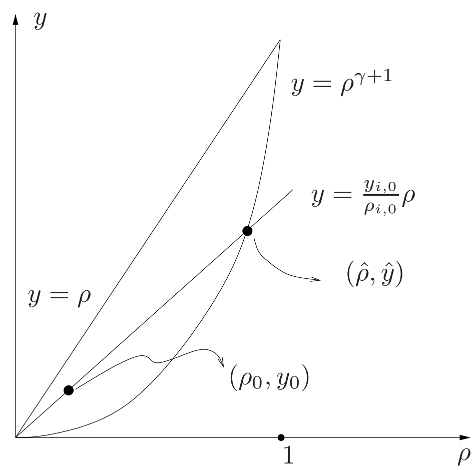

FIGURE 2. Solution of the initial-boundary value problem of Proposition 6.

Proof. First, one can prove that the domain $\mathcal{D}$ is invariant with respect to the solution to Riemann problems. Hence **P1** easily follows.

Consider now a point $(\bar{\rho}, \bar{y}) \in \mathcal{D}$ such that $0 < \bar{\rho} < 1$ and $\bar{y} = \bar{\rho}^{\gamma+1}$. The constant solution $(\bar{\rho}, \bar{y})$ describes a situation in which cars are stopped, but not at the maximum density. Thus **P2** does not hold.

Finally, by Proposition 6, also **P3** does not hold. □

3.2. **Hyperbolic phase transition model.** The hyperbolic phase transition model for traffic was introduced by Colombo in 2002; see [7, 8]. The complete model is described by

Free flow

$$\begin{cases} (\rho, q) \in \Omega_f, \\ \rho_t + [\rho \cdot v_f(\rho)]_x = 0, \\ v_f(\rho) = \left(1 - \frac{\rho}{\rho_{max}}\right) \cdot V, \end{cases}$$

Congested flow

$$\begin{cases} (\rho, q) \in \Omega_c, \\ \rho_t + [\rho \cdot v_c(\rho, q)]_x = 0, \\ q_t + [(q - Q) \cdot v_c(\rho, q)]_x = 0, \\ v_c(\rho, q) = \left(1 - \frac{\rho}{\rho_{max}}\right) \cdot \frac{q}{\rho}, \end{cases} \tag{8}$$

where ρ is the car density, v is the car speed, q is a weighted flow, ρ_{max} and V are respectively the maximal vehicle density and speed and finally Q is the weighted flow at the equilibrium value. Moreover the sets Ω_f and Ω_c (free and the congested phases; see Figure 3) are defined by

$$\Omega_f = \{(\rho, q) \in [0, \rho_{max}] \times [0, +\infty[: v_f(\rho) \geq V_f,\, q = \rho \cdot V\} \tag{9}$$

and

$$\Omega_c = \left\{(\rho, q) \in [0, \rho_{max}] \times [0, +\infty[: v_c(\rho, q) \leq V_c, \frac{Q^- - Q}{\rho_{max}} \leq \frac{q - Q}{\rho} \leq \frac{Q^+ - Q}{\rho_{max}}\right\}, \tag{10}$$

where $V_f < V$ and $V_c < V$ are threshold speed constants and the parameters $Q^- \in]0, Q[$, $Q^+ \in]Q, +\infty[$ depend on environmental conditions.

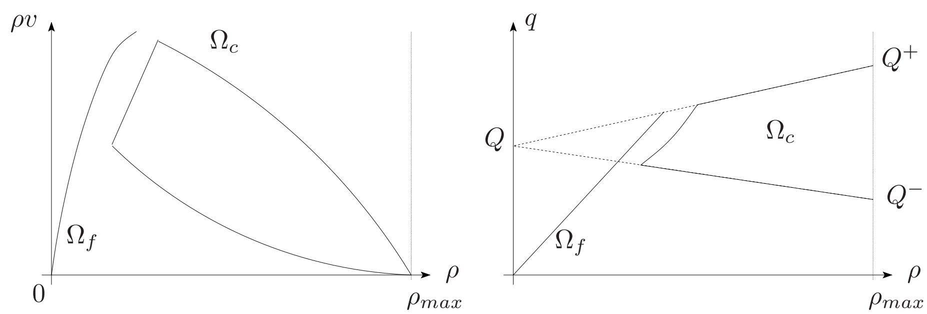

FIGURE 3. Free and congested phases.

It is assumed that the various parameters are strictly positive and satisfy the following conditions:

(PT1): $0 < V_c < V_f < V$;
(PT2): $0 < Q^- \leq Q \leq Q^+$;

Condition (PT1) implies that, for every $(\rho, q) \in \Omega_f \cup \Omega_c$, the velocity $v(\rho, q)$ cannot take value in the not empty interval $]V_c, V_f[$. Instead (PT2) says that Q^- and Q^+ are the bounds for the weighted flow Q.

(PT3): $\frac{Q^+ - Q}{\rho_{max} V} < 1$;
(PT4): $V_f = \frac{V - Q^+/\rho_{max}}{1 - (Q^+ - Q)/(\rho_{max} V)}$;
(PT5): $\left(1 - \frac{Q^+}{\rho_{max} V}\right) \cdot \left(\frac{Q^+}{Q} - 1\right) < 1$.

The following result about Riemann problems holds; see [7] for details and proof.

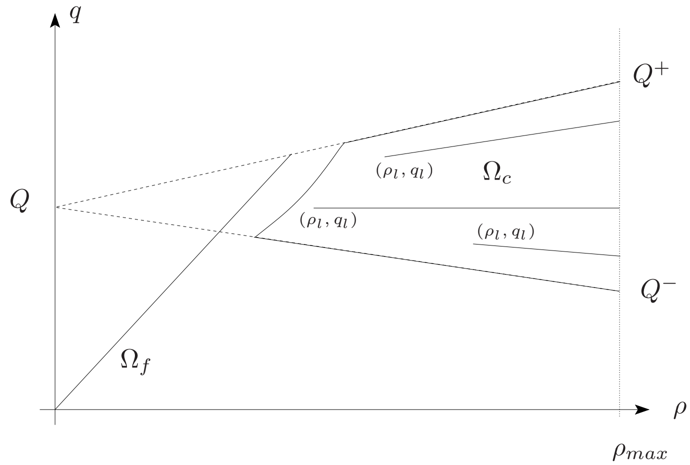

FIGURE 4. The solution in case 1.

Proposition 8. *Assume (PT1)–(PT5). Consider system* (8) *coupled with the initial condition*

$$\begin{cases} (\rho_l, q_l), & \text{if } x < 0, \\ (\rho_r, q_r), & \text{if } x > 0, \end{cases} \tag{11}$$

where both (ρ_l, q_l) *and* (ρ_r, q_r) *belong to* $\Omega_f \cup \Omega_c$. *The Riemann problem* (8)- (11) *admits a self-similar solution with values in* $\Omega_f \cup \Omega_c$. *In general, the solution is formed by a wave in* Ω_f, *by a phase-transition wave and by waves in* Ω_c.

Proposition 9. *Assume (PT1)–(PT5). The hyperbolic phase transition model satisfies* ***P1****,* ***P2*** *and* ***P3****.*

Proof. If $(\rho, q) \in \Omega_f \cup \Omega_c$, then the velocity v of cars is positive; hence **P1** holds. By (8), $v = 0$ implies $\rho = \rho_{max}$, since q is strictly positive. Therefore **P2** holds. Let us now consider **P3**. Fix $(\rho_l, q_l) \in \Omega_f \cup \Omega_c$ and consider the Riemann problem for (8) with initial datum

$$\begin{cases} (\rho_l, q_l), & \text{if } x < 0, \\ (\rho_{max}, q_r), & \text{if } x > 0, \end{cases}$$

where $q_r \in [Q^-, Q^+]$. We want to show that there exists $q_r \in [Q^-, Q^+]$ such that this Riemann problem is solved by waves with negative speed; see [7] for a detailed analysis. We have some different possibilities.

1. $(\rho_l, q_l) \in \Omega_c$. In this case a Lax curve of the first family is produced; see Figure 4. The wave is a shock, a contact discontinuity or a rarefaction respectively when $q_l > Q$, $q_l = Q$ or $q_l < Q$.
2. $(\rho_l, q_l) \in \Omega_f$, $q_l \geq Q$. In this case a single wave is produced, that is a shock-like phase transition or a phase boundary acting as a contact discontinuity; see Figure 5.
3. $(\rho_l, q_l) \in \Omega_f$, $(Q^- - Q)\frac{\rho}{\rho_{max}} + Q \leq q_l < Q$. In this case a phase transition followed by a rarefaction wave is produced; see Figure 6.
4. $(\rho_l, q_l) \in \Omega_f$, $0 \leq q_l < (Q^- - Q)\frac{\rho}{\rho_{max}} + Q$. In this case either a single phase transition wave is produced or a phase transition wave followed by a rarefaction wave of the first family is produced. The two possibilities depend on q_l; see Figure 7.

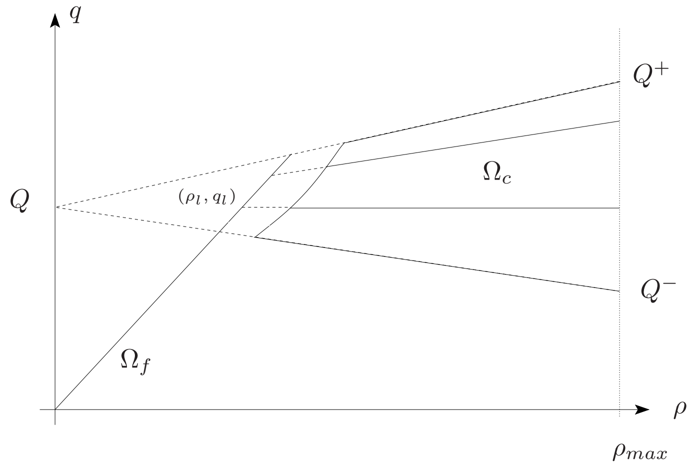

FIGURE 5. The solution in case 2.

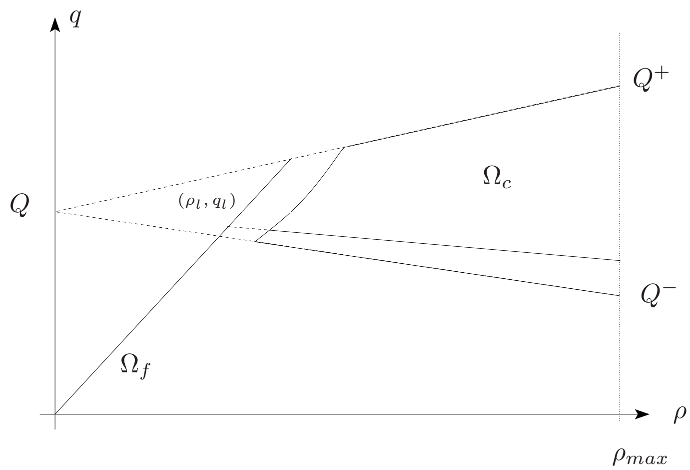

FIGURE 6. The solution in case 3.

Therefore **P3** holds. □

Remark 2. Notice that in presence of a traffic light the solution to the Riemann problem is qualitative different respect to the LWR model. In fact for the LWR model a single shock wave is produced, while in the phase-transition model the Riemann problem can be solved by a phase-transition wave followed by a rarefaction wave; see also [7].

3.3. **The Aw-Rascle-Zhang model with phase transition.** This model was introduced by Goatin [20] in 2005. In analogy with the phase transition model proposed by Colombo, this model combines the LWR model and the Aw-Rascle-Zhang model, in order to describe the free and the congested flow. The model is the following one.

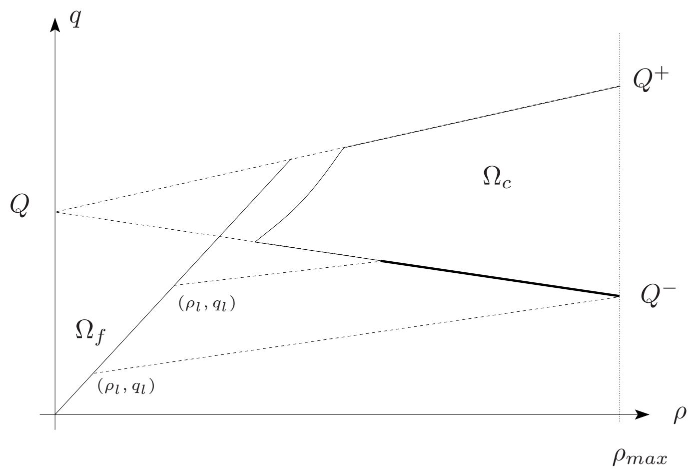

FIGURE 7. The solution in case 4.

Free flow **Congested flow**

$$\begin{cases} (\rho, v) \in \Omega_f, \\ \rho_t + [\rho \cdot v(\rho)]_x = 0, \\ v(\rho) = \left(1 - \frac{\rho}{\rho_{max}}\right) \cdot V, \end{cases} \qquad \begin{cases} (\rho, v) \in \Omega_c, \\ \rho_t + [\rho \cdot v]_x = 0, \\ [\rho(v + p(\rho))]_t + [\rho v(v + p(\rho))]_x = 0. \end{cases} \tag{12}$$

Here ρ and v denote respectively the density and the average speed of cars, while V is the maximum speed, ρ_{max} is the maximal possible car density and $p(\rho)$ is the "pressure", as in the Aw-Rascle-Zhang model. Moreover the sets Ω_f and Ω_c are defined by

$$\Omega_f = \left\{(\rho, v) \in [0, R_f] \times [V_f, V] : v(\rho) = \left(1 - \frac{\rho}{\rho_{max}}\right) V\right\} \tag{13}$$

$$\Omega_c = \{(\rho, v) \in [0, \rho_{max}] \times [0, V_c] : p(r) \le v + p(\rho) \le p(\rho_{max})\}, \tag{14}$$

where V_f, V_c, R_f and r are threshold parameters.

This model inherits the properties of the Aw-Rascle-Zhang model. In particular, the following proposition holds.

Proposition 10. *The Aw-Rascle-Zhang model with phase-transition (12) satisfies property **P1**, but not properties **P2** and **P3**.*

The proof is exactly the same of the proof of Proposition 7. These characterizations are also contained in [20].

3.4. **Siebel-Mauser balanced vehicular traffic model.** The Siebel-Mauser balanced vehicular traffic (BVT) model generalizes the Aw-Rascle-Zhang model and was introduced by Siebel and Mauser in [37]. The model is described by the hyperbolic system of balance laws

$$\begin{cases} \rho_t + (v\rho)_x = 0, \\ (\rho(v - u(\rho)))_t + (\rho v(v - u(\rho)))_x = b(\rho, v)\rho(u(\rho) - v), \end{cases} \tag{15}$$

where ρ is the density of traffic, v is the velocity, $u(\rho)$ is the equilibrium velocity and $b(\rho, v) > 0$ is a relaxation coefficient. We assume that $0 \le \rho \le \rho_{max}$ and we put $\rho_{max} = 1$. Moreover the function $u(\rho)$ is strictly decreasing and satisfies $u(\rho_{max}) = 0$. We suppose for simplicity that $u(\rho) = 1 - \rho$.

Introducing the variable $w = \rho(v - u(\rho))$, the system becomes

$$\begin{cases} \rho_t + (w + u(\rho)\rho)_x = 0, \\ w_t + \left(w \left(\frac{w}{\rho} + u(\rho) \right) \right)_x = -\tilde{b}(\rho, w) w, \end{cases} \tag{16}$$

where $\tilde{b}(\rho, w) = b\left(\rho, \frac{w}{\rho} + u\right)$.

Proposition 11. *If $\rho \neq 0$, then the system (16) is strictly hyperbolic with eigenvalues $\lambda_1(\rho, w) = u(\rho) + \frac{w}{\rho} + \rho u'(\rho)$ and $\lambda_2(\rho, w) = u(\rho) + \frac{w}{\rho}$.*

Proof. The Jacobian matrix of the flux function for (16) is

$$\begin{pmatrix} u(\rho) + \rho u'(\rho) & 1 \\ -\frac{w^2}{\rho^2} + u'(\rho) w & 2\frac{w}{\rho} + u(\rho) \end{pmatrix} \tag{17}$$

and so its eigenvalues are

$$\lambda_1(\rho, w) = u(\rho) + \frac{w}{\rho} + \rho u'(\rho), \quad \lambda_2(\rho, w) = u(\rho) + \frac{w}{\rho}.$$

This completes the proof. □

It is clear that **P1** holds by construction. Moreover, the relaxation term forces the system to reach the equilibrium configuration, i.e. $v = u(\rho)$. This, however, is not sufficient to deduce that $\rho = \rho_{max}$ from $v = 0$.

Proposition 12. *The BVT model (15) satisfies property **P1**, but not property **P2**.*

Property **P3** is difficult to be verified, since system (15) is not in conservation form. Therefore we try to partially analyze **P3**, by using traveling waves for (16).

Proposition 13. *Consider $\rho_l \in [0, \rho_{max}]$. There exist smooth real functions $\varphi : \mathbb{R} \to [0, \rho_{max}]$ and $\psi : \mathbb{R} \to [0, v_{max}]$ such that*

1. *$\lim_{x \to -\infty} \varphi(x) = \rho_l$ and $\lim_{x \to +\infty} \varphi(x) = \rho_{max}$;*
2. *$\lim_{x \to -\infty} \phi(x) = \lim_{x \to +\infty} \phi(x) = 0$;*
3. *the functions $(\rho, w)(t, x) = (\varphi, \psi)(x - \sigma t)$ provide a solution to (16), where the speed $\sigma < 0$ is given by the Rankine-Hugoniot relation*

$$\sigma = -\frac{\rho_l u(\rho_l)}{\rho_{max} - \rho_l}. \tag{18}$$

For a proof see [34, Theorem 4.1]. In view of this result, we conclude that every equilibrium for (16) can be connected by a traveling wave with negative speed with the equilibrium $(\rho_{max}, 0)$. We can not conclude that **P3** holds, but, at a red traffic light, the state goes asymptotically to the equilibrium $(\rho_{max}, 0)$, i.e. **P3** holds asymptotically.

3.5. **The Greenberg-Klar-Rascle multilane model.** Here we consider the multilane model, introduced by Greenberg, Klar and Rascle in [19]; see also [18].

We consider an unidirectional one-dimensional road with n lanes. In a multilane road, one can often observe different traffic behavior depending on the density of traffic. When traffic level is low, changing lane and overcome cars is easy and so the equilibrium speed for cars is high. When traffic level is high, these actions

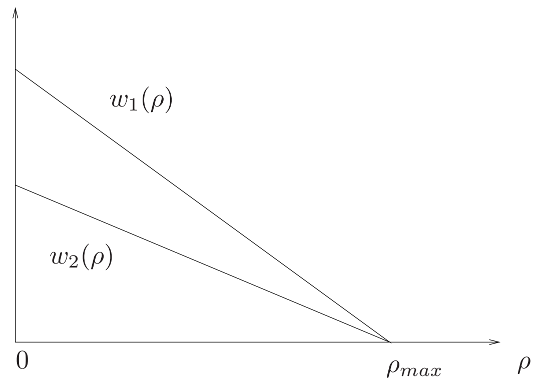

FIGURE 8. Graphs of equilibrium velocity functions w_1 and w_2.

become complicate and difficult, so that the equilibrium speed for cars is low. The typical situation involves two distinct equilibria for the average speed of cars. This is described by two functions $w_1(\rho)$ and $w_2(\rho)$ defined on $[0, \rho_{max}]$ such that $w_1(\rho) > w_2(\rho)$ for every $\rho \in [0, \rho_{max}[$ and $w_1(\rho_{max}) = w_2(\rho_{max}) = 0$. Interest situations are when w_1 and w_2 are linear functions; see Figure 8.

When the density ρ is less than a critical value $\bar{\rho}_1$, then the average speed is described by the function w_1, while when the density ρ is greater than a value $\bar{\rho}_2$, then the average speed is described by the function w_2.

Defining the variable $\alpha = v - w_1(\rho)$, the multilane model is described by the system

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ \alpha_t + v\alpha_x = \begin{cases} -\frac{\alpha}{\varepsilon}, & \rho < R(v), \\ \frac{(w_2(\rho) - w_1(\rho)) - \alpha}{\varepsilon}, & \rho \geq R(v), \end{cases} \end{cases} \tag{19}$$

where $R(v)$ is a monotone non-decreasing function defined on $\mathbb{R}^+$ satisfying

$$R(v) = \bar{\rho}_2, \quad \forall 0 \leq v \leq w_2(\bar{\rho}_2)$$

and

$$R(v) = \bar{\rho}_1, \quad \forall v \geq w_1(\bar{\rho}_1),$$

and ε is a small positive constant.

Notice that (19) can be written in the form

$$\begin{cases} \rho_t + (\rho\,(\alpha + w_1(\rho)))_x = 0, \\ (\rho\alpha)_t + (\rho\alpha\,(\alpha + w_1(\rho)))_x = b_\varepsilon(\rho, \alpha), \end{cases} \tag{20}$$

where

$$b_\varepsilon(\rho, \alpha) = \begin{cases} \frac{-\rho\alpha}{\varepsilon}, & \text{if } \rho < R(\alpha + w_1(\rho)), \\ \frac{\rho(w_2(\rho) - w_1(\rho) - \alpha)}{\varepsilon}, & \text{if } \rho \geq R(\alpha + w_1(\rho)). \end{cases} \tag{21}$$

Therefore the multilane model (19) has exactly the same properties of the BVT model, i.e. it has property **P1**, but not **P2**. Moreover it satisfies the property **P3** asymptotically.

4. **The Helbing third order model.** A third order model was introduced in 1995 by Dirk Helbing [22]. He considered not only equations for density and velocity, but also for the variance θ, defined as

$$\theta(t,x) = \int_{\mathbb{R}} \tilde{P}(s,t,x)\,(v(t,x)-s)^2\;ds,$$

where $\tilde{P}$ is a velocity distribution; see [22] for a detailed description. The θ variable becomes important to describe and predict traffic jams. In fact fast increment of the variance describes queue formation in car traffic. The model is the following one.

$$\begin{cases} \rho_t + (v\rho)_x = 0, \\ v_t + vv_x + \frac{1}{\rho}(\rho\theta)_x = \frac{1}{\tau}(v_e(\rho) - v) + \frac{\mu}{\rho}v_{xx}, \\ \theta_t + v\theta_x + 2\theta v_x = 2\frac{\mu}{\rho}(v_x)^2 + \frac{k}{\rho}\theta_{xx} + \frac{2}{\tau}(\theta_e(\rho) - \theta), \end{cases} \tag{22}$$

where θ_e and v_e are given smooth functions of the density ρ, while μ, k, τ are positive constants. The term $\frac{2}{\tau}(\theta_e(\rho) - \theta)$ results from the drivers' attempt to drive with their desired velocities and from drivers' interactions, i.e. from deceleration in a situation when a fast car can not overtake a slower one.

In this section we consider the following assumptions:

(T1): $\rho \in [0,1]$, i.e. the maximum density ρ_{max} is 1;
(T2): $v_e : [0,1] \to \mathbb{R}^+$ is a bounded, strictly decreasing function such that $v_e(1) = 0$;
(T3): $\theta_e : [0,1] \to \mathbb{R}^+$ is a bounded, strictly decreasing function such that $\theta_e(1) = 0$.

We note that, for general initial data and for general functions v_e and θ_e, **P1** does not hold. However, under reasonable assumptions, Helbing showed that **P1** is satisfied; see [22] for details.

4.1. **Preliminary results.** This subsection contains some preliminary results about the third order model (22). We start proving that, for classical constant solutions, zero velocity implies maximum density.

Proposition 14. *Let (T1), (T2) and (T3) hold. Let A be an open subset of $]0,+\infty[\times\mathbb{R}$ and let (ρ, v, θ) be a classical smooth solution to (22) in A. Assume that $v = 0$ in A and $\rho = c$ in A, where c is a strictly positive constant. Then $c = 1$, i.e. the density is the maximum possible in A.*

Proof. By assumptions, in A system (22) reduces to

$$\begin{cases} \theta_x = \frac{1}{\tau}v_e(c), \\ \theta_t = \frac{k}{c}\theta_{xx} + \frac{2}{\tau}\left(\theta_e(c) - \theta\right). \end{cases}$$

Differentiating the first equation with respect to x, we deduce that $\theta_{xx} = 0$ in A, and so

$$\begin{cases} \theta_x = \frac{1}{\tau}v_e(c), \\ \theta_t = \frac{2}{\tau}\left(\theta_e(c) - \theta\right). \end{cases}$$

Differentiating the first equation with respect to t and the second one with respect to x, we obtain the system

$$\begin{cases} \theta_{xt} = 0, \\ \theta_{tx} = -\frac{2}{\tau^2}v_e(c). \end{cases}$$

This implies that $v_e(c) = 0$ and so $c = 1$ by the assumptions on the function v_e. ☐

Proposition 15. *Let (T1), (T2) and (T3) hold. In general* ***P2*** *does not hold for the Helbing-third order model.*

Proof. Define $A =]0, +\infty[\times\mathbb{R}$ and assume that $\tau = k = 1$. If $v = 0$ in A, then the system (22) reduces to

$$\begin{cases} \rho_t = 0, \\ \frac{1}{\rho}(\rho\theta)_x = v_e(\rho), \\ \theta_t = \frac{1}{\rho}\theta_{xx} + 2(\theta_e(\rho) - \theta). \end{cases} \tag{23}$$

Stationary solutions to system (23) satisfy

$$\begin{cases} \frac{1}{\rho}(\rho\theta)_x = v_e(\rho), \\ \frac{1}{\rho}\theta_{xx} + 2(\theta_e(\rho) - \theta) = 0. \end{cases} \tag{24}$$

Calling $z_1 = \rho$, $z_2 = \theta$ and $z_3 = \theta_x$, we have

$$\begin{cases} z_1' = \frac{z_1(v_e(z_1) - z_3)}{z_2}, \\ z_2' = z_3, \\ z_3' = -2z_1\left(\theta_e(z_1) - z_2\right). \end{cases} \tag{25}$$

The previous system admits a unique solution, at least locally in x, for every initial datum $(z_1(0), z_2(0), z_3(0))$ with $z_2(0) \neq 0$; hence the hypothesis $v = 0$ in A does not imply that the density of cars is the maximum possible; i.e. **P2** does not hold. ☐

Remark 3. We observe that, in Proposition 14, the hypothesis that ρ is constant in A is fundamental in order to conclude that the density is the maximum possible.

4.2. **Traveling waves.** In this section we look for traveling wave solutions to (22) of the form

$$(\rho(t,x), v(t,x), \theta(t,x)) = (\psi_1(x - c_1t), \psi_2(x - c_2t), \psi_3(x - c_3t))$$

with strictly negative speeds c_1, c_2, c_3 and with the boundary condition at infinity

$$\lim_{x\to+\infty}\psi_1(x) = 1, \quad \lim_{x\to+\infty}\psi_2(x) = 0, \quad \lim_{x\to+\infty}\psi_3(x) = 0. \tag{26}$$

Then the functions ψ_1, ψ_2 and ψ_3 satisfy the ordinary differential equations

$$\begin{cases} -c_1\psi_1' + (\psi_1\psi_2)' = 0, \\ -c_2\psi_1\psi_2' + \psi_1\psi_2\psi_2' + (\psi_1\psi_3)' = \frac{\psi_1}{\tau}(v_e(\psi_1) - \psi_2) + \mu\psi_2'', \\ \psi_1\psi_2\psi_3' - c_3\psi_1\psi_3' + 2\psi_1\psi_2'\psi_3 = 2\mu(\psi_2')^2 + k\psi_3'' + \frac{2\psi_1}{\tau}(\theta_e(\psi_1) - \psi_3). \end{cases} \tag{27}$$

Introducing the variables $z_1 = \psi_1$, $z_2 = \psi_2$, $z_3 = \psi_2'$, $z_4 = \psi_3$, $z_5 = \psi_3'$, we derive the system

$$\begin{cases} z_1 = \frac{-c_1}{z_2 - c_1}, \\ z_2' = z_3, \\ \mu z_3' = \frac{c_1c_2}{z_2 - c_1}z_3 - \frac{c_1z_2z_3}{z_2 - c_1} + \frac{c_1z_3z_4}{(z_2 - c_1)^2} - \frac{c_1z_5}{z_2 - c_1} + \frac{1}{\tau}\frac{c_1}{z_2 - c_1}\left(v_e\left(\frac{-c_1}{z_2 - c_1}\right) - z_2\right), \\ z_4' = z_5, \\ kz_5' = \frac{c_1c_3z_5}{z_2 - c_1} - \frac{c_1z_2z_5}{z_2 - c_1} - \frac{2c_1z_3z_4}{z_2 - c_1} - 2\mu z_3^2 + \frac{2}{\tau}\frac{c_1}{z_2 - c_1}\left(\theta_e\left(-\frac{c_1}{z_2 - c_1}\right) - z_4\right). \end{cases} \tag{28}$$

All equilibria for the previous system are given by the relations

$$\begin{cases} z_2 = v_e\left(-\frac{c_1}{z_2-c_1}\right), \\ z_3 = 0, \\ z_4 = \theta_e\left(-\frac{c_1}{z_2-c_1}\right), \\ z_5 = 0. \end{cases} \tag{29}$$

Solutions to (29) clearly depend on the function v_e and θ_e. The next proposition shows that, for special functions v_e, there exist equilibria where the first component could be chosen in a suitable interval.

Proposition 16. *Fix $0 < \bar{\rho} < 1$ and consider any bounded and strictly positive function v_e such that*

$$v_e(\rho) = c_1\left(1 - \frac{1}{\rho}\right) \tag{30}$$

for every $\rho \in [\bar{\rho}, 1]$. Then, for every $\tilde{\rho} \in [\bar{\rho}, 1]$, there exists an equilibrium (z_2, z_3, z_4, z_5) for system (28) such that $z_2 = v_e(\tilde{\rho})$.

Proof. If $\tilde{\rho} \in [\bar{\rho}, 1]$, then $z_2 = v_e(\tilde{\rho}) = c_1\left(1 - \frac{1}{\tilde{\rho}}\right)$ and so

$$v_e\left(-\frac{c_1}{z_2 - c_1}\right) = c_1\left(1 + \frac{z_2 - c_1}{c_1}\right) = z_2.$$

This completes the proof. □

Remark 4. Notice that the number of equilibria for system (27) could be not countable. Indeed, if the function v_e is defined by

$$v_e(\rho) = \begin{cases} 4c_1\rho - 3c_1, & \text{if } 0 \le \rho \le \frac{1}{2}, \\ -\frac{c_1}{\rho} + c_1, & \text{if } \frac{1}{2} \le \rho \le 1, \end{cases} \tag{31}$$

then every $z_2 \in [0, -c_1]$ satisfies

$$z_2 = v_e\left(-\frac{c_1}{z_2 - c_1}\right);$$

see Figure 9.

If the function v_e is linear, then the number of equilibria of (29) is finite.

Proposition 17. *If $v_e'(\rho)$ is a negative constant, then equation*

$$z_2 = v_e\left(-\frac{c_1}{z_2 - c_1}\right)$$

has at most three solutions.

Proof. Define the function

$$g(x) := v_e\left(-\frac{c_1}{x - c_1}\right) - x.$$

We have

$$g'(x) = \frac{c_1}{(x - c_1)^2}\, v_e'\left(-\frac{c_1}{x - c_1}\right) - 1.$$

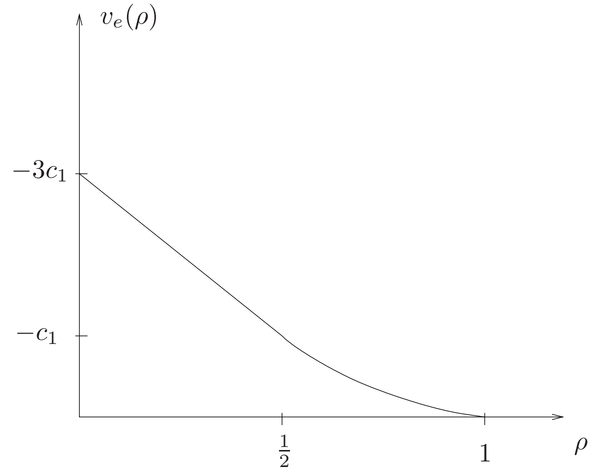

FIGURE 9. The equilibrium function of equation (31).

By hypothesis, we deduce that

$$g'(x) = -\frac{c_1}{(x-c_1)^2}\, M - 1,$$

where $M > 0$. Thus $g'(x) > 0$ if and only if

$$c_1 - \sqrt{-c_1 M} < x < c_1 + \sqrt{-c_1 M},$$

and this permits to conclude the proof. □

Consider now the equilibrium $(0,0,0,0)$, which corresponds to zero velocity of cars. The linearized system around this equilibrium is given by

$$\begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix}' = A_{(0,0,0,0)} \cdot \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix}$$

where the matrix $A_{(0,0,0,0)}$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{c_1 - v_e'(1)}{\mu\tau c_1} & -\frac{c_2}{\mu} & 0 & \frac{1}{\mu} \\ 0 & 0 & 0 & 1 \\ -\frac{2\theta_e'(1)}{k\tau c_1} & 0 & \frac{2}{k\tau} & -\frac{c_3}{k} \end{pmatrix}.$$

The characteristic polynomial of $A_{(0,0,0,0)}$ is

$$p_{A_{(0,0,0,0)}}(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d$$

where

$$\begin{aligned} a &= \frac{kc_2 + \mu c_3}{\mu k} \\ b &= -\frac{2\mu c_1 + kc_1 - kv_e'(1) - c_1 c_2 c_3 \tau}{\mu k \tau c_1} \\ c &= \frac{-2c_1 c_2 - c_1 c_3 + c_3 v_e'(1) + 2\theta_e'(1)}{\mu k c_1 \tau} \\ d &= \frac{2c_1 - 2v_e'(1)}{\mu k c_1 \tau^2}. \end{aligned}$$

4.2.1. *Explicit system.* The choice of the equilibrium function v_e influences the equilibria of the system (28). This implies that property **P3**, from the asymptotically point of view, could depend on the function v_e. Let us consider the particular case, where $\theta_e(\rho) = v_e(\rho) = 1 - \rho$, $\tau = \mu = k = 1$, $c_1 = c_2 = c_3 = -\frac{1}{2}$. We these assumptions, system (28) reads

$$\begin{cases} z_2' = z_3, \\ z_3' = \frac{1}{4z_2+2} z_3 + \frac{z_2 z_3}{2z_2+1} - \frac{2z_3 z_4}{(2z_2+1)^2} + \frac{z_5}{2z_2+1} - \frac{z_2(1-2z_2)}{(2z_2+1)^2}, \\ z_4' = z_5, \\ z_5' = \frac{z_5}{4z_2+2} + \frac{z_2 z_5}{2z_2+1} + \frac{2z_3 z_4}{2z_2+1} - 2z_3^2 - \frac{2}{2z_2+1}\left(1 - z_4 - \frac{1}{2z_2+1}\right). \end{cases} \tag{32}$$

We have the following proposition.

Proposition 18. *The previous system has exactly two unstable equilibria* $(0,0,0,0)$ *and* $\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$. *The stable and unstable manifold for* $(0,0,0,0)$ *have dimension respectively* 1 *and* 3. *The stable and unstable manifold for* $\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ *have dimension respectively* 2 *and* 2.

Proof. The characteristic polynomial of the linear system around $(0,0,0,0)$ is

$$p_{(0,0,0,0)}(\lambda) = \lambda^4 - \lambda^3 - \frac{3}{4}\lambda^2 + \frac{9}{2}\lambda - 2.$$

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of this polynomial, then $\lambda_1 < 0$ and $\lambda_2 > 0$ are real, while $\lambda_3, \lambda_4 \in \mathbb{C} \setminus \mathbb{R}$ with strictly positive real part.

The characteristic polynomial of the linear system around $\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ is

$$p_{\left(\frac{1}{2},0,\frac{1}{2},0\right)}(\lambda) = \lambda^4 - \frac{3}{4}\lambda^3 - \frac{11}{8}\lambda^2 + \frac{5}{8}\lambda + \frac{1}{4}.$$

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of this polynomial, then $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, $\lambda_1, \lambda_2 < 0$ and $\lambda_3, \lambda_4 > 0$. □

Corollary 2. *There exist at most two heteroclinic orbits for system (32) connecting the equilibrium* $\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ *to the equilibrium* $(0,0,0,0)$.

Proof. By Proposition 18, the stable manifold for $(0,0,0,0)$ has dimension 1; hence there are at most two non constant solutions to (32), which tend to $(0,0,0,0)$ when time goes to $+\infty$. This observation concludes the proof. □

Corollary 3. *Fix* $(\bar{z}_2, \bar{z}_3, \bar{z}_4, \bar{z}_5)$ *with* $\bar{z}_2 \geq 0$. *If* $(\bar{z}_2, \bar{z}_3, \bar{z}_4, \bar{z}_5) \neq \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ *and* $(\bar{z}_2, \bar{z}_3, \bar{z}_4, \bar{z}_5) \neq (0,0,0,0)$, *then there are not traveling waves for system* (32) *connecting* $(\bar{z}_2, \bar{z}_3, \bar{z}_4, \bar{z}_5)$ *to* $(0,0,0,0)$; *i.e. the property* ***P3*** *does not hold asymptotically.*

Proof. The proof easily follows from the fact that the system (32) has exact two equilibria. □

5. **Comments about property (P2).** Property **P2** prescribes that cars stop only at maximum density. From the mathematical point of view, this is clearly a good property for a car model. In reality, **P2** is not strictly satisfied. Therefore one can relax the property **P2** to a less stringent one. For example we can consider the following:

P2': There exists $0 < \rho_* < \rho_{max}$ such that, if the velocity v is 0, then the density ρ is bigger than ρ_*.

It is clear that if a model satisfies **P2**, then it satisfies also **P2'**. There are models which do not satisfy both **P2** and **P2'**.
P2' does not hold for the Aw-Rascle-Zhang model. Indeed the proof of Proposition 7 says that $v(\bar{\rho}, \bar{y}) = 0$ for an arbitrary point $(\bar{\rho}, \bar{y}) \in \mathcal{D}$ such that $0 < \bar{\rho} < 1$ and $\bar{y} = \bar{\rho}^{\gamma+1}$.
Instead the Goatin phase transition model satisfies property **P2'**. This is due to the fact that, when the density is small, then the state is in the free phase and the model behaves as the LWR one.
The Siebel-Mauser BVT model contains a relaxation term, which forces the system to reach the equilibrium $\rho = \rho_{max}$ in the case $v = 0$, but this is not sufficient for **P2'**. The same considerations hold also for the Greenberg-Klar-Rascle multilane model, since it can be written in the form (20).
Finally **P2'** does not hold in general for the Helbing third order model. The proof is exactly the same of that of Proposition 15.

6. **Conclusions.** In next table we resume the analysis of the properties for the various considered traffic models.

MODEL	**P1**	**P2**	**P3**
Lighthill-Whitham-Richards model	yes	yes	yes
Multipopulation model	yes	yes	yes
Aw-Rascle-Zhang model	yes	no	no
Colombo phase transition model	yes	yes	yes
Goatin phase transition model	yes	no	no
Siebel-Mauser BVT model	yes	no	asymptotically
Greenberg-Klar-Rascle multilane model	yes	no	asymptotically
Helbing third order model	yes	no	–

The main result is the following: the LWR model, the multipopulation models and the Colombo's phase transition hyperbolic model are the most appropriate to model urban traffic, since they satisfy all the three basic properties. However, also the Siebel-Mauser BVT model and the Greenberg-Klar-Rascle multilane one share at least two properties. Finally, the debate on the Helbing model is more delicate, since the properties depend on the choice of some functions. However, property P3 is hardly verified.

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