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STABILITY AND RIESZ BASIS PROPERTY OF A STAR-SHAPED NETWORK OF EULER-BERNOULLI BEAMS WITH JOINT DAMPING

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Abstract. In this paper we study a star-shaped network of Euler-Bernoulli beams, in which a new geometric condition at the common node is imposed. For the network, we give a method to assert whether or not the system is asymptotically stable. In addition, by spectral analysis of the system operator, we prove that there exists a sequence of its root vectors that forms a Riesz basis with parentheses for the Hilbert state space.

1. **Introduction.** In the past decades, many authors have studied the controllability and observability as well as stabilization of networks of the elastic structures. For example, Ali Mehmeti in [1] and Below in [5] [6] studied regularity of solutions and eigenvalue problems of the wave equation on networks, respectively; Rolewicz [26] and Schmidt [27] studied controllability of networks of vibrating strings; Schmidt, Leugering and Lagnese (see [16] [17] [18]) studied multi-link elastic structure and derived nonlinear and linearized equations (the detail see [17]); Leugering and Zuazua in [19] studied the exact controllability of generic trees; Deckoninck and Nicaise in [13] [12] studied control and eigenvalue problems of networks of Euler-Bernoulli beams; Dager and Zuazua in [8] [9] [10] studied controllability and observability of tree-shaped and star-shaped networks of strings (a complete result can be found in [11]); Ammari and Jellouli in [2] [3] studied the stabilization problem of tree-shaped networks of strings; Nicaise and Zair in [22] studied the identification problem for heterogeneous trees; Nicaise and Valein in [23] studied stabilization of the wave equation on 1-d networks with a delay term in the nodal feedbacks. Beside these concrete models, Pokornyi and Borovskikh studied more general differential equations on graphs (see, [24],[7] and references therein). Xu et al in [32] studied an abstract second order hyperbolic system and applied the result to controlled networks of strings.

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Although there are a lot of papers on networks, we observe that in these papers, the equations do not include the structure and topology of the network. As suggested by Borovshikh and Lazarew in [7], we must take the structure and topology of networks into account when we study it. Generally speaking, a network is essentially differential equations on a graph, which consist of the following three parts:

1) a geometric graph G (also called network): it has a vertex set V and an edge set E;

2) a group of partial differential equations defined on E: they describe the dynamical behavior of the network on E (deformation of the structure);

3) the junction region conditions: they describe the geometric and dynamic multiple node condition of the network which provide highly coupled information on the network.

Note that, for a network, the joint conditions at interior vertices play an essential role, in that the geometrical multiplicity of node conditions restricts deformation and rotation of the structure at the junction and that the dynamical multiplicity of node conditions represents the balance of forces and moments. Given a graph, there are two tasks to do: i) to model the junction region between two or more elements; ii) to solve the highly coupled partial differential equations including the control theoretic properties (see, [20]).

For the elastic network, some nice results have been obtained under the rigid joints at all interior nodes and some geometric conditions at the exterior vertices, for instance, the observability and controllability for tree-shaped and star-shaped networks of strings (see, [11]) and hybrid networks of strings and beams (see, [28]), and stabilization of star-shaped tree networks of strings (see, [2] [3]). Note that the rigid connections at the junction imply the displacement and rotation of beams are continuous. In the present paper, we shall consider a star-shaped network of Euler-Bernoulli beams, in which the displacement of the network at the interior node is continuous but the rotation is not, however, there is a geometrical constraint set for the rotation angles of the network. Obviously, this model is different from those discussed in [14] [15] and [13].

To precisely describe the network under consideration, let us recall some notations. Let $G=(V,E)$ be a planar graph of star shape with vertices $V=\{a,a_1,a_2,a_3,\cdots a_n\}$ and edges $E=\{\gamma_1,\gamma_2,\cdots,\gamma_n\}$, where edge γ_j joins the vertices a and a_j, and a is a common node. Suppose that every edge γ_j is straight and has length ℓ_j,$j=1,2,\cdots,n$. We define the parameterized map $\pi_j:[0,\ell_j]\to\gamma_j$ such that $\pi_j(0)=a,\pi_j(\ell_j)=a_j$. Then G is a metric graph induced by the parameterized map.

Suppose that there is an elastic structure whose equilibrium position coincides with G. The elastic structure is hinged at the exterior vertices $a_j, j=1,2\cdots,n$ and is pin-jointed with rotation angle constraint set $\{s_1,s_2,\cdots,s_n\}$, where $s_j\neq 0, j=1,2,\cdots,n$ are real numbers at the common vertex a on which there are viscous and rotation viscous damping.

Let $w(x,t)$ denote deflection of the elastic structure at position $x\in G$ at time t. Let

$$w^j(s,t)=w(\pi_j(s),t)\big|_{\gamma_j},\quad s\in(0,\ell_j),\quad \pi_j(s)\in\gamma_j.$$

The motion of the elastic structure on each edge γ_j is governed by the Euler-Bernoulli beam equation

$$\rho_j w^j_{tt}(s,t) + EI_j w^j_{ssss}(s,t) = 0, \quad s \in (0,\ell_j) \tag{1}$$

where ρ_j is the mass density and EI_j the flexural rigidity of the beam on γ_j. Since all elements of the elastic structure are hinged at exterior vertices, so we have the boundary conditions

$$w^j(\ell_j,t) = w^j_{ss}(\ell_j,t) = 0, \quad j = 1,2,\cdots,n. \tag{2}$$

At the common vertex a, the elements are pint-jointed, which implies $w(a,t) = w^j(0,t) = w^i(0,t), i,j = 1,2,\cdots,n$; the rotation angle constraint set $\{s_1,s_2,\cdots,s_n\}$ means that $s_j w^j_s(0,t)$ coincide for all j. Therefore the beams satisfy the joint-conditions at a:

$$\begin{cases} w(a,t) = w^j(0,t) = w^i(0,t), \\ s_i w^i_s(0,t) = s_j w^j_s(0,t), \quad \forall i,j = 1,2,\cdots,n. \end{cases} \tag{3}$$

For convenience, we denote $R(w)(a,t) = s_j w^j_s(0,t) = s_i w^i_s(0,t)$.

Due to the viscousness and rotation viscousness of the elastic structure at a, $w^j(s,t), j = 1,2,\cdots,n$ satisfy the dynamic conditions:

$$\begin{cases} -\sum\limits_{j=1}^{n} \frac{EI_j}{s_j} w^j_{ss}(0,t) + R(w)(a,t) = -\beta R(w)_t(a,t), \\ \sum\limits_{j=1}^{n} EI_j w^j_{sss}(0,t) + w(a,t) = -\alpha w_t(a,t) \end{cases} \tag{4}$$

where α and β are viscous damping coefficients. In addition, we assume that the initial condition of the system is given by

$$\begin{cases} w(x,0) = w_0(x), \quad x \in G \\ w_t(x,0) = w_1(x), \quad x \in G. \end{cases} \tag{5}$$

Thus a whole description of dynamic behavior of a star-shaped network of the Euler-Bernoulli beams is

$$\begin{cases} \rho_j w^j_{tt}(s,t) + EI_j w^j_{ssss}(s,t) = 0, \quad s \in (0,\ell_j),\ t > 0 \\ w^j(\ell_j,t) = w^j_{ss}(\ell_j,t) = 0, j = 1,2,\cdots,n, \\ w(a,t) = w^j(0,t), \quad R(w)(a,t) = s_j w^j_s(0,t), j = 1,2,\cdots,n \\ -\sum\limits_{j=1}^{n} \frac{EI_j}{s_j} w^j_{ss}(0,t) + R(w)(a,t) = -\beta R(w)_t(a,t), \\ \sum\limits_{j=1}^{n} EI_j w^j_{sss}(0,t) + w(a,t) = -\alpha w_t(a,t) \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in G. \end{cases} \tag{6}$$

We remark that the pinned connection of the structure at the junction leads to rotation angles relaxed, whereas the rigid connection has the continuity of rotation angles. In our model, the junction is neither rigid-joint nor pin-joint. Although our model is simple, some surprising things occur for this multi-link structure. We shall see that the stability of the system is improved due to the existence of the geometrical constraints $\{s_1,s_2,\cdots,s_n\}$ and viscous damping. In addition, the approach we used in the present paper is also different from the ones used in [11] and [13]. Herein we mainly apply the frequency method to giving a complete analysis for the system. In particular, we obtain the expansion property of the solution of the system according to its root vectors.

The rest is as follows. In section 2, we formulate (6) in a Hilbert state space and then investigate the wellposedness of the system. We show that the operator $\mathcal{A}$ determined by system (6) is dissipative and generates a C_0 semigroup on the Hilbert state space. In particular, $\mathcal{A}^{-1}$ is a compact operator. In section 3, we carry out a complete spectral analysis of $\mathcal{A}$. By employing the asymptotic analysis technique, we prove that the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis. Further, we discuss the condition that there is no eigenvalue on the imaginary axis, from which we can assert asymptotic stability of the system. In section 4, we discuss the completeness and basis property of root vectors (eigenvectors and generalized eigenvectors) of $\mathcal{A}$. We show that the system of root vectors of $\mathcal{A}$ is complete in the Hilbert state space and that there is a sequence of root vectors that forms a Riesz basis with parentheses for the Hilbert state space. Hence the solution of the system can be expanded according to its root vectors.

2. **Well-posed-ness of the network.** In this section we shall discuss the well-posedness of system (6). At first we formulate problem (6) in a Hilbert state space.

Let $L^2(G)$ and $C(G)$ be the linear spaces defined as usual. Denote by $H^k(E), k \geq 1$, the set

$$H^k(E) = \left\{ f \in L^2(G) \;\middle|\; f|_{\gamma_j} = f^j \in H^k(0, \ell_j) \right\}$$

where $H^k(0, \ell_j)$ is the usual Sobolev space. We define the linear space $H^k(G), k \in \mathbb{N}$, by

$$H^k(G) = C(G) \cap H^k(E) = \left\{ f \in C(G) \;\middle|\; f|_{\gamma_j} = f^j \in H^k(0, \ell_j), j = 1, 2, \cdots, n \right\}.$$

We observe that, for a function defined on a graph G, there only exist derivatives along edges at the interior vertex. A function $w(x, t)$ is said to be a solution to (6), if for each $t > 0$, $w(x, t) \in H^4(G)$ and $w(x, t)$ is continuously differentiable with respect to t, and $w_{tt}(x, t)$ exists and belongs to $L^2(G)$ such that the conditions in equations (6) are verified.

Denote $H_E^k(0, \ell_j) = \{f \in H^k(0, \ell_j) \mid f(\ell_j) = 0\}$ and let the state space be

$$\mathcal{H} = \left\{ (f, g) \in H^2(G) \times L^2(G) \;\middle|\; \begin{array}{l} (f^j, g^j) \in H_E^2(0, \ell_j) \times L^2[0, \ell_j], \\ f(a) := f^j(0) = f^i(0), \\ R(f)(a) := s_j f_s^j(0) = s_i f_s^i(0), \\ i, j = 1, 2, \ldots, n \end{array} \right\}$$

equipped with the inner product

$$\begin{aligned} ((f, g), (u, v))_{\mathcal{H}} : \; &= \sum_{j=1}^{n} \int_0^{\ell_j} [EI_j f_{ss}^j(s) \overline{u_{ss}^j(x)} + \rho_j g^j(s) \overline{v^j(s)}] ds \\ &\quad + f(a)\overline{u(a)} + R(f)(a)\overline{R(u)(a)}. \end{aligned}$$

Clearly, $\mathcal{H}$ is a Hilbert space.

Define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ (w, z) \in \mathcal{H} \;\middle|\; \begin{array}{l} (w^j, z^j) \in H_E^4(0, \ell_j) \times H_E^2(0, \ell_j), w_{ss}^j(\ell_j) = 0, \\ -\sum\limits_{j=1}^{n} \frac{EI_j}{s_j} w_{ss}^j(0) + R(w)(a) = -\beta R(z)(a), \\ \sum\limits_{j=1}^{n} EI_j w_{sss}^j(0) + w(a) = -\alpha z(a) \end{array} \right\} \tag{7}$$

$$\mathcal{A}(w, z) = \left\{ \left(z^j, -\frac{EI_j}{\rho_j} w_{ssss}^j \right) \right\} \in \mathcal{H}, \quad \forall (w, z) \in \mathcal{D}(\mathcal{A}). \tag{8}$$

With the help of these notations we can rewrite (6) into an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dW(t)}{dt} = \mathcal{A}W(t), \quad t > 0 \\ W(0) = W_0, \end{cases} \tag{9}$$

where $W(t) = (w(x,t), w_t(x,t))$ and $W(0) = (w_0(x), w_1(x)) \in \mathcal{H}$.

Theorem 2.1. *Let $\mathcal{A}$ be defined by (7–8). Then $\mathcal{A}$ is dissipative, $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact. Hence the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues of finite multiplicity.*

Proof. It is easy to check that $\mathcal{A}$ is a closed and densely defined linear operator, here we omit the detail.

For any $(w,z) \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned} \Re\left(\mathcal{A}(w,z),(w,z)\right)_{\mathcal{H}} &= \Re\sum_{j=1}^{n}\int_0^{\ell_j}[EI_j z_{ss}^j(s)\overline{w_{ss}^j(s)} - EI_j w_{ssss}^j(s)\overline{z^j(s)}]ds \\ &\quad +\Re\left\{z(a)\overline{w(a)} + R(z)(a)\overline{R(w)(a)}\right\} \\ &= \Re\left\{\sum_{j=1}^{n} EI_j w_{sss}^j(0)\overline{z^j(0)} - \sum_{j=1}^{n} EI_j w_{ss}^j(0)\overline{z_s^j(0)}\right\} \\ &\quad +\Re z(a)\overline{w(a)} + \Re R(z)(a)\overline{R(w)(a)} \\ &= \overline{z(a)}\left[\sum_{j=1}^{n} EI_j w_{sss}^j(0) + w(a)\right] \\ &\quad +\overline{R(z)(a)}\left[-\sum_{j=1}^{n}\frac{EI_j}{s_j}w_{ss}^j(0) + R(w)(a)\right] \\ &= -\alpha|z(a)|^2 - \beta|R(z)(a)|^2. \end{aligned}$$

So $\mathcal{A}$ is dissipative.

For any fixed $(f,g) \in \mathcal{H}$, we consider the resolvent equation $\mathcal{A}(w,z) = (f,g)$, i.e.,

$$\begin{cases} z(x) = f(x), \quad x \in G \\ -EI_j w_{ssss}^j(s) = \rho_j g^j(s), \quad s \in (0,\ell_j), \\ w^j(\ell_j) = w_{ss}^j(\ell_j) = 0, \quad j = 1,2,\cdots,n, \\ w(a) = w^j(0), \quad R(w)(a) = s_j w_s^j(0), \quad j = 1,2,\cdots,n, \\ -\sum\limits_{j=1}^{n}\frac{EI_j}{s_j}w_{ss}^j(0) + R(w)(a) = -\beta R(z)(a), \\ \sum\limits_{j=1}^{n} EI_j w_{sss}^j(0) + w(a) = -\alpha z(a). \end{cases} \tag{10}$$

In what follows, we shall find a solution to equation (10). Firstly, solving the differential equation in (10) yields

$$w^j(s) = w^j(0) + s w_s^j(0) + \frac{s^2}{2}w_{ss}^j(0) + \frac{s^3}{3!}w_{sss}^j(0) - \int_0^s \frac{(s-r)^3}{3!}\frac{\rho_j}{EI_j}g^j(r)dr \tag{11}$$

and

$$w_{ss}^j(s) = w_{ss}^j(0) + s w_{sss}^j(0) - \int_0^s \frac{\rho_j(s-r)}{EI_j}g^j(r)dr. \tag{12}$$

From the boundary conditions $w^j(\ell_j)=w^j_{ss}(\ell_j)=0$, we get

$$\begin{cases} w^j(0)+\ell_j w^j_s(0)+\frac{\ell_j^2}{2}w^j_{ss}(0)+\frac{\ell_j^3}{3!}w^j_{sss}(0)-\int_0^{\ell_j}\frac{(\ell_j-r)^3}{3!}\frac{\rho_j}{EI_j}g^j(r)dr=0,\\ w^j_{ss}(0)+\ell_j w^j_{sss}(0)-\int_0^{\ell_j}\frac{\rho_j(\ell_j-r)}{EI_j}g^j(r)dr=0.\end{cases}$$

Using the conditions $w(a)=w^j(0), R(w)(a)=s_j w^j_s(0)$, the above algebraic equations become

$$\begin{cases} \frac{\ell_j^2}{2}w^j_{ss}(0)+\frac{\ell_j^3}{3!}w^j_{sss}(0)=\int_0^{\ell_j}\frac{(\ell_j-r)^3}{3!}\frac{\rho_j}{EI_j}g^j(r)dr-[w(a)+\frac{\ell_j}{s_j}R(w)(a)],\\ w^j_{ss}(0)+\ell_j w^j_{sss}(0)=\int_0^{\ell_j}\frac{\rho_j(\ell_j-r)}{EI_j}g^j(r)dr.\end{cases}$$

Solving these algebraic equations yield

$$\begin{aligned} w^j_{ss}(0) &= \frac{3}{\ell_j^2}\left[\int_0^{\ell_j}\frac{(\ell_j-r)^3}{3!}\frac{\rho_j}{EI_j}g^j(r)dr-[w(a)+\frac{\ell_j}{s_j}R(w)(a)]\right]\\ &\quad -\frac{3}{\ell_j^2}\frac{\ell_j^2}{3!}\int_0^{\ell_j}\frac{\rho_j(\ell_j-r)}{EI_j}g^j(r)dr \end{aligned}\tag{13}$$

and

$$\begin{aligned} w^j_{sss}(0) &= -\frac{3}{\ell_j^3}\left[\int_0^{\ell_j}\frac{(\ell_j-r)^3}{3!}\frac{\rho_j}{EI_j}g^j(r)dr-[w(a)+\frac{\ell_j}{s_j}R(w)(a)]\right]\\ &\quad +\frac{3}{\ell_j^3}\frac{\ell_j^2}{2}\int_0^{\ell_j}\frac{\rho_j(\ell_j-r)}{EI_j}g^j(r)dr. \end{aligned}\tag{14}$$

Thus we have

$$\begin{aligned} \sum_{j=1}^n EI_j w^j_{sss}(0) &= -\sum_{j=1}^n\frac{\rho_j}{\ell_j^3}\int_0^{\ell_j}\frac{(\ell_j-r)^3}{2}g^j(r)dr+w(a)\sum_{j=1}^n\frac{3EI_j}{\ell_j^3}\\ &\quad +R(w)(a)\sum_{j=1}^n\frac{3EI_j}{s_j\ell_j^2}+\sum_{j=1}^n\frac{3\rho_j}{2\ell_j}\int_0^{\ell_j}(\ell_j-r)g^j(r)dr \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^n \frac{EI_j}{s_j} w^j_{ss}(0) &= \sum_{j=1}^n\frac{\rho_j}{s_j\ell_j^2}\int_0^{\ell_j}\frac{(\ell_j-r)^3}{2}g^j(r)dr-w(a)\sum_{j=1}^n\frac{3EI_j}{s_j\ell_j^2}\\ &\quad -R(w)(a)\sum_{j=1}^n\frac{3EI_j}{s_j^2\ell_j}-\sum_{j=1}^n\frac{\rho_j}{2s_j}\int_0^{\ell_j}(\ell_j-r)g^j(r)dr. \end{aligned}$$

Set

$$\mathcal{G}_1(g)=-\sum_{j=1}^n\frac{\rho_j}{\ell_j^3}\int_0^{\ell_j}\frac{(\ell_j-r)^3}{2}g^j(r)dr+\sum_{j=1}^n\frac{3\rho_j}{2\ell_j}\int_0^{\ell_j}(\ell_j-r)g^j(r)dr,$$

$$\mathcal{G}_2(g)=\sum_{\nu=1}^n\frac{\rho_j}{s_j\ell_j^2}\int_0^{\ell_j}\frac{(\ell_j-r)^3}{2}g^j(r)dr-\sum_{j=1}^n\frac{\rho_j}{2s_j}\int_0^{\ell_j}(\ell_j-r)g^j(r)dr.$$

Substituting them into the last two conditions in (10) leads to

$$\begin{cases} w(a)\left[1+\sum_{j=1}^{n}\frac{3EI_j}{\ell_j^3}\right]+R(w)(a)\sum_{j=1}^{n}\frac{3EI_j}{s_j\ell_j^2}=-\alpha f(a)-\mathcal{G}_1(g),\\ w(a)\sum_{j=1}^{n}\frac{3EI_j}{s_j\ell_j^2}+R(w)(a)\left[1+\sum_{j=1}^{n}\frac{3EI_j}{s_j^2\ell_j}\right]=-\beta R(z)(a)+\mathcal{G}_2(g),\end{cases}$$

where we have used the conditions:

$$z(a)=f(a)=f^j(0),\quad \text{and}\quad R(z)(a)=R(f)(a)=s_jf_s^j(0),\ j=1,2,\cdots,n.$$

Solving the algebraic equations above we get

$$w(a)=\frac{-1}{\Delta}\left[\left(1+\sum_{j=1}^{n}\frac{3EI_j}{s_j^2\ell_j}\right)(\alpha f(a)+\mathcal{G}_1(g))-\sum_{j=1}^{n}\frac{3EI_j}{s_j\ell_j^2}(\beta R(f)(a)+\mathcal{G}_2(g)\right],\quad (15)$$

$$R(w)(a)=\frac{-1}{\Delta}\left[\left(1+\sum_{j=1}^{n}\frac{3EI_j}{\ell_j^3}\right)(\beta R(z)(a)-\mathcal{G}_2(g))-\sum_{j=1}^{n}\frac{3EI_j}{s_j\ell_j^2}(\alpha f(a)-\mathcal{G}_1(g))\right],\quad (16)$$

where

$$\Delta=\left[1+\sum_{j=1}^{n}\frac{3EI_j}{\ell_j^3}\right]\left[1+\sum_{j=1}^{n}\frac{3EI_j}{s_j^2\ell_j}\right]-\left(\sum_{j=1}^{n}\frac{3EI_j}{s_j\ell_j^2}\right)^2\neq 0.$$

Plugging (15) and (16) into (13) and (14), we can determine $w_{ss}^j(0)$ and $w_{sss}^j(0)$ of the form

$$\begin{cases} w_{ss}^j(0)=\mathbf{K}_j^1(g)+a_{11}^j\beta R(f)(a)+a_{12}^j\alpha f(a)\\ w_{sss}^j(0)=\mathbf{K}_j^2(g)+a_{21}^j\beta R(f)(a)+a_{22}^j\alpha f(a)\end{cases}\quad (17)$$

where $\mathbf{K}_j^k, k=1,2, j=1,2,\cdots,n$, denote the integral operators. The coefficients a_{ik}^j depend only on the physical and geometrical parameters of beams and the graph, and $R(f)(a)=s_jf_s^j(0), f(a)=f^j(0), j=1,2,\cdots,n$. Moreover, we have

$$w^j(0)=w(a),\quad w_s^j(0)=\frac{R(w)(a)}{s_j},\quad j=1,2,\cdots,n.\quad (18)$$

Inserting (17) and (18) into (11), we can determine uniquely the functions $w^j(s), j=1,2,\cdots,n$.

Now we define a function on G by

$$w(x)=\begin{cases} w^j(\pi_j^{-1}(x)),\quad x=\pi_j(s)\in\gamma_j,\quad s\in(0,\ell_j],\\ w(a),\quad x=a.\end{cases}$$

Clearly, $w(x)\in C(G)$ and $w(x)\big|_{\gamma_j}=w^j(s)\in H^4(0,\ell_j)$. Also we have $z=f\in H^2(G)$. So $(w,z)=(w,f)\in\mathcal{D}(\mathcal{A})$ and $\mathcal{A}(w,z)=(f,g)$. So the Closed Graph Theorem asserts that $\mathcal{A}^{-1}$ is bounded and hence $0\in\rho(\mathcal{A})$. From the expressions in (11), (17) and (18) as well as $f_j\in H^2(0,\ell_j)$, we deduce from the Sobolev's Embedding Theorem that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. □

As a direct consequence of Theorem 2.1 and the Lumer-Phillips Theorem (e.g. see, [25]), we have the following result.

Corollary 1. *Let $\mathcal{A}$ be defined by (7)–(8). Then $\mathcal{A}$ generates a C_0 semigroup of contraction on $\mathcal{H}$. Hence the system (9) is well-posed on $\mathcal{H}$.*

3. **Eigenvalue problem.** In this section we shall discuss the eigenvalue problem. We shall investigate the distribution of eigenvalues of $\mathcal{A}$ and the existence of eigenvalues on the imaginary axis.

For $\lambda \in \mathbb{C}$, we seek the necessary condition for λ to be an eigenvalue of $\mathcal{A}$. We consider the eigenvalue problem $\mathcal{A}(w,z) = \lambda(w,z)$ in $\mathcal{H}$. From the representation of $\mathcal{A}$, we know that $\mathcal{A}(w,z) = \lambda(w,z)$ implies that $z = \lambda w$ and $w(x)$ satisfies the differential equations

$$\begin{cases} EI_j w^j_{ssss}(s) = -\rho_j \lambda^2 w^j(s), \quad s \in (0,\ell_j) \\ w^j(\ell_j) = w^j_{ss}(\ell_j) = 0, \quad j = 1,2,\cdots,n, \\ w(a) = w^j(0), \quad R(w)(a) = s_j w^j_s(0), \quad j = 1,2,\cdots,n, \\ -\sum\limits_{j=1}^n \frac{EI_j}{s_j} w^j_{ss}(0) + R(w)(a) = -\beta\lambda R(w)(a), \\ \sum\limits_{j=1}^n EI_j w^j_{sss}(0) + w(a) = -\alpha\lambda w(a). \end{cases} \tag{19}$$

Set $\omega_j = \sqrt[4]{\frac{\rho_j}{EI_j}}$. The general solution of the differential equation

$$w^j_{ssss}(s) = (i\lambda)^2 \omega_j^4 w^j(s), \quad s \in (0,\ell_j), \quad w^j(\ell_j) = w^j_{ss}(\ell_j) = 0$$

is of the form

$$w^j(s) = b^j_1 \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) + b^j_2 \sin \sqrt{i\lambda}\omega_j(\ell_j - s), \quad s \in (0,\ell_j).$$

Thus we have

$$\begin{cases} w(a) = b^j_1 \sinh \sqrt{i\lambda}\omega_j\ell_j + b^j_2 \sin \sqrt{i\lambda}\omega_j\ell_j, \\ R(w)(a) = -s_j\sqrt{i\lambda}\omega_j[b^j_1 \cosh \sqrt{i\lambda}\omega_j\ell_j + b^j_2 \cos \sqrt{i\lambda}\omega_j\ell_j], \end{cases} \tag{20}$$

$$w^j_{ss}(0) = (\sqrt{i\lambda}\omega_j)^2[b^j_1 \sinh \sqrt{i\lambda}\omega_j\ell_j - b^j_2 \sin \sqrt{i\lambda}\omega_j\ell_j] \tag{21}$$

and

$$w^j_{sss}(0) = -(\sqrt{i\lambda}\omega_j)^3[b^j_1 \cosh \sqrt{i\lambda}\omega_j\ell_j - b^j_2 \cos \sqrt{i\lambda}\omega_j\ell_j]. \tag{22}$$

Substituting (21) and (22) into the last two equalities of (19) leads to

$$-\sum_{j=1}^n \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[b^j_1 \sinh \sqrt{i\lambda}\omega_j\ell_j - b^j_2 \sin \sqrt{i\lambda}\omega_j\ell_j] = -(1+\beta\lambda)R(w)(a),$$

$$-\sum_{j=1}^n EI_j(\sqrt{i\lambda}\omega_j)^3[b^j_1 \cosh \sqrt{i\lambda}\omega_j\ell_j - b^j_2 \cos \sqrt{i\lambda}\omega_j\ell_j] = -(1+\alpha\lambda)w(a),$$

or equivalently

$$\sum_{j=1}^n \begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2 \end{pmatrix} \begin{pmatrix} \cosh \sqrt{i\lambda}\omega_j\ell_j & -\cos \sqrt{i\lambda}\omega_j\ell_j \\ \sinh \sqrt{i\lambda}\omega_j\ell_j & -\sin \sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \begin{pmatrix} b^j_1 \\ b^j_2 \end{pmatrix}$$
$$= \begin{pmatrix} (1+\alpha\lambda) & 0 \\ 0 & (1+\beta\lambda) \end{pmatrix} \begin{pmatrix} w(a) \\ R(w)(a) \end{pmatrix}.$$

We rewrite (20) into the matrix form

$$\begin{pmatrix} 1 & 0 \\ 0 & -s_j\sqrt{i\lambda}\omega_j \end{pmatrix} \begin{pmatrix} \sinh \sqrt{i\lambda}\omega_j\ell_j & \sin \sqrt{i\lambda}\omega_j\ell_j \\ \cosh \sqrt{i\lambda}\omega_j\ell_j & \cos \sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \begin{pmatrix} b^j_1 \\ b^j_2 \end{pmatrix} = \begin{pmatrix} w(a) \\ R(w)(a) \end{pmatrix}.$$

Set

$$M_j(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -s_j\sqrt{i\lambda}\omega_j \end{pmatrix} \begin{pmatrix} \sinh \sqrt{i\lambda}\omega_j\ell_j & \sin \sqrt{i\lambda}\omega_j\ell_j \\ \cosh \sqrt{i\lambda}\omega_j\ell_j & \cos \sqrt{i\lambda}\omega_j\ell_j \end{pmatrix},$$

$$N_j(\lambda)=\begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2 \end{pmatrix}\begin{pmatrix} \cosh\sqrt{i\lambda}\omega_j\ell_j & -\cos\sqrt{i\lambda}\omega_j\ell_j \\ \sinh\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix},$$

and

$$Q(\lambda)=\begin{pmatrix} (1+\alpha\lambda) & 0 \\ 0 & (1+\beta\lambda) \end{pmatrix},\quad \vec{B}_j=\begin{pmatrix} b_1^j \\ b_2^j \end{pmatrix},\quad \vec{W}=\begin{pmatrix} w(a) \\ R(w)(a) \end{pmatrix}.$$

Thus we have

$$\begin{cases} M_j(\lambda)\vec{B}_j=\vec{W}, \quad j=1,2,\cdots,n; \\ \sum\limits_{j=1}^{n} N_j(\lambda)\vec{B}_j=Q(\lambda)\vec{W}. \end{cases} \tag{23}$$

The algebraic equation (23) has a nonzero solution $(\vec{B}_1,\vec{B}_2,\cdots,\vec{B}_n,\vec{W})^\tau$ if, and only if,

$$\Delta(\lambda)=\det\begin{vmatrix} M_1(\lambda) & 0 & 0 & \cdots & 0 & -I \\ 0 & M_2(\lambda) & 0 & \cdots & 0 & -I \\ 0 & 0 & M_3(\lambda) & \cdots & 0 & -I \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & M_n(\lambda) & -I \\ N_1(\lambda) & N_2(\lambda) & N_3(\lambda) & \cdots & N_n(\lambda) & -Q(\lambda) \end{vmatrix}=0$$

where I denotes the identity matrix. Therefore we can prove the following result.

Theorem 3.1. *Let $\mathcal{A}$ be defined as before, then the spectrum, $\sigma(\mathcal{A})$, of $\mathcal{A}$ distributes symmetrically with respect to the real axis. In particular, we have*

$$\sigma(\mathcal{A})=\{\lambda\in\mathbb{C}\mid\Delta(\lambda)=0\}.$$

Proof. From Theorem 2.1 we know that $\sigma(\mathcal{A})=\sigma_p(\mathcal{A})$. Let $\lambda\in\mathbb{C}$ be an eigenvalue of $\mathcal{A}$ and $(w,z)\in\mathcal{D}(\mathcal{A})$ be a corresponding eigenvector. From (19) we see that $\overline{\lambda}$ is also an eigenvalue of $\mathcal{A}$, a corresponding eigenvector is $(\overline{w},\overline{z})$. Therefore, $\sigma(\mathcal{A})$ distributes symmetrically with respect to the real axis.

If $\lambda\in\sigma(\mathcal{A})$, the previous discussion has shown that $\Delta(\lambda)=0$. We shall prove below that $\lambda\in\mathbb{C}$ such that $\Delta(\lambda)=0$ also implies that λ is an eigenvalue of $\mathcal{A}$.

Suppose that $\lambda\in\mathbb{C}$ such that $\Delta(\lambda)=0$. Then the algebraic equation (23) has a nonzero solution $(\vec{B}_1,\vec{B}_2,\cdots,\vec{B}_n,\vec{W})^\tau$. We set

$$w^j(s)=[\sinh\sqrt{i\lambda}\omega_j(\ell_j-s),\sin\sqrt{i\lambda}\omega_j(\ell_j-s)]\vec{B}_j,\quad j=1,2,\cdots,n.$$

Clearly, $w^j(s), j=1,2,\cdots,n$, satisfy the equation

$$w^j_{ssss}(s)=-\lambda^2\omega_j^4 w^j(s),\quad w^j(\ell_j)=w^j_{ss}(\ell_j)=0,\quad j=1,2,\cdots,n.$$

In addition, we have

$$\begin{aligned}\begin{pmatrix} w^j(0) \\ s_j w^j_s(0) \end{pmatrix} &= \begin{pmatrix} \sinh\sqrt{i\lambda}\omega_j\ell_j & \sin\sqrt{i\lambda}\omega_j\ell_j \\ -s_j\sqrt{i\lambda}\omega_j\cosh\sqrt{i\lambda}\omega_j\ell_j & -s_j\sqrt{i\lambda}\omega_j\cos\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix}\vec{B}_j \\ &= M_j(\lambda)\vec{B}_j=\vec{W}\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{n}\begin{pmatrix} -EI_j w^j_{sss}(0) \\ \frac{EI_j}{s_j} w^j_{ss}(0)\end{pmatrix} \\
=\ & \sum_{j=1}^{n}\begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2\end{pmatrix}\begin{pmatrix} \cosh\sqrt{i\lambda}\omega_j\ell_j & -\cos\sqrt{i\lambda}\omega_j\ell_j \\ \sinh\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j\end{pmatrix}\vec{B}_j \\
=\ & Q(\lambda)\vec{W}.
\end{aligned}$$

So, $w^j(s), j = 1, 2, \cdots, n$, satisfy (19). Therefore $(w, \lambda w) \in \mathcal{D}(\mathcal{A})$ is an eigenvector corresponding to λ. The desired result follows. □

3.1. **Distribution of eigenvalues.** From Theorem 3.1 we see that the eigenvalues of $\mathcal{A}$ are entirely given by the zeros of the function $\Delta(\lambda)$. In this subsection we shall discuss the distribution of zeros of $\Delta(\lambda)$. Here we shall employ the asymptotic analysis technique to get an asymptotic estimate of $\Delta(\lambda)$ in λ with sufficiently large modulus.

Let us consider the four functions in λ:

$$\sinh\sqrt{i\lambda}\omega_j\ell_j, \quad \cosh\sqrt{i\lambda}\omega_j\ell_j, \quad \sin\sqrt{i\lambda}\omega_j\ell_j, \quad \cos\sqrt{i\lambda}\omega_j\ell_j.$$

For $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$,

$$\sqrt{i\lambda} = \sqrt{|\lambda|}e^{i\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}, \quad \arg\lambda = \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and

$$-i\sqrt{i\lambda} = e^{-\frac{\pi}{2}i}\sqrt{|\lambda|}e^{i\left(\frac{\pi}{4}+\frac{\theta}{2}\right)} = \sqrt{|\lambda|}e^{i\left(-\frac{\pi}{4}+\frac{\theta}{2}\right)}, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Hence $\Re\sqrt{i\lambda} > 0$ and $\Re(-i\sqrt{i\lambda}) > 0$. When $\Re\lambda \to +\infty$, we have

$$\sinh\sqrt{i\lambda}\omega_j\ell_j = \frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1]_0, \quad \cosh\sqrt{i\lambda}\omega_j\ell_j = \frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1]_0,$$

$$\sin\sqrt{i\lambda}\omega_j\ell_j = \frac{i}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]_0, \quad \cos\sqrt{i\lambda}\omega_j\ell_j = \frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]_0$$

where the notation $[a]_0$ denotes the asymptotic expression of the function $f(z) = a + O(z^{-1})$ in which a is a constant.

Therefore, we have the following asymptotic expressions:

$$M_j(\lambda) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -s_j\sqrt{i\lambda}\omega_j\end{pmatrix}\begin{pmatrix} e^{\sqrt{i\lambda}\omega_j\ell_j}[1]_0 & ie^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]_0 \\ e^{\sqrt{i\lambda}\omega_j\ell_j}[1]_0 & e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]_0\end{pmatrix},$$

and

$$N_j(\lambda) = \frac{1}{2}\begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2\end{pmatrix}\begin{pmatrix} e^{\sqrt{i\lambda}\omega_j\ell_j}[1]_0 & -e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]_0 \\ e^{\sqrt{i\lambda}\omega_j\ell_j}[1]_0 & -ie^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]_0\end{pmatrix}$$

as $\Re\lambda \to +\infty$,

For $\lambda \in \mathbb{C}$ with $\Re\lambda < 0$,

$$\sqrt{i\lambda} = \sqrt{|\lambda|}e^{i\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}, \quad \arg\lambda = \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

and

$$i\sqrt{i\lambda} = e^{\frac{\pi}{2}i}\sqrt{|\lambda|}e^{i\left(\frac{3\pi}{4}+\frac{\theta}{2}\right)} = \sqrt{|\lambda|}e^{i\left(-\frac{\pi}{4}+\frac{\theta}{2}\right)}, \quad \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

we have $\Re(-\sqrt{i\lambda}) > 0$ and $\Re(i\sqrt{i\lambda}) > 0$. Hence, as $\Re\lambda \to -\infty$, it holds that

$$\sinh\sqrt{i\lambda}\omega_j\ell_j = -\frac{1}{2}e^{-\sqrt{i\lambda}\omega_j\ell_j}[1]_0, \quad \cosh\sqrt{i\lambda}\omega_j\ell_j = \frac{1}{2}e^{-\sqrt{i\lambda}\omega_j\ell_j}[1]_0,$$

$$\sin\sqrt{i\lambda}\omega_j\ell_j = -\frac{i}{2}e^{i\sqrt{i\lambda}\omega_j\ell_j}[1]_0, \quad \cos\sqrt{i\lambda}\omega_j\ell_j = \frac{1}{2}e^{i\sqrt{i\lambda}\omega_j\ell_j}[1]_0.$$

Therefore, when $\Re\lambda \to -\infty$, we get the following asymptotical expression:

$$M_j(\lambda) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -s_j\sqrt{i\lambda}\omega_j \end{pmatrix}\begin{pmatrix} -e^{-\sqrt{i\lambda}\omega_j\ell_j}[1]_0 & -ie^{i\sqrt{i\lambda}\omega_j\ell_j}[1]_0 \\ e^{-\sqrt{i\lambda}\omega_j\ell_j}[1]_0 & e^{i\sqrt{i\lambda}\omega_j\ell_j}[1]_0 \end{pmatrix},$$

and

$$N_j(\lambda) = \frac{1}{2}\begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2 \end{pmatrix}\begin{pmatrix} e^{-\sqrt{i\lambda}\omega_j\ell_j}[1]_0 & -e^{i\sqrt{i\lambda}\omega_j\ell_j}[1]_0 \\ -e^{-\sqrt{i\lambda}\omega_j\ell_j}[1]_0 & ie^{i\sqrt{i\lambda}\omega_j\ell_j}[1]_0 \end{pmatrix}.$$

At first we estimate $\Delta(\lambda)$ in the right-half plane. When $\Re\lambda \to +\infty$, we get

$$\Delta(\lambda) = \frac{e^{(1-i)\sqrt{i\lambda}\sum_{j=1}^{n}\omega_j\ell_j}}{2^{2n}}\det\begin{vmatrix} M_1^0(\lambda) & 0 & 0 & \cdots & 0 & -I \\ 0 & M_2^0(\lambda) & 0 & \cdots & 0 & -I \\ 0 & 0 & M_3^0(\lambda) & \cdots & 0 & -I \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & M_n^0(\lambda) & -I \\ N_1^0(\lambda) & N_1^0(\lambda) & N_3^0(\lambda) & \cdots & N_n^0(\lambda) & -Q(\lambda) \end{vmatrix},$$

where

$$M_j^0(\lambda) = \begin{pmatrix} [1]_0 & i[1]_0 \\ -s_j\sqrt{i\lambda}\omega_j[1]_0 & -s_j\sqrt{i\lambda}\omega_j[1]_0 \end{pmatrix}, \quad j = 1, 2, \cdots, n$$

and

$$N_j^0(\lambda) = \begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3[1]_0 & -EI_j(\sqrt{i\lambda}\omega_j)^3[1]_0 \\ \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[1]_0 & -i\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[1]_0 \end{pmatrix}, \quad j = 1, 2, \cdots, n.$$

Since

$$\begin{aligned} &\det\begin{vmatrix} M_1^0(\lambda) & 0 & 0 & \cdots & 0 & -I \\ 0 & M_2^0(\lambda) & 0 & \cdots & 0 & -I \\ 0 & 0 & M_3^0(\lambda) & \cdots & 0 & -I \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & M_n^0(\lambda) & -I \\ N_1^0(\lambda) & N_2^0(\lambda) & N_3^0(\lambda) & \cdots & N_n^0(\lambda) & -Q(\lambda) \end{vmatrix} \\ = &\prod_{j=1}^{n}\det|M_j^0|\det\left|Q(\lambda) - \sum_{j=1}^{n}N_j^0(\lambda)(M_j^0)^{-1}\right| \\ = &\prod_{j=1}^{n}(-1+i)s_j(\sqrt{i\lambda}\omega_j)\times \\ &\det\begin{vmatrix} \left(1+\alpha\lambda-(1+i)\sum_{j=1}^{n}EI_j(\sqrt{i\lambda}\omega_j)^3\right)[1]_0 & -(i\sum_{j=1}^{n}\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2)[1]_0 \\ -i\sum_{j=1}^{n}\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[1]_0 & \left((1+\beta\lambda)-(i-1)\sum_{j=1}^{n}\frac{EI_j}{s_j^2}(\sqrt{i\lambda}\omega_j)\right)[1]_0 \end{vmatrix} \end{aligned}$$

where

$$
\begin{aligned}
&\det\begin{vmatrix} \left(1+\alpha\lambda-(1+i)\sum_{j=1}^{n} EI_j(\sqrt{i\lambda}\omega_j)^3\right)[1]_0 & -(i\sum_{j=1}^{n}\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2([1]_0 \\ -i\sum_{j=1}^{n}\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[1]_0 & \left((1+\beta\lambda)-(i-1)\sum_{j=1}^{n}\frac{EI_j}{s_j^2}(\sqrt{i\lambda}\omega_j)\right)[1]_0 \end{vmatrix} \\
= \ &\det\begin{vmatrix} \left(1-i\alpha(\sqrt{i\lambda})^2-(1+i)(\sqrt{i\lambda})^3\sum_{j=1}^{n} EI_j\omega_j^3\right)[1]_0 & -i(\sqrt{i\lambda})^2\sum_{j=1}^{n}\frac{EI_j\omega_j^2}{s_j}[1]_0 \\ -i(\sqrt{i\lambda})^2\sum_{j=1}^{n}\frac{EI_j\omega_j^2}{s_j}[1]_0 & \left(1-i\beta(\sqrt{i\lambda})^2-(i-1)(\sqrt{i\lambda})\sum_{j=1}^{n}\frac{EI_j\omega_j}{s_j^2}\right)[1]_0 \end{vmatrix} \\
= \ &(\sqrt{i\lambda})^5\left[(-1+i)\beta\sum_{j=1}^{n} EI_j\omega_j^3\right][1]_0,
\end{aligned}
$$

so, we have

$$
\lim_{\Re\lambda\to+\infty}\frac{\Delta(\lambda)}{(\sqrt{i\lambda})^{n+5}e^{(1-i)\sqrt{i\lambda}\sum_{j=1}^{n}\omega_j\ell_j}} = \frac{(i-1)^{n+1}}{2^{2n}}\beta\prod_{j=1}^{n} s_j\omega_j\sum_{j=1}^{n} EI_j\omega_j^3. \tag{24}
$$

As $\Re\lambda\to-\infty$, we derive

$$
\Delta(\lambda) = \frac{e^{(-1+i)\sqrt{i\lambda}\sum_{j=1}^{n}\omega_j\ell_j}}{2^{2n}}\det\begin{vmatrix} M_1^1(\lambda) & 0 & 0 & \cdots & 0 & -I \\ 0 & M_2^1(\lambda) & 0 & \cdots & 0 & -I \\ 0 & 0 & M_3^1(\lambda) & \cdots & 0 & -I \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & M_n^1(\lambda) & -I \\ N_1^1(\lambda) & N_2^1(\lambda) & N_3^1(\lambda) & \cdots & N_n^1(\lambda) & -Q(\lambda) \end{vmatrix}
$$

where

$$
M_j^1(\lambda) = \begin{pmatrix} -[1]_0 & -i[1]_0 \\ -s_j\sqrt{i\lambda}\omega_j[1]_0 & -s_j\sqrt{i\lambda}\omega_j[1]_0 \end{pmatrix}, \quad j=1,2,\cdots,n
$$

and

$$
N_j^1(\lambda) = \begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3[1]_0 & -EI_j(\sqrt{i\lambda}\omega_j)^3[1]_0 \\ -\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[1]_0 & i\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[1]_0 \end{pmatrix}, \quad j=1,2,\cdots,n.
$$

In this case, we have

$$
\begin{aligned}
&\det\begin{vmatrix}
M_1^1(\lambda) & 0 & 0 & \cdots & 0 & -I \\
0 & M_2^1(\lambda) & 0 & \cdots & 0 & -I \\
0 & 0 & M_3^1(\lambda) & \cdots & 0 & -I \\
\vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & M_n^1(\lambda) & -I \\
N_1^1(\lambda) & N_2^1(\lambda) & N_3^1(\lambda) & \cdots & N_n^1(\lambda) & -Q(\lambda)
\end{vmatrix} \\
= \;& \prod_{j=1}^{n} \det|M_j^1| \det\left| Q(\lambda) - \sum_{j=1}^{n} N_j^1(\lambda)(M_j^1)^{-1} \right| \\
= \;& \prod_{j=1}^{n} (1-i)s_j(\sqrt{i\lambda}\omega_j) \times \\
& \det\begin{vmatrix}
\left(1+\alpha\lambda+(1+i)\sum_{j=1}^{n} Ei_j(\sqrt{i\lambda}\omega_j)^3\right)[1]_0 & -i\sum_{j=1}^{n}\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[1]_0 \\
-i\sum_{j=1}^{n}\frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2[1]_0 & \left((1+\beta\lambda)+(1-i)\sum_{j=1}^{n}\frac{EI_j}{s_j^2}(\sqrt{i\lambda}\omega_j)\right)[1]_0
\end{vmatrix} \\
= \;& (1-i)^n(\sqrt{i\lambda})^n \prod_{j=1}^{n} s_j\omega_j \times \\
& \det\begin{vmatrix}
\left(1-i\alpha(\sqrt{i\lambda})^2+(1+i)(\sqrt{i\lambda})^3\sum_{j=1}^{n} Ei_j\omega_j^3\right)[1]_0 & -i(\sqrt{i\lambda})^2\sum_{j=1}^{n}\frac{EI_j\omega_j^2}{s_j}[1]_0 \\
-i(\sqrt{i\lambda})^2\sum_{j=1}^{n}\frac{EI_j\omega_j^2}{s_j}[1]_0 & \left((1-i\beta(\sqrt{i\lambda})^2+(1-i)(\sqrt{i\lambda})\sum_{j=1}^{n}\frac{EI_j\omega_j}{s_j^2}\right)[1]_0
\end{vmatrix} \\
= \;& (\sqrt{i\lambda})^{n+5}(1-i)^{n+1}\prod_{j=1}^{n} s_j\omega_j \left[\beta\sum_{j=1}^{n} Ei_j\omega_j^3\right][1]_0.
\end{aligned}
$$

This yeilds

$$
\lim_{\Re\lambda\to-\infty} \frac{\Delta(\lambda)}{(\sqrt{i\lambda})^{n+5} e^{(-1+i)\sqrt{i\lambda}\sum_{j=1}^{n}\omega_j\ell_j}} = \frac{(1-i)^{n+1}}{2^{2n}}\beta\prod_{j=1}^{n} s_j\omega_j \sum_{j=1}^{n} EI_j\omega_j^3 \neq 0. \tag{25}
$$

From (24) and (25) we get that there exist positive constants C_1, C_2 and h such that when $|\Re\lambda| \geq h$,

$$
\begin{aligned}
&C_1\left|(\sqrt{i\lambda})^{n+5} e^{\operatorname{sign}(\Re\lambda)(1-i)\sqrt{i\lambda}\sum_{j=1}^{n}\omega_j\ell_j}\right| \\
&\leq |\Delta(\lambda)| \leq C_2\left|(\sqrt{i\lambda})^{n+5} e^{\operatorname{sign}(\Re\lambda)(1-i)\sqrt{i\lambda}\sum_{j=1}^{n}\omega_j\ell_j}\right|.
\end{aligned} \tag{26}
$$

The above inequality implies that the zeros of $\Delta(\lambda)$ are in the region $-h < \Re\lambda < h$, if they exist. In order to show that there is at least one zero of $\Delta(\lambda)$, we observe that $\Delta(\lambda)$ is an entire function and it is of the form

$$
\Delta(\lambda) = \sum_{j=1}^{n} c_j e^{b_j\sqrt{i\lambda}} p_j(\sqrt{i\lambda}) + \sum_{k=1}^{n} d_k e^{-b_k\sqrt{i\lambda}} q_k(\sqrt{i\lambda})
$$

where p_j,q_k, $j, k = 1, 2\cdots, n$ are polynomials, $c_j, d_k, j, k = 1, 2, 3\cdots, n$ are complex constants and $b_j, j = 1, 2, \cdots, n$ are complex numbers with $|b_j| > |b_{j+1}|$. (24) and

(25) show that the high order terms of $\Delta(\lambda)$ are given by

$$\begin{aligned} & c_1p_1(\sqrt{i\lambda})e^{b_1\sqrt{i\lambda}} \\ = \quad & \frac{(i-1)^{n+1}}{2^{2n}}\beta\prod_{j=1}^{n}s_j\omega_j\sum_{j=1}^{n}EI_j\omega_j^3(\sqrt{i\lambda})^{n+5}e^{(1-i)\sqrt{i\lambda}\sum\limits_{j=1}^{n}\omega_j\ell_j}[1+O((\sqrt{i\lambda})^{-1})] \end{aligned}$$

and

$$\begin{aligned} & d_1q_1(\sqrt{i\lambda})e^{-b_1\sqrt{i\lambda}} \\ = \quad & \frac{(1-i)^{n+1}}{2^{2n}}\beta\prod_{j=1}^{n}s_j\omega_j\sum_{j=1}^{n}EI_j\omega_j^3(\sqrt{i\lambda})^{n+5}e^{-(1-i)\sqrt{i\lambda}\sum\limits_{j=1}^{n}\omega_j\ell_j}[1+O((\sqrt{i\lambda})^{-1})]. \end{aligned}$$

The theory of entire function asserts that $\Delta(\lambda)$ has infinitely many zeros in the complex plane.

To obtain the detailed distribution of zeros of $\Delta(\lambda)$, we need the following notions (see, [4, Definition II.1.17, II.1.27, pp.52-61]).

Definition 3.2. A set $\sigma \subset \mathbb{C}$ is said to be separable if $\inf\limits_{\lambda,\mu\in\sigma,\lambda\neq\mu}|\lambda-\mu|>0$.

Let $S \subset \mathbb{C}$ be an infinite set. S is said to be a finite union of separable sets if there exist a sequence, $\{O_p, p\in\mathbb{N}\}$, of bounded open sets and an integer N such that

$$S\subset\bigcup_{p=1}^{\infty}O_p,\qquad \inf_{p,r\in\mathbb{N},p\neq r}\operatorname{dist}(O_p,O_r)>0,\quad \text{and}\quad \sup_{p\in\mathbb{N}}\#\{O_p\cap S\}\leq N$$

where $\#O$ denotes the number of elements in set O (taking the multiplicity into account).

Definition 3.3. An entire function f of exponential type is said to be of sine type if

(a). the zeros of f lie in a strip $\{z\in\mathbb{C} \mid |y|\leq h, z=x+iy\}$ for some $h>0$;
(b). there is a $y_0\in\mathbb{R}$ such that $\sup\limits_{x\in\mathbb{R}}|f(x+iy_0)|<\infty$ holds.

For the sine-type function, the following result holds (see, [4, Proposition 11.1.28, pp-61]).

Proposition 1. *(Levin Theorem) If f is a sine-type function, then the set of its zeros is a finite union of separable sets.*

Based on the previous discussion, we have the following result.

Theorem 3.4. *Let $\mathcal{A}$ be defined as before, then the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis, that is, there is a positive constant h such that*

$$\sigma(\mathcal{A})\subset\{\lambda\in \mid -h\leq\Re\lambda\leq 0\}.$$

In particular, $\sigma(\mathcal{A})$ is a finite union of separable sets.

Proof. From inequality (26) we see that there exists a constant $h>0$ such that λ is not a zero of $\Delta(\lambda)$ for $|\Re\lambda|>h$. This together with dissipative property of $\mathcal{A}$ asserts that the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis. In particular, (26) implies that $\Delta(\lambda)$ is a sine-type function in $\sqrt{i\lambda}$. The Levin's Theorem asserts that the set of zeros of $\Delta(\lambda)$ is a finite union of separable sets. $\square$

3.2. **Existence of eigenvalues on the imaginary axis.** In the previous subsection we have shown that the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis. In this subsection, we are interested in whether or not there exist eigenvalues of $\mathcal{A}$ on the imaginary axis. For this purpose, we begin with studying the following eigenvalue problem:

$$\left\{\begin{array}{l} w^j_{ssss}(s) = -\lambda^2\omega_j^4 w^j(s), \quad s \in (0, \ell_j) \\ w^j(\ell_j) = w^j_{ss}(\ell_j) = 0, \\ w^j(0) = 0, \quad w^j_s(0) = 0. \end{array}\right. \tag{27}$$

Set

$$w^j(s) = b_1^j \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) + b_2^j \sin \sqrt{i\lambda}\omega_j(\ell_j - s),$$

then $w^j(s)$ satisfies the equation and the boundary conditions $w^j(\ell_j) = w^j_{ss}(\ell_j) = 0$. So the boundary conditions $w^j(0) = w^j_s(0) = 0$ read

$$\left\{\begin{array}{l} b_1^j \sinh \sqrt{i\lambda}\omega_j\ell_j + b_2^j \sin \sqrt{i\lambda}\omega_j\ell_j = 0, \\ b_1^j \cosh \sqrt{i\lambda}\omega_j\ell_j + b_2^j \cos \sqrt{i\lambda}\omega_j\ell_j = 0. \end{array}\right.$$

Note that $\lambda = 0$ is not an eigenvalue. Therefore, for $\lambda \neq 0$, the equation (27) has a nonzero solution if, and only if,

$$\begin{aligned} \Delta_j(\lambda) &= \det \begin{pmatrix} \sinh \sqrt{i\lambda}\omega_j\ell_j & \sin \sqrt{i\lambda}\omega_j\ell_j \\ \cosh \sqrt{i\lambda}\omega_j\ell_j & \cos \sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \\ &= \sinh \sqrt{i\lambda}\omega_j\ell_j \cos \sqrt{i\lambda}\omega_j\ell_j - \cosh \sqrt{i\lambda}\omega_j\ell_j \sin \sqrt{i\lambda}\omega_j\ell_j = 0. \end{aligned}$$

Obviously, $\Delta_j(\lambda) = 0$ is equivalent to the function equation

$$\tan \sqrt{i\lambda}\omega_j\ell_j = \tanh \sqrt{i\lambda}\omega_j\ell_j$$

whose zeros are given by

$$\lambda = \pm i \frac{(k\pi + \nu_k)^2}{\omega_j^2 \ell_j^2}, \quad \forall k \in \mathbb{Z}$$

where $\nu_k \in (0, \frac{\pi}{4})$ satisfy $\tan(k\pi + \nu_k) = \tanh(k\pi + \nu_k)$. One can prove that $\{\nu_k\}_{k\in\mathbb{N}}$ is an increasing sequence and $\lim\limits_{k\to\infty} \nu_k = \frac{\pi}{4}$. Let us denote the set of zeros of $\Delta_j(\lambda)$ by

$$\sigma_j = \left\{ \lambda = \pm i \frac{(k\pi + \nu_k)^2}{\omega_j^2 \ell_j^2} \;\Big|\; \forall k \in \mathbb{Z} \right\}.$$

Then σ_j is the set of all eigenvalues of (27).

Now we consider the existence of eigenvalues of $\mathcal{A}$ on the imaginary axis. If there is a nonzero $(w, z) \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}(w, z) = \lambda(w, z)$ for $\lambda \in i\mathbb{R}$, then we have

$$\Re(\mathcal{A}(w, z), (w, z))_{\mathcal{H}} = -\alpha|z(a)|^2 - \beta|R(z)(a)|^2 = 0,$$

from which we get that $z(a) = R(z)(a) = 0$, $z(x) = \lambda w(x)$ and $w(x)$ satisfies the equations:

$$\left\{\begin{array}{l} w^j_{ssss}(s) = -\lambda^2\omega_j^4 w^j(s), \quad s \in (0, \ell_j) \quad j = 1, 2, \cdots, n, \\ w^j(\ell_j) = w^j_{ss}(\ell_j) = 0, \quad j = 1, 2, \cdots, n, \\ w(a) = w^j(0) = 0, \quad R(w)(a) = s_j w^j_s(0) = 0, \quad j = 1, 2, \cdots, n, \\ \sum\limits_{j=1}^n \frac{EI_j}{s_j} w^j_{ss}(0) = 0, \\ \sum\limits_{j=1}^n EI_j w^j_{sss}(0) = 0. \end{array}\right. \tag{28}$$

From (27) we see that if $\lambda \notin \sigma_j$, then $(b_1^j, b_2^j) = 0$; if $\lambda \in \sigma_j$ then a corresponding eigenfunction is given by

$$w^i(s) = b_1^j \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) + b_2^j \sin \sqrt{i\lambda}\omega_j(\ell_j - s)].$$

So we have

$$w_{ss}^j(0) = -2b_2^j(-\sqrt{i\lambda}\omega_j)^2 \sin \sqrt{i\lambda}\omega_j\ell_j,$$

and

$$w_{sss}^j(0) = -2b_2^j(-\sqrt{i\lambda}\omega_j)^3 \cos \sqrt{i\lambda}\omega_j\ell_j$$

inhere we have used the equality

$$b_1^j = -b_2^j \frac{\sin \sqrt{i\lambda}\omega_j\ell_j}{\sinh \sqrt{i\lambda}\omega_j\ell_j} = -b_2^j \frac{\cos \sqrt{i\lambda}\omega_j\ell_j}{\cosh \sqrt{i\lambda}\omega_j\ell_j}.$$

We define an index set $J(\lambda)$ for $\lambda \in i\mathbb{R}$ by

$$J(\lambda) = \left\{ j \in \{1, 2, \cdots, n\} \,\middle|\, \lambda \in \sigma_j \right\}.$$

Obviously, if $J(\lambda) = \emptyset$, then $\Delta_j(\lambda) \neq 0, j = 1, 2, \cdots, n$, which implies $(b_1^j, b_2^j) = 0$ for any j, so $\lambda \in \rho(\mathcal{A})$.

When $J(\lambda) \neq \emptyset$, we have

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \frac{EI_j}{s_j} w_{ss}^j(0) = \sum_{j\in J(\lambda)} -2\frac{EI_j}{s_j}(-\sqrt{i\lambda}\omega_j)^2 b_2^j \sin \sqrt{i\lambda}\omega_j\ell_j \\ &= -2(-\sqrt{i\lambda})^2 \sum_{j\in J(\lambda)} EI_j\omega_j^3 b_2^j \frac{\sin \sqrt{i\lambda}\omega_j\ell_j}{s_j\omega_j}, \end{aligned}$$

$$\begin{aligned} 0 &= \sum_{j=1}^{n} EI_j w_{sss}^j(0) = \sum_{j\in J(\lambda)} -2EI_j(-\sqrt{i\lambda}\omega_j)^3 b_2^j \cos \sqrt{i\lambda}\omega_j\ell_j \\ &= -2(-\sqrt{i\lambda})^3 \sum_{j\in J(\lambda)} EI_j\omega_j^3 b_2^j \cos \sqrt{i\lambda}\omega_j\ell_j. \end{aligned}$$

The above two equalities can be rewritten as

$$\sum_{j\in J(\lambda)} EI_j\omega_j^3 b_2^j \begin{pmatrix} \frac{\sin \sqrt{i\lambda}\omega_j\ell_j}{s_j\omega_j} \\ \cos \sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} = 0. \tag{29}$$

Denote by $\#(J(\lambda)$ the number of elements in $J(\lambda)$. If $\#J(\lambda) = 1$, then (29) implies $b_2^j = 0$ for $j \in J(\lambda)$, and hence for all j. This leads to $b_1^j = 0$ for all j. So equation (28) has no nonzero solution. And λ is not an eigenvalue of $\mathcal{A}$.

If $\#J(\lambda) = 2$, which means that there exist indices j and r such that $\lambda \in \sigma_j \cap \sigma_r$, we consider the matrix

$$\begin{pmatrix} \frac{1}{s_j\omega_j} \sin \sqrt{i\lambda}\omega_j\ell_j & \frac{1}{s_r\omega_r} \sin \sqrt{i\lambda}\omega_r\ell_r \\ \cos \sqrt{i\lambda}\omega_j\ell_j & \cos \sqrt{i\lambda}\omega_r\ell_r \end{pmatrix}.$$

Since

$$\begin{aligned} &\det \begin{vmatrix} \frac{1}{s_j\omega_j} \sin \sqrt{i\lambda}\omega_j\ell_j & \frac{1}{s_r\omega_r} \sin \sqrt{i\lambda}\omega_r\ell_r \\ \cos \sqrt{i\lambda}\omega_j\ell_j & \cos \sqrt{i\lambda}\omega_r\ell_r \end{vmatrix} \\ = \; &\cos \sqrt{i\lambda}\omega_j\ell_j \cos \sqrt{i\lambda}\omega_r\ell_r \left[\frac{1}{s_j\omega_j} \tan \nu_k - \frac{1}{s_r\omega_r} \tan \nu_m \right] \end{aligned}$$

where $\sqrt{i\lambda}\omega_j\ell_j = (k\pi + \nu_k)$ and $\sqrt{i\lambda}\omega_r\ell_r = (m\pi + \nu_m)$, so when $s_j\omega_j \tan\nu_m \neq s_r\omega_r \tan\nu_k$, the vectors

$$\begin{pmatrix} \frac{1}{s_j\omega_j}\sin\sqrt{i\lambda}\omega_j\ell_j \\ \cos\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{s_r\omega_r}\sin\sqrt{i\lambda}\omega_r\ell_r \\ \cos\sqrt{i\lambda}\omega_r\ell_r \end{pmatrix}$$

are linearly independent. Therefore we have $b_2^j = b_2^r = 0$ and hence $b_1^j = b_2^j = 0$ for all j. This shows that λ is not an eigenvalue of $\mathcal{A}$. When $s_j\omega_j \tan\nu_m = s_r\omega_r \tan\nu_k$, there exist nonzero numbers b_2^j and b_2^r such that (29) holds. This will lead to $b_1^j, b_1^r \neq 0$. Hence (28) has a nonzero solution, and λ is an eigenvalue of $\mathcal{A}$.

If $\#J(\lambda) \geq 3$, then (29) always has a nonzero solution. This is because (29) is equivalent to the vector equation

$$\sum_{j\in J(\lambda)} \tilde{\tilde{b}}_j \vec{V}_j = 0$$

where $\vec{V}_j, j \in J(\lambda)$ are two-dimensional vectors defined by

$$\vec{V}_j = \begin{pmatrix} \frac{1}{s_j\omega_j}\sin\sqrt{i\lambda}\omega_j\ell_j \\ \cos\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix}.$$

So (28) has a nonzero solution, and λ is an eigenvalue of $\mathcal{A}$.

Summarizing the above discussion and applying the stability theorem in [21], we have achieved the following result.

Theorem 3.5. *Let σ_j be the eigenvalue set of (27), then the following statements hold*

1). If $\sigma_j \cap \sigma_r = \emptyset$ for any $j, r = 1, 2, \cdots, n$, then there is no eigenvalue of $\mathcal{A}$ on the imaginary axis. In this case, system (9) is asymptotically stable;

2). If there exist indices j and r such that $\sigma_j \cap \sigma_r \neq \emptyset$, then there is no eigenvalue of $\mathcal{A}$ on the imaginary axis provided $s_j\omega_j \tan\nu_k \neq s_r\omega_r \tan\nu_m$. If $s_j\omega_j \tan\nu_k = s_r\omega_r \tan\nu_m$, then $\lambda \in \sigma_j \cap \sigma_r$ is an eigenvalue of $\mathcal{A}$. In the first case, the system (9) is asymptotically stable, in the second case, the system is not stable;

3). If there exist indices i, j, r such that $\sigma_j \cap \sigma_i \cap \sigma_r \neq \emptyset$, then $\lambda \in \sigma_j \cap \sigma_i \cap \sigma_r$ is an eigenvalue of $\mathcal{A}$. In this case, the system is never asymptotically stable.

Remark 1. Theorem 3.5 is very important in the design of star-shaped networks of Euler-Bernoulli beams. Usually we know the frequency, σ_j, of each beam γ_j. Note that if $\lambda \in \sigma_j \cap \sigma_r \neq \emptyset$, then there exist integers k and m such that

$$\frac{(k\pi + \nu_k)^2}{\omega_j^2\ell_j^2} = \frac{(m\pi + \nu_m)^2}{\omega_r^2\ell_r^2},$$

which implies that

$$\frac{\omega_j\ell_j}{\omega_r\ell_r} = \frac{(m\pi + \nu_m)}{(k\pi + \nu_k)}.$$

This equality is very precise. So we can change the length of beams such that $\sigma_j \cap \sigma_r = \emptyset$ for any $r, j = 1, 2, \cdots, n$. Also we can adjust the rotation angle constraint set, $\{s_1, s_2, \cdots, s_n\}$, of the star-shaped networks to strengthen the stability of the system.

4. **Completeness and basis property of root vectors.** In this section we shall study the completeness and basis property of root vectors (eigenvectors and generalized eigenvectors) of $\mathcal{A}$. To discuss the completeness of root vectors, we consider the nonhomogeneous node problem of differential equations on network.

Theorem 4.1. *Let $\lambda \in \mathbb{C}$ such that $\Delta(\lambda) \neq 0$, and let $M_j(\lambda), N_j(\lambda)$ and $Q(\lambda)$ be defined as before. Then for $\xi, \eta \in \mathbb{C}$, the equations*

$$\begin{cases} EI_j w^j_{ssss}(s) = -\rho_j \lambda^2 w^j(s), \quad s \in (0, \ell_j) \\ w^j(\ell_j) = w^j_{ss}(\ell_j) = 0, \quad j = 1, 2, \cdots, n, \\ w(a) = w^j(0), \quad R(w)(a) = s_j w^j_s(0), \quad j = 1, 2, \cdots, n, \\ -\sum\limits_{j=1}^{n} \frac{EI_j}{s_j} w^j_{ss}(0) + (1 + \beta\lambda) R(w)(a) = \eta, \\ \sum\limits_{j=1}^{n} EI_j w^j_{sss}(0) + (1 + \alpha\lambda) w(a) = \xi \end{cases} \tag{30}$$

have a unique solution

$$w^j(s) = b^j_1 \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) + b^j_2 \sin \sqrt{i\lambda}\omega_j(\ell_j - s), \quad j = 1, 2, \cdots, n, \tag{31}$$

where

$$\begin{pmatrix} b^j_1 \\ b^j_2 \end{pmatrix} = M_j^{-1}(\lambda) \begin{pmatrix} w(a) \\ R(w)(a) \end{pmatrix}, \quad \begin{pmatrix} w(a) \\ R(w)(a) \end{pmatrix} = D^{-1}(\lambda) \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{32}$$

$$D(\lambda) = \sum_{j=1}^{n} N_j(\lambda) M_j^{-1}(\lambda) - Q(\lambda). \tag{33}$$

In particular, we have an estimate for $D(\lambda)$:

$$||\sqrt{i\lambda} D^{-1}(\lambda)|| \leq M, \quad |\lambda| > h, \ \lambda \in \mathbb{R} \tag{34}$$

where $||D(\lambda)||$ denotes the operator norm in $\mathbb{C}^2$.

Proof. Let $\lambda \in \mathbb{C}$. We shall solve the following equations:

$$\begin{cases} EI_j w^j_{ssss}(s) = -\rho_j \lambda^2 w^j(s), \quad s \in (0, \ell_j) \\ w^j(\ell_j) = w^j_{ss}(\ell_j) = 0, \quad j = 1, 2, \cdots, n, \\ w(a) = w^j(0), \quad R(w)(a) = s_j w^j_s(0), \quad j = 1, 2, \cdots, n, \\ -\sum\limits_{j=1}^{n} \frac{EI_j}{s_j} w^j_{ss}(0) + (1 + \beta\lambda) R(w)(a) = \eta, \\ \sum\limits_{j=1}^{n} EI_j w^j_{sss}(0) + (1 + \alpha\lambda) w(a) = \xi. \end{cases} \tag{35}$$

Set $\omega_j = \sqrt[4]{\frac{\rho_j}{EI_j}}$ and

$$w^j(s) = b^j_1 \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) + b^j_2 \sin \sqrt{i\lambda}\omega_j(\ell_j - s), \quad s \in (0, \ell_j).$$

Then $w^j(s)$ satisfies the differential equation

$$w^j_{ssss}(s) = (i\lambda)^2 \omega_j^4 w^j(s), \quad s \in (0, \ell), \quad w^j(\ell_j) = w^j_{ss}(\ell_j) = 0.$$

From the connective conditions in (35) we get

$$\begin{pmatrix} 1 & 0 \\ 0 & -s_j\sqrt{i\lambda}\omega_j \end{pmatrix} \begin{pmatrix} \sinh \sqrt{i\lambda}\omega_j\ell_j & \sin \sqrt{i\lambda}\omega_j\ell_j \\ \cosh \sqrt{i\lambda}\omega_j\ell_j & \cos \sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \begin{pmatrix} b^j_1 \\ b^j_2 \end{pmatrix} = \begin{pmatrix} w(a) \\ R(w)(a) \end{pmatrix}$$

and

$$\sum_{j=1}^{n}\begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2 \end{pmatrix}\begin{pmatrix} \cosh\sqrt{i\lambda}\omega_j\ell_j & -\cos\sqrt{i\lambda}\omega_j\ell_j \\ \sinh\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix}\begin{pmatrix} b_1^j \\ b_2^j \end{pmatrix}$$
$$-\begin{pmatrix} (1+\alpha\lambda) & 0 \\ 0 & (1+\beta\lambda) \end{pmatrix}\begin{pmatrix} w(a) \\ R(w)(a) \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

As before, we denote

$$M_j(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -s_j\sqrt{i\lambda}\omega_j \end{pmatrix}\begin{pmatrix} \sinh\sqrt{i\lambda}\omega_j\ell_j & \sin\sqrt{i\lambda}\omega_j\ell_j \\ \cosh\sqrt{i\lambda}\omega_j\ell_j & \cos\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix},$$

$$N_j(\lambda)=\begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2 \end{pmatrix}\begin{pmatrix} \cosh\sqrt{i\lambda}\omega_j\ell_j & -\cos\sqrt{i\lambda}\omega_j\ell_j \\ \sinh\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix},$$

$$Q(\lambda) = \begin{pmatrix} (1+\alpha\lambda) & 0 \\ 0 & (1+\beta\lambda) \end{pmatrix}, \quad \vec{B}_j = \begin{pmatrix} b_1^j \\ b_2^j \end{pmatrix},$$

and

$$\vec{W} = \begin{pmatrix} w(a) \\ R(w)(a) \end{pmatrix}, \quad \vec{V}_j = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Thus we have

$$\begin{cases} M_j(\lambda)\vec{B}_j = \vec{W}, \quad j = 1, 2, \cdots, n; \\ \sum\limits_{j=1}^{n} N_j(\lambda)\vec{B}_j - Q(\lambda)\vec{W} = \vec{V}. \end{cases} \tag{36}$$

If $\lambda \notin \sigma_j, j = 1, 2, \cdots, n$, then $\vec{B}_j = M_j^{-1}(\lambda)\vec{W}$ and $\vec{W}$ satisfies the equation:

$$D(\lambda)\vec{W} = \sum_{j=1}^{n} N_j(\lambda)M_j^{-1}(\lambda)\vec{W} - Q(\lambda)\vec{W} = \vec{V}.$$

Hence for $\lambda \in \mathbb{C}$ with $\Delta(\lambda) \neq 0$ and $\lambda \notin \sigma_j, j = 1, 2, \cdots, n$, we have

$$\vec{W} = D^{-1}(\lambda)\vec{V}, \quad \vec{B}_j = M_j^{-1}(\lambda)D^{-1}(\lambda)\vec{V}.$$

Therefore, we get a unique solution to (30)

$$w^j(s) = [\sinh\sqrt{i\lambda}\omega_j(\ell_j - s), \sin\sqrt{i\lambda}\omega_j(\ell_j - s)]\vec{B}_j, \quad j = 1, 2, \cdots, n.$$

Note that

$$\begin{aligned} N_j(\lambda)M_j^{-1}(\lambda) \quad &= \quad \begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2 \end{pmatrix}\begin{pmatrix} \cosh\sqrt{i\lambda}\omega_j\ell_j & -\cos\sqrt{i\lambda}\omega_j\ell_j \\ \sinh\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \\ &\quad \times\frac{1}{\Delta_j(\lambda)}\begin{pmatrix} \cos\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \\ -\cosh\sqrt{i\lambda}\omega_j\ell_j & \sinh\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{-s_j\sqrt{i\lambda}\omega_j} \end{pmatrix}, \end{aligned}$$

$$\begin{pmatrix} \cosh\sqrt{i\lambda}\omega_j\ell_j & -\cos\sqrt{i\lambda}\omega_j\ell_j \\ \sinh\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix}\begin{pmatrix} \cos\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \\ -\cosh\sqrt{i\lambda}\omega_j\ell_j & \sinh\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix}$$
$$= \begin{pmatrix} [\cosh(1+i)\sqrt{i\lambda}\omega_j\ell_j + \cosh(1-i)\sqrt{i\lambda}\omega_j\ell_j] & -\frac{1-i}{2}\sinh(1+i)\sqrt{i\lambda}\omega_j\ell_j - \frac{1+i}{2}\sinh(1-i)\sqrt{i\lambda}\omega_j\ell_j \\ \frac{1-i}{2}\sinh(1+i)\sqrt{i\lambda}\omega_j\ell_j + \frac{1+i}{2}\sinh(1-i)\sqrt{i\lambda}\omega_j\ell_j & i[\cosh(1+i)\sqrt{i\lambda}\omega_j\ell_j - \cosh(1-i)\sqrt{i\lambda}\omega_j\ell_j] \end{pmatrix}$$

and

$$\Delta_j(\lambda) = \frac{1+i}{2}\sinh(1+i)\sqrt{i\lambda}\omega_j\ell_j + \frac{1-i}{2}\sinh(1-i)\sqrt{i\lambda}\omega_j\ell_j.$$

For $\lambda = -\tau^2, \tau \in \mathbb{R}_+$,

$$\frac{\sqrt{2}}{2}(1+i)\sqrt{i\lambda} = \tau e^{i\pi} = -\tau, \quad \frac{\sqrt{2}}{2}(1-i)\sqrt{i\lambda} = \tau e^{i\frac{\pi}{2}} = i\tau,$$

we have the asymptotic estimate

$$\begin{aligned}
&\begin{pmatrix} \cosh\sqrt{i\lambda}\omega_j\ell_j & -\cos\sqrt{i\lambda}\omega_j\ell_j \\ \sinh\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \begin{pmatrix} \cos\sqrt{i\lambda}\omega_j\ell_j & -\sin\sqrt{i\lambda}\omega_j\ell_j \\ -\cosh\sqrt{i\lambda}\omega_j\ell_j & \sinh\sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \\
= &\begin{pmatrix} [\cosh\sqrt{2}\tau\omega_j\ell_j + \cos\sqrt{2}\tau\omega_j\ell_j] & \frac{1-i}{2}\sinh\sqrt{2}\tau\omega_j\ell_j + \frac{1-i}{2}\sin\sqrt{2}\tau\omega_j\ell_j \\ -\frac{1-i}{2}\sinh\sqrt{2}\tau\omega_j\ell_j - \frac{1-i}{2}\sin\sqrt{2}\tau\omega_j\ell_j & i[\cosh\sqrt{2}\tau\omega_j\ell_j - \cos\sqrt{2}\tau\omega_j\ell_j] \end{pmatrix} \\
= &\begin{pmatrix} \frac{1}{2}e^{\sqrt{2}\tau\omega_j\ell_j}[1]_0 & \frac{1-i}{4}e^{\sqrt{2}\tau\omega_j\ell_j}[1]_0 \\ -\frac{1-i}{4}e^{\sqrt{2}\tau\omega_j\ell_j}[1]_0 & \frac{i}{2}e^{\sqrt{2}\tau\omega_j\ell_j}[1]_0 \end{pmatrix}
\end{aligned}$$

and

$$\Delta_j(\lambda) = -\frac{1+i}{2}\sinh\sqrt{2}\tau\omega_j\ell_j + \frac{1+i}{2}\sin\sqrt{2}\tau\omega_j\ell_j = -\frac{1+i}{4}e^{\sqrt{2}\tau\omega_j\ell_j}[1]_0$$

where $[1]_0 = 1 + o(1)$. So we derive the asymptotic expression

$$\begin{aligned}
&N_j(\lambda)M_j^{-1}(\lambda) \\
= &\begin{pmatrix} EI_j(\sqrt{i\lambda}\omega_j)^3 & 0 \\ 0 & \frac{EI_j}{s_j}(\sqrt{i\lambda}\omega_j)^2 \end{pmatrix} \begin{pmatrix} -(1-i)[1]_0 & i[1]_0 \\ -i[1]_0 & (1+i)[1]_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{-s_j\sqrt{i\lambda}\omega_j} \end{pmatrix} \\
= &\begin{pmatrix} p_3(\sqrt{i\lambda}) & p_2(\sqrt{i\lambda}) \\ \widetilde{p}_2(\sqrt{i\lambda}) & p_1(\sqrt{i\lambda}) \end{pmatrix}
\end{aligned}$$

where $p_k(x)$ denotes a polynomial of degree k. Since

$$Q(\lambda) = \begin{pmatrix} (1+\alpha\lambda) & 0 \\ 0 & (1+\beta\lambda) \end{pmatrix} = \begin{pmatrix} q_2(\sqrt{i\lambda}) & 0 \\ 0 & \widetilde{q}_2(\sqrt{i\lambda}) \end{pmatrix},$$

we get the asymptotical formula

$$D(\lambda) = \begin{pmatrix} \widehat{p}_3(\sqrt{i\lambda}) & \widehat{p}_2(\sqrt{i\lambda}) \\ \widetilde{\widehat{p}}_2(\sqrt{i\lambda}) & \widehat{p}_2(\sqrt{i\lambda}) \end{pmatrix}$$

and $\det(D(\lambda)) = P_5(\sqrt{i\lambda})$. Hence we have

$$||\det(D(\lambda))D^{-1}(\lambda)|| \leq 4\max\left\{|\widehat{p}_2(\sqrt{i\lambda})|, |\widehat{p}_3(\sqrt{i\lambda})|, |\widehat{p}_1(\sqrt{i\lambda})|, |\widetilde{\widehat{p}}_2(\sqrt{i\lambda})|\right\}.$$

For $\lambda = \tau^2, \tau \in \mathbb{R}_+$ we can get a similar estimate. The inequality (34) follows. □

To obtain the completeness of root vectors of $\mathcal{A}$, we need the following result (see [29, Theorem 4, pp.970]).

Lemma 4.2. *Let $\mathcal{A}$ be the generator of a C_0-semigroup in a Hilbert space $\mathcal{H}$. Assume that $\mathcal{A}$ is discrete and for $\lambda \in \rho(\mathcal{A}^*)$, $R(\lambda, \mathcal{A}^*)$ is of the form*

$$R(\lambda, \mathcal{A}^*)x = \frac{G(\lambda)x}{F(\lambda)}, \quad \forall x \in \mathcal{H}$$

where $G(\lambda)x$ is an H-valued entire function for each $x \in \mathcal{H}$ with order less than or equal to ρ_1 and $F(\lambda)$ is a scalar entire function of order ρ_2. Let $\rho = \max\{\rho_1, \rho_2\} < \infty$ and m be an integer such that $m - 1 \leq \rho < m$. If there are $m+1$ rays $\Gamma_j, j = 0, 1, \cdots, m$, in the complex plane

$$\arg\Gamma_0 = \frac{\pi}{2} < \arg\Gamma_1 \leq \arg\Gamma_2 \leq \cdots \leq \arg\Gamma_m = \frac{3\pi}{2}$$

with

$$\arg\Gamma_{j+1} - \arg\Gamma_j \leq \frac{\pi}{m}, \quad 0 \leq j \leq m-1$$

such that $R(\lambda,\mathcal{A}^*)x$ is uniformly bounded for every $x\in\mathcal{H}$ and any $\lambda\in\Gamma_j, 0\le j\le m$, then $\overline{Sp(\mathcal{A})}=\overline{Sp(\mathcal{A}^*)}=\mathcal{H}$ where $Sp(\mathcal{A})$ is the subspace spanned by all root vectors of $\mathcal{A}$.

In what follows, we shall use Lemma 4.2 to prove the completeness of root vectors of $\mathcal{A}$.

Theorem 4.3. *Let $\mathcal{A}$ be defined by (7–8). Then the system of root vectors of $\mathcal{A}$ is complete in $\mathcal{H}$, i.e, $\overline{Sp(\mathcal{A})}=\mathcal{H}$.*

Proof. We prove the assertion of Theorem 4.3 by the following four steps:

Step 1. The adjoint operator $\mathcal{A}^*$ has the form

$$\mathcal{D}(\mathcal{A}^*)=\left\{(w,z)\in\prod_{j=1}^{n}H_E^4(0,\ell_j)\times H_E^2(0,\ell_j)\;\middle|\;\begin{array}{l}-\sum_{j=1}^{n}\frac{EI_j}{s_j}w_{ss}^j(0)+R(w)(a)=\beta z_x(a)\\ w_{ss}^j(\ell_j)=0\\ \sum_{j=1}^{n}EI_jw_{sss}^j(0)+w(a)=\alpha z(a)\end{array}\right\}, \tag{37}$$

$$\mathcal{A}^*(w,z)=-\left\{\left(z^j,-\frac{EI_j}{\rho_j}w_{ssss}^j\right)\right\}\in\mathcal{H},\quad \forall(w,z)\in\mathcal{D}(\mathcal{A}^*). \tag{38}$$

Since this is a direct verification, we omit the detail. Note that $\sigma(\mathcal{A}^*)=\overline{\sigma(\mathcal{A})}$ always holds, we have $\sigma(\mathcal{A}^*)=\sigma(\mathcal{A})$ due to Theorem 3.1.

Step 2. Let $\mathcal{A}_0$ denote the operator $\mathcal{A}$ under the restriction $\alpha=\beta=0$. Then $\mathcal{A}_0$ is a skew adjoint operator, i.e., $\mathcal{A}_0^*=-\mathcal{A}_0$. This assertion follows directly from (7–8) and (37–38).

Step 3. Let $\Delta(\lambda),\lambda\in\mathbb{C}$, be defined as before, then $\Delta(\lambda)$ is an entire function of order $\rho_2\le 1$. For any $F\in\mathcal{H}$, the $\mathcal{H}$-valued functions $\Delta(\lambda)R(\lambda,\mathcal{A})F$ and $\Delta(\lambda)R(\lambda,\mathcal{A}^*)F$ can be extended to entire functions of order at most 1, i.e., $\rho_1\le 1$.

Since $R(\lambda,\mathcal{A}^*)F$, $R(\lambda,\mathcal{A})F$ and $\Delta(\lambda)$ consist of functions of type

$$\sinh\sqrt{i\lambda}\omega_j(\ell_j-s),\quad \cosh\sqrt{i\lambda}\omega_j(\ell_j-s),\quad \sin\sqrt{i\lambda}\omega_j(\ell_j-s),\quad \cos\sqrt{i\lambda}\omega_j(\ell_j-s)$$

multiplying $\sqrt{i\lambda}$,$(\sqrt{i\lambda})^2$ and $(\sqrt{i\lambda})^3$ as well as their integral, we get that $\Delta(\lambda)$, $\Delta(\lambda)R(\lambda,\mathcal{A})F$ and $\Delta(\lambda)R(\lambda,\mathcal{A}^*)F$ are entire functions of finite exponential type.

Step 4. The root vectors of $\mathcal{A}$ are complete in $\mathcal{H}$.

To prove this, we can assume without loss of generality that $\mathbb{R}_-\subset\rho(\mathcal{A})$. For $\lambda\in\mathbb{R}_-$, $F\in\mathcal{H}$, set $Y_1=R(\lambda,\mathcal{A}^*)F$, $Y_2=R(\lambda,\mathcal{A}_0^*)F$ and $\Phi=Y_1-Y_2$. Writing

$$\Phi=\begin{pmatrix}w\\ z\end{pmatrix},\quad Y_2=\begin{pmatrix}u\\ v\end{pmatrix},$$

we have that $z=-\lambda w$ and w satisfies the equations:

$$\begin{cases}EI_jw_{ssss}^j(s)=-\rho_j\lambda^2w^j(s),\quad s\in(0,\ell_j)\\ w^j(\ell_j)=w_{ss}^j(\ell_j)=0,\quad j=1,2,\cdots,n,\\ w(a)=w^j(0),\quad R(w)(a)=s_jw_s^j(0),\quad j=1,2,\cdots,n,\\ -\sum_{j=1}^{n}\frac{EI_j}{s_j}w_{ss}^j(0)+(1+\beta\lambda)R(w)(a)=R(v)(a),\\ \sum_{j=1}^{n}EI_jw_{sss}^j(0)+(1+\alpha\lambda)w(a)=v(a).\end{cases} \tag{39}$$

Therefore,

$$\begin{aligned}
&||\Phi||^2\\
= \ & \sum_{j=1}^{n}\int_0^{\ell_j}(EI_j|w_{ss}^j(s)|^2+\rho_j|\lambda|^2|w^j(s)|^2)ds+w(a)\overline{w(a)}+R(w)(a)\overline{R(w)(a)}\\
= \ & \sum_{j=1}^{n}\int_0^{\ell_j}(EI_jw_{ssss}^j(s)+\rho_j\lambda^2w^j(s)^2)\overline{w^j(s)}ds+w(a)\overline{w(a)}+R(w)(a)\overline{R(w)(a)}\\
& +\sum_{j=1}^{n}EI_jw_{sss}^j(0)\overline{w^j(0)}-\sum_{j=1}^{n}EI_jw_{ss}^j(0)\overline{w_s^j(0)}\\
= \ & -\alpha\lambda|w(a)|^2-\beta\lambda|R(w)(a)|^2+v(a)\overline{w(a)}+R(v)(a)\overline{R(w)(a)}\\
\leq \ & \max\{\alpha,\beta\}\left(|\lambda||w(a)|^2+|\lambda||R(w)(a)|^2\right)+v(a)\overline{w(a)}+R(v)(a)\overline{R(w)(a)}.
\end{aligned}$$

According to Theorem 4.1, we have

$$\begin{pmatrix} w(a)\\ R(w)(a)\end{pmatrix}=D^{-1}(\lambda)\begin{pmatrix} v(a)\\ R(v)(a)\end{pmatrix},$$

which means

$$|w(a)|^2+|R(w)(a)|^2\leq ||D^{-1}(\lambda)||^2[|v(a)|^2+|R(v)(a)|^2].$$

Therefore, using the estimate (34) in Theorem 4.1, we get

$$\begin{aligned}
||\Phi||^2 \ \leq \ & \max\{\alpha,\beta\}(|\lambda|||D^{-1}(\lambda)||^2[|v(a)|^2+|R(v)(a)|^2])\\
& +||D^{-1}(\lambda)||[|v(a)|^2+|R(v)(a)|^2]\\
\leq \ & \max\{\alpha,\beta,1\}(||\sqrt{i\lambda}D^{-1}(\lambda)||^2+||D^{-1}(\lambda)||)[|v(a)|^2+|R(v)(a)|^2]\\
\leq \ & (2M)^2\max\{\alpha,\beta,1\}[|v(a)|^2+|R(v)(a)|^2].
\end{aligned}$$

Since the operator $V:\mathcal{H}\to\mathbb{C}^2$ defined by

$$V(R(\lambda,\mathcal{A}_0^*)F):=\begin{pmatrix} v(a)\\ R(v)(a)\end{pmatrix}$$

is a bounded linear operator on $\mathcal{H}$, there is a positive constant M_1 such that

$$[|v(a)|^2+|R(v)(a)|^2]=\left\|\begin{pmatrix} v(a)\\ R(v)(a)\end{pmatrix}\right\|^2\leq M_1^2||F||^2.$$

Therefore we derive

$$||\Phi||\leq 2M_1M\sqrt{\max\{\alpha,\beta,1\}}||F||.$$

Now for $\lambda\in\mathbb{R}_-$, we have

$$\begin{aligned}
||R(\lambda,\mathcal{A}^*)F|| \ = \ & ||Y_1||=||Y_2+\Phi||\leq ||Y_2||+||\Phi||\\
\leq \ & ||R(\lambda,A_0^*)F||+2M_1M\sqrt{\max\{\alpha,\beta,1\}}||F||\\
\leq \ & \left(\frac{1}{|\lambda|}+2M_1M\sqrt{\max\{\alpha,\beta,1\}}\right)||F||,\quad |\lambda|>h.
\end{aligned}$$

This means that $||R(\lambda,\mathcal{A}^*)F||$ is bounded on the negative real axis. The completeness of root vectors of $\mathcal{A}$ follows from Lemma 4.2. □

To study the basis property of the root vectors of $\mathcal{A}$, we need the following notion.

Definition 4.4. Let $\mathcal{H}$ be a Hilbert space, and $\{\mathcal{H}_j, j \in \mathbb{N}\}$ be a subspace sequence of $\mathcal{H}$. $\{\mathcal{H}_j, j \in \mathbb{N}\}$ is said to be a subspace Riesz basis for $\mathcal{H}$ if for each $x \in \mathcal{H}$, there are unique $x_j \in \mathcal{H}_j$, $j \in \mathbb{N}$, such that $x = \sum_{j=1}^{\infty} x_j$ and there exist constants C_1 and C_2 such that

$$C_1 \sum_{j=1}^{\infty} ||x_j||^2 \leq ||x||^2 \leq C_2 \sum_{j=1}^{\infty} ||x_j||^2, \quad \forall x \in \mathcal{H}.$$

A sequence $\{\varphi_n, n \in \mathbb{N}\} \subset \mathcal{H}$ is said to be a Riesz basis with parentheses if there is an increasing subsequence in $\mathbb{N}$, $\{n_k, k \in \mathbb{N}\}$, $\lim_{k\to\infty} n_k = \infty$, such that the subspace sequence

$$\mathcal{H}_k = \mathrm{span}\{\varphi_j, n_k \leq j \leq n_{k+1} - 1\}, \quad k \in \mathbb{N}$$

forms a subspace Riesz basis for $\mathcal{H}$.

For a linear operator $\mathcal{A}$ with discrete spectrum, it is very difficult to verify its root subspaces forming a subspace Riesz basis for $\mathcal{H}$. If an operator $\mathcal{A}$ generates a C_0 semigroup and its spectrum satisfies certain conditions, we can assert the Riesz basis property of the root subspaces (see,[29],[30]). The following lemma that comes from [31] is a more general result.

Lemma 4.5. *Let $\mathcal{A}$ be the generator of a C_0-semigroup on a separable Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied:*

1). The spectrum of $\mathcal{A}$ has the decomposition

$$\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \bigcup \sigma_2(\mathcal{A});$$

2). There exists a real number $\alpha \in \mathbb{R}$ such that

$$\sup\{\Re\lambda : \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{\Re\lambda : \lambda \in \sigma_2(\mathcal{A})\};$$

3). The set $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k\in\mathbb{N}}$ consists of eigenvalues of $\mathcal{A}$ and is a finite union of separable sets.

Then there exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\mathcal{H}_1 = \{f \in \mathcal{H} : E(\lambda, \mathcal{A})f = 0, \forall\lambda \in \sigma_2(\mathcal{A})\},$$
$$\mathcal{H}_2 = \overline{\mathrm{span}\left\{\sum_{k=1}^{m} E(\lambda_k, \mathcal{A})f : \forall f \in \mathcal{H}, \forall m \in \mathbb{N}\right\}},$$

and $\mathcal{H}_1 \bigcap \mathcal{H}_2 = \{0\}$ with the property that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$. Moreover, there exists a sequence $\{\Omega_k, k \in \mathbb{N}\}$ such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k\in\mathbb{N}}$ forms a subspace Riesz basis for $\mathcal{H}_2$, where $\bigcup_{k=1}^{\infty} \Omega_k = \sigma_2(\mathcal{A})$ and each Ω_k includes only finitely many elements of $\sigma_2(\mathcal{A})$.

Applying Lemma 4.5 to our problem, we can achieve the following result.

Theorem 4.6. *Let $\mathcal{A}$ be defined by (7–8). Then there exists a sequence of root vectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$.*

Proof. Setting $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$ and $\sigma_1(A) = \{\infty\}$, Theorem 3.1 and Theorem 3.4 ensure that all conditions in Lemma 4.5 are satisfied. By Lemma 4.5, there is a subset sequence $\{\Omega_k, k \in \mathbb{N}\}$ of $\sigma(\mathcal{A})$ such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k\in\mathbb{N}}$ forms a subspace Riesz basis for $\mathcal{H}_2$. The completeness in Theorem 4.3 implies $\mathcal{H} = \mathcal{H}_2$. So $\{E(\Omega_k, \mathcal{A})\mathcal{H}\}_{k\in\mathbb{N}}$ is also a subspace Riesz basis for $\mathcal{H}$. Note that there are only finitely many elements in each Ω_k. Therefore there is a sequence of root vectors of $\mathcal{A}$ that forms a Riesz basis with parentheses according to $\{\Omega_k, k \in \mathbb{N}\}$ for $\mathcal{H}$. $\square$

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