NETWORKS AND HETEROGENEOUS MEDIA
doi:10.3934/nhm.2008.3.691
©American Institute of Mathematical Sciences
Volume **3**, Number **4**, December **2008**
pp. **691–707**

MODELLING AND SIMULATION OF FIRES IN TUNNEL NETWORKS

Ingenuin Gasser

Department Mathematik, Universität Hamburg
Bundesstraße 55, 20146 Hamburg, Germany

Marcus Kraft

Institut für Numerische Simulation, Technische Universität Hamburg-Harburg
Schwarzenbergstraße 95, 21073 Hamburg, Germany

(Communicated by Roberto Natalini)

Abstract. A known model for describing tunnel fires is extended to the case of tunnel networks. Physically motivated coupling conditions are formulated. A numerical realisation of these conditions and of the full network problem is presented. Finally, numerical simulation of realistic tunnel fires in networks are performed.

1. **Introduction.** Due to some serious fire accidents in the near past tunnel fires have become an interesting topic not only for CFD engineers. In the last decades various mathematical models based on a gas-dynamic description of the air or the air/smoke mixture in the tunnel have been proposed. A good overview on the modeling approaches is given in [7, 15]. General information on tunnel fires can be found on the information platform [3]. A recent survey on computer codes for tunnel fire simulations can be found in [14] (and in [2]). Almost all these approaches refer to single tunnels and not to tunnel networks.

Most of the approaches for single tunnels in [14] are based on standard gas dynamic CFD tools. Such (3 dimensional) simulations are known to be complex and expensive. In addition many data are needed and the interpretation of the results is a non-trivial task. From an application point of view a detailed knowledge of the three-dimensional flow is in general not necessary. The main questions arising from the application can often be answered by one-dimensional models (one longitudinal spatial dimension). In addition one-dimensional approaches have important advantages with respect to higher-dimensional models:

- There are simple ways to include turbulence models.
- Numerical simulations are in general not very expensive.
- There is a chance to extend the model to tunnel networks.
- Optimization strategies can be applied and solved in reasonable time.

Therefore in the following we only consider one-dimensional models.

In addition to the question about the number of (spatial) dimensions there are mainly three difficult issues:

2000 *Mathematics Subject Classification.* Primary: 76D05, 76N15, 35Q30, 80A20, 80A25;
Key words and phrases. Gasdynamics, Low-Mach-number, Fire in Tunnels, Tunnelnetworks.
The first author was partially supported by the *Berufsfeuerwehr Bozen.*

- Very small Mach numbers.
- Large temperature differences such that significant heat transport takes place.
- The turbulent character of the flow.

Especially the combination of the first two points induces a serious problem. The second point excludes a pure incompressible description (with the standard Boussinesq approximation for the energy transport). On the other hand there are the known problems of fully compressible approaches in the small Mach number regime. Although this facts are known (even in the tunnel-fire literature, see [7]) they are often ignored due to the lack of alternatives.

When considering tunnel networks the additional problem lies in the coupling conditions of the various "tunnel pieces" at a vertex, where different tunnels meet. This is a known problem and the solution depends on the models used in the tunnels between the vertices. This question concerns well posedness of the conditions, but also the numerical realization of the conditions at a vertex. Similar problems arise in traffic flow networks (when using a fluiddynamic description) (see e.g [8]), in gas pipeline networks (see e.g. [1]), in hydraulic networks (see e.g. [13, 17]), etc..

In this paper we consider a one dimensional model which was introduced in [11] for a single tunnel. To our knowledge this is one of the first models derived from underlying fluid dynamic equations. This model is derived in such a way that it allows to combine both a good description in the low Mach number regime and significant heat transport. The results obtained by now show that the model seems to keep the main features of tunnel fires. A good (at least qualitative) agreement with results from experiments has been obtained (see [5, 6]). In [4] the stationary model was discussed. In [9] a global existence result of solutions of the transient model was presented and in [10] a bifurcation analysis of the model faces the question on the stability of certain stationary solutions. Higher dimensional versions of the model have been employed in [12] and [16].

The paper is organized as follows. In section 2 we briefly present the tunnel fire model for a single tunnel. In section 3 we discuss the case of a network and the coupling conditions. In section 4 we present a numerical approach for the tunnel network problem. In section 5 we give some numerical examples.

2. The model for a single tunnel.

In this chapter we shortly describe the model for a single tunnel derived in [11]. The purpose of the model is to describe fire events in a long tunnel.

We give a short outline of the derivation of the model. The starting point are the fully compressible Navier-Stokes equations for the gas (air or air-smoke mixture) in the tunnel (in one space dimension). Let ρ, u, p, T be the dimensionless density, velocity, pressure and the temperature of the flow in the tunnel, respectively. Let x and t denote the dimensionless space coordinate along the tunnel axis and the dimensionless time, respectively.

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
u_t + u u_x + \left(\frac{1}{\gamma M^2}\right)\frac{1}{\rho} p_x &= \eta \frac{1}{\rho} u_{xx} - p_{dv}\frac{u|u|}{2} - f_d \sin(\alpha) + s_d s, \\
(\rho T)_t + (u\rho T)_x + (\gamma - 1) p u_x &= \lambda T_{xx} + \frac{q_r L}{u_r p_r}(\gamma - 1) q + \hat{\eta} u_x^2
\end{aligned}
\tag{1}
$$

The fourth equation (for the four unknowns) is the scaled ideal gas law $p = \rho T$. L and d are the length and diameter of the tunnel. We note that we choose the reference value for the heat source as $q_r = \frac{u_r p_r}{L}$, and the dimensionless constants η, λ, f_d are given by (values with a subscript r are reference values)

$$\eta = \frac{4}{3}\frac{1}{Re}\frac{d}{L}, \quad \lambda = \frac{1}{Pr}\frac{1}{Re}\frac{d}{L}, \quad f_d = \frac{gL}{u_r^2}, \quad s_d = \frac{s_r L}{u_r^2}, \quad p_{dv} = \frac{p_{lr} L}{u_r^2}, \quad p_{lr} = \frac{\xi u_r^2}{d}, \tag{2}$$

where γ is the adiabatic exponent, c the sound speed, M the Mach number, Re the Reynolds number and Pr the Prandtl number. These numbers are defined by

$$\gamma = \frac{c_p}{c_v},\ c = \sqrt{\frac{\gamma p_r}{\rho_r}},\ M^2 = \frac{\rho_r u_r^2}{\gamma p_r},\ Re = \frac{\rho_r u_r d}{\tilde{\eta}},\ Pr = \frac{\tilde{\eta} c_p}{\tilde{\lambda}},\ \hat{\eta} = \eta \frac{u_r}{p_r L}(\gamma - 1), \tag{3}$$

where c_p, c_v, $\tilde{\eta}$, $\tilde{\lambda}$ the specific heats at constant pressure and volume, the viscosity and the heat conductivity, respectively. The quantities $p_{dv} = p_{dv}(x)$ and $\alpha = \alpha(x)$ are the x-dependent pressure loss coefficient and the slope profile of the tunnel. Note that $p_{dv} = \xi L/d$ is the pressure loss coefficient of a fully developed pipe flow which due to $\xi = \xi(Re_x)$ is a function of the (local) Reynolds number Re_x only. The constant g is the gravitational acceleration. The term s models an additional force, i.e. ventilation fans. Finally, q is the (scaled) heat source due to the fire.

Typical values corresponding to the above scaling are

$$\gamma = \frac{7}{5}, \quad c = 341\, ms^{-1}, \quad M^2 = 8.6 \cdot 10^{-6},\ Re = 6.7 \cdot 10^5,\ Pr = 0.72. \tag{4}$$

Due to $M \approx 3 \cdot 10^{-3}$ we are in the low–Mach number regime. Also, it is important to note that the calculated Reynolds number indicates that the flow in the tunnel is turbulent.

The small Mach number is the reason for the above mentioned difficulties in the numerical treatment of the equations (1). The advantage in our application is that the Mach number is always small (in time and in space). Therefore a small Mach number asymptotic of the form (setting $\varepsilon = \gamma M^2$)

$$p = p_0 + \varepsilon p_1 + \mathcal{O}(\varepsilon^2). \tag{5}$$

is reasonable. This leads to $p_0 = p_0(t)$. Since the tunnel is an open region the leading order pressure (which corresponds to the mean outside pressure) will not change in time. Therefore $p_0 = const$. We have $p_0 = 1$ (such that the unscaled leading order pressure is equal p_r). Then in leading order we have $T = \frac{1}{\rho}$.

Since η and λ are very small in our scaling, we (formally) neglect those terms. The leading order equations (in ϵ, η, λ) are given by

$$\rho_t + u\rho_x = -\rho q, \tag{6}$$

$$u_t + uu_x + \frac{1}{\rho}p_x = -p_{dv}\frac{u|u|}{2} - f_d \sin(\alpha) + s_d s \tag{7}$$

$$u_x = q \tag{8}$$

with $q = q(x,t)$ as time and space dependent (scaled) heat source. For more details on the derivation of the model and scaling see [11, 5, 6]. The term s models ventilation fans.

At the boundary (i.e the tunnel entrance and exit) we prescribe Dirichlet data for the pressure p and standard inflow boundary conditions for the density for $t > 0$

$$p(t,0) = p_l(t), \qquad p(t,1) = p_r(t), \tag{9}$$

$$\rho(t,0) = \rho_l(t) \text{ if } u(t,0) > 0, \qquad \rho(t,1) = \rho_r(t) \text{ if } u(t,1) < 0. \tag{10}$$

Initial data are prescribed for the density and the velocity

$$u(0,x) = u_0(x), \qquad \rho(0,x) = \rho_0(x), \quad \forall x \in [0,1]. \tag{11}$$

Thus, our model consists of the equations (6)-(8), the boundary conditions (9)-(10) and the initial conditions (11).

A first problem lies in the fact that we have two boundary conditions for the pressure but only a first derivative of the pressure in the model. There are at least two ways to get rid of this problem ([11],[9]). We only present one reformulation which will be used in the following. We eliminate the pressure by multiplying the equation (7) by ρ and integrating over $x \in [0,1]$. This gives

$$\int_0^1 \rho u_t dx + \int_0^1 \rho u u_x dx + p_r - p_l = -\int_0^1 p_{dv}\rho \frac{u|u|}{2} dx - \int_0^1 (f_d \sin(\alpha) + s_d s)\rho dx. \tag{12}$$

Equation (8) gives

$$u(x,t) = v(t) + \int_0^x q(y,t)dy = v(t) + Q(x,t) \tag{13}$$

where $Q(x,t)$ is a known function. Then we obtain the system

$$\rho_t + (v+Q)\rho_x = -\rho q, \tag{14}$$

$$Rv_t + R_q v + \int_0^1 p_{dv}\rho \frac{(v+Q)|v+Q|}{2} dx = -R_{Q_t + Qq + f_d \sin(\alpha) + s_d s} - p_r + p_l \tag{15}$$

for ρ and v, where R, R_q, $R_{Q_t+Qq+f_d \sin(\alpha)+s_d s}$ denote functionals applied to $\rho(x,t)$ defined as (for an $f = f(x,t)$)

$$R_f(t) = \int_0^1 \rho(x,t) f(x,t) dx. \tag{16}$$

The system (14)-(15) consists of an ODE for v and a PDE for the density ρ. Remember that the temperature is given by $T = \frac{1}{\rho}$. The only boundary conditions needed are the inflow conditions (10) for the continuity equation. The conditions on the pressure appear as parameters in (15). In the following we use the formulation (14)-(15) for the network studies.

3. **Tunnel networks.** In this chapter we consider tunnel networks. The case of a single tunnel becomes rare, since most of the modern tunnels are equipped with additional ventilation systems and channels, which lead to a tunnel network. In addition tunnel networks are common in subway systems, in cities, etc. In most of the tunnel fire applications the fire will only be located in one single tunnel of the network. This doesn't make any problem, since in the case of no fire the tunnel fire model in chapter 2 reduces to an incompressible fluid model with variable density for the air or air/smoke mixture.

The idea is to use the model from chapter 2 for the single tunnels in the network an to define (physically) meaningful conditions at the vertices. Boundary conditions for a network structure are only allowed at the (real) entrances and exits, i.e. at the boundaries to the outside world. Since we know, that the model for a single tunnel needs boundary conditions at entrance and exit, we have to define quantities like the pressure and the density (in the inflow case) at a vertex for every tunnel meeting this vertex.

3.1. **The network notation.** Before defining the condition at a vertex we have to fix the network notations. Suppose we have a set of vertices $N_N = \{1, ..., n_V\}$ and a set of tunnels $N_T = \{1, ..., n_T\}$ where $n_V, n_T \in \mathbb{N}$ are the numbers of vertices and tunnels, respectively. In every tunnel we have still a left and a right exit such that the model of section 2 can be used without changes. We denote the quantities of the ith tunnel by a superscript $i \in N_T$ and values at the left (right) boundary by a subscript $l(r)$ as in section 2. To summarize, the quantities of the ith tunnel are given by

A^i	cross section,
$\rho^i, u^i = v^i + Q^i, p^i, T^i$	density, velocity, pressure, temperature
$M^i = \int_x \rho^i(x)dx$	total mass,
p^i_l, p^i_r	left and right pressure values
ρ^i_l, ρ^i_r	left and right density values
T^i_l, T^i_r	left and right temperature values
$u^i_l = v^i + Q^i_l, u^i_r = v^i + Q^i_r$	left and right velocitiy values

For every vertex $j \in N_V$ the set $T^j \subset N_T$ it the set of those indices of tunnels which enter the vertex j. We define the two maps

$$T : N_T \times N_V \to \{l, r, 0\}, T(i,j) = \begin{cases} l & \text{vertex } j \text{ lies on the left of tunnel } i \\ r & \text{vertex } j \text{ lies on the right of tunnel } i \\ 0 & \text{vertex } j \text{ is not connected to tunnel } i \end{cases}$$

$$\text{sgn} : N_T \times N_V \to \{-1, 0, 1\},\ \text{sgn}(i,j) = \begin{cases} -1 & T(i,j) = r \\ 0 & T(i,j) = 0 \\ 1 & T(i,j) = l \end{cases}$$

This seemingly complicated definitions are necessary to write down the vertex conditions in a simple and consistent way.

3.2. **The coupling conditions at a vertex.** Let us consider a single vertex $j \in N_V$. The tunnels with index $i \in T^j$ enter this vertex. Every tunnel is described by the model (14)-(15) with inflow boundary conditions (10) and left and right pressure values as parameters (or time dependent functions) in the equations. This model is valid for constant cross section. The only additional information in the network case are the cross sections $A^i, i \in N_T$. The only value in the model which depends on the cross section – or better on the relation between longitudinal extension and extension in the cross section – is the factor p_{dv} in front of the pressure loss term. But this factor includes also the surface properties and has to be determined anyway for every single tunnel. As a consequence the cross sections enter as constant multipliers for the various flows of the corresponding tunnels.

At the vertex j at every time t, some of the tunnels entering this vertex will transport air or an air/smoke mixture into this vertex (we denote those inflow tunnels) and the rest of the tunnels will transport air or an air/smoke mixture away from this vertex (we denote those outflow tunnels). A tunnel with no flow can be counted for as inflow or as outflow tunnel. Clearly when time evolves we have to allow that tunnels may change from inflow to outflow and viceversa. In fact, these are the cases of special interest in the applications.

Now we discuss the coupling conditions in a vertex. We know from chapter 2 that all tunnels - inflow and outflow - need a value for the pressure (as "boundary condition") at the vertex. In addition the outflow tunnels need a value for the density (as "inflow boundary condition" for the density).

It is easy to understand that we ask the mass to be conserved at every vertex $j \in N_V$, i.e. the sum of the mass fluxes $\text{sgn}(i,j)\rho^i_{T(i,j)} u^i_{T(i,j)} A^i$ in the ith tunnel has to be zero

$$\sum_{i \in T^j} \text{sgn}(i,j)\rho^i_{T(i,j)} u^i_{T(i,j)} A^i = 0 \qquad \forall j \in N_V. \tag{17}$$

This is our first condition at every vertex.

It is evident, that in general the momentum will not be conserved at a vertex. In reality at a vertex a complex three-dimensional flow will be such, that a non negligible part of the momentum is absorbed by the walls (and/or by viscosity in case of vortices etc.).

It is also clear, that energy is not conserved at a vertex. However, the non conserved energy is mainly the kinetic energy due to the motion of the flow. In fact, if we look at the full energy density (in the ith tunnel)

$$e^i = (M^2 \rho^i (u^i)^2 + \rho^i T^i) A^i,$$

we realize, that in the small Mach number regime in leading order the kinetic energy can be neglected. Therefore, the leading order term of the energy flux density is the internal energy flux $e^i = u^i \rho^i T^i A^i$ which we assume to be conserved at the vertex

$$\sum_{i \in T^j} \text{sgn}(i,j) T^i_{T(i,j)} \rho^i_{T(i,j)} u^i_{T(i,j)} A^i = 0 \qquad \forall j \in N_V.$$

In other words, no other then kinetic energy is lost in a vertex. Since we know, that in leading order all the terms $\rho^i T^i$ satisfy $\rho^i T^i = p^i = 1$, we obtain a simplified second condition at the vertex

$$\sum_{i \in T^j} \text{sgn}(i,j) u^i_{T(i,j)} A^i = 0 \qquad \forall j \in N_V. \tag{18}$$

Splitting into a sum of outflow and a sum of inflow tunnels gives

$$\sum_{\substack{i \in T^j \\ \text{sgn}(i,j) u^i_{T(i,j)} > 0}} \text{sgn}(i,j) u^i_{T(i,j)} A^i + \sum_{\substack{i \in T^j \\ \text{sgn}(i,j) u^i_{T(i,j)} < 0}} \text{sgn}(i,j) u^i_{T(i,j)} A^i = 0. \tag{19}$$

The conditions (17) and (18) (or (19)) deduced form the conservation of the mass flux and the conservation of the non-kinetic energy flux are not sufficient to define all required quantities at a vertex (the pressures for all tunnels meeting in a vertex and densities for all outgoing tunnels).

Therefore we assume that in a vertex there is a good "mixing" of the flow in the vertex, such that all outgoing tunnels (from a single vertex) have the same inflow boundary condition for the density $\forall j \in N_V \quad \rho^j_{\text{inflow}} = \rho^i_{T(i,j)}$ for all i such that $\text{sgn}(i,j) u^i_{T(i,j)} > 0$. Then from (17) we obtain

$$\rho^j_{\text{inflow}} = - \frac{\sum_{\substack{i \in T^j \\ \text{sgn}(i,j) u^i_{T(i,j)} < 0}} \text{sgn}(i,j) \rho^i_{T(i,j)} u^i_{T(i,j)} A^i}{\sum_{\substack{i \in T^j \\ \text{sgn}(i,j) u^i_{T(i,j)} > 0}} \text{sgn}(i,j) u^i_{T(i,j)} A^i} > 0 \quad \forall j \in N_V,$$

and using (18)

$$\rho^j_{\text{inflow}} = + \frac{\sum_{\substack{i\in T^j \\ \text{sgn}(i,j)u^i_{T(i,j)}<0}} \text{sgn}(i,j)\rho^i_{T(i,j)} u^i_{T(i,j)} A^i}{\sum_{\substack{i\in T^j \\ \text{sgn}(i,j)u^i_{T(i,j)}<0}} \text{sgn}(i,j) u^i_{T(i,j)} A^i} > 0 \quad \forall j \in N_V, \tag{20}$$

Note that at every vertex j the density ρ^j_{inflow} is defined only by (known) inflow quantities.

It remains to define the pressures at a vertex $j \in N_V$. We assume, that again the good "mixing" of the flow at the vertex causes a single pressure (the subscript N stays for values at a vertex) in the vertex (as boundary condition for all inflowing and outflowing tunnels). In fact, this is the pressure condition at the vertices

$$p^j_N = p^i_{T(i,j)} \quad \forall j \in N_V, \forall i \in T^j. \tag{21}$$

Since the pressure in the vertex influences all the inflowing and outflowing quantities it cannot be calculated directly from inflowing and/or outflowing quantities.

In the following we will see, that (at least numerically) the above mentioned conditions are sufficient for the definition of the network problem. The total tunnel network problem therefore consists of the following equations and conditions. N_O denotes the number of exits to the outside world (remember that N_T, N_V denote the number of tunnels and the number of vertices, respectively).

- a number N_T of single tunnel equations of type (14)-(15),
- a number N_V of coupling conditions of type (18) for the velocities,
- a number $N_{N,inflow}$ of inflow conditions for the densities of type (20) with $0 \le N_{N,inflow} \le 2N_T - N_O - N_V$,
- a number $2N_T - N_O$ of pressure conditions of type (21),
- a number N_O of pressures at all exits to the outside world,
- a number $N_{O,inflow}$ of densities at all inflowing exits to the outside world with $N_{O,inflow} \le N_O$.

The question of (unique) solvability of the total tunnel network problem is completely open and will not be answered in this paper. Some remarks on the solvability will be made and the numerical solvability of the problem will be discussed in the next section.

3.3. **A (formal) monotonicity argument.** We have seen that the pressures at the vertices are additional unknowns in the tunnel network problem. However, we expect a unique pressure value at every vertex due to the following formal "monotonicity" argument. We consider the total volume flux q^j_T at a vertex j (which according to (18) should vanish) as a function of the pressure p^j_N at this vertex

$$q^j_T(p^j_N) = \sum_{i\in T^j} \left(\text{sgn}(i,j) u^i_{T(i,j)} A^i\right)(p^j_N)$$

Monotonicity property of the Pressure : *When neglecting the condition (18) at a vertex $j \in N_V$ then the total volume flux q^j_T at this vertex is a monoton function of the pressure p^j_N.*

In fact, increasing the pressure p^j_N in a vertex (by taking the outside boundary conditions constant), we increase the outflow and we decrease the inflow such that the total volume flux in the vertex decreases. On the other hand, decreasing the pressure p_N^j in the vertex, the total volume flux is increased. The last two conjectures could be proved studying the dependence of the volume flux on the pressure boundary data in the single tunnel fire model. As a consequence of this seemingly monotonicity property we expect a unique value of the pressure at the vertex with vanishing total volume flux. In fact, this (formal) monotonicity property will be used in the numerical solution of the problem in section 4.

3.4. **Comparison to other network problems.** Here we would like to compare our network model to some known network problems, which are time dependent and involve differential equations. As mentioned in the introduction similar network problems arise in modeling gas pipelines, traffic flow, hydraulic networks, etc. From a physical point of view the gas pipeline networks seem to be very close. However, also the traffic flow models are very similar to the gas pipeline models. In fact, the more sophisticated models in both applications are based on 2×2 systems of hyperbolic equations with a "source term" (see [1] for the pipeline application and [8] for the traffic networks). The gas flow in the pipeline is described by the isentropic gas dynamic equations with a friction term in the momentum balance. In the case of traffic the situation is slightly different. Anyway, both are compressible models. There are two main differences between the tunnel fire network problem and the corresponding gas pipeline and traffic network problems:

- Like in tunnel fire network case, at a vertex the conservation of the volume flux (18) and the pressure condition (21) is assumed. Since momentum flux is not conserved at a vertex and since no low Mach number argument seems to be applicable the remaining necessary conditions have to be found using different arguments. As a consequence, there is a big variety of different additional conditions which seem not to be comparable to our physically motivated conditions.
- In the gas pipeline case (and in the case of traffic) waves (and information) in the underlying hyperbolic system travel with finite speed. Therefore the problem at a single vertex in the network comes out to be a local (in space) problem. In the tunnel fire network case we performed a small Mach number limit and therefore the resulting problem is not hyperbolic. In fact we can formulate an elliptic equation for the pressure. This means that we have infinite speed of information and as consequence the problem at a single vertex becomes a global problem with instantaneous influences from all over the network. Therefore, we need to solve the full problem instantaneously.

From this point of view the (incompressible) hydraulic network problems seem to be much closer to our problem. In fact, there the instantaneous solution of the full problem is a known procedure (see [13]). However, two main differences to the standard approach in hydraulic networks have to be mentioned:

- In our case $u_x \neq 0$ at the position of the fires. In an incompressible model the condition $u_x = 0$ is fullfilled everywhere and cancels out the nonlinear convective term $(\frac{u^2}{2})_x = uu_x = 0$ and the related difficulties.
- In our case the density is varying, where as in most of the hydraulic applications the density is constant. This makes the network problem simpler.

Nevertheless there are hydraulic applications where weakly compressible models are used. Such models are supposed to be closer to the above gas pipeline (or traffic) network problems (see [17]).

4. **Numerical simulation.** The numerical approximation of the tunnel network problem consists in solving the one dimensional model (14)-(15) for every tunnel $i \in N_T$, to fulfill the vertex conditions (18),(20),(21) and to fulfill the boundary conditions on the exits to the outside world.

The problem for a single tunnel with index $i \in N_T$ consisting of an ODE for the time dependent velocity part v^i and a continuity equation for ρ^i can be solved by standard schemes (see [5, 9]). In the simplest case this can be an explicit Euler-scheme (or Runge-Kutta scheme) for the ODE and an upwind scheme for the PDE. Even the simplest method is known to give reasonable results.

In the case of a network a naive approach would be to solve in every time step the problems in every tunnel $i \in N_T$ using the pressures $p_N^j, j \in N_V$ at the vertices as unknown parameters and trying to obtain (e.g. by an iterative approach) good pressure values at the vertices (satisfying (21)) such that the vertex conditions (18) (and also 20) are fulfilled. This is equivalent to a N_V dimensional fixedpoint problem (for every time step), which is expected to be numerically very expensive.

Here we propose a much faster alternative approach. For simplicity we start with a simple explicit Euler scheme to solve the ODE. One explicit Euler step gives for $i \in N_T$

$$v^i(t+\Delta t) = v^i(t) + \Delta t\, f^i[v,\rho,Q](t) + \Delta t \frac{p_l^i - p_r^i}{M^i}.$$

where the functional f^i depends only on the state of the system at time t and not on the pressure values of the boundary. We plug the value of v at time $t+\Delta t$ into the vertex condition (18) and obtain for every $j \in N_V$ the condition

$$\sum_{i\in T^j} \operatorname{sgn}(i,j)(v^i(t) + \Delta t\, f^i[v,\rho,Q](t) + \Delta t \frac{p_l^i - p_r^i}{M^i} + Q^i_{T(i,j)}(t+\Delta t))A^i = 0.$$

Even before using the pressure condition (21) we see that this are apparently linear equations for the pressure values. Rearranging gives

$$\begin{aligned} &\frac{1}{\Delta t}\sum_{i\in T^j}\big(\operatorname{sgn}(i,j)(v^i(t) + \Delta t f^i[v,\rho,Q](t) + Q^i_{T(i,j)}(t+\Delta t))A^i\big) \\ &= \; -\sum_{i\in T^j}\operatorname{sgn}(i,j)\frac{p_l^i - p_r^i}{M^i}A^i \\ &= \; -\sum_{i\in T^j}\operatorname{sgn}(i,j)p_l^i\frac{A^i}{M^i} + \sum_{i\in T^j}\operatorname{sgn}(i,j)p_r^i\frac{A^i}{M^i} \end{aligned} \tag{22}$$

Now we use the pressure condition (21) and obtain for every $j \in N_V$

$$\begin{aligned} &\frac{1}{\Delta t}\sum_{i\in T^j}\big(\operatorname{sgn}(i,j)(v^i(t) + \Delta t f^i[v,\rho,Q](t) + Q^i_{T(i,j)}(t+\Delta t))A^i\big) \qquad (23) \\ &+ \sum_{\{i\in T^j | T(i,k)\neq l\,\forall k\}}\operatorname{sgn}(i,j)p_l^i\frac{A^i}{M^i} - \sum_{\{i\in T^j | T(i,k)\neq r\,\forall k\}}\operatorname{sgn}(i,j)p_r^i\frac{A^i}{M^i} \\ &= \; -p_N^j\sum_{\{i\in T^j | T(i,j)=l\}}\operatorname{sgn}(i,j)\frac{A^i}{M^i} - \sum_{k=1,k\neq j}^{n_V} p_N^k \sum_{\{i\in T^j | T(i,k)=l\}}\operatorname{sgn}(i,j)\frac{A^i}{M^i} \end{aligned}$$

$$
\begin{aligned}
& +p_N^j \sum_{\{i\in T^j | T(i,j)=r\}} \operatorname{sgn}(i,j)\frac{A^i}{M^i} + \sum_{k=1,k\neq j}^{n_V} p_N^k \sum_{\{i\in T^j | T(i,k)=r\}} \operatorname{sgn}(i,j)\frac{A^i}{M^i} \\
= \quad & -p_N^j \sum_{i\in T^j} \frac{A^i}{M^i} \\
& + \sum_{k=1,k\neq j}^{n_V} p_N^k \left(- \sum_{\{i\in T^k | T(i,j)=r\}} \operatorname{sgn}(i,j)\frac{A^i}{M^i} + \sum_{\{i\in T^k | T(i,j)=l\}} \operatorname{sgn}(i,j)\frac{A^i}{M^i} \right) \\
= \quad & -p_N^j \sum_{i\in T^j} \frac{A^i}{M^i} + \sum_{k=1,k\neq j}^{n_V} p_N^k \sum_{i\in T^j\cap T^k} \frac{A^i}{M^i}
\end{aligned}
$$

Now it is evident, that the pressure boundary conditions under use of an explicit scheme are determined by a linear system of equations where the left hand side of the equation is a well known constant at the time t. Further simplification gives

$$
\begin{aligned}
& -\frac{1}{\Delta t} \sum_{i\in T^j} \Big(\operatorname{sgn}(i,j)(v^i(t) + \Delta t f^i[v,\rho,Q](t) + Q^i_{T(i,j)}(t+\Delta t))A^i \Big) \qquad (24) \\
+ \quad & \sum_{\{i\in T^j | T(i,k)\neq l \, \forall k\}} p_l^i \frac{A^i}{M^i} + \sum_{\{i\in T^j | T(i,k)\neq r \, \forall k\}} p_r^i \frac{A^i}{M^i} \\
= \quad & p_N^j \sum_{i\in T^j} \frac{A^i}{M^i} - \sum_{k=1,k\neq j}^{n_V} p_N^k \sum_{i\in T^j\cap T^k} \frac{A^i}{M^i}
\end{aligned}
$$

Note that for fixed k the set $T^j \cap T^k$ usually is empty or consists of one element (whenever two vertices are connected by at most one tunnel). Reinterpreting the right hand side of the equation to give the j-th row of the pressure matrix, we conclude the following properties:

- (P1) All diagonal elements of the pressure matrix are positive.
- (P2) All non-diagonal elements of the pressure matrix are non-positive.
- (P3) The pressure matrix is symmetric (neighbored vertices influence each other directly).
- (P4) All rows of the pressure matrix are at least weakly diagonal dominant.
- (P5) If one tunnel $i \in T^j$ is connected to the outside (not to another vertex) then the j-th row is strictly diagonal dominant.
- (P6) Let us reinterpret the tunnel network as a graph. Assume this graph to be connected, i.e. one can get from each tunnel in the network to any other. Then a vertex creating a strictly diagonal dominant row influences all other vertices at least indirectly.

Form this we conclude the following Lemma.

Lemma 4.1. *Suppose the graph corresponding to the tunnel network is connected. Then the corresponding matrix of the linear system for the pressure values at the vertices is*

- *regular, if there is at least one tunnel connecting the network to the outside.*
- *singular, if there is no connection to the outside. However, the eigenspace of the zero eigenvalue is one dimensional and corresponds to a possible additive constant for all pressures in the (closed) network.*

Proof. Due to the theorem of Gerschgorin, the positivity of the diagonal elements, the symmetry and the weak diagonal dominance, the pressure matrix is at least positive semi-definite. Because the graph of the tunnel network is connected and the entries outside the diagonal are non-positive, the only possible eigenvector of 0 would be a multiple of $(1, \ldots, 1)^T$, but due to the strictly diagonal dominant row of the matrix this vector cannot be an eigenvector to the proposed eigenvalue 0. Thus the pressure matrix is positive definite and hence regular.

Furthermore if we do not require one tunnel to be connected to the atmosphere then the pressure matrix is genuinely weakly diagonal dominant and one eigenvalue of the pressure matrix is zero. Hence the space of possible pressure values is a one dimensional affine linear space or empty. This corresponds to a possible additive constant for all pressure values in the network. □

Therefore, for realistic tunnel networks (with a connection to the outside) we obtain in every time step unique values for the pressures in the vertices. Remember, that right now we only used the coupling conditions (18) and the pressure conditions (21). The missing step is a time step in the continuity equation where we need the inflow data for the density which are given uniquely by the inflow conditions for the densities (20).

The overall algorithm for each time step we used here is given by the following description.

1. Network step:
   (a) Calculate pressure boundary values at the vertices by (24).
   (b) Calculate inflow densities if necessary by (20).
2. Calculate the dynamics in each tunnel segment separately. Integrate for this aim all necessary integrals by trapezoidal rule.
   (a) Use first order finite difference upwind scheme to solve the continuity equation (14).
   (b) Use explicit Euler scheme to solve (15).
   (c) Calculate u by (13).
   (d) Calculate T by $T = 1/\rho$.
3. Repeat these steps up to final time.

This algorithm is very efficient due to its simplicity. Numerical tests with up to 5 vertices on a standard personal computer indicated that reasonable computations run easily 10 times faster then real time (keeping in mind that we only use simple standard schemes). But this could be fastened easily by one or even two orders of magnitude. One can for instance coarsen the grids, especially those belonging to tunnel segments that are not affected by hot air. Additionally, the algorithm is naturally accessible to parallelized computation strategies.

5. **Examples.** In this section we present numerical simulations of a tunnel network using the model and the numerics presented in sections 3 and 4. We choose a real tunnel network, the Young Dong tunnel in Korea. This is a about $16.3\,km$ long railway tunnel with 3 additional ventilation tunnels going outside (see table 1, table 5 and figure 1). Since we do not have all necessary data for this tunnel we have to guess some of them. Therefore, our results can only have a qualitative character. The idea is to have a fire for a certain time period somewhere in the tunnel and to try – by using a ventilation system – to keep certain areas in the tunnel free of smoke (which in our model means free of hot air). Clearly, this is in general an even

more complicated problem, since on top of the tunnel network model one has to do even a control or an optimization problem for the ventilation strategy.

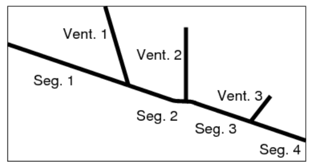

FIGURE 1. The profile of the Young Dong tunnel.

In our model the Young Dong tunnel is a network with 3 vertices and every vertex has 3 entering tunnels. Let us refer to segments of the main tunnel as segments numbered from left to right. The corresponding parameters are given in table 1. We

Parameter	Value
Length Segment 1	6580 m
Length Segment 2	3130 m
Length Segment 3	3510 m
Length Segment 4	3020 m
Slope Segment 1	-2.45%
Slope Segment 2 at $0-2285$ m	-2.45%
Slope Segment 2 at $2285-2685$ m	Polynomial $\mathcal{C}^1$ blending of degree 3
Slope Segment 2 at $2685-3130$ m	-0.3%
Slope Segment 3 at $0-100$ m	-0.3%
Slope Segment 3 at $100-500$ m	Polynomial $\mathcal{C}^1$ blending of degree 3
Slope Segment 3 at $500-3510$ m	-2.45%
Slope Segment 4	-2.45%
Cross section area of all segments	64 m^2

TABLE 1. Main-Tunnel parameters of the Young-Dong-Model

denote those tunnels not belonging to the main tunnel by ventilation channels. The corresponding data are given in table 5. In the succeeding figures of the examples a positive velocity in a segment reflects flow from left to right, a positive velocity in a ventilation channel reflects a fluid flow from up to down. Additionally, arrows in the temperature illustrations indicate the flow direction. For the following simulations we have chosen a set of parameters and conditions reported in table 5. Now we make two different numerical experiments. In both cases we put on a fire in segment 3 of the main tunnel. In case of unforced pressure conditions we expect smoke (hot air) to go upwards due to the slope profile of the tunnel and to leave the tunnel towards the left main exit and towards the ventilation channels 1 and 2. Clearly, the ventilation channel 2 would act like a chimney and probably transport most of the smoke out of the tunnel. In both examples we would like to prevent the smoke from going to the left from the fire by using a ventilation system.

Therefore we have to built up an over-pressure from the left of the fire in order to force the smoke to leave the tunnel towards the right exit and the ventilation

Parameter	Value
Length of ventilation channel 1	1300 m
Length of ventilation channel 2	290 m
Length of ventilation channel 3	1097 m
Slope of ventilation channel 1	-25.53%
Slope of ventilation channel 2	perpendicular to horizon
Slope of ventilation channel 3	9.03%
Cross section area of all ventilation channels	7.84 m^2

TABLE 2. Ventilation channel parameters of the Young-Dong-Model

Parameter	Value
Spatial discretization	0.5 m
ξ of all channels/tunnels	0.01
Pressure boundary conditions	atmospheric
Onset of the fire	after 60 s up to 160 s
Fire in the mid of segment 3	20 MW for 40 minutes

TABLE 3. Common parameters for the fire simulations

channel 3. From simulation with the single tunnel fire model [11, 5, 9] we know that this is not possible in general, it depends on the slope profile, on the outside pressures, on the ventilation system and on the strength of the fire. In the following two examples we present two different ventilation strategies.

Example 1: This example uses a simple ventilation strategy. We put on the fans in segment 2 and 3 and in the ventilation channel 2 in such a way that they produce an over-pressure on the left side of the fire. In table 5 we present the details of this first strategy. The result is interesting and needs some explanation (see figure 5 and

Position of ventilator	Force	Pressure accordingly	Activation time
Segment 2	7200 N	1.125 $mbar$	after 1 Minute
Ventilation channel 2	1200 N	1.531 $mbar$	after 1 Minute
Segment 3	8400 N	1.313 $mbar$	after 1 Minute

TABLE 4. Ventilation strategy for example 1

5, the illustrations are given approximately at minute 7, 10, 28, 40, 46, 54, 62 and 72). At the beginning the smoke is pushed downwards and a (small) part leaves the tunnel towards ventilation channel 3. With ongoing fire there is more an more smoke in the main channel on the right side of the fire producing big buoyancy forces. After about 31 minutes these buoyancy forces are bigger then the forces of the over pressure from the left and the smoke goes also to the left side of the fire. Then a certain amount of smoke leaves the tunnel towards ventilation channel 2 and the buoyancy forces in the tunnel are reduced. We conclude that this ventilation strategy was not able to prevent smoke from going to the left side of the fire, because there was a to large amount of hot air in segment 4 of the main tunnel.

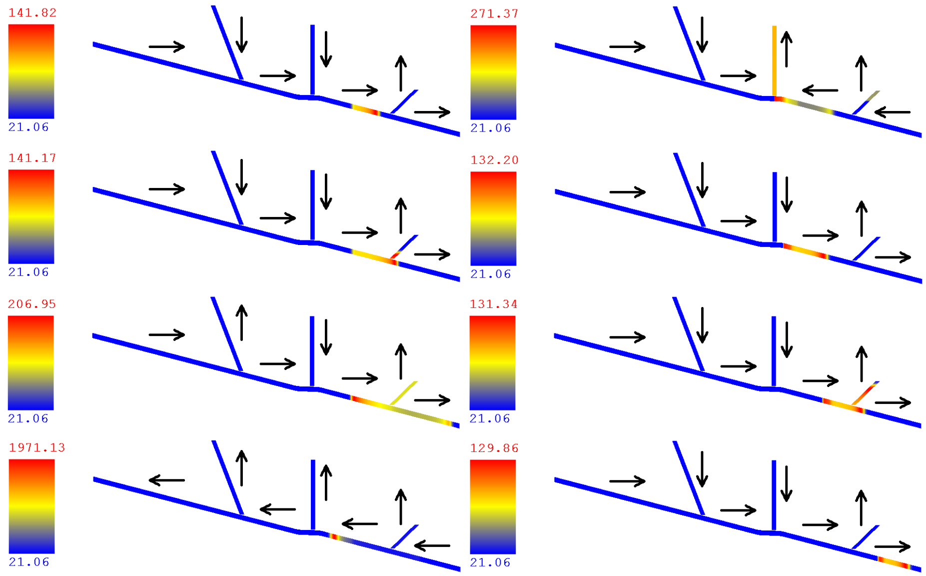

FIGURE 2. Example 1: Temperature at different times (read column after column)

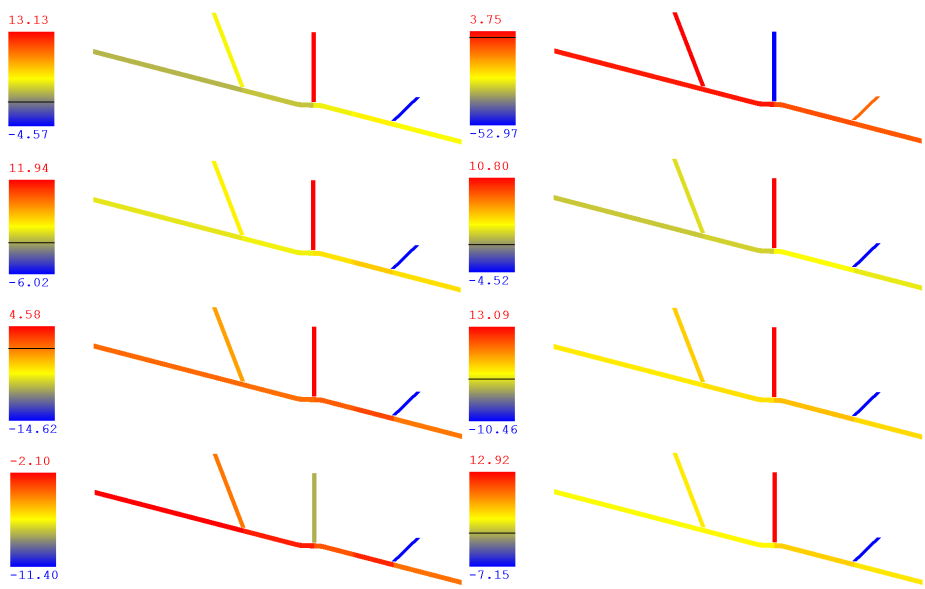

FIGURE 3. Example 1: Velocity at different times (read column after column)

Example 2: Now, we apply a more sophisticated ventilation strategy (see table 5 and 6 for details). In example 1 the problem was, that the amount of hot smoke in the main tunnel on the right side of the tunnel was to big after some time. To avoid this we would like to push the hot smoke out of the tunnel through the third ventilation channel, such that the heated air leaves the tunnel system as fast as possible. We still produce an over pressure from the left side of the tunnel. Even though it seems to be counterproductive we push air from the fourth segment to the left hoping the hot air will leave the system through ventilation channel 3 instead of entering segment four by a significantly large amount. In fact, the simulation shows (see figure 4 and 5, the illustrations are given approximately at minute 7, 12, 25, 33, 46, 50 and 60) that in this way we can keep the buoyancy forces of the smoke on the right side of the fire small enough. Since the pressure produced in the fourth segment acts against the ventilation from the other parts of the main tunnel, this ventilator has to be deactivated from time to time. But then the hot air is against forced to enter the fourth segment of the main tunnel. We therefore activated and deactivated ventilators manually, such that both counteracting goals, namely protection of the area left to the fire and protection of the fourth tunnel segment, can be achieved.

Position of ventilator	Force	Pressure accordingly
Segment 2	7200 N	1.1250 $mbar$
Ventilation channel 2	1200 N	1.5306 $mbar$
Segment 3	8400 N	1.3125 $mbar$
Ventilation channel 3	-2400 N	-3.0612 $mbar$
Segment 4	-3600 N	-0.5625 $mbar$

TABLE 5. Ventilation data for example 2

Position of ventilator	Minute 1	Minute 7	Minute 13	Minute 32
Segment 2	on	off	on	on
Ventilation channel 2	on	off	on	on
Segment 3	on	on	on	on
Ventilation channel 3	on	on	on	on
Segment 4	on	on	on	off

Position of ventilator	Minute 46	Minute 50	Minute 56	Minute 70
Segment 2	off	off	off	off
Ventilation channel 2	off	off	off	off
Segment 3	on	on	off	off
Ventilation channel 3	on	on	on	off
Segment 4	on	off	off	off

TABLE 6. Ventilation strategy for example 2

6. **Conclusions.**

We present a modeling approach for tunnel fires in tunnel networks. We use a known model for single tunnels and built up a model for the case of a network. We formulate consistent vertex conditions. We propose an efficient numerical scheme. The results show that it should be possible to study the underlying optimal control problem with reasonable effort.

Acknowledgements. The first author was partially supported by the *Berufsfeuerwehr Bozen.*

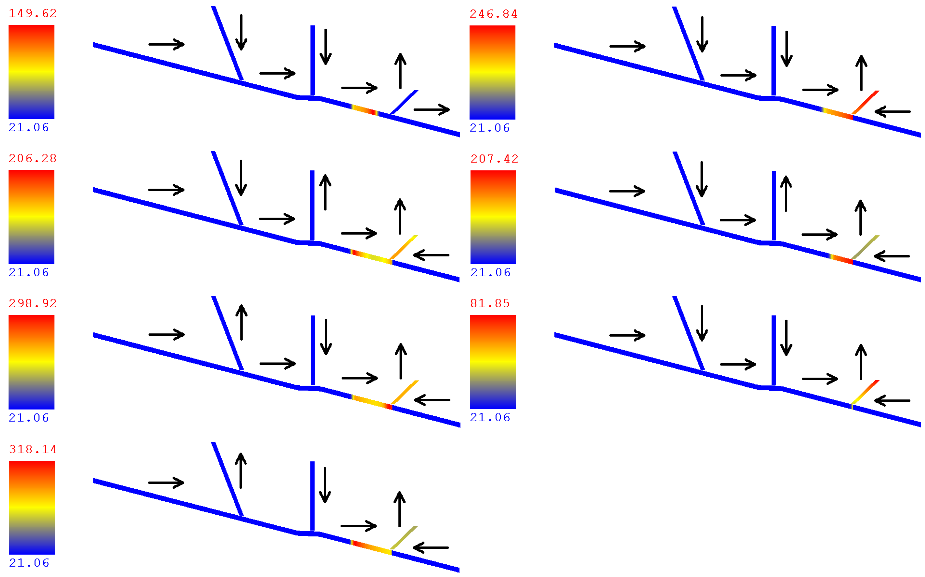

FIGURE 4. Example 2: Temperature at different times (read column after column)

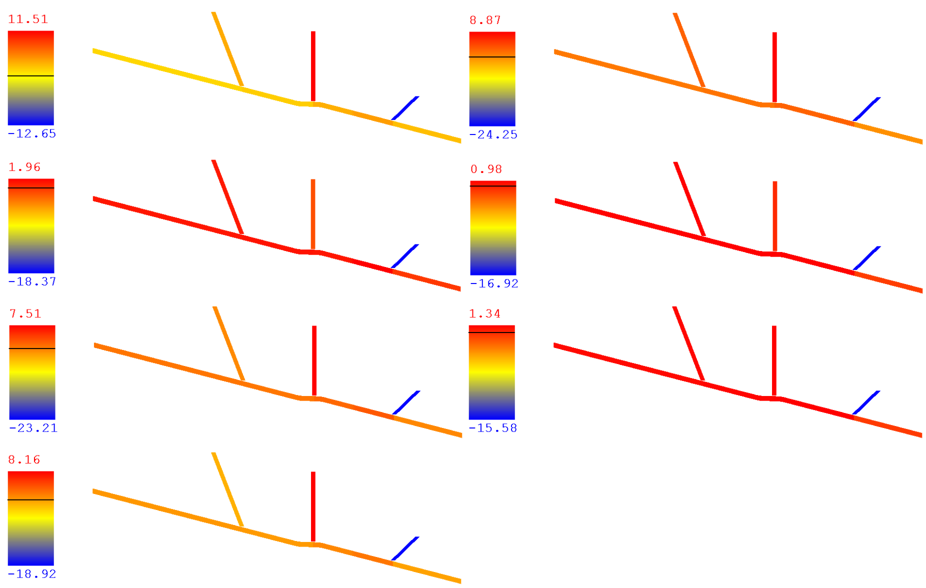

FIGURE 5. Example 2: Velocities at different times (read column after column)

REFERENCES

[1] M. K. Banda, M. Herty and A. Klar, *Gas flow in pipeline networks*, Networks and Heterogeneous Media NHM, **1** (2006), 41–56.

[2] R. Friedman, *An international survey of computer models for fire and smoke*, J. of Fire Prot. Engr., **4** (1992), 81–92.

[3] http://www.etnfit.net/, 10th march 2008.

[4] I. Gasser, *An asymptotic-induced one-dimensional model to describe fires in tunnels II: The stationary problem*, Math. Meth. in the Appl. Sci. M^2AS, **26** (2003), 1327–1347.

[5] I. Gasser, *On the mathematics of tunnel fires*, Mitteilungen der GAMM, **26** (2003), 109–126.

[6] I. Gasser, *One-dimensional modelling of tunnel fires*, Mitt. Math. Ges. Hamburg, **23** (2004), 49–61.

[7] G.B. Brandt, S.F. Jagger and C.J. Lea, *Fires in tunnels*, Phil. Trans. R. Soc. Lond. A, **356** (1998), 2873–2906.

[8] M. Garavello and B. Piccoli, "Traffic Flow on Networks. Conservation Laws Models," AIMS Series on Applied Mathematics No. 1, American Institute of Mathematical Sciences (AIMS), 2006.

[9] I. Gasser and H. Steinrück, *An asymptotic one–dimensional model to describe fires in tunnels III: The transient problem*, SIAM Journal of Applied Mathematics, **66** (2006), 2187–2203.

[10] I. Gasser and H. Steinrück, *On the existence of transient solutions of a tunnel fire model*, Communications in Mathematical Sciences, **4** (2006), 609–619.

[11] I. Gasser and J. Struckmeier, *An asymptotic-induced one-dimensional model to describe fires in tunnels*, Math. Meth. in the Appl. Sci. M^2AS, **25** (2002), 1231–1249.

[12] I. Gasser, J. Struckmeier and I. Teleaga, *Modelling and simulation of fires in vehicle tunnels*, International Journal for Numerical Methods of Fluids, **44** (2004), 277–296.

[13] B. E. Larock, A. W. Jeppson and G. Z. Watters, "Hydraulics of Pipeline Systems," CRC Press, Boca Raton, 2000.

[14] S. M. Olenick and D. J. Carpenter, *An updated international survey of computer models for fire and smoke*, J. of Fire Prot. Engr., **13** (2003), 87–110.

[15] "PIARC 1999 Fire and Smoke Control in Road Tunnels," PIARC Commettee on road tunnels (C 5), 05.05.B, World Road Association, 1999.

[16] I. Teleaga, M. Seaid, I. Gasser, A. Klar and J. Struckmeier, *Radiation models for thermal flows at low mach number*, Journal of Computational Physics, **215** (2006), 506–525.

[17] W. Zielke, *Mathematische simulation der druckschwankungen in rohrleitungen und rohrnetzen*, in "Elektronische Berechnung von Rohr und Gerinneströmungen" (eds. Zielke et al.), Erich Schmidt Verlag, (1974), 119–185.

Received March 2008; revised June 2008.

E-mail address: gasser@math.uni-hamburg.de
E-mail address: kraft@tu-harburg.de