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HOMOGENIZATION PROBLEM FOR A PARABOLIC VARIATIONAL INEQUALITY WITH CONSTRAINTS ON SUBSETS SITUATED ON THE BOUNDARY OF THE DOMAIN

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Abstract. This paper is aimed at a homogenization problem for a parabolic variational inequality with unilateral constraints. The constraints on solutions are imposed on disk-shaped subsets belonging to the boundary of the domain and forming a periodic structure, so that one has a problem with rapidly oscillating boundary conditions on a part of the boundary. Under certain conditions on the relation between the period of the structure and the radius of the disks, the homogenized problem is obtained. With the help of special auxiliary functions, the solutions of the original variational inequalities are shown to converge to the solution of the homogenized problem in Sobolev space as the period of the structure tends to zero.

Homogenization problems for variational inequalities were studied in many papers (see, for example, [1],[2], [4]-[10]). In these papers one can find an extensive bibliography on this subject. In the present paper we study the homogenization problem for a variational inequality with one-sided constraints on ε-periodically situated subsets G_ε of the boundary Σ of a domain Q_T, $Q_T = \Omega \times (0,T)$, as $\varepsilon \to 0$. We consider the so-called "critical case" in which a new term appears in the limit (homogenized) problem.

Let Ω be a bounded domain in R^3 belonging to the halfspace $x_1 > 0$ and having a piecewise smooth boundary $\partial\Omega$ such that $\partial\Omega \cap \{x_1 = 0\} = \Gamma_1 \neq \emptyset$; $\partial\Omega \setminus \Gamma_1 = \Gamma_2$. We introduce the sets $G_\varepsilon^0 = \{x \in R^3 : x_1 = 0,\ x_2^2 + x_3^2 < a_\varepsilon^2\}$, $\hat{G}_\varepsilon = \sum_{z \in Z'} (G_\varepsilon^0 + 2\varepsilon z) = \cup_{j=1}^{\infty} G_\varepsilon^j$, where Z' is the set of vectors $z = (0, z_2, z_3)$, with integer $z_j\ (j = 2, 3)$. The union of the sets $G_\varepsilon^j \subset \hat{G}_\varepsilon$ such that $\overline{G_\varepsilon^j} \subset \Gamma_1^\varepsilon = \{x \in \Gamma_1 : \varrho(x, \partial\Gamma_1) \geq 2\varepsilon\}$ is denoted by G_ε, i.e., $G_\varepsilon = \cup_{j=1}^{N_\varepsilon} G_\varepsilon^j$. Let $Q_T = \Omega \times (0,T)$ be a cylinder with the lateral surface $\Sigma = \partial\Omega \times (0,T)$.

Consider the sets

$$K_\varepsilon = \{v \in H_1(\Omega, \Gamma_2) |\, v(x) \geq 0 \text{ almost everywhere on } G_\varepsilon\},$$
$$\mathcal{K}_\varepsilon = \{g(x,t) \in L_2(0,T; H_1(\Omega, \Gamma_2)) \mid g(\cdot,t) \in K_\varepsilon \text{ for almost all } t \in (0,T)\}.$$

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Here and in what follows, $H_m(\Omega, \gamma)$ stands for the space obtained by completion (with respect to norm $H_m(\Omega)$) of the set of functions that are infinitely differentiable in $\overline{\Omega}$ and vanish in some neighborhood of γ, where γ is an s-dimensional manifold in $\overline{\Omega}$.

Let $u_\varepsilon \in \mathcal{K}_\varepsilon$, $\frac{\partial u_\varepsilon}{\partial t} \in L_2(0,T;H^{-1}(\Omega,\Gamma_2))$, $u_\varepsilon(x,0) = 0$, be a solution of the inequality

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial u_\varepsilon}{\partial t}, v - u_\varepsilon \right\rangle dt + \int_{Q_T} \nabla u_\varepsilon \nabla (v - u_\varepsilon) dxdt \\ &\geq \int_{Q_T} f(v - u_\varepsilon) dxdt, \end{aligned} \tag{1}$$

where v is an arbitrary function in $\mathcal{K}_\varepsilon$; $\nabla u \nabla g \equiv \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial g}{\partial x_j}$ and $\langle a, b\rangle$ is the value of a functional $a \in H^{-1}(\Omega, \Gamma_2)$ on an element $b \in H_1(\Omega, \Gamma_2)$. Suppose that $f, \frac{\partial f}{\partial t} \in L_2(Q_T)$. It is known [13] that the problem (1) has the unique solution u_ε such that $u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t} \in L_2(0,T;H_1(\Omega,\Gamma_2))$

Note that $u_\varepsilon(x,t)$ is a "strong" generalized solution of the following problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f \quad \text{for } (x,t) \in Q_T; \\ u_\varepsilon(x,0) = 0,\ x \in \Omega;\ u_\varepsilon = 0 \text{ on } \Gamma_2 \times (0,T);\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ on } (\Gamma_1 \setminus G_\varepsilon) \times (0,T); \\ u_\varepsilon \geq 0,\ \frac{\partial u_\varepsilon}{\partial \nu} \geq 0,\ u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ for } x \in G_\varepsilon \times (0,T), \end{cases} \tag{2}$$

where ν is the unit exterior normal to $\partial\Omega \times (0,T)$.

Let us prove the following estimate:

$$\max_{[0,T]} \|u_\varepsilon\|_{L_2(\Omega)} + \|u_\varepsilon\|_{L_2(0,T;H_1(\Omega,\Gamma_2))} + \|\frac{\partial u_\varepsilon}{\partial t}\|_{L_2(Q_T)} \leq K, \tag{3}$$

where K is a constant independent of ε. Here and in what follows, all constants independent of ε are denoted by K.

Taking into account that $u_\varepsilon \in L_2(0,T;H_1(\Omega,\Gamma_2))$, $\frac{\partial u_\varepsilon}{\partial t} \in L_2(0,T;H_1(\Omega,\Gamma_2))$, it can be shown that (see [3]) $u_\varepsilon \in C([0,T];L_2(\Omega))$, the mapping $t \to \|u_\varepsilon(t)\|_{L_2(\Omega)}$ is absolutely continuous, $\langle \frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon \rangle = \frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t)\|^2_{L_2(\Omega)}$ for almost all $t \in [0,T]$, and

$$\max_{[0,T]} \|u_\varepsilon\|_{L_2(\Omega)} \leq K\{\|u_\varepsilon\|_{L_2(0,T;H_1(\Omega,\Gamma_2))} + \|\frac{\partial u_\varepsilon}{\partial t}\|_{L_2(Q_T)}\}, \tag{4}$$

where $K = K(T,\Omega)$. Therefore, to prove the estimate (3) it suffices to show that the right-hand side of (4) is bounded by a constant independent of ε.

Taking $v = 0$ in (1), we obtain

$$\begin{aligned} &\|\nabla u_\varepsilon\|^2_{L_2(Q_T)} \leq \|f\|_{L_2(Q_T)} \|u_\varepsilon\|_{L_2(Q_T)} \leq \\ &\leq K\|\nabla u_\varepsilon\|_{L_2(Q_T)}, \end{aligned}$$

and thus,

$$\|\nabla u_\varepsilon\|_{L_2(Q_T)} \leq K.$$

In order to estimate $\|\frac{\partial u_\varepsilon}{\partial t}\|_{L_2(Q_T)}$, we consider an auxiliary problem:

$$\begin{cases} \frac{\partial u_\varepsilon^\delta}{\partial t} - \Delta u_\varepsilon^\delta = f \text{ for } x \in Q_T; \\ u_\varepsilon^\delta(x,0) = 0,\ x \in \Omega;\ u_\varepsilon^\delta = 0 \text{ on } \Gamma_2 \times (0,T); \quad \frac{\partial u_\varepsilon^\delta}{\partial \nu} = 0 \text{ on } (\Gamma_1 \setminus G_\varepsilon) \times (0,T); \\ \frac{\partial u_\varepsilon^\delta}{\partial \nu} = -\frac{1}{\delta}(u_\varepsilon^\delta)^- \text{ for } x \in G_\varepsilon; \end{cases} \tag{5}$$

where $\delta > 0$.

The weak solution of this problem is a function $u_\varepsilon^\delta \in L_2(0,T;H_1(\Omega,\Gamma_2))$ such that $u_\varepsilon^\delta(x,0) = 0$, $\frac{\partial u_\varepsilon^\delta}{\partial t} \in L_2(0,T;H_1(\Omega,\Gamma_2))$ (see [3]) and following integral identity holds:

$$\int_0^T (\frac{\partial u_\varepsilon^\delta}{\partial t}, h)dt + \int_{Q_T} \nabla u_\varepsilon^\delta \nabla h dx dt + \frac{1}{\delta}\int_0^T \int_{G_\varepsilon} (u_\varepsilon^\delta)^- h d\hat{x} dt = \\ = \int_{Q_T} fh dx dt \tag{6}$$

for an arbitrary function $h \in L_2(0,T;H_1(\Omega,\Gamma_2))$. Here $(\cdot,\cdot)$ is an scalar product in $L_2(\Omega)$. Taking $h = \frac{\partial u_\varepsilon^\delta}{\partial t}$ in (6), we obtain for almost all $t \in [0,T]$

$$\|\frac{\partial u_\varepsilon^\delta}{\partial t}\|^2_{L_2(Q_t)} + \frac{1}{2}\|\nabla u_\varepsilon^\delta\|^2_{L_2(\Omega_t)} + \frac{1}{\delta}\int_0^t \int_{G_\varepsilon} \frac{\partial u_\varepsilon^\delta}{\partial t}(u_\varepsilon^\delta)^- d\hat{x} dt = \\ = \int_{Q_t} f \frac{\partial u_\varepsilon^\delta}{\partial t} dx dt \le \|f\|_{L_2(Q_t)} \|\frac{\partial u_\varepsilon^\delta}{\partial t}\|_{L_2(Q_t)},$$

Where $Q_\tau = \{Q_T \cap \{t < \tau\}\}$, $\Omega_\tau = \{Q_T \cap \{t = \tau\}\}$. Since $\int_0^t \int_{G_\varepsilon} \frac{\partial u_\varepsilon^\delta}{\partial t}(u_\varepsilon^\delta)^- d\hat{x} dt \ge 0$ (see [3]) the following estimate is valid

$$\|\frac{\partial u_\varepsilon^\delta}{\partial t}\|_{L_2(Q_T)} \le K.$$

Using the estimates established above, we deduce that there is a function $w_\varepsilon \in L_2(0,T;H_1(\Omega,\Gamma_2))$, $\frac{\partial w_\varepsilon}{\partial t} \in L_2(Q_T)$, such that $u_\varepsilon^\delta \rightharpoonup w_\varepsilon$ in $L_2(0,T;H_1(\Omega,\Gamma_2))$, $\frac{\partial u_\varepsilon^\delta}{\partial t}$ weakly converges to $\frac{\partial w_\varepsilon}{\partial t}$ in $L_2(Q_T)$ as $\delta \to 0$. Let us prove that w_ε is a solution of the same problem as u_ε. Let us pass to the limit as $\delta \to 0$ in (6) with an arbitrary test function $v \in \mathcal{K}_\varepsilon$. Taking into account that $\frac{1}{\delta}\int_0^T \int_{G_\varepsilon} (u_\varepsilon^\delta)^- v d\hat{x} dt \le 0$ we obtain

$$\int_0^T (\frac{\partial w_\varepsilon}{\partial t}, v)dt + \int_{Q_T} \nabla w_\varepsilon \nabla v dx dt \ge \int_{Q_T} fv dx dt. \tag{7}$$

Using (6) with $h = u_\varepsilon^\delta$ and noting that

$$\int_0^T (\frac{\partial w_\varepsilon}{\partial t}, w_\varepsilon)dt + \int_{Q_T} \mid \nabla w_\varepsilon \mid^2 dxdt \leq$$

$$\leq \lim_{\delta\to 0} \left\{ \int_0^T (\frac{\partial u_\varepsilon^\delta}{\partial t}, u_\varepsilon^\delta)dt + \int_{Q_T} \mid \nabla u_\varepsilon^\delta \mid^2 dxdt \right\} \leq \lim_{\delta\to 0} \int_{Q_T} f u_\varepsilon^\delta dxdt,$$

we find that w_ε satisfies the inequality

$$\int_0^T (\frac{\partial w_\varepsilon}{\partial t}, w_\varepsilon)dt + \int_{Q_T} \mid \nabla w_\varepsilon \mid^2 dxdt \leq \int_{Q_T} f w_\varepsilon dxdt. \tag{8}$$

Subtracting the inequality (8) from (7), we see that w_ε is a solution of the following inequality:

$$\int_0^T (\frac{\partial w_\varepsilon}{\partial t}, v - w_\varepsilon)dt + \int_{Q_T} \nabla w_\varepsilon \nabla (v - w_\varepsilon)dxdt \geq$$

$$\geq \int_{Q_T} f(v - w_\varepsilon)dxdt,$$

where $v \in \mathcal{K}_\varepsilon$.

Since problem (1) has a unique solution, we see that $w_\varepsilon = u_\varepsilon$ and $u_\varepsilon \in L_2(0,T; H_1(\Omega,\Gamma_2))$, $\frac{\partial u_\varepsilon}{\partial t} \in L_2(Q_T)$. From the fact that $\|u_\varepsilon\|_W \leq K$, where $W = \{v \mid v \in L_2(0,T; H_1(\Omega,\Gamma_2)), \frac{\partial v}{\partial t} \in L_2(Q_T)\}$, we conclude that there is a function $u_0 \in W$ such that u_ε weakly converges in W to u_0 as $\varepsilon \to 0$ (for a subsequence). Moreover, using the imbedding theorem, we can assume that $u_\varepsilon \to u_0$ in $L_2(Q_T)$.

It is easy to see that the following inequality for the solution u_ε of problem (1) is valid:

$$\int_0^T (\frac{\partial u_\varepsilon}{\partial t}, v - u_\varepsilon)dt + \int_0^T \int_\Omega \nabla v \nabla (v - u_\varepsilon)dxdt \geq \int_0^T \int_\Omega f(v - u_\varepsilon)dxdt, \tag{9}$$

where $v \in \mathcal{K}_\varepsilon$. We pass to the limit in this inequality as $\varepsilon \to 0$.

Suppose that

$$\lim_{\varepsilon\to 0} a_\varepsilon \varepsilon^{-2} = C = const \geq 0. \tag{10}$$

Consider a function $w_\varepsilon(x)\,(x = (x_1,x_2,x_3))$ which is a solution of the problem:

$$\begin{cases} \Delta w_\varepsilon = 0 \text{ for } x \in T_\varepsilon \setminus \overline{G_\varepsilon^0},\ T_\varepsilon = \{x \in R^3 : \frac{x_1^2}{1-a_\varepsilon^2\varepsilon^{-2}} + x_2^2 + x_3^2 < \varepsilon^2\}, \\ w_\varepsilon = 0 \text{ for } x \in G_\varepsilon^0, \\ w_\varepsilon = 1 \text{ for } x \in \partial T_\varepsilon. \end{cases} \tag{11}$$

Problem (11) admits a solution that can be easily found in explicit form. For this purpose, we use ellipsoidal coordinates ψ, θ_1, θ_2:

$$\begin{aligned} x_1 &= a_\varepsilon \sinh\psi \cos\theta_1, \\ x_2 &= a_\varepsilon \cosh\psi \sin\theta_1 \cos\theta_2, \\ x_3 &= a_\varepsilon \cosh\psi \sin\theta_1 \sin\theta_2, \end{aligned}$$

where $0 \le \psi < +\infty$; $0 \le \theta_1 \le \pi$; $0 \le \theta_2 \le 2\pi$. Let us seek a solution of problem (11) in the set of functions depending on $\sigma = \sinh\psi$ only, i.e., $w_\varepsilon(x) = V_\varepsilon(\sigma)$, where $V_\varepsilon(\sigma)$ satisfies the equation $\frac{\partial}{\partial\sigma}((1+\sigma^2)\frac{\partial V}{\partial\sigma}) = 0$ on the interval $0 < \sigma < \sqrt{a_\varepsilon^{-2}\varepsilon^2 - 1}$ and the boundary conditions $V(0) = 0$, $V(\sqrt{a_\varepsilon^{-2}\varepsilon^2 - 1}) = 1$. It is easy to check that

$$w_\varepsilon(x) = V_\varepsilon(\sinh\psi) = \frac{\arctan\sinh\psi}{\arctan\sinh\psi_\varepsilon},$$

where

$$\sinh\psi = \sqrt{\frac{|x|^2 - a_\varepsilon^2 + \sqrt{(a_\varepsilon^2 - |x|^2)^2 + 4x_1^2 a_\varepsilon^2}}{2a_\varepsilon^2}}$$

$$\sinh\psi_\varepsilon = \sqrt{a_\varepsilon^{-2}\varepsilon^2 - 1}.$$

Since $w_\varepsilon(x)$ is an even function of the variable x_1 and $\frac{\partial w_\varepsilon}{\partial x_1} = 0$ for $x_1 = 0$, $a_\varepsilon^2 < x_2^2 + x_3^2 < \varepsilon^2$, we find that

$$\int\limits_{T_\varepsilon^+} |\nabla w_\varepsilon|^2 dx = \int\limits_{S_\varepsilon^+} \frac{\partial w_\varepsilon}{\partial\nu} ds, \tag{12}$$

where $T_\varepsilon^+ = T_\varepsilon \cap \{x_1 > 0\}$, $S_\varepsilon^+ = \partial T_\varepsilon \cap \{x_1 > 0\}$, ν is the unit exterior normal to ∂T_ε. The coordinates of $\nu = (\nu_1, \nu_2, \nu_3)$ have the form

$$\nu_1 = \frac{x_1}{\sqrt{|x|^2 + \alpha_\varepsilon(x_2^2 + x_3^2)}}, \quad \nu_i = \frac{x_i(1 - a_\varepsilon^2\varepsilon^{-2})}{\sqrt{|x|^2 + \alpha_\varepsilon(x_2^2 + x_3^2)}}, \quad i = 2, 3,$$

where $\alpha_\varepsilon = -2a_\varepsilon^2\varepsilon^{-2} + a_\varepsilon^4\varepsilon^{-4}$. Using this fact and the relation

$$\frac{\partial w_\varepsilon}{\partial x_j}\Big|_{\partial T_\varepsilon} = \frac{C_\varepsilon a_\varepsilon \varepsilon^{-3}}{4\sqrt{1 - a_\varepsilon^2\varepsilon^{-2}}} I_{x_j}^\varepsilon\Big|_{\partial T_\varepsilon},$$

where $I^\varepsilon \equiv |x|^2 - a_\varepsilon^2 + \sqrt{(a_\varepsilon^2 - |x|^2)^2 + 4x_1^2 a_\varepsilon^2}$, $C_\varepsilon \equiv \frac{1}{\arctan\sinh\psi_\varepsilon}$, we obtain

$$\frac{\partial w_\varepsilon}{\partial\nu}\Big|_{\partial T_\varepsilon} = \frac{1}{\sqrt{1 - a_\varepsilon^2\varepsilon^{-2}}} \frac{C_\varepsilon a_\varepsilon \varepsilon^{-2}}{\sqrt{1 + \frac{a_\varepsilon^2\varepsilon^{-4}x_1^2}{(1 - a_\varepsilon^2\varepsilon^{-2})^2}}}. \tag{13}$$

From (12), (13) and (7) we deduce that

$$\int\limits_{T_\varepsilon^+} |\nabla w_\varepsilon|^2 dx \le K|\partial T_\varepsilon| \le K\varepsilon^2. \tag{14}$$

We define a function $Q_\varepsilon(x)$ by

$$Q_\varepsilon(x)=\begin{cases} w_\varepsilon(x-P_\varepsilon^j),\ x\in T_\varepsilon^j,\ j=1,2,\dots,\\ 1,\ x\in R^3\setminus\cup_{j=1}^{\infty}\overline{T_\varepsilon^j}, \end{cases}\tag{15}$$

where $P_\varepsilon^j=(0,P_{\varepsilon,2}^j,P_{\varepsilon,3}^j)$ is the center of the ball G_ε^j, $T_\varepsilon^j=\{x\in R^3:\frac{x_1^2}{1-a_\varepsilon^2\varepsilon^{-2}}+(x_2-P_{\varepsilon,2}^j)^2+(x_3-P_{\varepsilon,3}^j)^2<\varepsilon^2\}$, $j=1,2,\dots$. Using the definition of $Q_\varepsilon(x)$ and the estimate (14), we obtain the inequality

$$\int\limits_\Omega|\nabla Q_\varepsilon|^2dx=\sum_{j=1}^{N(\varepsilon)}\int\limits_{T_\varepsilon^{j,+}}|\nabla w_\varepsilon(x-P_\varepsilon^j)|^2dx\le KN(\varepsilon)\varepsilon^2\le K,$$

and therefore,

$$Q_\varepsilon\rightharpoonup 1\ \text{in}\ H_1(\Omega)\ \text{as}\ \varepsilon\to 0.\tag{16}$$

Taking $v=h^++Q_\varepsilon(x)h^-\in\mathcal{K}_\varepsilon$ in the inequality (9), where $h^+=\sup\{h,0\}$, $h^-(x,t)=h-h^+$, and $h\in L_2(0,T;H_1(\Omega,\Gamma_2))$, we obtain

$$\begin{gathered}\int\limits_0^T(\frac{\partial u_\varepsilon}{\partial t},h^++Q_\varepsilon(x)h^--u_\varepsilon)dt+\int\limits_0^T\int\limits_\Omega\nabla[h^++Q_\varepsilon(x)h^-]\times\\ \times\nabla[h^++Q_\varepsilon(x)h^--u_\varepsilon]dxdt\ge\int\limits_0^T\int\limits_\Omega f(h^++Q_\varepsilon(x)h^--u_\varepsilon)dxdt.\end{gathered}\tag{17}$$

Denote the left-hand side of (17) by L_ε. It is easy to see that L_ε has the form

$$\begin{aligned}L_\varepsilon\equiv&\int\limits_0^T(\frac{\partial u_\varepsilon}{\partial t},h^++Q_\varepsilon(x)h^--u_\varepsilon)dt+\int\limits_0^T\int\limits_\Omega|\nabla h^+|^2dxdt+\\ &+\int\limits_0^T\int\limits_\Omega|\nabla[Q_\varepsilon h^-]|^2dxdt-\int\limits_0^T\int\limits_\Omega\nabla h^+\nabla u_\varepsilon dxdt-\\ &-\int\limits_0^T\int\limits_\Omega\nabla Q_\varepsilon\nabla(u_\varepsilon h^-)dxdt+\int\limits_0^T\int\limits_\Omega u_\varepsilon\nabla Q_\varepsilon\nabla h^-dxdt-\int\limits_0^T\int\limits_\Omega Q_\varepsilon\nabla h^-\nabla u_\varepsilon dxdt.\end{aligned}\tag{18}$$

Taking into account that

$$\int\limits_\Omega\nabla Q_\varepsilon\nabla(u_\varepsilon h^-)dx=-\sum_{j=1}^{N(\varepsilon)}\int\limits_{G_\varepsilon^j}\frac{\partial w_\varepsilon}{\partial x_1}h^-u_\varepsilon dx_2dx_3+\sum_{j=1}^{N(\varepsilon)}\int\limits_{\partial T_\varepsilon^j\cap\Omega}\frac{\partial w_\varepsilon}{\partial\nu}h^-u_\varepsilon ds$$

and $\frac{\partial w_\varepsilon}{\partial x_1}\Big|_{x\in G_\varepsilon^j}>0$, $h^-\le 0$, $u_\varepsilon\Big|_{G_\varepsilon^j}\ge 0$, we obtain

$$\int\limits_0^T\int\limits_\Omega\nabla Q_\varepsilon\nabla(u_\varepsilon h^-)dxdt\ge\sum_{j=1}^{N(\varepsilon)}\int\limits_0^T\int\limits_{\partial T_\varepsilon^j\cap\Omega}\frac{\partial w_\varepsilon}{\partial\nu}h^-u_\varepsilon dsdt.\tag{19}$$

From (18) and (19) it follows that

$$
\begin{aligned}
L_\varepsilon \leq & \int_0^T (\frac{\partial u_\varepsilon}{\partial t}, h^+ + Q_\varepsilon(x)h^- - u_\varepsilon)dt + \int_0^T \int_\Omega |\nabla h^+|^2 dxdt + \\
& + \int_0^T \int_\Omega |\nabla(Q_\varepsilon h^-)|^2 dxdt - \int_0^T \int_\Omega \nabla h^+ \nabla u_\varepsilon dxdt - \\
& - \sum_{j=1}^{N(\varepsilon)} \int_0^T \int_{\partial T_\varepsilon^j \cap \Omega} \frac{\partial w_\varepsilon}{\partial \nu} h^- u_\varepsilon dsdt + \int_0^T \int_\Omega \nabla Q_\varepsilon \nabla h^- u_\varepsilon dxdt - \int_0^T \int_\Omega Q_\varepsilon \nabla h^- \nabla u_\varepsilon dxdt.
\end{aligned} \tag{20}
$$

To obtain the homogenized problem we need the following lemma.

Lemma 0.1. *Suppose that $h,\ g \in H_1(\Omega, \Gamma_2)$. Then*

$$
\begin{aligned}
& |\sum_{j=1}^{N(\varepsilon)} \int_{\partial T_\varepsilon^j \cap \Omega} \frac{\partial w_\varepsilon}{\partial \nu}(x - P_\varepsilon^j) hg ds - C \int_{\Gamma_1} hg dx_2 dx_3| \leq \\
& \qquad \leq K\{\|a_\varepsilon \varepsilon^{-2} - C| + \sqrt{\varepsilon}\} \|g\|_{H_1(\Omega)} \|h\|_{H_1(\Omega)},
\end{aligned}
$$

where ν is the unit exterior normal to ∂T_ε^j.

Proof. We have

$$
\begin{aligned}
& \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_\varepsilon^j \cap \Omega} \frac{\partial w_\varepsilon}{\partial \nu} hg ds = D_\varepsilon \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_\varepsilon^j \cap \Omega} gh(1 + \frac{a_\varepsilon^2 \varepsilon^{-4} x_1^2}{(1 - a_\varepsilon^2 \varepsilon^{-2})^2})^{-1/2} ds = \\
& = D_\varepsilon \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_\varepsilon^j \cap \Omega} gh(\frac{1 - a_\varepsilon^2 \varepsilon^{-2}}{1 - a_\varepsilon^2 \varepsilon^{-4} |\hat{x}|^2})^{1/2} ds,
\end{aligned} \tag{21}
$$

where $D_\varepsilon = C_\varepsilon a_\varepsilon \varepsilon^{-2}(1 - a_\varepsilon^2 \varepsilon^{-2})^{-1/2}$, $\hat{x} = (x_2 - P_{\varepsilon,2}^j, x_3 - P_{\varepsilon,3}^j)$.

Let $Q = \{y \in R^3 | 0 < y_1 < 2, -1 < y_j < 1,\ j = 2,3\}$, $T^+ = \{y \in Q || y| < 1\}$. Consider an auxiliary problem:

$$
\begin{cases}
\Delta_y \xi = 0 \text{ for } y \in Q \setminus \overline{T^+}, \\
\frac{\partial \xi}{\partial \nu}\Big|_{\partial T^+ \cap Q} = 1, \\
\frac{\partial \xi}{\partial y_1}\Big|_{\partial Q \cap \{y_1 = 2\}} = -\mu, \\
\frac{\partial \xi}{\partial \nu} = 0 \text{ on the rest of } \partial Q.
\end{cases} \tag{22}
$$

Problem (22) has a solution uniquely defined up to an additive constant. We choose the constant from the condition $\frac{1}{|Q \setminus T^+|} \int_{Q \setminus T^+} \xi dy = 0$. From the solvability conditions for problem (22), it follows that $\mu = \pi/2$.

We introduce a vector-valued function $A^\varepsilon(x) = (A_1^\varepsilon(x), A_2^\varepsilon(x), A_3^\varepsilon(x))^t$ by

$$A_1^\varepsilon(x) = \sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\,\xi_{y_1}\Big(\frac{x_1}{\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big),$$
$$A_2^\varepsilon(x) = \xi_{y_2}\Big(\frac{x_1}{\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big),\ A_3^\varepsilon(x) = \xi_{y_3}\Big(\frac{x_1}{\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big).$$

The vector $A^\varepsilon(x)$ satisfies the relation $\operatorname{div}A^\varepsilon(x) = 0$ for $x \in Y_\varepsilon$, where

$$Y_\varepsilon = \{x \in R^3 | x_1 \in (0, 2\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}),\ |x_j| < \varepsilon,\ j = 2,3\} \setminus \overline{T_\varepsilon}.$$

Using the definition of $A^\varepsilon(x)$, we obtain

$$\int\limits_{Y_\varepsilon} \operatorname{div}(A^\varepsilon\Phi_\varepsilon(x)) = \int\limits_{Y_\varepsilon} A^\varepsilon(x)\nabla\Phi_\varepsilon(x)dx = \int\limits_{\partial Y_\varepsilon} (A^\varepsilon, \nu)\Phi_\varepsilon(x)ds, \tag{23}$$

where $\Phi_\varepsilon(x) \equiv g(x)h(x)(\frac{1-a_\varepsilon^2\varepsilon^{-2}}{1-a_\varepsilon^2\varepsilon^{-4}|\hat{x}|^2})^{1/2}$.

Note that the right-hand side of (23) has the form

$$\int\limits_{\partial Y^\varepsilon} (A^\varepsilon, \nu)\Phi_\varepsilon(x)ds = \sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\int\limits_{\gamma_\varepsilon} \xi_{y_1}(2, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon})\Phi_\varepsilon(x)ds + \int\limits_{(\partial T_\varepsilon)^+} (A^\varepsilon, \nu)\Phi_\varepsilon ds, \tag{24}$$

where $(\partial T_\varepsilon)^+ = \partial T_\varepsilon \cap \{x | x_1 > 0\}$.

The second term in right-hand side of (24) can be written in the form

$$\begin{aligned}
&\int\limits_{(\partial T_\varepsilon)^+} (A^\varepsilon, \nu)\Phi_\varepsilon(x)ds = \\
&= \int\limits_{(\partial T_\varepsilon)^+} \sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\,\xi_{y_1}\Big(\frac{x_1}{\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\frac{-y_1}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}|^2}}\Phi_\varepsilon(x)ds + \\
&+ \int\limits_{(\partial T_\varepsilon)^+} \xi_{y_2}\Big(\frac{x_1}{\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\frac{-y_2\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}|^2}}\Phi_\varepsilon(x)ds + \\
&+ \int\limits_{(\partial T_\varepsilon)^+} \xi_{y_3}\Big(\frac{x_1}{\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\frac{-y_3\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}|^2}}\Phi_\varepsilon ds = \\
&= \sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\int\limits_{(\partial T_\varepsilon)^+} \frac{\Phi_\varepsilon(x)}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}|^2}}\frac{\partial\xi}{\partial\nu_y}ds = \\
&= \sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\int\limits_{(\partial T_\varepsilon)^+} \frac{\Phi_\varepsilon(x)}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}|^2}}ds,
\end{aligned} \tag{25}$$

where $\hat{y} = \frac{\hat{x}}{\varepsilon}$.

Thus, from (23)-(25) we obtain

$$D_\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\{\int\limits_{(\partial T_\varepsilon)^+}\frac{\Phi_\varepsilon(x)}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}|^2}}ds-\mu\int\limits_{\gamma_\varepsilon}\Phi_\varepsilon(x)ds\}=D_\varepsilon\int\limits_{Y_\varepsilon}A^\varepsilon(x)\nabla\Phi_\varepsilon(x)dx. \tag{26}$$

From (26) we deduce that

$$D_\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\{\int\limits_{(\partial T_\varepsilon^j)^+}\frac{\Phi_\varepsilon(x)}{\sqrt{1-a_\varepsilon^2\varepsilon^{-4}[(x_2-P_{2,\varepsilon}^j)^2+(x_3-P_{3,\varepsilon}^j)^2]}}ds- \\ -\mu\int\limits_{\gamma_\varepsilon^j}\Phi_\varepsilon^j(x)dx_2dx_3\}=D_\varepsilon\int\limits_{Y_\varepsilon^j}A^{\varepsilon,j}(x)\nabla\Phi_\varepsilon^j(x)dx, \tag{27}$$

where $A^{\varepsilon,j}(x)=A^\varepsilon(x_1,x_2-P_{2,\varepsilon}^j,x_3-P_{3,\varepsilon}^j)$, $\Phi_\varepsilon^j(x)=g(x)\phi(x)(\frac{1-a_\varepsilon^2\varepsilon^{-2}}{1-a_\varepsilon^2\varepsilon^{-4}|\hat{x^j}|^2})^{1/2}$, $\hat{x^j}=(x_2-P_{2,\varepsilon}^j,x_3-P_{3,\varepsilon}^j)$, $Y_\varepsilon^j=\{x\in R^3|x_1\in(0,2\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}),|x_i-P_i^j|<\varepsilon,\ i=2,3\}$, $\gamma_\varepsilon^j=\partial Y_\varepsilon^j\cap\{x_1=2\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\}$, $\cup_{j=1}^{N(\varepsilon)}\gamma_\varepsilon^j=\hat{\Gamma}_1^\varepsilon$, $(\partial T_\varepsilon^j)^+=(\partial T_\varepsilon^j)\cap\Omega$, $j=1,\dots,N(\varepsilon)$.

From (27), using the relation

$$\int\limits_{(\partial T_\varepsilon^j)^+}\frac{\Phi_\varepsilon^j(x)}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}^j|^2}}ds-\int\limits_{(\partial T_\varepsilon^j)^+}\Phi_\varepsilon^j(x)ds=\int\limits_{(\partial T_\varepsilon^j)^+}\Phi_\varepsilon^j(x)\frac{1-\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}^j|^2}}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}^j|^2}}ds$$

with $\hat{y}^j=\varepsilon^{-1}\hat{x}^j$, we deduce that

$$D_\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\{\int\limits_{(\partial T_\varepsilon^j)^+}\Phi_\varepsilon^j(x)ds-\mu\int\limits_{\gamma_\varepsilon^j}\Phi_\varepsilon^j(x)d\hat{x}\}= \\ =-D_\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\int\limits_{(\partial T_\varepsilon^j)^+}\Phi_\varepsilon^j(x)\frac{1-\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}^j|^2}}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}^j|^2}}ds+D_\varepsilon\int\limits_{Y_\varepsilon^j}A^{\varepsilon,j}(x)\nabla\Phi_\varepsilon^j(x)dx. \tag{28}$$

Since $|\hat{y}^j|\le 1$,$j=1,\dots,N(\varepsilon)$, from (27) we obtain

$$D_\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\sum_{j=1}^{N(\varepsilon)}\int\limits_{(\partial T_\varepsilon^j)^+}|\Phi_\varepsilon^j|\frac{1-\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}^j|^2}}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}|\hat{y}^j|^2}}ds\le \\ \le Ka_\varepsilon^2\varepsilon^{-2}\{\int\limits_{\Gamma_1}|g||h|d\hat{x}+\int\limits_{\Pi_\varepsilon}(|\nabla(gh)|+|gh|)d\hat{x}\}, \tag{29}$$

From (28), (29), it follows that

$$
\begin{aligned}
&|D_\varepsilon \sum_{j=1}^{N(\varepsilon)} \int\limits_{(\partial T_\varepsilon^j)^+} \Phi_\varepsilon^j(x)ds - D_\varepsilon\mu \int\limits_{\Gamma_1} ghd\hat{x}| \le \\
&\le D_\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}|\sum_{j=1}^{N(\varepsilon)} \int\limits_{(\partial T_\varepsilon^j)^+} \Phi_\varepsilon^j(x)ds - \frac{1}{\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}}\sum_{j=1}^{N(\varepsilon)} \int\limits_{(\partial T_\varepsilon^j)^+} \Phi_\varepsilon^j(x)ds|+ \\
&+ D_\varepsilon\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}|\sum_{j=1}^{N(\varepsilon)} \int\limits_{(\partial T_\varepsilon^j)^+} \Phi_\varepsilon^j ds - \mu\sum_{j=1}^{N(\varepsilon)} \int\limits_{\gamma_\varepsilon^j} ghds|+ \\
&+ |D_\varepsilon\mu(\sqrt{1-a_\varepsilon^2\varepsilon^{-2}}\sum_{j=1}^{N(\varepsilon)} \int\limits_{\gamma_\varepsilon^j} ghds - \int\limits_{\Gamma_1} ghd\hat{x})| \equiv J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3.
\end{aligned}
\tag{30}
$$

Let us estimate J_ε^j, $j=1,2,3$. From (29) we conclude that

$$
\begin{aligned}
&J_\varepsilon^1 \le Ka_\varepsilon^2\varepsilon^{-2}\sum_{j=1}^{N(\varepsilon)} \int\limits_{(\partial T_\varepsilon^j)^+} |\Phi_\varepsilon^j(x)|ds \le \\
&\le Ka_\varepsilon^2\varepsilon^{-2}\{\int\limits_{\Gamma_1} |g||h|d\hat{x} + \int\limits_{\Pi_\varepsilon}(|\nabla(gh)| + |gh|dx\} \le Ka_\varepsilon^2\varepsilon^{-2}\|g\|_{H_1(\Omega)}\|h\|_{H_1(\Omega)}.
\end{aligned}
\tag{31}
$$

From (28) we obtain

$$
J_\varepsilon^2 \le Ka_\varepsilon^2\varepsilon^{-2}\|g\|_{H_1(\Omega)}\|h\|_{H_1(\Omega)}, \tag{32}
$$

and similarly

$$
J_\varepsilon^3 \le K\sqrt{\varepsilon}\|g\|_{H_1(\Omega)}\|h\|_{H_1(\Omega)}. \tag{33}
$$

To complete the proof we note that

$$
|D_\varepsilon\mu - C|\int\limits_{\Gamma_1} |hg|d\hat{x} \le K\{|a_\varepsilon\varepsilon^{-2} - C| + \sqrt{\varepsilon}\}\|h\|_{H_1(\Omega)}\|g\|_{H_1(\Omega)}.
$$

This concludes the proof of the lemma. □

Using the assertions of this lemma, we find the limits of some terms of the right - hand part of (20):

$$
\lim_{\varepsilon\to 0}\sum_{j=1}^{N(\varepsilon)}\int\limits_0^T \int\limits_{\partial T_\varepsilon^j\cap\Omega} \frac{\partial w_\varepsilon}{\partial\nu}h^- u_\varepsilon dsdt = C\int\limits_0^T\int\limits_{\Gamma_1} h^- u_0 d\hat{x}dt
$$

and

$$
\lim_{\varepsilon\to 0}\int\limits_0^T\int\limits_\Omega |\nabla(Q_\varepsilon h^-)|^2 dxdt = \int\limits_0^T\int\limits_\Omega |\nabla h^-|^2 dxdt + C\int\limits_0^T\int\limits_{\Gamma_1}(h^-)^2 d\hat{x}dt.
$$

It is easy to see that

$$\lim_{\varepsilon\to 0}\int_0^T \left(\frac{\partial u_\varepsilon}{\partial t}, h^+ + Q_\varepsilon(x)h^-\right)dt = \int_0^T \left(\frac{\partial u_0}{\partial t}, h\right)dt. \tag{34}$$

We need the following lemma.

Lemma 0.2. *Suppose that $h \in H_1(\Omega)$ and the function Q_ε is given by (15). Then*

$$\int_\Omega |\nabla Q_\varepsilon|^2 h^2 dx \le K\|h\|^2_{H_1(\Omega)}. \tag{35}$$

Proof. It is easy to see that

$$2\int_\Omega (\nabla Q_\varepsilon, \nabla h)Q_\varepsilon h dx = 2\sum_{j=1}^{N(\varepsilon)} \int_{(T_\varepsilon^j)^+} (\nabla w_\varepsilon, \nabla h) w_\varepsilon h dx =$$

$$= -\sum_{j=1}^{N(\varepsilon)} \int_{(T_\varepsilon^j)^+} |\nabla w_\varepsilon|^2 h^2 dx + \sum_{j=1}^{N(\varepsilon)} \int_{(\partial T_\varepsilon^j)^+} \frac{\partial w_\varepsilon}{\partial \nu} h^2 ds.$$

Estimating the left-hand side of this relation with the help of the Cauchy inequality ($ab \le \delta a^2 + C_\delta b^2$, where $\delta \in (0,1)$), we get

$$\int_\Omega |\nabla Q_\varepsilon|^2 h^2 dx \le K\{\int_\Omega |\nabla h|^2 dx + |\sum_{j=1}^{N(\varepsilon)} \int_{(\partial T_\varepsilon^j)^+} \frac{\partial w_\varepsilon}{\partial \nu} h^2 ds |\}.$$

Applying Lemma 0.1 to the last term in the right-hand side of this inequality, we obtain (35). Lemma 0.2 is proved. □

From the estimate (4), it follows that $\|u_\varepsilon(x,T)\|_{L_2(\Omega)} \le K$. Consequently, there is a subsequence $\{\varepsilon\}$ and a function $w(x,T) \in L_2(\Omega)$, such that $u_\varepsilon(x,T) \rightharpoonup w(x,T)$ in $L_2(\Omega)$ as $\varepsilon \to 0$. Let us show that $w(x,T) = u_0(x,T)$. Taking an arbitrary $v \in L_2(\Omega)$, integrating by parts the expression

$$\int_0^T \left(\frac{\partial u_\varepsilon(x,t)}{\partial t}, v(x)\right)dt = \int_\Omega u_\varepsilon(x,T)v(x)dx,$$

and then passing to the limit as $\varepsilon \to 0$, we obtain

$$\int_\Omega w(x,T)v(x)dx = \int_0^T \left(\frac{\partial u_0(x,t)}{\partial t}, v(x)\right)dt = \int_\Omega u_0(x,T)v(x)dx.$$

Therefore, $u_\varepsilon(x,T) \rightharpoonup u_0(x,T)$ in $L_2(\Omega)$ as $\varepsilon \to 0$ and

$$\|u_0(x,T)\|^2_{L_2(\Omega)} \le \underline{\lim}_{\varepsilon\to 0}\|u_\varepsilon(x,T)\|^2_{L_2(\Omega)}.$$

Thus we have

$$\int_0^T (\frac{\partial u_0}{\partial t}, u_0)dt \le \lim_{\varepsilon\to 0}\int_0^T (\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon)dt.$$

Using Lemma 0.1 and the estimate (20), we find that u_0 satisfies the inequality

$$\int_0^T \left\{ \int_\Omega \frac{\partial u_0}{\partial t}(h-u_0)dx + \int_\Omega \nabla h \nabla (h-u_0)dx + C\int_{\Gamma_1}(h-u_0)h^- d\hat{x} \right\} dt \ge$$
$$\ge \int_0^T\int_\Omega f(h-u_0)dxdt$$

for any function $h \in L_2(0,T;H_1(\Omega,\Gamma_2))$.

Taking $h = u_0+\lambda v$ with an arbitrary $v \in L_2(0,T;H_1(\Omega,\Gamma_2))$ and passing to the limit as $\lambda \to +0$ and $\lambda\to -0$, we obtain an integral identity for the function u_0:

$$\int_0^T (\frac{\partial u_0}{\partial t}, v)dt + \int_0^T\int_\Omega \nabla u_0 \nabla v dxdt + C\int_0^T\int_{\Gamma_1} u_0^- v d\hat{x}dt = \int_0^T\int_\Omega f v dxdt. \tag{36}$$

Therefore, $u_0 \in L_2(0,T;H_1(\Omega,\Gamma_2))$, $\frac{\partial u_0}{\partial t} \in L_2(Q_T)$, and u_0 is a weak solution of the problem

$$\begin{cases} \frac{\partial u_0}{\partial t} - \Delta u_0 = f \text{ for } x\in Q_T, \\ u_0(x,0)=0,\ x\in\Omega,\ u_0=0, \text{ on } \Gamma_2\times[0,T] \\ \frac{\partial u_0}{\partial \nu} = -Cu_0^- \text{ on } \Gamma_1\times[0,T]. \end{cases}$$

Let us show that $\|u_\varepsilon - u_0^+ - Q_\varepsilon u_0^-\|_{L_2(0,T;H_1(\Omega,\Gamma_2))} \to 0$ as $\varepsilon\to 0$.

Setting $v = u_0^+ + Q_\varepsilon u_0^-$ in (1) and $v = u_0^+ + Q_\varepsilon u_0^- - u_\varepsilon$ in (36) and subtracting (1) from (36), we get

$$\begin{aligned} \|\nabla(u_0^+ + Q_\varepsilon u_0^- - u_\varepsilon)\|^2_{L_2(Q_T)} \le -\int_0^T (\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon)dt + \int_0^T(\frac{\partial u_0}{\partial t}, u_\varepsilon)dt + \\ + \int_0^T (\frac{\partial u_\varepsilon}{\partial t}, u_0^+ + Q_\varepsilon u_0^-)dt - \int_0^T (\frac{\partial u_0}{\partial t}, u_0^+ + Q_\varepsilon u_0^-)dt + \\ + \int_0^T\int_\Omega |\nabla(u_0^-(Q_\varepsilon - 1))|^2 dxdt + \int_0^T\int_\Omega \nabla[u_0^-(Q_\varepsilon-1)]\nabla(u_0-u_\varepsilon)dxdt - \\ - C\int_0^T\int_{\Gamma_1} u_0^-(Q_\varepsilon u_0 - u_\varepsilon)d\hat{x}dt. \end{aligned} \tag{37}$$

Denote the right-hand side of the inequality (37) by R_ε. Let us show that $R_\varepsilon \to 0$ as $\varepsilon \to 0$. Taking into account that

$$\lim_{\varepsilon\to 0}\left(\int_0^T \Big(\frac{\partial u_0}{\partial t}, u_\varepsilon\Big)dt - \int_0^T \Big(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon\Big)dt\right) \le 0, \tag{38}$$

we deduce that

$$\begin{aligned}
0 \le R_\varepsilon = &\int_0^T \Big(\frac{\partial(u_\varepsilon - u_0)}{\partial t}, u_0^+ + Q_\varepsilon u_0^- - u_0\Big)dt + \\
&+\int_0^T\int_\Omega |\nabla u_0{}^-|^2 (Q_\varepsilon - 1)^2 dx dt + \int_0^T\int_\Omega |u_0^-|^2 |\nabla Q_\varepsilon|^2 dx dt + \\
&+2\int_0^T\int_\Omega u_0 (Q_\varepsilon - 1)\nabla u_0^- \nabla Q_\varepsilon dx dt + \int_0^T\int_\Omega (Q_\varepsilon - 1)\nabla u_0^- \nabla (u_0 - u_\varepsilon) dx dt + \\
&+\int_0^T\int_\Omega u_0^- \nabla Q_\varepsilon \nabla u_0 dx dt - \int_0^T\int_\Omega u_0^- \nabla Q_\varepsilon \nabla u_\varepsilon dx dt - C\int_0^T\int_{\Gamma_1} u_0^- (Q_\varepsilon u_0 - u_\varepsilon) d\hat{x} dt = \\
&= \int_0^T \Big(\frac{\partial(u_\varepsilon - u_0)}{\partial t}, u_0^+ + Q_\varepsilon u_0^- - u_0\Big)dt + \\
&+\left\{\sum_{j=1}^{N(\varepsilon)} \int_0^T \int_{\partial(T_\varepsilon^j\cap\Omega)} \frac{\partial Q_\varepsilon}{\partial \nu} Q_\varepsilon |u_0^-|^2 ds dt - C\int_0^T\int_{\Gamma_1} |\, u_0^- \,|^2 \, d\hat{x} dt\right\} - \\
&-\left\{\sum_{j=1}^{N(\varepsilon)} \int_0^T \int_{\partial(T_\varepsilon^j\cap\Omega)} \frac{\partial Q_\varepsilon}{\partial \nu} u_0^- u_\varepsilon ds dt - C\int_0^T\int_{\Gamma_1} u_0^- u_\varepsilon d\hat{x} dt\right\} + \\
&+C\int_0^T\int_{\Gamma_1} u_0^- (u_0 - Q_\varepsilon u_0) d\hat{x} dt,
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
Z_\varepsilon = &\int_0^T\int_\Omega |\nabla u_0^-|^2 (Q_\varepsilon - 1)^2 dx dt - \int_0^T\int_\Omega u_0^- \nabla u_0 \nabla Q_\varepsilon dx dt + \int_0^T\int_\Omega u_\varepsilon \nabla Q_\varepsilon \nabla u_0{}^- dx dt + \\
&+\int_0^T\int_\Omega (Q_\varepsilon - 1)\nabla u_0^- \nabla (u_0 - u_\varepsilon) dx dt.
\end{aligned}$$

Lemma 0.2 and the properties of Q_ε ensure that

$$\lim_{\varepsilon\to 0} Z_\varepsilon = 0.$$

By Lemma 0.1, the right-hand side of (39) tends to zero as $\varepsilon \to 0$, and therefore,

$$\lim_{\varepsilon\to 0} R_\varepsilon = 0.$$

Thus $\|u_\varepsilon - u_0^+ - Q_\varepsilon u_0^-\|_{L_2(0,T;H_1(\Omega,\Gamma_2))} \to 0$ as $\varepsilon \to 0$.

Remark 1. To prove that $\lim\limits_{\varepsilon\to 0} R_\varepsilon = 0$ we use the following relation:

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\Gamma_1} u_0^-(u_0 - Q_\varepsilon u_0) d\hat{x} dt = 0.$$

In fact, by the Gauss–Ostrogradskii formula we have

$$\int_0^T \int_{\Gamma_1} u_0^-(u_0 - Q_\varepsilon u_0) d\hat{x} dt = \int_0^T \int_\Omega \mid u_0^- \mid^2 \frac{\partial Q_\varepsilon}{\partial x_1} dx dt - \int_0^T \int_\Omega u_0^- \frac{\partial u_0}{\partial x_1}(1 - Q_\varepsilon) dx dt. \tag{40}$$

Using Lemma 0.2 and the properties of Q_ε, we deduce that the right-hand side of (40) tends to zero as $\varepsilon \to 0$.

Theorem 0.3. *Let u_ε be a solution of problem (1) and u_0 a solution of problem (36). Suppose that $a_\varepsilon \varepsilon^{-2} \to C$ as $\varepsilon \to 0$ and $C = const > 0$. Then*

$$\|u_\varepsilon - u_0^+ - Q_\varepsilon u_0^-\|_{L_2(0,T;H_1(\Omega,\Gamma_2))} \to 0 \text{ as } \varepsilon \to 0,$$

where Q_ε is defined by (15).

Remark 2. Using similar technique we can study the following problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f \quad \text{for } (x,t) \in Q_T; \\ u_\varepsilon(x,0) = 0,\ x \in \Omega;\ u_\varepsilon = 0 \text{ on } \Gamma_2 \times (0,T);\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ on } (\Gamma_1 \setminus G_\varepsilon) \times (0,T); \\ u_\varepsilon \geq g_0,\ \frac{\partial u_\varepsilon}{\partial \nu} \geq g_1,\ u_\varepsilon(\frac{\partial u_\varepsilon}{\partial \nu} - g_1) = 0 \text{ for } x \in G_\varepsilon \times (0,T), \end{cases} \tag{41}$$

REFERENCES

[1] H. Attouch and C. Picard, *Variational inequalities with varying obstacles*, J. Funct. Anal., **50** (1983), 329–386.

[2] L. Carbone and F. Colombini, *On convergence of functionals with unilateral constraints*, J. Math. Pures Appl., **59** (1980), 465–500.

[3] J. L. Lions, “Quelques Méthodes de Résolution de Problèmes aux Limites non Linéaires,” Dunod, Paris, 1969.

[4] G. Dal Maso and G. Paderni, *Variational inequalities for the biharmonic operator with variable obstacles*, Ann. Mat. Pura Appl., **4**, (1989), 203–227.

[5] G. Dal Maso and L. Modica, *A general theory of variational functionals*, “Topics in Functional Analysis 1980-81,” Scuola Normale Superiore, Pisa, (1981)

[6] G. Dal Maso and P. Longo, Γ - *limits of obstacles*, Ann. Mat. Pura Appl., **128** (1981), 1–50.

[7] G. Dal Maso, *On the integral representation of certain local functionals*, Ricerche Mat., **32** (1983), 85–113.

[8] C. Picard, *Problème biharmonique avec obstacles variables*, Thèse, Universitè Paris-Sud, 1984.

[9] G. A. Yosifian, *Homogenization of some problems with rapidly oscillating constraints*, J. Math. Sci., New York 120, **3** (2003), 1353–1363.

[10] G. V. Sandrakov, *Homogenization of variational inequalities for problems with an obstacle*, Sb. Math. 196, **4** (2005), 541–560.

[11] M. N. Zubova and T. A. Shaposhnikova, *Homogenization of the variational inequality corresponding to a problem with rapidly varying boundary conditions*, (Russian) Mat. Zametki, **82** (2007), 538–549; translation in Math. Notes, **82** (2007), 481–491.
[12] M. N. Zubova, *On the homogenization of variational inequalities for the biharmonic operator with constraints on ϵ-periodically arranged subsets*, Differ. Uravn., **42** (2006), 801–813, 862; translation in Differ. Equ., **42** (2006), 853–866.
[13] R. Glowinski, J. L. Lions and R. Tremoliere, "Numerical Analysis of Variational Inequalities," Am-. sterdam: North-Holland 1980. 17.

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