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ASYMPTOTIC ANALYSIS OF A NON-PERIODIC FLOW IN A THIN CHANNEL WITH VISCO-ELASTIC WALL

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Abstract. In this paper we continue the study of a fluid-structure interaction problem with the non periodic case. We consider the non stationary flow of a viscous fluid in a thin rectangle with an elastic membrane as the upper part of the boundary. The physical problem which corresponds to non homogeneous boundary conditions is stated. By using a boundary layer method, an asymptotic solution is proposed. The properties of the boundary layer functions are established and an error estimate is obtained.

1. **Introduction.** Many physical phenomena involve a fluid interacting with a moving or deformable structure. This kind of problems finds practical use in many areas of engineering and pure science. Some areas of applications are: biomechanics, hydroelasticity, aeroelasticity, etc.

In the last years, there was an increasing interest in the study of such problems: [3], [4], [8], [11], [12], [2] are only a few examples of works dealing with the fluid-structure interaction.

The purpose of this paper is to continue the asymptotic analysis of the interaction between a viscous fluid and an elastic membrane in the case of non homogeneous boundary conditions. We consider, as in [13], the case of small enough deformations of the elastic structure, inducing negligible deformations of the fluid domain. This problem is a simplified model for blood motion through an artery.

Recently, we published some results concerning the flow through the bloodstream. In [5], [6], [7] we considered a more complicated model for the fluid motion, but the flow domain was taken with rigid boundaries.

An asymptotic approach of a viscous quasi-static flow through a narrow elastic tube was performed in [1]. An error estimate between the exact solution and the first term of the asymptotic solution is obtained when the displacement of the elastic wall is described by Navier equation.

We consider a non steady state viscous flow in a thin channel with a visco-elastic wall. The fluid motion is simulated by the Stokes equations, the wall behaviour is described by the Sophie Germain fourth order in space non-steady state equation

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for the transversal displacements of the elastic wall (the plate model), while the longitudinal wall displacements are disregarded. The fluid-structure interaction is simulated by the equality of the fluid velocity at the boundary and the time derivative of the wall displacement (the longitudinal velocity is taken equal to zero). In a previous paper, [13], we studied the periodic case. We extend here the results obtained in [13] for the non-periodic problem.

The problem contains two small parameters: one of them is the ratio ε of the thickness of the channel to its length; the second, δ, is the ratio of the linear density to the stiffness of the wall. For various ratios of these two small parameters, an asymptotic expansion of a periodic solution is constructed. Parameter δ is taken as some power of ε, namely, $\delta = \varepsilon^\gamma$, $\gamma \geq 3$.

The outline of this paper is as follows: in Section 2 we describe the physical problem. In order to approximate the solution of the considered problem with more regular functions, we propose an asymptotic expansion. The main difference between the present paper and [13] is the following: in [13] the asymptotic solution was introduced to approximate a periodic flow in a semi infinite rectangle. We proposed an asymptotic expansion with the terms verifying the same boundary conditions as the solution of the initial problem. The aim of this paper is to approximate the solution of the fluid-structure interaction problem in a finite rectangle. Consequently, the traces of the asymptotic solution on the lateral sides of the boundary may be different from those of the solution for the initial problem. To overcome this difficulty, we introduce the boundary layer correctors which allow us to estimate the error between the asymptotic solution and the macroscopic one. The next section deals with the study of the boundary layer problems. The boundary layer problem for the velocity/pressure is not a classical one; hence, the results of [9] can not be applied in this case. For obtaining the exponential decay at infinity for the boundary layers, we propose a method based on the construction of several functions. In the last section we establish the error between the exact solution and the asymptotic solution of order K.

2. **The physical problem.**

We consider a small parameter ε, $\varepsilon = \dfrac{1}{q}$, $q \in \mathbb{N}^*$ and we define the thin domain

$$D_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\ -\frac{\varepsilon}{2} < x_2 < \frac{\varepsilon}{2}\}.$$

Let Γ_ε be the elastic part of the boundary of D_ε, given by:

$$\Gamma_\varepsilon = \{(x_1, \frac{\varepsilon}{2}) : 0 < x_1 < 1\}.$$

The other part of the boundary is rigid.

We suppose that the incompressible, viscous fluid fills the domain D_ε and interacts with the elastic structure Γ_ε. The interaction between the fluid and the elastic boundary produces the displacement $dd(x_1, t)$ of this boundary in Ox_2 direction. We neglect the longitudinal displacement. We study this problem for $t \in [0, T]$, with T an arbitrary positive constant and we assume that the membrane is not very elastic so that the displacement of the boundary is small enough. Consequently, at each time t, we can consider with a good approximation the fluid flow equations in the initial configuration. For the case when the equations for the fluid are set in the deformed configuration we can refer, for instance, to [11] for the stationary case and to [2] for the non stationary one; in these papers the existence and the uniqueness

of the solution were studied.

Let $\mathbf{f}$ be the exterior force applied to the fluid, $g\,\mathbf{e}_2$ the exterior force applied on the elastic boundary and $\big(T_f\,\mathbf{n}\big)_2$ the surface force exerted by the fluid on the structure, with T_f the stress tensor and $\mathbf{n}$ the outer unit normal on the boundary of D_ε.

The non stationary problem described above, with non homogeneous boundary conditions for the velocity is modelised by the following coupled system:

$$\begin{cases} \rho_f \dfrac{\partial \mathbf{u}}{\partial t} - \mu\ \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } D_\varepsilon \times (0,T), \\ \text{div } \mathbf{u} = 0 \text{ in } D_\varepsilon \times (0,T), \\ \mathbf{u} = \mathbf{0} \text{ on } (\partial D_\varepsilon \cap \{x_2 = -\dfrac{\varepsilon}{2}\}) \times (0,T), \\ \mathbf{u} = \varepsilon^2 \boldsymbol{\psi}^\varepsilon \text{ on } (\partial D_\varepsilon \cap \{x_1 = i\}) \times (0,T),\ i \in \{0,1\}, \\ u_1 = 0 \text{ and } u_2 = \dfrac{\partial d}{\partial t} \text{ on } \Gamma_\varepsilon \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{0} \text{ in } D_\varepsilon, \\ \rho h \dfrac{\partial^2 d}{\partial t^2} + \dfrac{h^3 E}{12}\ \dfrac{\partial^4 d}{\partial x_1^4} + \bar{\nu} \dfrac{\partial^4}{\partial x_1^4}\Big(\dfrac{\partial d}{\partial t}\Big) = g(x_1,t) + \big(T_f \mathbf{n}\big)_2 \text{ on } \Gamma_\varepsilon \times (0,T), \\ d(0,t) = d(1,t) = \dfrac{\partial d}{\partial x_1}(0,t) \dfrac{\partial d}{\partial x_1}(1,t) = 0 \text{ in } (0,T), \\ d(x_1,0) = \dfrac{\partial d}{\partial t}(x_1,0) = 0 \text{ in } (0,1), \end{cases} \tag{1}$$

where $\rho_f,\ \rho,\ \mu,\ \bar{\nu},\ E$ represent positive given constants in connection with the properties of the materials. The positive constant h is the thickness of the elastic membrane. We consider that the elastic boundary is clamped. The non homogeneous boundary conditions for the velocity are given by the function $\boldsymbol{\psi}^\varepsilon$ which is defined as follows :

$$\boldsymbol{\psi}^\varepsilon(x,t) = \boldsymbol{\psi}(\xi,t),$$

with $\xi = x/\varepsilon$ and $\boldsymbol{\psi}$ is 1-periodic in ξ_1 and satisfies the problem

$$\begin{cases} \text{div } \boldsymbol{\psi} = 0 \text{ in } (0,1) \times (-1/2,1/2) \times (0,T), \\ \boldsymbol{\psi} = \mathbf{0} \text{ on } \{\xi_2 = \pm 1/2\} \times (0,T), \\ \boldsymbol{\psi}(\xi,0) = \mathbf{0}. \end{cases}$$

The unknowns of the system (1) are: the velocity of the fluid, $\mathbf{u}$, the pressure of the fluid, p, and the displacement of the elastic membrane, d. The fluid flow is described by the non stationary Stokes equations. A "viscous" type term, $\bar{\nu}\ \dfrac{\partial^4}{\partial x_1^4}\Big(\dfrac{\partial d}{\partial t}\Big)$, was added to the usual forth-order equation for the normal displacement. This additional term will ensure that the velocity of the structure is smooth enough. The coefficient $\dfrac{h^3 E}{12}$ will play an important role for our problem. Usually, the Young's modulus, E, has a very big value (E is of order $10^6 Pa$) and this value becomes more important if the elastic medium is more rigid. On the other hand, we assume that the characteristic longitudinal space scale for vessels is of order of cm

and the time scale is of order of seconds. Let us use the SI system of units. This leads us to the necessity of scaling of every derivative in x_1 by the factor 10^2, i. e. the fourth derivative will contain the additional factor 10^8. If h is of order 10^{-3} m or 10^{-2} m, then the coefficient ρh can be taken in the further analysis equal to one. The coefficient $h^3E/12$ in equation $(1)_7$ will be replaced (after scaling in x_1) by a great coefficient δ^{-1} with the value of δ of order from 10^{-7} to 10^{-4}. If the ratio of the thickness and the length of the vessel ε is of order 10^{-2}, then δ is of order from ε^2 to ε^4. We assume that the "viscous" term is much smaller than the term with coefficient δ^{-1} considering that the new coefficient ν, obtained after scaling in x_1, $(\nu 10^8\,\bar{\nu})$ is of order of one. The coefficient of the term $\dfrac{\partial \mathbf{u}}{\partial t}$ in $(1)_1$ is not too important, so for the sake of simplicity we shall take it in the sequel equal to one.

The action of the viscous fluid on the elastic membrane is represented by the stress tensor $T_f = T_f(\mathbf{u}, p)$ which is defined by

$$T_f(\mathbf{u}, p) = pI - \mu\,\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^t\big).$$

On the boundary Γ_ε $\mathbf{n} = \mathbf{e}_2$; hence

$$\big(T_f \mathbf{n}\big)_2 = p - 2\mu\,\frac{\partial u_2}{\partial x_2} \text{ on } \Gamma_\varepsilon \times (0, T).$$

If we formally consider div $\mathbf{u} = 0$ on $\Gamma_\varepsilon \times (0, T)$, from $(1)_{5_1}$ it follows that :

$$\frac{\partial u_2}{\partial x_2} = 0.$$

Hence, the surface force exerted by the fluid on the elastic boundary can be defined by:

$$\big(T_f \mathbf{n}\big)_2 = p. \tag{2}$$

Due to the periodicity of the function $\boldsymbol{\psi}$, the compatibility condition for the coupled system which describes the physical problem is the same as in [13], i. e.

$$0 = \int_{\partial D_\varepsilon} \mathbf{u}(x,t)\cdot \mathbf{n}\,\mathrm{d}\gamma_x = \int_{\Gamma_\varepsilon} u_2(x_1, \frac{\varepsilon}{2}, t)\,\mathrm{d}x_1 = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 d(x_1,t)\mathrm{d}x_1\Big).$$

It follows that $\displaystyle\int_0^1 d(x_1,t)\mathrm{d}x_1 =$ constant for all $t \in (0,T)$. Using next the initial condition for d, we obtain the constant equal to zero.

Hence, the compatibility condition for the above coupled system becomes:

$$\int_0^1 d(x_1,t)\mathrm{d}x_1 = 0 \text{ for all } t \in (0,T). \tag{3}$$

This condition states that the global area of the flow domain is preserved.

We consider the following regularity for the data: $\mathbf{f} \in L^2(0,\ T;\ (L^2(D_\varepsilon))^2)$, $g \in L^2((0,1)\times(0,T))$ and $\boldsymbol{\psi} \in H^1(0,T;(H^2((0,1)\times(-1/2,1/2)))^2)$ with the properties stated above.

For obtaining the variational formulation of (1) and (3) we introduce the following spaces:

$$\begin{cases} V^\varepsilon = \{\mathbf{v} \in (H^1(D_\varepsilon))^2 : \operatorname{div}\mathbf{v} = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \partial D_\varepsilon \backslash \Gamma_\varepsilon,\ v_1 = 0 \text{ on } \Gamma_\varepsilon\}, \\ B_0 = \{b \in H_0^2(0,1) : \displaystyle\int_0^1 b(x_1)\,\mathrm{d}x_1 = 0\} \end{cases}$$

If we replace the system (1) with a homogeneous one by changing the function $\mathbf{u}$ with $\mathbf{v} = \mathbf{u} - \varepsilon^2 \boldsymbol{\psi}^\varepsilon$, then the variational formulation of the homogeneous system is given by:

$$\left\{\begin{array}{l} \text{Find } (\mathbf{v}, d) \in L^2(0,T;V^\varepsilon) \times H^1(0,T;B_0), \\ \text{ with } (\mathbf{v}', d') \in L^2(0,T;(V^\varepsilon)') \times H^1(0,T;(B_0)'), \\ \text{ which satisfies a. e. in } (0,T): \\ \dfrac{d}{dt}\displaystyle\int_{D_\varepsilon} \mathbf{v}\cdot\boldsymbol{\varphi} + \mu \int_{D_\varepsilon} \nabla\mathbf{v} : \nabla\boldsymbol{\varphi} + \frac{d}{dt}\int_0^1 \frac{\partial d}{\partial t}\, b + \frac{1}{\delta}\int_0^1 \frac{\partial^2 d}{\partial x_1^2}\frac{\partial^2 b}{\partial x_1^2} \\ \quad +\nu\displaystyle\int_0^1 \frac{\partial^3 d}{\partial x_1^2 \partial t}\frac{\partial^2 b}{\partial x_1^2} \int_{D_\varepsilon} \tilde{\mathbf{f}}\cdot\boldsymbol{\varphi} + \int_0^1 gb,\ \forall\, \boldsymbol{\varphi} \in V^\varepsilon,\ b \in B_0,\ \varphi_2 = b \text{ on } \Gamma_\varepsilon \\ v_2 = \dfrac{\partial d}{\partial t} \text{ on } \Gamma_\varepsilon, \\ \mathbf{v}(0) = \mathbf{0}, d(x_1,0) = \dfrac{\partial d}{\partial t}(x_1,0) = 0. \end{array}\right.$$

The results concerning the existence, the uniqueness and the regularity of the solution and some *a priori* estimates can be found in [13]. Since we shall need the *a priori* estimates for obtaining the error between the exact solution and the asymptotic one we give without proof the following result:

Proposition 2.1. *Let* $(\mathbf{u}_i,\ p_i,\ d_i)$ *be the solution of the problem (1) corresponding to the data* $\mathbf{f}_i,\ g_i,\ i = 1,2$. *Then the following estimates hold:*

$$\left\{\begin{array}{l} \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)} \le C(T)E(\mathbf{f}_1,\mathbf{f}_2,g_1,g_2), \\ \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} \le C(T,\mu)E(\mathbf{f}_1,\mathbf{f}_2,g_1,g_2), \\ \left\|\dfrac{\partial(d_1-d_2)}{\partial t}\right\|_{L^\infty(0,T;L^2(0,1))} \le C(T)E(\mathbf{f}_1,\mathbf{f}_2,g_1,g_2), \\ \left\|\dfrac{\partial^2(d_1-d_2)}{\partial x_1^2}\right\|_{L^\infty(0,T;L^2(0,1))} \le \sqrt{\delta}C(T)E(\mathbf{f}_1,\mathbf{f}_2,g_1,g_2) \\ \left\|\dfrac{\partial^3(d_1-d_2)}{\partial x_1^2\partial t}\right\|_{L^2((0,1)\times(0,T))} \le C(T,\nu)E(\mathbf{f}_1,\mathbf{f}_2,g_1,g_2), \end{array}\right. \tag{4}$$

where $E(\mathbf{f}_1,\mathbf{f}_2,g_1,g_2) = \|\mathbf{f}_1 - \mathbf{f}_2\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} + \|g_1 - g_2\|_{L^2((0,1)\times(0,T))}$

3. **Asymptotic approach.** In the sequel, we introduce the asymptotic expansions for the problem described in the previous section. We consider more regular data than in Section 2. We suppose that

$$\left\{\begin{array}{l} \boldsymbol{\psi} = \psi_1(\xi_2,\ t)\mathbf{e}_1,\ \psi_1 \in C^\infty([-1/2,\ 1/2]\times[0,T]), \\ \mathbf{f} = f_1(x_1,\ t)\ \mathbf{e}_1,\ f_1,\ g \in C^\infty([0,1]\times[0,T]), \\ \exists\, t^* < T \text{ such that } f_1(x_1,t) = g(x_1,t) = \psi_1(\xi_2,t) = 0 \\ \qquad \forall\ (x_1,\xi_2,t) \in (0,1)\times(-\tfrac{1}{2},\tfrac{1}{2})\times(0,t^*). \end{array}\right. \tag{5}$$

We shall approximate the functions $\mathbf{u}$, p, d which verify the system (1). In the sequel we take $\delta = \varepsilon^\gamma$, with $\gamma \in \mathbb{N}^*,\ \gamma \ge 3$.

In this case, we consider a more complicated asymptotic solution than in the periodic case, since it is necessary to introduce the boundary layer correctors.

We define an asymptotic solution by:

$$\begin{cases} \hat{\mathbf{u}}^{(K)}(x_1,x_2,t) = \mathbf{u}^{(K)}(x_1,\frac{x_2}{\varepsilon},t) + \mathbf{u}_{BL}^{(K)\,0}\Big(\frac{x}{\varepsilon},t\Big) + \mathbf{u}_{BL}^{(K)\,1}\Big(\frac{x}{\varepsilon}-\frac{1}{\varepsilon}\mathbf{e}_1,t\Big), \\ \hat{p}^{(K)}(x_1,x_2,t) = p^{(K)}(x_1,\frac{x_2}{\varepsilon},t) + p_{BL}^{(K)\,0}\Big(\frac{x}{\varepsilon},t\Big) + p_{BL}^{(K)\,1}\Big(\frac{x}{\varepsilon}-\frac{1}{\varepsilon}\mathbf{e}_1,t\Big), \\ \hat{d}^{(K)}(x_1,t) = d^{(K)}(x_1,t) + d_{BL}^{(K)\,0}\Big(\frac{x_1}{\varepsilon},t\Big) + d_{BL}^{(K)\,1}\Big(\frac{x_1-1}{\varepsilon},t\Big). \end{cases} \tag{6}$$

The expressions of $\mathbf{u}^{(K)},\ p^{(K)},\ d^{(K)}$ are the same as in the periodic case, i. e.

$$\begin{cases} u_1^{(K)}(x_1,\ \frac{x_2}{\varepsilon},\ t) = \sum\limits_{j=0}^{K}\ \varepsilon^{j+2}\ u_{1,j}\big(x_1,\frac{x_2}{\varepsilon},t\big), \\ u_2^{(K)}(x_1,\ \frac{x_2}{\varepsilon},\ t) = \sum\limits_{j=0}^{K}\ \varepsilon^{j+3}\ u_{2,j}\big(x_1,\frac{x_2}{\varepsilon},t\big), \\ p^{(K)}(x_1,\ \frac{x_2}{\varepsilon},\ t) = \sum\limits_{j=0}^{K}\ \varepsilon^{j+1}\ p_j\big(x_1,\frac{x_2}{\varepsilon},t\big) + \sum\limits_{j=0}^{K}\ \varepsilon^{j}\ q_j\big(x_1,t\big), \\ d^{(K)}(x_1,\ t) = \sum\limits_{j=0}^{K}\ \varepsilon^{j+\gamma}\ d_j\big(x_1,t\big). \end{cases} \tag{7}$$

Since the functions given by $(7)_{1,2,4}$ do not satisfy the same boundary conditions as $\mathbf{u},\ d$ on the lateral sides of the boundary, we introduce the boundary layer correctors. They correspond to the left end for $i=0$ and to the right end for $i=1$ and their expressions are given by:

$$\begin{cases} \mathbf{u}_{BL}^{(K)\,i}\big(\frac{x}{\varepsilon},\ t\big) = \sum\limits_{j=0}^{K}\ \varepsilon^{j+2}\ \mathbf{u}_j^{(i)}\big(\frac{x}{\varepsilon},\ t\big), \\ p_{BL}^{(K)\,i}\big(\frac{x}{\varepsilon},\ t\big) = \sum\limits_{j=0}^{K}\ \varepsilon^{j+1}\ p_j^{(i)}\big(\frac{x}{\varepsilon},\ t\big), \\ d_{BL}^{(K)\,i}\big(\frac{x_1}{\varepsilon},\ t\big) \sum\limits_{j=0}^{K}\ \varepsilon^{j+\gamma}\ d_j^{(i)}\big(\frac{x_1}{\varepsilon},\ t\big). \end{cases} \tag{8}$$

Introducing the asymptotic expansions into (1) and into (3), identifying the coefficients of the powers of ε and denoting $\xi_2 = \dfrac{x_2}{\varepsilon}$ we are leaded to consider the

following problem:

$$\begin{cases} -\mu\ \dfrac{\partial^2 u_{1,j}}{\partial \xi_2^2} + \dfrac{\partial p_{j-1}}{\partial x_1} - \mu\ \dfrac{\partial^2 u_{1,j-2}}{\partial x_1^2} + \dfrac{\partial u_{1,j-2}}{\partial t} + \dfrac{\partial q_j}{\partial x_1} = f_1 \delta_{j0}, \\ -\mu\ \dfrac{\partial^2 u_{2,j-1}}{\partial \xi_2^2} + \dfrac{\partial p_j}{\partial \xi_2} - \mu\ \dfrac{\partial^2 u_{2,j-3}}{\partial x_1^2} + \dfrac{\partial u_{2,j-3}}{\partial t} = 0, \\ \dfrac{\partial u_{1,j}}{\partial x_1} + \dfrac{\partial u_{2,j}}{\partial \xi_2} = 0, \\ u_{1,j}\big(x_1, \pm\dfrac{1}{2},\ t\big) = 0, \\ u_{2,j}\big(x_1, -\dfrac{1}{2},\ t\big) = 0, \\ u_{2,j}\big(x_1, \dfrac{1}{2},\ t\big)\dfrac{\partial d_{j-\gamma+3}}{\partial t}, \\ \dfrac{\partial^4 d_j}{\partial x_1^4} + \nu\ \dfrac{\partial^5 d_{j-\gamma}}{\partial x_1^4 \partial t} + \ \dfrac{\partial^2 d_{j-\gamma}}{\partial t^2} = g\ \delta_{j0} + q_j + p_{j-1}/_{\xi_2=1/2}, \\ d_j(0,\ t) = -d_j^{(0)}(0,\ t), \\ d_j(1,\ t) = -d_j^{(1)}(0,\ t), \\ \dfrac{\partial d_j}{\partial x_1}(0,\ t) = -\dfrac{\partial d_{j+1}^{(0)}}{\partial \xi_1}(0,\ t), \\ \dfrac{\partial d_j}{\partial x_1}(1,\ t) - \dfrac{\partial d_{j+1}^{(1)}}{\partial \xi_1}(0,\ t), \\ \displaystyle\int_0^1 d_j(x_1,t)\mathrm{d}x_1 = -\int_0^\infty d_{j-1}^{(0)}(\xi_1,t)\mathrm{d}\xi_1 - \int_{-\infty}^0 d_{j-1}^{(1)}(\xi_1,t)\mathrm{d}\xi_1. \end{cases} \tag{9}$$

We associate to the problem $(9)_{1-6}$ the following compatibility condition:

$$-\int_{-1/2}^{1/2} u_{1,j}(0,\xi_2,t)\mathrm{d}\xi_2 + \int_{-1/2}^{1/2} u_{1,j}(1,\xi_2,t)\mathrm{d}\xi_2, + \int_0^1 \frac{\partial d_{j-\gamma+3}}{\partial t}(x_1,t)\mathrm{d}x_1 = 0. \tag{10}$$

For obtaining the problems for the boundary layers corresponding to the left side we introduce the domain $\Pi^+ = (0,\ \infty) \times (-1/2,\ 1/2)$. The problem:

$$\begin{cases} -\mu\Delta_\xi \mathbf{u}_j^{(0)} + \nabla_\xi p_j^{(0)} - \dfrac{\partial \mathbf{u}_{j-2}^{(0)}}{\partial t} \text{ in } \Pi^+ \times (0,T), \\ \mathrm{div}_\xi\ \mathbf{u}_j^{(0)} = 0 \text{ in } \Pi^+ \times (0,T), \\ \mathbf{u}_j^{(0)}(\xi_1, -\dfrac{1}{2},\ t) = \mathbf{0}, \\ \mathbf{u}_j^{(0)}(\xi_1,\ \dfrac{1}{2},\ t) = \dfrac{\partial d_{j-\gamma+2}^{(0)}}{\partial t}(\xi_1,\ t)\mathbf{e}_2, \\ \mathbf{u}_j^{(0)}(0,\ \xi_2,\ t) = -u_{1,j}(0,\ \xi_2,\ t)\mathbf{e}_1 - u_{2,j-1}(0,\ \xi_2,\ t)\mathbf{e}_2 + \delta_{j0}\psi_1(\xi_2,\ t)\mathbf{e}_1, \\ \mathbf{u}_j^{(0)} \to \mathbf{0},\ p_j^{(0)} \to 0 \text{ uniformely, when } \xi_1 \to \infty, \end{cases} \tag{11}$$

with the compatibility condition

$$\int_0^\infty \frac{\partial d_{j-\gamma+2}^{(0)}}{\partial t}(\xi_1,t)\mathrm{d}\xi_1 + \int_{-1/2}^{1/2} u_{1,j}(0,\xi_2,t)\mathrm{d}\xi_2 - \delta_{j0}\int_{-1/2}^{1/2}\psi_1(\xi_2,t)\mathrm{d}\xi_2 = 0 \quad (12)$$

will give the boundary layer correctors for the velocity and for the pressure corresponding to the left end.

The boundary layer corrector for the displacement, corresponding to the left side, is obtained as a solution to the following problem:

$$\begin{cases} \dfrac{\partial^4 d_j^{(0)}}{\partial\xi_1^4} = -\dfrac{\partial^2 d_{j-\gamma-4}^{(0)}}{\partial t^2} - \nu\dfrac{\partial^5 d_{j-\gamma}^{(0)}}{\partial\xi_1^4\partial t} + p_{j-5}^{(0)}/_{\xi_2=1/2} \text{ in } (0,\infty)\times(0,T), \\ \dfrac{\partial^a d_j^{(0)}}{\partial\xi_1^a} \to 0 \text{ uniformely, when } \xi_1\to\infty,\ \forall\ a\in\{0,1,2,3\}. \end{cases} \quad (13)$$

In a similar way we introduce the boundary layer correctors corresponding to the right side. The boundary layers for the velocity and pressure are defined on $\Pi^-\times(0,T)$ with $\Pi^- = (-\infty,0)\times(-1/2,1/2)$ and the boundary layer for the displacement is defined on $(-\infty,0)\times(0,T)$.

Remark 3.1. For $\gamma > 3$ all the problems (9), (11) and (13) are stationary, while for $\gamma = 3$ only problems (11) and (13) are stationary, the time variable appearing in these cases as a parameter. However, all the unknowns must satisfy the homogeneous conditions for $t = 0$. These conditions can be obtained due to the hypothesis $(5)_3$.

The last part of this section is devoted to an analysis of the problems (9), (11), (13). We shall present step by step the order of solving the previous problems for different values of γ and we shall analyse the leading terms.

For this purpose we introduce the functions:

$$N_1(\xi_2) = \frac{1}{2}(\xi_2^2 - \frac{1}{4}),$$

which satisfies $N_1'' = 1$, $N_1\big(\pm\frac{1}{2}\big) = 0$ and

$$N_2(\xi_2) = \int_{-1/2}^{\xi_2} N_1(\tau)\ \mathrm{d}\tau;$$

with $N_2\big(\frac{1}{2}\big) = \displaystyle\int_{-1/2}^{1/2} N_1(\tau)\mathrm{d}\tau = -\frac{1}{12}$.

We shall use the following notations:

$$D^{-1} : F \mapsto \int_{-1/2}^{\xi_2} F(x_1,\tau,t)\ \mathrm{d}\tau,$$

$$D^{-2} : F \mapsto \int_{-1/2}^{\xi_2}\int_{-1/2}^{\theta} F(x_1,\tau,t)\ \mathrm{d}\tau\ \mathrm{d}\theta - (\xi_2+\frac{1}{2})\int_{-1/2}^{1/2}\int_{-1/2}^{\theta} F(x_1,\tau,t)\ \mathrm{d}\tau\ \mathrm{d}\theta\ .$$

a) The case $\gamma > 3$

The steps of solving the previous problems are:

• We determine, up to two functions of t, $u_{1,j}$, $u_{2,j}$, p_j, q_j from $(9)_{1-6}$.

Proposition 3.1. *The unknowns $u_{1,j},\ u_{2,j},\ p_j,\ q_j$ are given, up to two functions of t, by the following relations:*

$$\left\{\begin{array}{l} u_{1,j} = \dfrac{1}{\mu}\Big(D^{-2}\Big(\dfrac{\partial p_{j-1}}{\partial x_1} - \mu\dfrac{\partial^2 u_{1,j-2}}{\partial x_1^2} + \dfrac{\partial u_{1,j-2}}{\partial t}\Big) + \Big(\dfrac{\partial q_j}{\partial x_1} - f_1\delta_{j0}\Big)N_1(\xi_2)\Big), \\ p_j = D^{-1}\Big(\mu\ \Big(\dfrac{\partial^2 u_{2,j-1}}{\partial \xi_2^2} + \dfrac{\partial^2 u_{2,j-3}}{\partial x_1^2}\Big) - \dfrac{\partial u_{2,j-3}}{\partial t}\Big), \\ u_{2,j} = -D^{-1}\Big(\dfrac{1}{\mu}\Big(D^{-2}\Big(\dfrac{\partial^2 p_{j-1}}{\partial x_1^2} - \mu\dfrac{\partial^3 u_{1,j-2}}{\partial x_1^3} + \dfrac{\partial^2 u_{1,j-2}}{\partial x_1\partial t}\Big)\Big)\Big) \\ \qquad -\dfrac{1}{\mu}\Big(\dfrac{\partial^2 q_j}{\partial x_1^2} - \dfrac{\partial f_1}{\partial x_1}\delta_{j0}\Big)N_2(\xi_2), \\ -\dfrac{1}{\mu}\displaystyle\int_{-1/2}^{1/2} D^{-2}\Big(\dfrac{\partial^2 p_{j-1}}{\partial x_1^2} - \mu\dfrac{\partial^3 u_{1,j-2}}{\partial x_1^3} + \dfrac{\partial^2 u_{1,j-2}}{\partial x_1\partial t}\Big)\mathrm{d}\xi_2 \\ \qquad -\dfrac{1}{12\mu}\Big(\dfrac{\partial f_1}{\partial x_1}\delta_{j0} - \dfrac{\partial^2 q_j}{\partial x_1^2}\Big) = \dfrac{\partial d_{j-\gamma+3}}{\partial t}. \end{array}\right. \tag{14}$$

Proof. Integrating twice $(9)_1$ with respect to ξ_2 and using the boundary conditions $(9)_4$ we get $(14)_1$. This relation will give the unknown $u_{1,j}$ after determining q_j. The other functions contained by this relation are either known from previous computations or equal to zero. We integrate next the incompressibility condition $(9)_3$ with respect to ξ_2 with the boundary condition $(9)_5$ and we obtain $(14)_3$. The pressure approximations are given by $(9)_2$ since all the functions which appear in this relation, except p_j, are already known. The integration of $(9)_2$ in ξ_2 yields that the functions p_j are unique up to an additive function depending on $x_1,\ t$. In $(14)_2$ we took this function equal to zero since we consider (in the expansion $(7)_3$) that any function depending only on $x_1,\ t$ is contained in q_j. Finally, the equation $(14)_4$ is obtained introducing the expression of $u_{2,j}$ into the boundary condition $(9)_6$. The unknown of $(14)_4$ is q_j, since $j-\gamma+3<j$. The integration of $(14)_4$ with respect to x_1 introduces two unknown functions of t. At the end of this proof we have $p_j,\ u_{2,j}$ uniquely determined, $u_{1,j}$ given up to one function of t and q_j known up to two functions of t. □

For $j=0$ the system (14) becomes:

$$\left\{\begin{array}{l} u_{1,0} = \dfrac{1}{\mu}\Big(\dfrac{\partial q_0}{\partial x_1} - f_1\Big)\ N_1(\xi_2), \\ p_0 = 0 \\ u_{2,0} = -\dfrac{1}{\mu}\Big(\dfrac{\partial^2 q_0}{\partial x_1^2} - \dfrac{\partial f_1}{\partial x_1}\Big)N_2(\xi_2), \\ \dfrac{\partial^2 q_0}{\partial x_1^2} - \dfrac{\partial f_1}{\partial x_1} = 0. \end{array}\right. \tag{15}$$

It follows that $u_{2,0}(x_1,\ \xi_2,\ t)=0$.

The integration of $(15)_4$ yields:

$$q_0(x_1,t) = q_0(0,t) + \int_0^{x_1} f_1(s,t)\mathrm{d}s + \Big(\frac{\partial q_0}{\partial x_1}(0,t) - f_1(0,t)\Big)x_1. \tag{16}$$

The expression of $u_{1,0}$ is given by:

$$u_{1,0}(x_1,\xi_2,t)=\frac{1}{\mu}\Big(\frac{\partial q_0}{\partial x_1}(0,t)-f_1(0,t)\Big)N_1(\xi_2). \tag{17}$$

• We determine the function of t contained by the expression of $u_{1,j}$ $(\frac{\partial q_j}{\partial x_1}(0,t))$ from the compatibility condition (12).

We get from (12) written for $j=0$ (since $2-\gamma<0$):

$$\int_{-1/2}^{1/2}u_{1,0}(0,\xi_2,t)\mathrm{d}\xi_2\int_{-1/2}^{1/2}\psi_1(\xi_2,t)\mathrm{d}\xi_2.$$

Using next (17) it follows that

$$\frac{\partial q_0}{\partial x_1}(0,t)=-12\mu\int_{-1/2}^{1/2}\psi_1(\xi_2,t)\mathrm{d}\xi_2+f_1(0,t)$$

and hence

$$u_{1,0}(x_2,\xi_2,t)=-12\Big(\int_{-1/2}^{1/2}\psi_1(\xi_2,t)\mathrm{d}\xi_2\Big)N_1(\xi_2).$$

Remark 3.2. The compatibility condition for the boundary layers correctors corresponding to the right end gives the same expression for $\frac{\partial q_0}{\partial x_1}(0,t)$ due to $(9)_{12}$ and to (10).

• We solve (13) and the corresponding problem for the right end for j and for $j+1$. The right hand side of $(13)_1$ is known, both for j and for $j+1$ from previous computations.

Remark 3.3. $d_j^{(i)}=0,\ i=0,1$ for $j<5$ since for these values of j the right hand side of $(13)_1$ is equal to zero.

• We solve (11) and the corresponding problem for the right end. Since $j-\gamma+2<j$, all the right hand sides of (11) are known from previous computations.

The next section is devoted to the study of the boundary layer problems.

• We solve $(9)_{7-12}$ which gives the function d_j, and the last unknown function of t, contained by q_j, $q_j(0,t)$. For $j=0$ $(9)_{7-12}$ becomes:

$$\begin{cases}\dfrac{\partial^4 d_0}{\partial x_1^4}=g+q_0,\\ d_0(0,t)=0,\\ d_0(1,t)=0,\\ \dfrac{\partial d_0}{\partial x_1}(0,t)=0,\\ \dfrac{\partial d_0}{\partial x_1}(1,t)=0,\\ \displaystyle\int_0^1 d_0(x_1,t)\mathrm{d}x_1=0.\end{cases} \tag{18}$$

We integrate $(18)_1$ four times with respect to x_1 which introduces four new functions of t. For these functions and for $q_0(0,t)$ we have exactly five conditions, $(18)_{2-6}$.

b) The case $\gamma = 3$

In this case, the steps of solving the problems are:

• We express, from $(9)_{1-5}$, $\mathbf{u}_j$, p_j depending on q_j. For $j=0$ the expressions of $\mathbf{u}_0$ and p_0 with respect to q_0 are given by $(15)_{1-3}$.

• We solve (13) and the corresponding problem for the right end for j and for $j+1$.

• We solve (11) and the corresponding problem for the right end.

• We solve the problem satisfied by d_j.

Theorem 3.1. *The approximations of the displacement, d_j, are obtained as solutions of the following parabolic problems of the sixth order in the space variable:*

$$
\begin{cases}
\dfrac{\partial d_j}{\partial t} - \dfrac{1}{12\mu}\dfrac{\partial^6 d_j}{\partial x_1^6} = -\dfrac{1}{12\mu}\dfrac{\partial f_1}{\partial x_1}\delta_{j0} + A_{2,j-1}(x_1, 1/2, t) \\
-\dfrac{1}{12\mu}\dfrac{\partial^2 g}{\partial x_1^2}\delta_{j0} + \dfrac{1}{12\mu}\dfrac{\partial^4 d_{j-3}}{\partial x_1^2 \partial t^2} + \dfrac{\nu}{12\mu}\dfrac{\partial^7 d_{j-3}}{\partial x_1^6 \partial t} - \dfrac{1}{12\mu}\dfrac{\partial^2 p_{j-1}}{\partial x_1^2}\Big/_{\xi_2=1/2}, \\
d_j(0,t) = -d_j^{(0)}(0,t), \\
d_j(1,t) = -d_j^{(1)}(0,t), \\
\dfrac{\partial d_j}{\partial x_1}(0,t) = -\dfrac{\partial d_{j+1}^{(0)}}{\partial \xi_1}(0,t), \\
\dfrac{\partial d_j}{\partial x_1}(1,t) = -\dfrac{\partial d_{j+1}^{(1)}}{\partial \xi_1}(0,t), \\
\dfrac{\partial^5 d_j}{\partial x_1^5}(0,t)\Big(\dfrac{\partial g}{\partial x_1}(0,t) + f_1(0,t)\Big)\delta_{j0} - \dfrac{\partial^3 d_{j-3}}{\partial x_1 \partial t^2}(0,t) - \nu\dfrac{\partial^6 d_{j-3}}{\partial x_1^5 \partial t}(0,t) \\
\quad + \dfrac{\partial p_{j-1}}{\partial x_1}(0,1/2,t) + 12\mu\displaystyle\int_0^\infty \frac{\partial d_{j-1}^{(0)}}{\partial t}(\xi_1,t)\mathrm{d}\xi_1 \\
\quad + 12\mu\displaystyle\int_{-1/2}^{1/2} A_{1,j-1}(0,\xi_2,t)\mathrm{d}\xi_2 - 12\mu\delta_{j0}\int_{-1/2}^{1/2}\psi_1(\xi_2,t)\mathrm{d}\xi_2, \\
\dfrac{\partial^5 d_j}{\partial x_1^5}(1,t)\Big(\dfrac{\partial g}{\partial x_1}(1,t) + f_1(1,t)\Big)\delta_{j0} - \dfrac{\partial^3 d_{j-3}}{\partial x_1 \partial t^2}(1,t) - \nu\dfrac{\partial^6 d_{j-3}}{\partial x_1^5 \partial t}(1,t) \\
\quad + \dfrac{\partial p_{j-1}}{\partial x_1}(1,1/2,t) - 12\mu\displaystyle\int_0^\infty \frac{\partial d_{j-1}^{(1)}}{\partial t}(\xi_1,t)\mathrm{d}\xi_1 - 12\mu\int_{-1/2}^{1/2} A_{1,j-1}(1,\xi_2,t)\mathrm{d}\xi_2 \\
\quad - 12\mu\displaystyle\int_0^1 A_{2,j-1}(x_1,1/2,t)\mathrm{d}x_1 + 12\mu\delta_{j0}\int_{-1/2}^{1/2}\psi_1(\xi_2,t)\mathrm{d}\xi_2, \\
d_j(x_1,0) = 0,
\end{cases}
\tag{19}
$$

where $A_{1,j-1}$ and $A_{2,j-1}$ are known functions, depending on previous approximations.

Proof. From $(9)_7$ we get:

$$\frac{\partial^5 d_j}{\partial x_1^5}\frac{\partial g}{\partial x_1}\delta_{j0} - \frac{\partial^3 d_{j-3}}{\partial x_1 \partial t^2} - \nu\frac{\partial^6 d_{j-3}}{\partial x_1^5 \partial t} + \frac{\partial p_{j-1}}{\partial x_1}/_{\xi_2=1/2} + \frac{\partial q_j}{\partial x_1} \tag{20}$$

On the other hand, $(14)_1$ can be written as

$$u_{1,j}(x_1,\xi_2,t) = A_{1,j-1}(x_1,\xi_2,t) + \frac{1}{\mu}\Big(\frac{\partial q_j}{\partial x_1} - f_1\delta_{j0}\Big)N_1(\xi_2). \tag{21}$$

We introduce $u_{1,j}(0,\xi_2,t)$ given by (21) into the compatibility condition (12) and we obtain $\dfrac{\partial q_j}{\partial x_1}(0,t)$ which leads, together with (20), to the boundary condition $(19)_6$. The boundary condition for $x_1 = 1$ can be obtained in a similar way. □

The first approximation d_0 is the unique solution of the following system:

$$\begin{cases} \dfrac{\partial d_0}{\partial t} - \dfrac{1}{12\mu}\dfrac{\partial^6 d_0}{\partial x_1^6} = -\dfrac{1}{12\mu}\Big(\dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial^2 g}{\partial x_1^2}\Big), \\ d_0(0,t) = d_0(1,t) = \dfrac{\partial d_0}{\partial x_1}(0,t)\dfrac{\partial d_0}{\partial x_1}(1,t) = 0, \\ \dfrac{\partial^5 d_0}{\partial x_1^5}(0,t)\dfrac{\partial g}{\partial x_1}(0,t) + f_1(0,t) - 12\mu\displaystyle\int_{-1/2}^{1/2}\psi_1(\xi_2,t)\mathrm{d}\xi_2, \\ \dfrac{\partial^5 d_0}{\partial x_1^5}(1,t)\dfrac{\partial g}{\partial x_1}(1,t) + f_1(1,t) + 12\mu\displaystyle\int_{-1/2}^{1/2}\psi_1(\xi_2,t)\mathrm{d}\xi_2, \\ d_0(x_1,0) = 0. \end{cases} \tag{22}$$

- We determine q_j from $(9)_7$ and then $\mathbf{u}_j$ and p_j from $(14)_{1-3}$.

Remark 3.4. The asymptotic velocity in the transversal direction is greater than for $\gamma > 3$ since, for $\gamma = 3$, $u_{2,0}$ given by $(15)_3$ is not equal to zero.

4. **Boundary layer problems.** In the sequel we study the problems for the boundary layers corresponding to the left side, (11) and (13). The next theorem gives the exponentially decay to zero of the boundary layers at infinity.

Theorem 4.1. *Let* $(\mathbf{u}_j^{(0)},\ p_j^{(0)})$ *be the solution of (11) and* $d_j^{(0)}$ *the solution of (13). Then* $\forall j \in \mathbb{N}$ *there exist* $c_j = c_j(t),\ d_j = d_j(t),\ \sigma_j = \sigma_j(t),\ c_j,\ d_j,\ \sigma_j > 0$ *such that:*

$$\begin{cases} \|\nabla \mathbf{u}_j^{(0)}\|_{L^2(\Pi^+\cap\{\xi_1>R\}))^4} \le c_j(t)\exp(-\sigma_j(t)\,R), \\ \|\nabla p_j^{(0)}\|_{L^2(\Pi^+\cap\{\xi_1>R\}))^2} \le c_j(t)\exp(-\sigma_j(t)\,R),\ \forall\ R>0, \\ \Big|\dfrac{\partial^m d_j^{(0)}}{\partial \xi_1^m}\Big| \le c_j(t)\exp(-\sigma_j(t)\,\xi_1),\ \forall\ m\in\mathbb{N},\ \xi_1 > 1. \end{cases} \tag{23}$$

Moreover, the property $p_j^{(0)} \to 0$ *when* $\xi_1 \to \infty$ *yields:*

$$|p_j^{(0)}(\xi_1,\xi_2,t)| \le d_j(t)\exp(-\sigma_j(t)\,\xi_1),\ \forall\ \xi_1 > 1,\ \xi_2 \in (-1/2,1/2). \tag{24}$$

Proof. The existence and uniqueness of $\mathbf{u}_j^{(0)}$ and the existence of $p_j^{(0)}$ are obtained in a classical way. The uniqueness of $p_j^{(0)}$ is a consequence of $(11)_6$. Moreover, $(11)_6$ ensures enough regularity for $\mathbf{u}_j^{(0)}$ and $p_j^{(0)}$ so that the left hand side of (23) makes sense. We shall prove the estimates (23) recursively with respect to j. For $j = 0$

(11) represents the classical problem of [9]. Hence, $(23)_{1,2}$ are obtained by using the technique of [9].

(13) for j=0 yields $d_0^{(0)} = 0$.

We suppose that the estimates (23) are satisfied for $0, 1, ..., j-1$ and we prove them for j.

We notice that $(11)_1$ and $(11)_4$ for a general value of j have a non homogeneous right hand side. For this reason, the technique of [9] cannot be applied any more.

We define a new function

$$\mathbf{v}_j^{(0)}(\xi_1,\xi_2,t) = \mathbf{u}_j^{(0)}(\xi_1,\xi_2,t) - (\xi_2+1/2)\frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1,t)\mathbf{e}_2. \tag{25}$$

Denoting

$$\begin{cases} \tilde{\mathbf{f}}_{j-1}(\xi_1,\xi_2,t) = -\dfrac{\partial \mathbf{u}_{j-2}^{(0)}}{\partial t}(\xi_1,\xi_2,t) + \mu(\xi_2+\dfrac{1}{2})\dfrac{\partial^3 d_{j+2-\gamma}^{(0)}}{\partial \xi_1^2 \partial t}(\xi_1,t)\mathbf{e}_2, \\ \boldsymbol{\varphi}_{j-1}(\xi_2,t) = \boldsymbol{\psi}_1(\xi_2,t)\delta_{j0}\mathbf{e}_1 - u_{1,j}(0,\xi_2,t)\mathbf{e}_1 - u_{2,j-1}(0,\xi_2,t)\mathbf{e}_2 \\ \qquad\qquad -(\xi_2+\dfrac{1}{2})\dfrac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(0,t)\mathbf{e}_2, \end{cases} \tag{26}$$

we obtain for $(\mathbf{v}_j^{(0)},\ p_j^{(0)})$ the following problem:

$$\begin{cases} -\mu\Delta \mathbf{v}_j^{(0)} + \nabla p_j^{(0)} = \tilde{\mathbf{f}}_{j-1} \text{ in } \Pi^+ \times (0,T), \\ \operatorname{div} \mathbf{v}_j^{(0)} - \dfrac{\partial d_{j+2-\gamma}^{(0)}}{\partial t} \text{ in } \Pi^+ \times (0,T), \\ \mathbf{v}_j^{(0)}(\xi_1, \pm\dfrac{1}{2},\ t) = \mathbf{0}, \\ \mathbf{v}_j^{(0)}(0,\ \xi_2,\ t) = \boldsymbol{\varphi}_{j-1}(\xi_2,t) \\ \mathbf{v}_j^{(0)} \to \mathbf{0},\ p_j^{(0)} \to 0 \text{ uniformely, when } \xi_1 \to \infty. \end{cases} \tag{27}$$

For the sake of simplicity, in the sequel we will not write the dependence on t.

For any $R > 0$ we define $\Pi_R^+ = \Pi^+ \cap \{\xi_1 > R\}$. Multiplying $(27)_1$ by $\mathbf{v}_j^{(0)}$ and integrating on Π_R^+ we get

$$\begin{aligned} &\mu\int_{\Pi_R^+} |\nabla \mathbf{v}_j^{(0)}|^2 \mathrm{d}\xi + \frac{\mu}{2}\int_{-1/2}^{1/2} \frac{\partial (\mathbf{v}_j^{(0)})^2}{\partial \xi_1}(R,\xi_2)\mathrm{d}\xi_2 - \int_{-1/2}^{1/2} p_j^{(0)}(R,\xi_2)(v_j^{(0)})_1(R,\xi_2)\mathrm{d}\xi_2 \\ &= \int_{\Pi_R^+} \tilde{\mathbf{f}}_{j-1}\cdot \mathbf{v}_j^{(0)}\mathrm{d}\xi_1\mathrm{d}\xi_2 - \int_{\Pi_R^+} p_j^{(0)}\frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}\mathrm{d}\xi_1\mathrm{d}\xi_2. \end{aligned} \tag{28}$$

We majorate next each term of the right hand side of (28).

For the first term we have obviously, after using the Poincaré inequality and the recurence hypothesis

$$\int_{\Pi_R^+} \tilde{\mathbf{f}}_{j-1}\cdot \mathbf{v}_j^{(0)}\mathrm{d}\xi_1\mathrm{d}\xi_2 \le \frac{\mu}{2}\int_{\Pi_R^+} |\nabla \mathbf{v}_j^{(0)}|^2 \mathrm{d}\xi_1\mathrm{d}\xi_2 + \alpha_1 \exp(-\lambda_1 R).$$

For the second one, taking into account the regularity of $p_j^{(0)}$ at ∞ and again the recurence hypothesis, it follows that

$$-\int_{\Pi_R^+} p_j^{(0)} \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t} \mathrm{d}\xi \leq \Big(\int_{\Pi_R^+} (p_j^{(0)})^2 \Big| \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t} \Big| \mathrm{d}\xi \Big)^{1/2} \Big(\int_{\Pi_R^+} \Big| \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t} \Big| \mathrm{d}\xi \Big)^{1/2}$$

$$\leq \alpha_2 \Big(\int_{\Pi^+} (p_j^{(0)})^2 \Big| \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t} \Big| \mathrm{d}\xi \Big)^{1/2} \exp(-\lambda_2 R) \leq \alpha_3 \exp(-\lambda_2 R).$$

Denoting by $y(R) = \displaystyle\int_{\Pi_R^+} |\nabla \mathbf{v}_j^{(0)}|^2$, with $y'(R) = -\displaystyle\int_{-1/2}^{1/2} |\nabla \mathbf{v}_j^{(0)}(R, \xi_2)|^2 \mathrm{d}\xi_2$ and introducing the previous computations into (28) we obtain:

$$\frac{\mu}{2} y(R) + \frac{\mu}{2} \int_{-1/2}^{1/2} \frac{\partial (\mathbf{v}_j^{(0)})^2}{\partial \xi_1}(R, \xi_2) \mathrm{d}\xi_2 - \int_{-1/2}^{1/2} p_j^{(0)} (v_j^{(0)})_1 (R, \xi_2) \mathrm{d}\xi_2 \leq \alpha_4 \exp(-\lambda_3 R). \quad (29)$$

Integrating (29) from s to ∞ with respect to R and using the Poincaré inequality for the second term of the left hand side we get:

$$\frac{\mu}{2} \int_s^\infty y(R) \mathrm{d}R + \alpha_5 y'(s) \leq \int_s^\infty \int_{-1/2}^{1/2} p_j^{(0)} (v_j^{(0)})_1 (R, \xi_2) \mathrm{d}\xi_2 \mathrm{d}R + \alpha_6 \exp(-\lambda_3 s). \quad (30)$$

It remains to estimate the first term of the right hand side of the above inequality. For this purpose, we define for any $l \in \mathbb{N}$, $D_{s,l} = (s + l, s + l + 1) \times (-1/2, 1/2)$.

The compatibility condition for the system (27) written in $\Pi_R^+ \times (0, T)$ is

$$\int_{-1/2}^{1/2} (v_j^{(0)})_1 (R, \xi_2) \mathrm{d}\xi_2 \int_R^\infty \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1) \mathrm{d}\xi_1. \quad (31)$$

Integrating (31) from $s + l$ to $s + l + 1$ and defining the new function

$$\zeta(R, \xi_2) = (v_j^{(0)})_1 (R, \xi_2) - \int_R^\infty \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1) \mathrm{d}\xi_1 \quad (32)$$

it follows that:

$$\int_{D_{s,l}} \zeta(R, \xi_2) \mathrm{d}R \mathrm{d}\xi_2 = 0, \quad (33)$$

and the first term of the right hand side of (30) becomes

$$I = \sum_{l \geq 0} \int_{D_{s,l}} p_j^{(0)} \zeta(R, \xi_2) \mathrm{d}R \mathrm{d}\xi_2$$

$$+ \sum_{l \geq 0} \int_{D_{s,l}} \Big(p_j^{(0)}(R, \xi_2) \int_R^\infty \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1) \mathrm{d}\xi_1 \Big) \mathrm{d}R \mathrm{d}\xi_2. \quad (34)$$

For the first term of (34) we consider the problem:

$$\begin{cases} \mathbf{w}_{s,l} \in (H_0^1(D_{s,l}))^2, \\ \operatorname{div} \mathbf{w}_{s,l} \zeta \text{ in } D_{s,l}, \\ \|\mathbf{w}_{s,l}\|_{(H^1(D_{s,l}))^2} \leq c_0 \|\zeta\|_{L^2(D_{s,l})}, \end{cases} \quad (35)$$

where c_0 is the constant which corresponds to the square $(0, 1) \times (-1/2, 1/2)$.

The previous problem has a solution, due to the compatibility condition (33).

In the sequel we use the new problem (35) and $(27)_1$ for writting the first sum of (34) as follows:

$$\begin{aligned}
&\sum_{l\geq 0}\int_{D_{s,l}} p_j^{(0)}(R,\xi_2)\zeta(R,\xi_2)\mathrm{d}R\mathrm{d}\xi_2 - \sum_{l\geq 0}\int_{D_{s,l}} \nabla p_j^{(0)}(R,\xi_2)\cdot \mathbf{w}_{s,l}(R,\xi_2)\mathrm{d}R\mathrm{d}\xi_2\\
&= \mu\sum_{l\geq 0}\int_{D_{s,l}} \nabla \mathbf{v}_j^{(0)}(R,\xi_2)\cdot\nabla \mathbf{w}_{s,l}(R,\xi_2)\mathrm{d}R\mathrm{d}\xi_2 - \sum_{l\geq 0}\int_{D_{s,l}} \tilde{\mathbf{f}}_{j-1}(R,\xi_2)\cdot \mathbf{w}_{s,l}(R,\xi_2)\mathrm{d}R\mathrm{d}\xi_2\\
&\leq \mu y(s) + \alpha_1\exp(-\lambda_1\, s) + \alpha_7\int_{\Pi_s^+}\zeta^2(R,\xi_2)\mathrm{d}R\mathrm{d}\xi_2\\
&\leq \mu y(s) + \alpha_1\exp(-\lambda_1\, s) + \alpha_8\int_{\Pi_s^+}\Big((v_j^{(0)})_1^2(R,\xi_2)\\
&\quad + \Big(\int_R^\infty \tfrac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1)\mathrm{d}\xi_1\Big)^2\Big)\mathrm{d}R\mathrm{d}\xi_2\\
&\leq \alpha_9 y(s) + \alpha_{10}\exp(-\lambda_4\, s).
\end{aligned}$$

The computations for the second sum of (34) are the following:

$$\begin{aligned}
&\sum_{l\geq 0}\int_{D_{s,l}} p_j^{(0)}(R,\xi_2)\int_R^\infty \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1)\mathrm{d}\xi_1 \int_{\Pi_s^+} p_j^{(0)}(R,\xi_2)\int_R^\infty \frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1)\mathrm{d}\xi_1\\
&\leq \Big(\int_{\Pi_s^+}(p_j^{(0)})^2(R,\xi_2)\int_R^\infty \Big|\frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1)\Big|\mathrm{d}\xi_1\Big)^{1/2}\Big(\int_{\Pi_s^+}\int_R^\infty \Big|\frac{\partial d_{j+2-\gamma}^{(0)}}{\partial t}(\xi_1)\Big|\mathrm{d}\xi_1\Big)^{1/2}\\
&\leq \alpha_{11}\exp(-\lambda_5\, s).
\end{aligned}$$

With the previous estimates (30) becomes:

$$y'(s) + a\int_s^\infty y(R)\mathrm{d}R \leq b y(s) + b_{j-1}\exp(-2\alpha_{j-1}\, s). \tag{36}$$

We define the new function $z(s) = y(s)\exp(-b\, s)$ and we multiply (36) by $\exp(-b\, s)$. It follows that:

$$z'(s) + a\exp(-b\, s)\int_s^\infty z(R)\exp(b\, R)\mathrm{d}R \leq b_{j-1}\exp(-(b+2\alpha_{j-1})s). \tag{37}$$

We define, for any $\delta > 0$, $F(s) = z(s) + \delta\exp(-b\, s)\displaystyle\int_s^\infty z(R)\exp(b\, R)\mathrm{d}R$. We obtain

$$F'(s)+\delta F(s) \leq (\delta^2 - \delta\, b - a)\exp(-b\, s)\int_s^\infty z(R)\exp(b\, R)\mathrm{d}R + b_{j-1}\exp(-(b+2\alpha_{j-1})\, s).$$

Multiplying the previous inequality by $\exp(\delta\, s)$, integrating it from 0 to θ, denoting $G(\theta)\displaystyle\int_0^\theta\Big(\exp((\delta-b)\, s)\int_s^\infty z(R)\exp(b\, R)\mathrm{d}R\Big)\mathrm{d}s$ and using $F(\theta) > z(\theta)\,\forall\,\theta > 0$ we are leaded to

$$\begin{aligned}
y(\theta) \leq \Big(F(0) + \frac{b_{j-1}}{2\alpha_{j-1} + b - \delta}\Big)\exp((b-\delta)\theta) + (\delta^2 - \delta\, b - a)\\
G(\theta)\exp((b-\delta)\theta) - \frac{b_{j-1}}{2\alpha_{j-1} + b - \delta}\exp(-2\alpha_{j-1}\theta).
\end{aligned} \tag{38}$$

For any δ satisfying the inequalities $\delta > b,\ 2\alpha_{j-1} + b - \delta > 0$ and $\delta^2 - \delta\, b - a < 0$ we get from (38)

$$\|\nabla \mathbf{v}_j^{(0)}(t)\|^2_{L^2(\Pi^+_\theta))^4} \leq \tilde{c}_j^2(t) \exp(-2\tilde{\sigma}_j(t)\,\theta), \tag{39}$$

with $\tilde{c}_j^2 = F(0) + \dfrac{b_{j-1}}{2\alpha_{j-1} + b - \delta}$ and $\tilde{\sigma}_j = \dfrac{\delta - b}{2}$. $(23)_1$ is now an obvious consequence of (39) and of the recurence hypothesis. We obtain then from $(23)_1$, in a classical way the estimate $(23)_2$. Then, with the same technique as in [9], we get from $(23)_2$

$$|\nabla p_j^{(0)}(\xi, t)| \leq c_j(t) \exp(-\sigma_j(t)\,\xi_1) \ \forall\ \xi_1 > 1. \tag{40}$$

For any $\alpha > 1$ and $\xi_1 \in (1, \alpha)$ we have

$$p_j^{(0)}(\alpha, \xi_2, t) - p_j^{(0)}(\xi_1, \xi_2, t) \int_{\xi_1}^{\alpha} \frac{\partial p_j^{(0)}}{\partial \tau}(\tau, \xi_2, t) \mathrm{d}\tau.$$

Making $\alpha \to \infty$ and using $(11)_6$ it follows that

$$-p_j^{(0)}(\xi_1, \xi_2, t) = \int_{\xi_1}^{\infty} \frac{\partial p_j^{(0)}}{\partial \tau}(\tau, \xi_2, t) \mathrm{d}\tau.$$

Hence $|p_j^{(0)}(\xi, t)| \leq \displaystyle\int_{\xi_1}^{\infty} |\nabla p_j^{(0)}(\tau, \xi_2, t)| \mathrm{d}\tau \leq \frac{c_j(t)}{\sigma_j(t)} \exp(-\sigma_j(t)\xi_1)$ which represents the inequality (24).

Finally, $(23)_3$ is obtained as a consequence of the recurence hypothesis and the proof is achieved. □

Remark 4.1. The functions c_j, d_j, σ_j which appear in the estimates (23) depend on time.

Since we need an exponentially decay to zero of the boundary layers at infinity independent on t, we shall prove the following result:

Proposition 4.1. *For any $j \in \mathbb{N}$, there exist two positive constants $m_j < M_j$ so that $m_j < c_j(t) < M_j$, $m_j < d_j(t) < M_j$, $m_j < \sigma_j(t) < M_j$, $\forall t \in [0, T]$.*

Proof. We shall prove this property recursively with respect to j.

For $j = 0$, as we previously said, (11) represents the classical problem of [9], with t appearing as a parameter; the problem (11) becomes:

$$\begin{cases} -\mu \Delta_\xi \mathbf{u}_0^{(0)} + \nabla_\xi p_0^{(0)} = 0 \\ \mathrm{div}_\xi\ \mathbf{u}_0^{(0)} = 0 \text{ in } \Pi^+ \times (0, T), \\ \mathbf{u}_0^{(0)}(\xi_1, -\dfrac{1}{2},\ t) = \mathbf{0} \\ \mathbf{u}_0^{(0)}(\xi_1,\ \dfrac{1}{2},\ t) = \mathbf{0} \\ \mathbf{u}_0^{(0)}(0,\ \xi_2,\ t) = -u_{1,0}(0,\ \xi_2,\ t)\mathbf{e}_1 + \psi_1(\xi_2,\ t)\mathbf{e}_1, \\ \mathbf{u}_0^{(0)} \to \mathbf{0},\ p_0^{(0)} \to 0 \text{ uniformely, when } \xi_1 \to \infty, \end{cases} \tag{41}$$

We multiply (41) by $\mathbf{u}_0^{(0)}$, we integrate on Π^+_R and we denote $y(R) = \int_{\Pi^+_R} |\nabla \mathbf{u}_0^{(0)}|^2$. It follows

$$\mu y(R) + \frac{\mu}{2} \int_{-1/2}^{1/2} \frac{\partial (\mathbf{u}_0^{(0)})^2}{\partial \xi_1}(R, \xi_2) \mathrm{d}\xi_2 - \int_{-1/2}^{1/2} p_0^{(0)}(R, \xi_2)(u_0^{(0)})_1 = 0 \tag{42}$$

Integrating (42) from s to ∞ we obtain

$$\mu\int_s^\infty y(R)dR-\frac{\mu}{2}\int_s^\infty\int_{-1/2}^{1/2}\frac{\partial(\mathbf{u}_0^{(0)})^2}{\partial R}(R,\xi_2)+\int_s^\infty\int_{-1/2}^{1/2}p_0^{(0)}(u_0^{(0)})_1(R,\xi_2). \quad (43)$$

The first term of the right hand side (T_1) becomes with the condition to ∞

$$T_1=\frac{\mu}{2}\int_{-1/2}^{1/2}(\mathbf{u}_0^{(0)})^2(s,\xi_2)d\xi_2$$

and using the Poincare inequality it follows

$$T_1\le p_0\frac{\mu}{2}\int_{-1/2}^{1/2}(\nabla u_0^{(0)})^2(s,\xi_2)d\xi_2=-p_0\frac{\mu}{2}y'(s)$$

with p_0 a constant corresponding to Poincare inequality (hence independent on t).

Hence (43) becomes

$$\mu\int_s^\infty y(R)dR+p_0\frac{\mu}{2}y'(s)\le\sum_{l\ge0}\int_{D_{s,l}}p_0^{(0)}(u_0^{(0)})_1dRd\xi_2 \quad (44)$$

We consider the problem:

$$\begin{cases}\mathbf{w}_{s,l}\in(H_0^1(D_{s,l}))^2,\\ \operatorname{div}\mathbf{w}_{s,l}(u_0^{(0)})_1 \text{ in } D_{s,l},\\ \|\mathbf{w}_{s,l}\|_{(H^1(D_{s,l}))^2}\le p_1\|(u_0^{(0)})_1\|_{L^2(D_{s,l})},\end{cases} \quad (45)$$

where p_1 is independent on t.

With the same computations as those of Theorem 4.1 we get from (45)

$$\sum_{l\ge0}\int_{D_{s,l}}p_0^{(0)}(R,\xi_2)(u_0^{(0)})_1(R,\xi_2)\mathrm{d}R\mathrm{d}\xi_2\le\mu p_1y(s) \quad (46)$$

From (44) and (46) we obtain

$$y'(s)+\frac{2}{p_0}\int_s^\infty y(R)\mathrm{d}R\le2p_1y(s)$$

We define (see [9], p. 315)

$$\sigma_0=\frac{1}{2}(\sqrt{(2p_1)^2+4\cdot\frac{2}{p_0}}-2p_1),k=\frac{1}{\sigma_0}\sqrt{(2p_1)^2+4\cdot\frac{2}{p_0}}$$

and

$$c_0=ky(0).$$

It follows for $j=0$ σ_0 independent on t; c_0 depends on t due to $y(0)=\int_{\Pi^+}|\nabla u_0^{(0)}|^2$.

Since $u_0^{(0)}$ depends on t continuously ($(41)_5$) the same property holds for c_0.

We suppose that the result is true for $0,1,...,j-1$ and we prove it for j.

Due to the recurence hypothesis, to the regularity of the solutions and to the definition of the functions c_j,d_j,σ_j (which depend continuously on the functions given by the previous steps of the recurence) the proof of the proposition is achieved. □

5. **Error estimates.** In the last section we establish the error between the exact solution and the asymptotic one. Due to the boundary layer functions, the estimates obtained in this case are not as good as those for the periodic case, given in [13].

In the sequel we obtain for the general asymptotic solution $(\hat{\mathbf{u}}^{(K)},\ \hat{p}^{(K)},\ \hat{d}^{(K)})$ given by (6) a problem of the same type as (1).

For any $\gamma \geq 3$ we introduce the following notations

$$
\begin{aligned}
&\mathbf{F}_K(x_1,x_2,t)\Big(\varepsilon^{K+1}\frac{\partial u_{1,K-1}}{\partial t}+\varepsilon^{K+2}\frac{\partial u_{1,K}}{\partial t}\Big)(x_1,\frac{x_2}{\varepsilon},t)\,\mathbf{e}_1\\
&+\Big(\varepsilon^{K+1}\frac{\partial u_{2,K-2}}{\partial t}+\varepsilon^{K+2}\frac{\partial u_{2,K-1}}{\partial t}+\varepsilon^{K+3}\frac{\partial u_{2,K}}{\partial t}\Big)(x_1,\frac{x_2}{\varepsilon},t)\,\mathbf{e}_2\\
&-\mu\Big(\varepsilon^{K+1}\frac{\partial^2 u_{1,K-1}}{\partial x_1^2}+\varepsilon^{K+2}\frac{\partial^2 u_{1,K}}{\partial x_1^2}\Big)(x_1,\frac{x_2}{\varepsilon},t)\,\mathbf{e}_1\\
&-\mu\Big(\varepsilon^{K+1}\frac{\partial^2 u_{2,K-2}}{\partial x_1^2}+\varepsilon^{K+2}\frac{\partial^2 u_{2,K-1}}{\partial x_1^2}+\varepsilon^{K+3}\frac{\partial^2 u_{2,K}}{\partial x_1^2}\Big)(x_1,\frac{x_2}{\varepsilon},t)\,\mathbf{e}_2\\
&-\mu\varepsilon^{K+1}\frac{\partial^2 u_{2,K}}{\partial \xi_2^2}(x_1,\frac{x_2}{\varepsilon},t)\,\mathbf{e}_2+\varepsilon^{K+1}\frac{\partial p_K}{\partial x_1}(x_1,\frac{x_2}{\varepsilon},t)\,\mathbf{e}_1+\\
&\Big(\varepsilon^{K+1}\frac{\partial \mathbf{u}^{(0)}_{K-1}}{\partial t}+\varepsilon^{K+2}\frac{\partial \mathbf{u}^{(0)}_{K}}{\partial t}\Big)(\frac{x}{\varepsilon},t)+\Big(\varepsilon^{K+1}\frac{\partial \mathbf{u}^{(1)}_{K-1}}{\partial t}+\varepsilon^{K+2}\frac{\partial \mathbf{u}^{(1)}_{K}}{\partial t}\Big)(\frac{x-\mathbf{e}_1}{\varepsilon},t)
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
A_K(x_1,t)=&-\Big(\varepsilon^{K+4}\frac{\partial d_{K+4-\gamma}}{\partial t}+\cdots+\varepsilon^{K+\gamma}\frac{\partial d_K}{\partial t}\Big)(x_1,t)-\\
&\Big(\varepsilon^{K+3}\frac{\partial d^{(0)}_{K+3-\gamma}}{\partial t}+\cdots+\varepsilon^{K+\gamma}\frac{\partial d^{(0)}_{K}}{\partial t}\Big)(\frac{x_1}{\varepsilon},t)\\
&-\Big(\varepsilon^{K+3}\frac{\partial d^{(1)}_{K+3-\gamma}}{\partial t}+\cdots+\varepsilon^{K+\gamma}\frac{\partial d^{(1)}_{K}}{\partial t}\Big)(\frac{x_1-1}{\varepsilon},t),
\end{aligned}
\tag{48}
$$

$$
\mathbf{r}^{(0)}_K(x_2,t)=\varepsilon^{K+3}u_{2,K}(0,\frac{x_2}{\varepsilon},t)\mathbf{e}_2+\sum_{j=0}^{K}\varepsilon^{j+2}\mathbf{u}^{(1)}_j(-\frac{1}{\varepsilon},\frac{x_2}{\varepsilon},t),
\tag{49}
$$

$$
\mathbf{r}^{(1)}_K(x_2,t)=\varepsilon^{K+3}u_{2,K}(1,\frac{x_2}{\varepsilon},t)\mathbf{e}_2+\sum_{j=0}^{K}\varepsilon^{j+2}\mathbf{u}^{(0)}_j(\frac{1}{\varepsilon},\frac{x_2}{\varepsilon},t),
\tag{50}
$$

$$
\begin{aligned}
&G_K(x_1,t)=\Big(\varepsilon^{K+1}\frac{\partial^2 d_{K+1-\gamma}}{\partial t^2}+\cdots+\varepsilon^{K+\gamma}\frac{\partial^2 d_K}{\partial t^2}\Big)(x_1,t)\\
&+\nu\Big(\varepsilon^{K+1}\frac{\partial^5 d_{K+1-\gamma}}{\partial x_1^4\partial t}+\cdots+\varepsilon^{K+\gamma}\frac{\partial^5 d_K}{\partial x_1^4\partial t}\Big)(x_1,t)\\
&-\varepsilon^{K+1}p_K(x_1,\frac{1}{2},t)+\Big(\varepsilon^{K-3}\frac{\partial^2 d^{(0)}_{K-3-\gamma}}{\partial t^2}+\cdots+\varepsilon^{K+\gamma}\frac{\partial^2 d^{(0)}_{K}}{\partial t^2}\Big)(\frac{x_1}{\varepsilon},t)\\
&+\nu\Big(\varepsilon^{K-3}\frac{\partial^5 d^{(0)}_{K+1-\gamma}}{\partial \xi_1^4\partial t}+\cdots+\varepsilon^{K+\gamma-4}\frac{\partial^5 d^{(0)}_{K}}{\partial \xi_1^4\partial t}\Big)(\frac{x_1}{\varepsilon},t)\\
&-(\varepsilon^{K-3}p^{(0)}_{K-4}+\cdots+\varepsilon^{K+1}p^{(0)}_K)(\frac{x_1}{\varepsilon},\frac{1}{2},t)
\end{aligned}
\tag{51}
$$

$$+\Big(\varepsilon^{K-3}\frac{\partial^2 d^{(1)}_{K-3-\gamma}}{\partial t^2}+\cdots+\varepsilon^{K+\gamma}\frac{\partial^2 d^{(1)}_{K}}{\partial t^2}\Big)(\frac{x_1-1}{\varepsilon},t)$$
$$+\nu\Big(\varepsilon^{K-3}\frac{\partial^5 d^{(1)}_{K+1-\gamma}}{\partial \xi_1^4\partial t}+\cdots+\varepsilon^{K+\gamma-4}\frac{\partial^5 d^{(1)}_{K}}{\partial \xi_1^4\partial t}\Big)(\frac{x_1-1}{\varepsilon},t)$$
$$-(\varepsilon^{K-3}p^{(1)}_{K-4}+\cdots+\varepsilon^{K+1}p^{(1)}_{K})(\frac{x_1-1}{\varepsilon},\frac{1}{2},t),$$

$$R^{(0)}_K(t)=\sum_{j=0}^{K}\varepsilon^{j+\gamma}d^{(1)}_j(-\frac{1}{\varepsilon},t), \tag{52}$$

$$R^{(1)}_K(t)=\sum_{j=0}^{K}\varepsilon^{j+\gamma}d^{(0)}_j(\frac{1}{\varepsilon},t), \tag{53}$$

$$R'^{(0)}_K(t)=\varepsilon^{K+\gamma}\frac{\partial d_K}{\partial x_1}(0,t)+\sum_{j=0}^{K}\varepsilon^{j+\gamma-1}\frac{\partial d^{(1)}_j}{\partial \xi_1}(-\frac{1}{\varepsilon},t), \tag{54}$$

$$R'^{(1)}_K(t)=\varepsilon^{K+\gamma}\frac{\partial d_K}{\partial x_1}(1,t)+\sum_{j=0}^{K}\varepsilon^{j+\gamma-1}\frac{\partial d^{(0)}_j}{\partial \xi_1}(\frac{1}{\varepsilon},t). \tag{55}$$

The problem satisfied by the asymptotic solution of order K is the following:

$$\begin{cases}
\dfrac{\partial \hat{\mathbf{u}}^{(K)}}{\partial t}-\mu\ \Delta\hat{\mathbf{u}}^{(K)}+\nabla\hat{p}^{(K)}=f_1\mathbf{e}_1+\mathbf{F}_K \text{ in } D_\varepsilon\times(0,T),\\
\text{div } \hat{\mathbf{u}}^{(K)}=0 \text{ in } D_\varepsilon\times(0,T),\\
\hat{\mathbf{u}}^{(K)}(x_1,-\dfrac{\varepsilon}{2},t)=\mathbf{0} \text{ in } (0,1)\times(0,T),\\
\hat{\mathbf{u}}^{(K)}(x_1,\dfrac{\varepsilon}{2},t)\dfrac{\partial \hat{d}^{(K)}}{\partial t}(x_1,t)\mathbf{e}_2+A_K(x_1,t)\mathbf{e}_2 \text{ in } (0,1)\times(0,T),\\
\hat{\mathbf{u}}^{(K)}(i,x_2,t)=\varepsilon^2\psi^\varepsilon_1(x_2,t)\mathbf{e}_1+\mathbf{r}^{(i)}_K \text{ in}\Big(-\dfrac{\varepsilon}{2},\dfrac{\varepsilon}{2}\Big)\times(0,T),\ i\in\{0,1\},\\
\hat{\mathbf{u}}^{(K)}(x,0)=\mathbf{0}\ \forall\ x\in D_\varepsilon,\\
\dfrac{\partial^2\hat{d}^{(K)}}{\partial t^2}+\dfrac{1}{\varepsilon^\gamma}\ \dfrac{\partial^4\hat{d}^{(K)}}{\partial x_1^4}+\nu\dfrac{\partial^5\hat{d}^{(K)}}{\partial x_1^4\partial t}=g+G_K+\hat{p}^{(K)}/_{x_2=\varepsilon/2} \text{ on } \Gamma_\varepsilon\times(0,T),\\
\hat{d}^{(K)}(i,t)=R^{(i)}_K(t) \text{ in } (0,T),\ i\in\{0,1\},\\
\dfrac{\partial\hat{d}^{(K)}}{\partial x_1}(i,t)R'^{(i)}_K(t) \text{ in } (0,T)\ i\in\{0,1\},\\
\hat{d}^{(K)}(x_1,0)\dfrac{\partial\hat{d}^{(K)}}{\partial t}(x_1,0)=0\ \forall\ x_1\in(0,1),
\end{cases} \tag{56}$$

We notice that the boundary conditions for $\hat{\mathbf{u}}^{(K)}$ and $\hat{d}^{(K)}$ on $x_1=0,1$ and on $x_2=\varepsilon/2$ are different from those for $\mathbf{u}$ and d, respectively. In the sequel we define new functions $\hat{\mathbf{U}}^{(K)}$ and $\hat{D}^{(K)}$ which satisfy the same boundary conditions as $\mathbf{u}$ and d, respectively, on $x_1=0,1$ and on $x_2=\varepsilon/2$.

5.1. **Construction of $\hat{D}^{(K)}$.** We consider a function $\tilde{d}^{(K)} : [0,1] \times [0,T] \mapsto \mathbb{R}$, $\tilde{d}^{(K)} = a_4^K(t)x_1^4 + a_3^K(t)x_1^3 + a_2^K(t)x_1^2 + a_1^K(t)x_1 + a_0^K(t)$, with $\tilde{d}^{(K)}(0,t) = R_K^{(0)}(t)$, $\tilde{d}^{(K)}(1,t) = R_K^{(1)}(t)$, $\frac{\partial \tilde{d}^{(K)}}{\partial x_1}(0,t) = R_K'^{(0)}(t)$, $\frac{\partial \tilde{d}^{(K)}}{\partial x_1}(1,t) = R_K'^{(1)}(t)$ and

$$\int_0^1 \tilde{d}^{(K)}(x_1,t)\mathrm{d}x_1 = \int_0^1 \hat{d}^{(K)}(x_1,t)\mathrm{d}x_1.$$

It is easy to prove that there exists at least one function $\tilde{d}^{(K)}$ with the above properties. We define

$$\hat{D}^{(K)} = \hat{d}^{(K)} - \tilde{d}^{(K)}, \tag{57}$$

which satisfies the same boundary conditions as d on $x_1 = 0, 1$.

5.2. **Construction of $\hat{\mathbf{U}}^{(K)}$.** Let $\mathbf{U}_\varepsilon^{(K)} : \bar{D}_\varepsilon \times [0,T] \mapsto \mathbb{R}^2$ a solution of the following problem:

$$\begin{cases} \mathbf{U}_\varepsilon^{(K)}(t) \in (H^1(D_\varepsilon))^2, \\ \text{div } \mathbf{U}_\varepsilon^{(K)}(t) = 0 \text{ in } D_\varepsilon, \\ \mathbf{U}_\varepsilon^{(K)}(x_1, -\dfrac{\varepsilon}{2}, t) = 0, \text{ in } (0,1)\times(0,T), \\ \mathbf{U}_\varepsilon^{(K)}(x_1, \dfrac{\varepsilon}{2}, t)\dfrac{\partial \tilde{d}^{(K)}}{\partial t}(x_1,t)\mathbf{e}_2 + A_K(x_1,t)\mathbf{e}_2, \text{ in } (0,1)\times(0,T), \\ \mathbf{U}_\varepsilon^{(K)}(i, x_2, t) = \mathbf{r}_K^{(i)}, \text{ in } \left(\dfrac{-\varepsilon}{2}, \dfrac{\varepsilon}{2}\right)\times(0,T),\ i \in \{0,1\}. \end{cases} \tag{58}$$

We can prove that

Proposition 5.1. *The problem (58) has at least a solution, with the property*

$$\|\mathbf{U}_\varepsilon^{(K)}(t)\|_{H^1(D_\varepsilon))^2} = O(\varepsilon^{K+3/2}). \tag{59}$$

Proof. The existence of a solution of (58) is a consequence of the choice of $\tilde{d}^{(K)}$. We define $\boldsymbol{\eta}_\varepsilon^{(K)} : D \times (0,T) \mapsto \mathbb{R}^2$, where $D = (0,1)\times(-1/2,1/2)$, by $\boldsymbol{\eta}_\varepsilon^{(K)}(x_1,y_2,t) = (U_\varepsilon^{(K)})_1(x_1,x_2,t)\mathbf{e}_1 + \dfrac{1}{\varepsilon}(U_\varepsilon^{(K)})_2(x_1,x_2,t)\mathbf{e}_2$, with $(y_1,y_2) = (x_1, \dfrac{x_2}{\varepsilon})$.

Obvious computations lead to the following problem for $\boldsymbol{\eta}_\varepsilon^{(K)}(t)$:

$$\begin{cases} \text{div}_y \boldsymbol{\eta}_\varepsilon^{(K)}(t) = 0 \text{ in } D, \\ \boldsymbol{\eta}_\varepsilon^{(K)}(y_1, -\dfrac{1}{2}, t) = \mathbf{0}, \\ \boldsymbol{\eta}_\varepsilon^{(K)}(y_1, \dfrac{1}{2}, t)\dfrac{1}{\varepsilon}\Big(\dfrac{\partial \tilde{d}^{(K)}}{\partial t}(x_1,t) + A_K(x_1,t)\Big)\mathbf{e}_2, \\ \boldsymbol{\eta}_\varepsilon^{(K)}(i, y_2, t)(r_K^{(i)})_1(y_2,t)\mathbf{e}_1 + \dfrac{1}{\varepsilon}(r_K^{(i)})_2(y_2,t)\mathbf{e}_2,\ i \in \{0,1\}. \end{cases}$$

As in [10] we can prove that there exists a function $\boldsymbol{\eta}_\varepsilon^{(K)}(t) \in (H^1(D))^2$ so that $\|\boldsymbol{\eta}_\varepsilon^{(K)}(t)\|_{(H^1(D))^2} \le C(D)\|\boldsymbol{\eta}_\varepsilon^{(K)}(t)\|_{(H^{1/2}(\partial D))^2}$.

Using the properties of the boundary layer correctors it follows that

$$\|\boldsymbol{\eta}_\varepsilon^{(K)}(t)\|_{(H^1(D))^2} = O(\varepsilon^{K+2}). \tag{60}$$

Standard computations give $\|\mathbf{U}_\varepsilon^{(K)}(t)\|_{(H^1(D_\varepsilon))^2} \leq \frac{1}{\varepsilon^{1/2}}\|\boldsymbol{\eta}_\varepsilon^{(K)}(t)\|_{(H^1(D))^2}$ and combining these two inequalities the proof is complete. □

The function

$$\hat{\mathbf{U}}^{(K)} = \hat{\mathbf{u}}^{(K)} - \mathbf{U}_\varepsilon^{(K)}, \tag{61}$$

satisfies the same boundary conditions as $\mathbf{u}$ on $x_1 = 0, 1$ and on $x_2 = \varepsilon/2$. The problem for the new functions $\hat{\mathbf{U}}^{(K)}$, $\hat{p}^{(K)}$, $\hat{D}^{(K)}$ is an obvious consequence of (56), (57) and (58):

$$\begin{cases}
\dfrac{\partial \hat{\mathbf{U}}^{(K)}}{\partial t} - \mu\ \Delta \hat{\mathbf{U}}^{(K)} + \nabla \hat{p}^{(K)} = f_1\mathbf{e}_1 + \mathbf{F}_K - \dfrac{\partial \mathbf{U}_\varepsilon^{(K)}}{\partial t} + \mu\ \Delta \mathbf{U}_\varepsilon^{(K)}, \\
\text{div } \hat{\mathbf{U}}^{(K)} = 0 \text{ in } D_\varepsilon \times (0,T), \\
\hat{\mathbf{U}}^{(K)}(x_1, -\dfrac{\varepsilon}{2}, t) = \mathbf{0} \text{ in } (0,1)\times(0,T), \\
\hat{\mathbf{U}}^{(K)}(x_1, \dfrac{\varepsilon}{2}, t)\dfrac{\partial \hat{D}^{(K)}}{\partial t}(x_1,t)\mathbf{e}_2 \text{ in } (0,1)\times(0,T), \\
\hat{\mathbf{U}}^{(K)}(i, x_2, t) = \varepsilon^2\psi_1^\varepsilon(x_2,t)\mathbf{e}_1,\ i \in \{0,1\}, \\
\hat{\mathbf{U}}^{(K)}(x,0) = \mathbf{0}\ \forall\ x \in D_\varepsilon, \\
\dfrac{\partial^2 \hat{D}^{(K)}}{\partial t^2} + \dfrac{1}{\varepsilon^\gamma}\dfrac{\partial^4 \hat{D}^{(K)}}{\partial x_1^4} + \nu\dfrac{\partial^5 \hat{D}^{(K)}}{\partial x_1^4 \partial t} = g + G_K + \hat{p}^{(K)} \\
\qquad\qquad - \dfrac{\partial^2 \tilde{d}^{(K)}}{\partial t^2} - \dfrac{1}{\varepsilon^\gamma}\dfrac{\partial^4 \tilde{d}^{(K)}}{\partial x_1^4} - \nu\dfrac{\partial^5 \tilde{d}^{(K)}}{\partial x_1^4 \partial t} \text{ on } \Gamma_\varepsilon \times (0,T), \\
\hat{D}^{(K)}(0,t) = \hat{D}^{(K)}(1,t) = 0,\ \text{ in } (0,T), \\
\dfrac{\partial \hat{D}^{(K)}}{\partial x_1}(0,t)\dfrac{\partial \hat{D}^{(K)}}{\partial x_1}(1,t) = 0,\ \text{ in } (0,T), \\
\hat{D}^{(K)}(x_1,0)\dfrac{\partial \hat{D}^{(K)}}{\partial t}(x_1,0) = 0\ \forall\ x_1 \in (0,1).
\end{cases} \tag{62}$$

The next theorem, which represents the main result of this section, will give the error between the exact solution of problem (1) and the asymptotic solution of order K, given by (6).

Theorem 5.1. *Let* $(\hat{\mathbf{u}}^{(K)},\ \hat{p}^{(K)},\ \hat{d}^{(K)})$ *be the asymptotic solution given by (6) and* $(\mathbf{u},\ p,\ d)$ *the exact solution of the physical problem. Then the following estimates*

hold:

$$\begin{cases} \|\mathbf{u}-\hat{\mathbf{u}}^{(K)}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)}\begin{cases} O(\varepsilon^{3/2}) \text{ for } K\in\{0,1,2,3\},\\ O(\varepsilon^{\min\{K-5/2,K+4-\gamma\}}) \text{ for } K\geq 4\end{cases}\\ \|\nabla(\mathbf{u}-\hat{\mathbf{u}}^{(K)})\|_{L^2(0,T;(L^2(D_\varepsilon))^4)}\begin{cases} O(\varepsilon^{3/2}) \text{ for } K\in\{0,1,2,3\},\\ O(\varepsilon^{\min\{K-5/2,K+4-\gamma\}}) \text{ for } K\geq 4\end{cases}\\ \Big\|\dfrac{\partial}{\partial t}(d-\hat{d}^{(K)})\Big\|_{L^\infty(0,T;(L^2(0,1))}\begin{cases} O(\varepsilon^{3/2}) \text{ for } K\in\{0,1,2,3\},\\ O(\varepsilon^{\min\{K-5/2,K+4-\gamma\}}) \text{ for } K\geq 4\end{cases}\\ \Big\|\dfrac{\partial^3}{\partial x_1^2\partial t}(d-\hat{d}^{(K)})\Big\|_{L^2((0,1)\times(0,T))}\begin{cases} O(\varepsilon^{3/2}) \text{ for } K\in\{0,1,2,3\},\\ O(\varepsilon^{\min\{K-5/2,K+4-\gamma\}}) \text{ for } K\geq 4\end{cases}\\ \Big\|\dfrac{\partial^2}{\partial x_1^2}(d-\hat{d}^{(K)})\Big\|_{L^\infty(0,T;(L^2(0,1))}\begin{cases} O(\varepsilon^{3/2+\gamma/2}) \text{ for } K\in\{0,1,2,3\},\\ O(\varepsilon^{\min\{K-5/2,K+4-\gamma\}+\gamma/2}) \text{ for } K\geq 4\end{cases}\\ \|\nabla(p-\hat{p}^{(K)})\|_{L^2(0,T;(H^{-1}(D_\varepsilon))^2)}\begin{cases} O(\varepsilon^{5/2-\gamma/2}) \text{ for } K\in\{0,1,2,3\},\\ O(\varepsilon^{\min\{K-3/2,K+5-\gamma\}-\gamma/2}) \text{ for } K\geq 4\end{cases}\end{cases} \tag{63}$$

Proof. We establish the estimate $(63)_2$. From (61) and (59) it follows that

$$\|\nabla(\mathbf{u}-\hat{\mathbf{u}}^{(K)})\|_{L^2(0,T;(L^2(D_\varepsilon))^4)}\leq \|\nabla(\mathbf{u}-\hat{\mathbf{U}}^{(K)})\|_{L^2(0,T;(L^2(D_\varepsilon))^4)}+O(\varepsilon^{K+3/2}).$$

An estimate for the first term of the right hand side of the previous inequality is given by $(4)_2$ if we compute $\|\mathbf{F}_K-\dfrac{\partial\mathbf{U}_\varepsilon^{(K)}}{\partial t}+\mu\ \Delta\ \mathbf{U}_\varepsilon^{(K)}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)}$ and $\|G_K-\dfrac{\partial^2\tilde{d}^{(K)}}{\partial t^2}-\dfrac{1}{\varepsilon^\gamma}\dfrac{\partial^4\tilde{d}^{(K)}}{\partial x_1^4}-\nu\dfrac{\partial^5\tilde{d}^{(K)}}{\partial x_1^4\partial t}\|_{L^2((0,1)\times(0,T))}$.

Taking into account the definitions of $\mathbf{F}_K, G_K, \tilde{d}^{(K)}, \mathbf{U}_\varepsilon^{(K)}$ and (59) it follows that the order of error is given by $\min\{\|G_K\|_{L^2((0,1)\times(0,T))},\dfrac{1}{\varepsilon^\gamma}\Big\|\dfrac{\partial^4\tilde{d}^{(K)}}{\partial x_1^4}\Big\|_{L^2((0,1)\times(0,T))}\}$.

We estimate first the previous norm for $K\in\{0,1,2,3\}$. For this purpose we compute $\displaystyle\int_0^T\int_0^1\Big(p_0^{(0)}\Big(\frac{x_1}{\varepsilon},\frac{1}{2},t\Big)\Big)^2\mathrm{d}x_1\mathrm{d}t=\varepsilon\int_0^T\int_0^{1/\varepsilon}\Big(p_0^{(0)}\Big(\xi_1,\frac{1}{2},t\Big)\Big)^2\mathrm{d}\xi_1\mathrm{d}t=$ $\displaystyle\varepsilon\Big(\int_0^T\int_0^1\Big(p_0^{(0)}\Big(\xi_1,\frac{1}{2},t\Big)\Big)^2\mathrm{d}\xi_1\mathrm{d}t+\int_0^T\int_1^{1/\varepsilon}\Big(p_0^{(0)}\Big(\xi_1,\frac{1}{2},t\Big)\Big)^2\mathrm{d}\xi_1\mathrm{d}t\Big)$.

Taking into account the estimate (24) we get $\displaystyle\Big(\int_0^T\int_0^1\Big(p_0^{(0)}\Big(\frac{x_1}{\varepsilon},\frac{1}{2},t\Big)\Big)^2\mathrm{d}x_1\mathrm{d}t\Big)^{1/2}$ $=O(\varepsilon^{1/2})$; hence $\|G_K\|_{L^2((0,1)\times(0,T))}=O(\varepsilon^{3/2})$.

Similar computations give, for $K\geq 4$ $\|G_K\|_{L^2((0,1)\times(0,T))}=O(\varepsilon^{K-5/2})$.

From the definition of the boundary layer functions we get $\tilde{d}^{(K)}=0$ for $K\leq 3$. Moreover, $\|\frac{\partial^4\tilde{d}^{(K)}}{\partial x_1^4}\|_{L^2((0,1)\times(0,T))}=O(\varepsilon^{\min\{K+4,K+\gamma\}})$ for $K\geq 4$; hence the estimate $(63)_2$ was established.

The estimates $(63)_{1,3,4,5}$ are proved with the same technique.

The proof is achieved if we obtain the last estimate, corresponding to the pressure. For this purpose, we introduce the notation $(\mathbf{U}^K,D^K,P^K)(\hat{\mathbf{U}}^K,\hat{D}^K,\hat{p}^K)-(\mathbf{u},d,p)$, we write the problem corresponding to $(\mathbf{U}^K,D^K,P^K)$ and we multiply the Stokes

equation for $(\mathbf{U}^K, P^K)$ by $\dfrac{\partial \mathbf{U}^K}{\partial t}$ integrating the result on D_ε and the Sophie-Germain equation for D^K by $\dfrac{\partial^2 D^K}{\partial t^2}$ integrating it on (0,1). Adding these two computations, we obtain, as in [13], the estimate

$$\left\|\frac{\partial \mathbf{U}^K}{\partial t}\right\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} \begin{cases} O(\varepsilon^{3/2-\gamma/2}) \text{ for } K \in \{0,1,2,3\}, \\ O(\varepsilon^{\min\{K-5/2, K+4-\gamma\}-\gamma/2}) \text{ for } K \geq 4, \end{cases}$$

which yields the desired estimate for the pressure. For more details concerning the estimate for the pressure we refer to [13]. □

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