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ON THE CONJUGATE OF PERIODIC PIECEWISE HARMONIC FUNCTIONS

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Abstract. The paper considers the conjugate of periodic functions which are piecewise harmonic. In particular, we consider the harmonic conjugate of the solution of the problem of stationary heat conduction through a periodic network of fibres and matrix of arbitrary shape. A numerical example is also presented.

1. **Introduction.** Many physical problems like heat conduction, electrical fields, stress distribution and fluid flow in composite materials and porous media involve the determination of periodic piecewise harmonic functions (see e.g. the books [5], [7] and [17]). In two dimensional problems much of the analysis of such problems is often significantly simplified by using analytic functions whose real parts are the sought harmonic functions. Examples of the use of such techniques related to heat conduction or electrical fields can be found e.g. in [3], [12], [15] and [16]. Similarly, in periodic elasticity problems the so-called Kolosov-Muskhelishvili analytic potentials $\varphi(z)$ and $\psi(z)$ can be used to determine the local stress tensor $\sigma(z)$ from the equations

$$\sigma_{xx}(z) + \sigma_{yy}(z) = 4\mathrm{Re}\,\varphi^{'}(z),$$
$$\sigma_{yy}(z) - \sigma_{xx}(z) = 2\mathrm{Re}\,[\overline{z}\varphi^{''}(z) + \psi^{'}(z)],$$
$$\sigma_{xy}(z) = \mathrm{Im}\,[\overline{z}\varphi^{''}(z) + \psi^{'}(z)].$$

For more information on the Kolosov-Muskhelishvili potentials, see e.g. [10], [19] and [20] and the references given therein. In all these problems it is helpful and sometime necessary to characterize the analytic functions involved in order to find the solutions to our problems. For example, if the real part is periodic we like to know whether the harmonic conjugate (i.e. the imaginary part) inherits this property. Moreover, from elementary courses in engineering mathematics it is known

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that in many cases the harmonic conjungate in itself provides important physical information.

The complex-variable technique, which is particularly popular in mathematical physics, is mostly based on standard results from Complex Analysis which where known before 1900, and much of the literature in this area never utilizes Hilbert Space techniques or Distribution Theory. This often makes it necessary to put strong smoothness requirements on the outer boundary and boundaries which separate the materials involved in the problems. In cases where the smoothness requirements are relaxed, the discussion concerning existence of solutions or regularity of solutions usually becomes more vague, or at least not related to the powerful tool of regularity theory developed in the 20th century. On the other hand, and maybe because of this, mathematicians working with composite materials and porous media from a more Functional Analysis point of view, seem to be unfamiliar with the potential of the complex-variable technique. Hence, there are reasons to believe that it could be fruitful to have a closer connection between these communities.

This paper is intended to be one step further in this direction. Therefore, in order to make the presentation readable to a broader audience we have made an effort to make the paper as self-contained as possible. Moreover, some of the results are even proved in two different ways by presenting proofs based on elements from duality theory and other proofs which are more based on standard complex analysis (see in particular Theorem 4.1 and Theorem 5.3). In order to focus on the main ideas we only consider the simplest possible case, namely the problem of stationary heat conduction through a periodic network of fibres and matrix of arbitrary shape. In particular, we present a link between the harmonic conjugate of the temperature, which is piecewise harmonic, and the solution of the corresponding dual problem. Since we aim to keep an interdisciplinary character of this paper we also present a computational example based on the finite element method which illustrates our theoretical results.

The paper is organized as follows. In Section 2 we prove a representation result for the harmonic conjugate. Section 3 is devoted to a discussion of the model problem. In Section 4 and 5 we prove some results on the harmonic conjugate of the solution of the model problem. Finally, in Section 6 we consider a computational example.

2. **Representation of the harmonic conjugate.** Let Y denote a rectangle $Y = \langle 0, t_1\rangle \times \langle 0, t_2\rangle$, and let S_2 denote an Y-periodic open subset of R^2 surrounding one single hole (a simply connected open set) in each period which is strictly contained within the period.

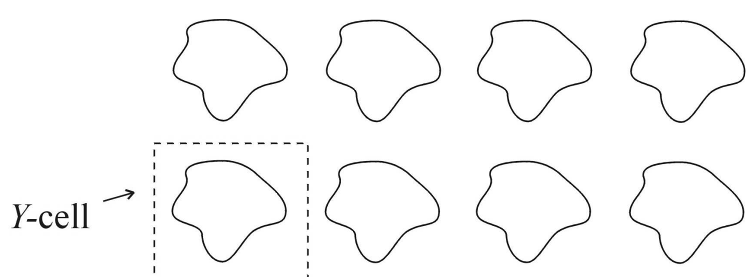

FIGURE 1. The set S_1 (the union of holes) and the set S_2 (the remaining connected set).

The union of all holes is denoted S_1 (see Figure 1).

Theorem 2.1. *Let $\varphi(z) = u(x,y) + iv(x,y)$ be an analytic function in S_2 and assume that $u = \operatorname{Re}\varphi$ is Y-periodic. Then v is of the form $v(x,y) = xr+ys+h(x,y)$, where r and s are constants and $h(x,y)$ is a Y-periodic function.*

Proof. Due to the periodicity of $\partial u/\partial x$ and $\partial u/\partial y$ in the x-direction and y-direction, respectively, we obtain from the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

that

$$\frac{\partial}{\partial x} v(x,y) - \frac{\partial}{\partial x} v(x+t_1,y) = 0, \tag{1}$$

$$\frac{\partial}{\partial y} v(x,y) - \frac{\partial}{\partial y} v(x+t_1,y) = 0, \tag{2}$$

and

$$\frac{\partial}{\partial x} v(x,y) - \frac{\partial}{\partial x} v(x,y+t_2) = 0, \tag{3}$$

$$\frac{\partial}{\partial y} v(x,y) - \frac{\partial}{\partial y} v(x,y+t_2) = 0. \tag{4}$$

In other words, $\operatorname{grad} f = \operatorname{grad} g = 0$, where $f(x,y) = v(x,y) - v(x+t_1,y)$ and $g(x,y) = v(x,y+t_2) - v(x,y)$. Hence, $v(x+t_1,y) - v(x,y) = t_1 r$ and $v(x,y+t_2) - v(x,y) = t_2 s$ for some constants r and s. These two identities imply that $v(x,y) = xr + ys + h(x,y)$ for some Y-periodic function $h(x,y)$. In order to see this, just put $h(x,y) = v(x,y) - xr - ys$ and observe that $h(x,y) - h(x+t_1,y) = 0$ and $h(x,y) - h(x,y+t_2) = 0$, i.e. $h(x,y) = h(x+t_1,y) = h(x,y+t_2)$, which shows that $h(x,y)$ is Y-periodic. This completes the proof. □

It is natural to ask whether the conclusion of the above theorem could be improved. With the theory of elliptic functions in mind (i.e. doubly periodic meromorphic functions) one might expect that an analytic function is periodic if its real part is periodic, in other words that the constants r and s always vanish. However, the following result shows that this is not the case.

Proposition 1. *For any pair of constants r and s there exists a function $\varphi(z) = u(x,y) + iv(x,y)$ which is analytic in S_2, $u = \operatorname{Re}\varphi$ is Y-periodic and v is of the form $v(x,y) = xr + ys + h(x,y)$ where $h(x,y)$ is Y-periodic.*

The above result will be proved in Section 4.

3. **A periodic boundary value problem.** Before we prove Proposition 1 we will first discuss a periodic boundary value problem associated with a function $\lambda = \lambda(x,y)$ (the conductivity function) defined by

$$\lambda(x,y) = \begin{cases} \lambda_1 & \text{on } S_1, \\ \lambda_2 & \text{on } S_2, \end{cases}$$

for some fixed strictly positive constants λ_1 and λ_2. Let ∂S_1 denote the boundary separating the sets S_1 and S_2 and let n denote the outward unit-normal vector of ∂S_1. The following problem admits a weak solution u which is unique within a positive constant in the space of Y-periodic members of the Sobolev space $W^{1,2}_{\text{loc}}(R^2)$

(which can be identified with the space $W_{\text{per}}(Y)$ = the completion of the space of smooth Y-periodic functions in the Sobolev space $W^{1,2}(Y)$):

$$\begin{cases} \text{div } (\lambda(x,y)\text{grad } (u(x,y)+x\xi_1+y\xi_2))=0, & (x,y)\in S_1\cup S_2, \\ \dfrac{\partial\,(u(x,y)+x\xi_1+y\xi_2)}{\partial n} \text{ continuous across } \partial S_1. \end{cases}$$

Here, $\partial/\partial n = n\cdot\text{grad}$ denotes the normal derivative and $\xi=(\xi_1,\xi_2)\in R^2$. It is not always possible to find solutions which satisfy the above continuity condition on the normal derivative across ∂S_1 in classical sense. In fact, unless ∂S_1 is sufficiently smooth (e.g. Lipschitz continuous) it is not even possible to define the normal n to this boundary in a reasonable way. Thus we are lead to the study of the corresponding weak formulation, which takes the following form: Find $u\in W_{\text{per}}(Y)$ such that

$$\int_Y \lambda\,(\text{grad}\,u+\xi)\cdot\text{grad}\,\varphi\;dxdy=0, \tag{5}$$

for all $\varphi\in W_{\text{per}}(Y)$, where $\int_Y=\int_0^{t_1}\int_0^{t_2}$. The existence and uniqueness of this problem (up to an arbitrary constant) can be proved by using the Lax-Milgram Theorem (see e.g. [5]).

4. The direct approach .

Theorem 4.1. *The solution u of the problem (5) is harmonic in the regions S_1 and S_2 separately. The harmonic conjugate v of u is single-valued in both regions. In particular, in S_2 it is of the form $v(x,y)=xr+ys+h(x,y)$ for some Y-periodic function $h(x,y)$ and some constants r and s given by*

$$\begin{bmatrix} s \\ -r \end{bmatrix} = \frac{1}{\lambda_2}(q-\lambda_2 I)\xi.$$

Here, $q=\{q_{ij}\}$ is the matrix given by

$$q_{ij}=\frac{1}{|Y|}\int_Y \lambda(x,y)\left(\frac{\partial u_j}{\partial z_i}+\delta_{ij}\right)\,dxdy, \qquad z_1=x,\quad z_2=y, \tag{6}$$

where u_1 and u_2 are the solutions of (5) in the special cases $\xi_1=1$, $\xi_2=0$ and $\xi_1=0$, $\xi_2=1$, respectively. Moreover, $q-\lambda_2 I$ is non-singular if $\lambda_1\neq\lambda_2$.

Proof. As indicated above, functions in $W_{\text{per}}(Y)$ can be extended Y-periodically to $W^{1,2}_{\text{loc}}(R^2)$ (for the proof, see e.g. [5]). For any point $(x_0,y_0)\in S_i$ the restriction of this extended function to $Y'=Y+(x_0,y_0)$ is then a member of $W_{\text{per}}(Y')$ and this restriction is certainly a solution of (5) with Y replaced by Y'. In particular this implies that

$$\int_D \lambda_i\left(\left(\frac{\partial u}{\partial x}+\xi_1\right)\frac{\partial\varphi}{\partial x}+\left(\frac{\partial u}{\partial y}+\xi_2\right)\frac{\partial\varphi}{\partial y}\right)\,dxdy=0$$

for all $\varphi\in W_0^{1,2}(D)$ where $D\subset S_i$ is a disk with centre at (x_0,y_0). Noting that by the generalized Gauss Theorem (valid for functions $\varphi\in W^{1,1}(D)$ and Lipschitz continuous boundary ∂D, see e.g. [2]),

$$\int_D \partial\varphi/\partial x\;dx=\int_{\partial D}\varphi n_1\;ds=0,\quad \int_D \partial\varphi/\partial y\;dx=\int_{\partial D}\varphi n_2\;ds=0, \tag{7}$$

(where again $n = (n_1, n_2)$ denotes the outward unit-normal of the boundary ∂D of D), we obtain that

$$\int_D \left(\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial y} \right) \, dxdy = 0$$

for all $\varphi \in W_0^{1,2}(D)$. Thus, u is a generalized solution of the Laplace equation in D, and by standard regularity results (see e.g. [6] Theorem 8.8 which states a much more general result), this gives that u is a solution of the Laplace equation in D in the classical sense. Hence, since the center (x_0, y_0) of D was chosen arbitrarily in S_i, we conclude that u is harmonic in the regions S_1 and S_2 separately.

Now, consider two disjoint intervals $I(x_1, r_1)$ and $I(x_2, r_2)$ in $\langle 0, t_1 \rangle$ of lengths $2r_1$ and $2r_2$ and with centers at some fixed points x_1 and x_2, respectively. Moreover, let $\varphi = \varphi(x)$ be a continuous periodic function of x defined in $\langle 0, t_1 \rangle$ by

$$\varphi'(x) = \begin{cases} s_1 & x \in \ I(x_1, r_1), \\ s_2 & x \in \ I(x_2, r_2), \\ 0 & \text{elsewhere}, \end{cases}$$

where s_1 and s_2 are constants satisfying the condition

$$r_1 s_1 + r_2 s_2 = 0 \tag{8}$$

(by this condition φ becomes $\langle 0, t_1 \rangle$-periodic). Putting

$$b_2(x) = \int_0^{t_2} \lambda(x, y) \left(\frac{\partial u}{\partial x} + \xi_1 \right) \, dy$$

and inserting φ into (5) we obtain that

$$s_1 \int_{I(x_1, r_1)} b_2(x) dx + s_2 \int_{I(x_2, r_2)} b_2(x) dx = 0.$$

Using (8) we now find that

$$\frac{1}{2r_1} \int_{I(x_1, r_1)} b_2(x) dx = \frac{1}{2r_2} \int_{I(x_2, r_2)} b_2(x) dx,$$

i.e.

$$\frac{1}{|I(x_1, r_1)|} \int_{I(x_1, r_1)} b_2(x) dx = \frac{1}{|I(x_2, r_2)|} \int_{I(x_2, r_2)} b_2(x) dx.$$

Since the intervals were chosen arbitrarily, this shows that the average value of $b_2(x)$ taken over any interval is equal to a constant k_2. In particular this gives that

$$\lim_{r \to 0} \frac{1}{|I(x_0, r)|} \int_{I(x_0, r)} b_2(x) dx = k_2,$$

at all points $x_0 \in \langle 0, t_1 \rangle$. According to Lebesgue differentiation theorem, almost all points in $\langle 0, t_1 \rangle$ are Lebesgue-points, i.e. points x_0 for which the above limit equals $b_2(x_0)$. Hence, $b_2(x) = k_2$ a.e., and since $\lambda(0, y) = \lambda_2$ (because $S_1 \cap Y$- is strictly contained in Y) this implies that

$$\int_Y \lambda(x, y) \left(\frac{\partial u}{\partial x} + \xi_1 \right) \, dxdy = t_1 b_2(0) = t_1 \int_0^{t_2} \lambda(0, y) \left(\frac{\partial u(0, y)}{\partial x} + \xi_1 \right) \, dy$$

$$= t_1 \lambda_2 \int_0^{t_2} \left(\frac{\partial u(0, y)}{\partial x} + \xi_1 \right) \, dy = t_1 \lambda_2 \int_0^{t_2} \frac{\partial u(0, y)}{\partial x} \, dy + t_1 t_2 \lambda_2 \xi_1. \tag{9}$$

Similarly, we can prove that $b_1(y) = k_1$ for some constant k_1, where

$$b_1(y) = \int_0^{t_1} \lambda(x,y)\left(\frac{\partial u}{\partial y} + \xi_2\right)\, dx,$$

and that

$$\int_Y \lambda(x,y)\left(\frac{\partial u}{\partial y} + \xi_2\right)\, dxdy = t_2\lambda_2 \int_0^{t_1} \frac{\partial u(x,0)}{\partial y}\, dx + t_1 t_2 \lambda_2 \xi_2. \tag{10}$$

Due to linearity it is easy to check that $u = \xi_1 u_1 + \xi_2 u_2$ is the solution of the general problem. Hence,

$$\frac{1}{|Y|}\int_Y \lambda\left(\operatorname{grad} u + \xi\right)\, dxdy = q\xi. \tag{11}$$

By formulating the equivalent minimum formulation of (5) it is easy to verify that

$$\xi \cdot q\xi = \min_{\varphi \in W_{\mathrm{per}}(Y)} \frac{1}{|Y|}\int_Y \lambda \left|\operatorname{grad}\varphi + \xi\right|^2\, dxdy. \tag{12}$$

Moreover, since φ is Y-periodic, $\int_Y \operatorname{grad}\varphi\, dx = 0$. Accordingly,

$$|Y|\,|\xi| = \left|\int_Y \left(\operatorname{grad}\varphi + \xi\right)\, dxdy\right| \le \int_Y \left|\left(\operatorname{grad}\varphi + \xi\right)\right|\, dxdy. \tag{13}$$

In addition, according to the Schwarz inequality,

$$\int_Y \lambda^{-\frac{1}{2}}\lambda^{\frac{1}{2}}\left|\left(\operatorname{grad}\varphi + \xi\right)\right|\, dxdy \le \left(\int_Y \lambda^{-1}\, dxdy\right)^{\frac{1}{2}}\left(\int_Y \lambda\left|\left(\operatorname{grad}\varphi + \xi\right)\right|^2\, dxdy\right)^{\frac{1}{2}}.$$

Hence, (12) and (13) gives

$$q_h |\xi|^2 \le \xi \cdot q\xi,$$

where

$$q_h = \left(\frac{1}{|Y|}\int_Y \lambda^{-1}\, dxdy\right)^{-1}.$$

In addition, putting $\varphi = 0$ into (12) we find that

$$\xi \cdot q\xi \le q_a |\xi|^2,$$

where

$$q_a = \frac{1}{|Y|}\int_Y \lambda\, dxdy.$$

Thus,

$$q_h |\xi|^2 \le \xi \cdot q\xi \le q_a |\xi|^2. \tag{14}$$

By (9), (10) and (11) we find that

$$\xi_1 q_{11} + \xi_2 q_{12} = \lambda_2 b_1 + \lambda_2 \xi_1,$$
$$\xi_1 q_{21} + \xi_2 q_{22} = \lambda_2 b_2 + \lambda_2 \xi_2,$$

where

$$b_1 = \frac{1}{t_2}\int_0^{t_2} \frac{\partial u(0,y)}{\partial x} dy, \quad b_2 = \frac{1}{t_1}\int_0^{t_1} \frac{\partial u(x,0)}{\partial y}\, dx, \tag{15}$$

i.e.

$$(q - \lambda_2 I)\xi = \lambda_2 b.$$

It is easy to see that $\min(\lambda_1, \lambda_2) < q_h < q_a < \max(\lambda_1, \lambda_2)$ if $\lambda_1 \ne \lambda_2$. Hence, the inequalities

$$(q_h - \lambda_2)|\xi|^2 \le \xi \cdot (q - \lambda_2 I)\xi \le (q_a - \lambda_2)|\xi|^2,$$

(deduced from (14), shows that $q - \lambda_2 I$ is either positive definite (if $\lambda_1 > \lambda_2$) or negative definite (if $\lambda_1 < \lambda_2$). This implies the non-singularity of the matrix $q - \lambda_2 I$.

Let M_0M denote an arbitrary smooth path which does not leave S_2 and connects some fixed point $M_0 = (x_0, y_0)$ with the variable point $M = (x, y)$. Moreover, let n be the normal towards the right of the path from M_0 to M. Then the harmonic conjugate v of u in S_2 is given by

$$v(x,y) = \int_{M_0M} \frac{du}{dn} ds + C, \tag{16}$$

where ds is the arc element of the path of integration and C is some arbitrary constant (this result can be found in most books on complex analysis, see e.g. Muskhelishvili [14], Appendix 3). Another well known result is that v is single-valued in any simply connected domain, hence in S_1. In order to show that v is single-valued in S_2, it is obviously enough to verify that

$$v(x,y) - v(x_0, y_0) = \int_{M_0M} \frac{du}{dn} ds = 0$$

for every simple closed contour M_0M in S_2. Note that the integral $\int_{M_0M}$ can be represented as a sum of integrals

$$\int_{M_0M} = \sum_{k=1}^{K} \int_{\Upsilon_k} + \sum_{k=1}^{N} \int_{\Gamma_k},$$

where Υ_k is a simple closed contour surrounding only points contained in S_2 and Γ_k is a simple closed contour contained in one period $Y + (nt_1, mt_2)$ (where n and m are integers) and surrounding the corresponding hole in that period. Due to the fact that v is single-valued in any simply connected domain, we easily see that

$$\int_{\Upsilon_k} \frac{du}{dn} ds = 0.$$

It remains to prove that the integral along Γ_k also vanishes. For simplicity, let $\Gamma_k \subset Y$ and let $\Gamma_k(w)$ denote a simple closed contour surrounding Γ_k at a constant distance w from Γ_k. We certainly assume that w is strictly less than the radius of all circles with centers outside Γ_k defining the curvature on Γ_k, otherwise such contours will be impossible to construct. Let $S \subset S_2$ be the strip between Γ_k and $\Gamma_k(w_0)$ for some fixed w_0, and let $\varphi \in W_{\text{per}}(Y)$ be given by $\varphi = 0$ at all points inside Γ_k, $\varphi = w$ on $\Gamma_k(w)$ for $0 \leq w \leq w_0$ and $\varphi = w_0$ at all points in Y outside $\Gamma_k(w_0)$. Obviously, $\operatorname{grad} \varphi = 0$ in $Y \backslash S$. Moreover, $\operatorname{grad} \varphi = n$, the outward unit normal vector on $\Gamma_k(w)$, for $0 \leq w \leq w_0$. Inserting φ into (5) and making use of (7) (with D replaced by S), we therefore obtain

$$\int_S \frac{\partial u}{\partial n}\ dxdy = \int_S (\operatorname{grad} u \cdot n)\ dxdy = \int_S (\operatorname{grad} u \cdot \operatorname{grad} \varphi)\ dxdy = 0. \tag{17}$$

The curve $\Gamma_k - \Gamma_k(w)$ may be written as a sum of two closed contours surrounding only points contained in S_2. Similarly as above we therefore find that (integrating in counter-clockwise direction)

$$\int_{\Gamma_k} \frac{du}{dn} ds - \int_{\Gamma_k(w)} \frac{du}{dn} ds = \int_{\Gamma_k - \Gamma_k(w)} \frac{du}{dn} ds = 0,$$

i.e.

$$\int_{\Gamma_k} \frac{du}{dn} ds = \int_{\Gamma_k(w)} \frac{du}{dn} ds$$

for all $0 \le w \le w_0$. Thus,

$$\int_S \frac{\partial u}{\partial n}\, dxdy = \int_0^{w_0} \left(\int_{\Gamma_k(w)} \frac{du}{dn} ds \right) dw = w_0 \int_{\Gamma_k} \frac{du}{dn} ds,$$

Hence, we obtain from (17) that

$$\int_{\Gamma_k} \frac{du}{dn} ds = w_0^{-1} \int_S \frac{\partial u}{\partial n}\, dxdy = 0.$$

This proves that v is single-valued.

Since $S_1 \cap Y$ is strictly contained in Y, the vertical line between $(0,0)$ and $(0,t_2)$ and the horizontal line between $(0,0)$ and $(t_1,0)$ lays in S_2. Hence, by (16) and (15)

$$\begin{aligned} v(0,t_2) - v(0,0) &= \int_{(0,0)}^{(0,t_2)} \frac{du}{dn} ds = \int_0^{t_2} \frac{\partial u(0,y)}{\partial x} dy = t_2 b_1, \\ v(t_1,0) - v(0,0) &= \int_{(0,0)}^{(t_1,0)} \frac{du}{dn} ds = -\int_0^{t_1} \frac{\partial u(x,0)}{\partial y} dx = -t_1 b_2. \end{aligned}$$

Moreover, Theorem 2.1 gives that v is of the form $v(x,y) = xr + ys + h(x,y)$, for some constants r and s, where $h(x,y)$ is a Y-periodic function. Accordingly,

$$\begin{aligned} v(0,t_2) - v(0,0) &= t_2 s, \\ v(t_1,0) - v(0,0) &= t_1 r. \end{aligned}$$

Thus, choosing, $r = -b_2$ and $s = b_1$, we obtain the desired result. The proof is complete. □

Remark 1. The upper and lower estimates of (14) in above proof is known as the Reuss-Voigt-Wiener bounds, which short proof is included here for completion. It should be noted that the proof of the lower bound is more direct and completely different from that normally used in the literature, which makes use of the dual problem (see e.g. [7]).

Remark 2. The problem (5) is usually referred to as the *cell problem* in the *homogenization theory*, a theory which was initiated by De Giorgi and Spagnolo in the late 60's and further developed in the 70's by Murat, Tartar, Bakhvalov, Bensoussan, Lions and Papanicolaou (concerning this, see the books [7] and [5], and the references given there). In the late 80's a new successful approach to this theory was initiated by Nguetseng and further developed by Allaire and others under the name of *two-scale convergence* (concerning this we refer to the article [11], which contains more than 80 references to this theory).

Proof of Proposition 1. The proof follows directly from Theorem 4.1 by choosing $\lambda_1 \neq \lambda_2$, then letting

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \lambda_2 (q - \lambda_2 I)^{-1} \begin{bmatrix} s \\ -r \end{bmatrix},$$

and considering the harmonic conjugate of the solution of the problem (5) in region S_2. □

5. **The dual approach.** We first observe that the space

$$V^2_{\text{pot}}(Y) = \{\operatorname{grad} u : u \in W_{\text{per}}(Y)\}$$

is closed in $\mathbf{L}^2(Y)$. Indeed, if $\{\operatorname{grad} u_h\}$ is a Cauchy sequence in $\mathbf{L}^2(Y)$, so is $\{\operatorname{grad} v_h\}$, where $v_h = u_h - \langle u_h \rangle$, ($\langle u_h \rangle$ denotes as usual the average value of u_h). By the Poincare inequality,

$$\int_Y |v_h - v_k|^2 \, dm \leq C \left(\left(\int_Y (v_h - v_k) \, dm \right)^2 + \int_Y |\text{grad } (v_h - v_k)|^2 \, dm \right),$$

in which $\int_Y (v_h - v_k) \, dm = 0$, we obtain that $\{v_h\}$ is a Cauchy sequence in $L^2(Y)$. Here, m denotes the Lebesgue measure in R^n (temporarily we do not restrict our discussion to the case $n = 2$). Combined with the fact that $\{\operatorname{grad} u_h\}$ is a Cauchy sequence in $\mathbf{L}^2(Y)$, we obtain that $\{v_h\}$ is a Cauchy sequence in $W_{\text{per}}(Y)$. Hence, by the completeness of $W_{\text{per}}(Y)$ we get that $v_h \to v$ in $W_{\text{per}}(Y)$ for some $v \in W_{\text{per}}(Y)$, which in particular implies that $\operatorname{grad} u_h = \operatorname{grad} v_h \to \operatorname{grad} v$.

Let $L^2_{\text{sol}}(Y)$ denote the orthogonal complement of $V^2_{\text{pot}}(Y)$ in $\mathbf{L}^2(Y)$, i.e.

$$L^2_{\text{sol}}(Y) = \left\{ p \in \mathbf{L}^2(Y) : \int_Y p \cdot \operatorname{grad} \varphi \; dm = 0 \;\; \forall \varphi \in C^\infty_{\text{per}}(Y) \right\}. \tag{18}$$

The reason why we can replace $W_{\text{per}}(Y)$ by $C^\infty_{\text{per}}(Y)$ in the definition of $L^2_{\text{sol}}(Y)$ is that $C^\infty_{\text{per}}(Y)$ is dense in $W_{\text{per}}(Y)$ (by definition) and the fact that the functional $F(\varphi) = \int_Y p(z) \cdot \operatorname{grad} \varphi(z) \, dz$ is continuous in $W_{\text{per}}(Y)$, which is easily seen by the Schwarz inequality.

Lemma 5.1. *The Y-periodic extension of $L^2_{sol}(Y)$ coincides with the space of solenoidal Y-periodic vector-fields in $\mathbf{L}^2_{loc}(R^n)$, i.e. the space*

$$\mathbf{L}^2_{Y\text{-}per,sol}(R^n) \equiv \left\{ p \in \mathbf{L}^2_{Y\text{-}per}(R^n) : \int_{R^n} p \cdot \operatorname{grad} \varphi \; dm = 0 \;\; \forall \varphi \in C^\infty_0(R^n) \right\}, \tag{19}$$

where $\mathbf{L}^2_{Y\text{-}per}(R^n)$ denotes the space of all Y-periodic extension to R^n of functions in $\mathbf{L}^2(Y)$.

The proof of Lemma 5.1 is briefly sketched in [7, p. 6], but we also present the proof here, which we believe is of independent interest since it is more detailed.

Proof. Let $\{O_k\}_{k=1}^\infty$ be a collection of open disks covering R^n with centers at equidistant points z_k with diameter less than the diameter of Y. Then, due to the theorem of infinitely differentiable partitions of unity (see e.g. [1, Theorem 3.14]), we can construct functions $\psi_k \in C^\infty_0(R^n)$, with support in O_k, such that $\psi(z) := \sum_{k=1}^\infty \psi_k(z) = 1$ for all $z \in R^n$. For any $\varphi \in C^\infty_0(R^n)$ we can find an integer $N < \infty$ such that $\sum_{k=1}^N \psi_k(z) = 1$ for all $z \in \operatorname{supp} \varphi$. Let $\overline{\varphi\psi}_k$ be the Y-periodic extension to R^n of $\varphi\psi_k|_{O_k}$ (note that such an extension exists thanks to the fact that $\operatorname{supp} \psi_k \subset O_k \subset Y + z_k$). Due to periodicity, we can translate the domain of integration in the following way:

$$\int_{Y+z_k} p \cdot \operatorname{grad} \varphi\psi_k \; dm = \int_{Y+z_k} p \cdot \operatorname{grad} \overline{\varphi\psi}_k \; dm = \int_Y p \cdot \operatorname{grad} \overline{\varphi\psi}_k \; dm = 0, \tag{20}$$

for all $p \in \mathbf{L}^2_{Y\text{-per}}(R^n)$. Assume that (18) holds. Then,

$$\int_{R^n} p \cdot \operatorname{grad} \varphi \; dm = \int_{R^n} p \cdot \operatorname{grad} \left(\sum_{k=1}^N \psi_k \varphi \right) \; dm = \sum_{k=1}^N \int_{Y+z_k} p \cdot \operatorname{grad} \varphi\psi_k \; dm = 0.$$

Hence, (18) implies (19). Now , let $\varphi \in C^\infty_{\text{per}}(Y)$, let $\overline{\varphi}$ denote its Y-periodic extension to R^n and assume (19) holds. For simplicity we let Y be centered at 0. Let N be a positive integer and let $\psi_N \in C^\infty_0((N+1)Y)$, $0 \le \psi_N \le 1$ such that $\psi_N = 1$ on NY and $|\text{grad}\, \psi_N| < C < \infty$ for some constant C which is independent of N (the latter is possible since the smallest distance between the sets $(N+1)Y$ and NY is independent of N) . Since $\psi_N \overline{\varphi} \in C^\infty_0(R^n)$ (19) gives that

$$0 = \int_{R^n} p \cdot \text{grad}\, \psi_N \overline{\varphi}\ dm = \int_{NY} p \cdot \text{grad}\, \overline{\varphi}\ dm + \int_{((N+1)Y)\setminus(NY)} p \cdot \text{grad}\ (\psi_N \overline{\varphi})\ dm \tag{21}$$

Clearly, $|\text{grad}\ (\psi_N \overline{\varphi})| < C' < \infty$ for some constant C', which is independent of N, so by the periodicity of p we obtain the estimate

$$\left| \int_{((N+1)Y)\setminus(NY)} p \cdot \text{grad}\ (\psi_N \overline{\varphi})\ dm \right| \le C' \int_{((N+1)Y)\setminus(NY)} |p|\ dm =$$

$$C'\left((N+1)^n - N^n\right) \int_Y |p|\ dm. \tag{22}$$

Moreover, using the periodicity of $p \cdot \text{grad}\, \overline{\varphi}$ we obtain that

$$\int_{NY} p \cdot \text{grad}\, \overline{\varphi}\ dm = N^n \int_Y p \cdot \text{grad}\, \varphi\ dm. \tag{23}$$

Hence, by (21), (22), (23) and the fact that

$$(N+1)^n - N^n = \sum_{k=0}^{n} \binom{n}{k} N^{n-k} - N^n = \sum_{k=1}^{n} \binom{n}{k} N^{n-k},$$

we obtain the estimate

$$\left| \int_Y p \cdot \text{grad}\, \varphi\ dm \right| \le C' \left(\sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{N^k} \right) \right) \int_Y |p|\ dm,$$

which goes to 0 as $N \to \infty$. Thus, $\int_Y p \cdot \text{grad}\, \varphi\ dm = 0$. This completes the proof. □

If $\varphi \in W^{1,2}(R^2)$ it is usual to define $\text{curl}\, \varphi$ as the vector-function $\text{curl}\, \varphi = (-\partial\varphi/\partial y, \partial\varphi/\partial x)$.

Lemma 5.2. *It holds that*

$$L^2_{sol}(Y) = R^2 \oplus \text{curl}\,(W^1_{per}(Y))$$

where $\text{curl}\,(W^1_{per}(Y)) = \left\{ \text{curl}\, \varphi : \varphi \in W^1_{per}(Y) \right\}$.

Proof. By Lemma 5.1 any function in $L^2_{\text{sol}}(Y)$ can be extended to some $p \in \mathbf{L}^2_{Y\text{-per,sol}}(R^2)$. Clearly

$$p|_\Omega \in H \equiv \left\{ q \in \mathbf{L}^2(Y) : \int_\Omega q \cdot \text{grad}\, \varphi\ dm = 0 \quad \forall \varphi \in C^\infty_0(\Omega) \right\}$$

for any bounded and simply connected domain Ω. Since

$$H = \left\{ \text{curl}\, \varphi : \varphi \in W^{1,2}(\Omega) \right\} \tag{24}$$

(see e.g. [18, p. 467]), $p = \text{curl}\, \varphi$ for some $\varphi \in W^{1,2}(\Omega)$. Hence, $\varphi(x,y) = xr + ys + h(x,y)$, where r and s are constants and $h \in W_{\text{per}}(Y)$. The proof of this fact is identical with that of Theorem 2.1 starting from eq. (1). Hence, p is of the form

$$p = (-s, r) + \operatorname{curl} h. \tag{25}$$

Moreover, (24) gives that any p of the form (25) is a member of H. Thus, $L^2_{\rm sol}(Y) = R^2 \oplus \operatorname{curl}(W_{\rm per}(Y))$, and the Lemma follows. □

Remark 3. Since $L^2_{\rm sol}(Y)$ is the orthogonal complement of $V^2_{\rm pot}(Y)$ in $\mathbf{L}^2(Y)$, Lemma 5.2 implies the following orthogonal decomposition of $\mathbf{L}^2(Y)$

$$\mathbf{L}^2(Y) = \operatorname{curl}(W_{\rm per}(Y)) \oplus V^2_{\rm pot}(Y) \oplus R^2. \tag{26}$$

By considering the dual problem to (5) we are able to present an independent proof of the fact that the harmonic conjugate is of the form $v(x, y) = xr+ys+h(x, y)$ in a more general situation. Moreover, we are going to show that $h(x, y)$ can be found by solving a weak problem of the same type as (5).

Theorem 5.3. *Let λ be of the form $\lambda = \sum \lambda_i \chi_{S_i}$, where $\{S_i\}$ is a disjoint open periodic cover of R^2, and let u be the solution of (5). Then u is harmonic in each of the regions S_i separately. Moreover, the harmonic conjugate v of u in the region S_i is single-valued and given by $v = -\lambda_i^{-1}\psi + (xr + ys)$, where the constants r and s are given by*

$$\begin{bmatrix} s \\ -r \end{bmatrix} = \frac{1}{\lambda_i}(q - \lambda_i I)\xi,$$

and ψ is the solution of the problem

$$\int_Y \frac{1}{\lambda}(\operatorname{grad}\psi + \sigma q \xi) \cdot \operatorname{grad}\varphi \; dxdy = 0, \tag{27}$$

where

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(i.e. a rotation of angle $\pi/2$). The matrix q is given by (6).

Proof. The fact that u is harmonic in each of the regions S_i separately is contained in the proof of Theorem 4.1. From the weak formulation (5) we see that

$$p = \lambda(\operatorname{grad} u + \xi) \in L^2_{\rm sol}(Y).$$

By Lemma 5.2, $p = \operatorname{curl}\psi + \eta$ for some function $\psi \in W^1_{\rm per}(Y)$ and some constant vector $\eta \in R^2$. Thus

$$\frac{1}{\lambda}(\operatorname{curl}\psi + \eta) = \frac{1}{\lambda} p = \operatorname{grad} u + \xi \in V^2_{\rm pot}(Y) \oplus R^2. \tag{28}$$

Therefore, by Remark 3

$$\int_Y \frac{1}{\lambda}(\operatorname{curl}\psi + \eta) \cdot \operatorname{curl}\varphi \; dxdy = 0,$$

which easily can be rewritten in the form

$$\int_Y \frac{1}{\lambda}(\operatorname{grad}\psi + \sigma\eta) \cdot \operatorname{grad}\varphi \; dxdy = 0.$$

Moreover, since

$$\langle \operatorname{curl}\psi + \eta \rangle = \langle p \rangle = \frac{1}{|Y|}\int_Y \lambda(\operatorname{grad} u + \xi) \; dxdy = q\xi$$

by (11) and $\langle \operatorname{curl} \psi \rangle = 0$ by periodicity, we find that $\eta = q\xi$. Thus, by (28) we obtain that

$$\operatorname{curl} \frac{1}{\lambda_i}\psi + \frac{1}{\lambda_i} q\xi - \xi = \operatorname{grad} u$$

on the set S_i. Letting r and s be given by

$$\begin{bmatrix} -s \\ r \end{bmatrix} = \frac{1}{\lambda_i}(q - \lambda_i I)\xi,$$

we find that

$$-\operatorname{curl} v = \operatorname{grad} u, \tag{29}$$

where

$$v = -\frac{1}{\lambda_i}\psi + (xr + ys),$$

which shows that v is the harmonic conjugate of u since (29) is nothing but the Cauchy-Riemann equations. □

6. **A numerical example.** In many physical problems the quantity $U = u + x\xi_1 + y\xi_2$ turns out to be more interesting than the solution u of (5). As a matter of fact, in a periodic composite material with average heat flux equal to (ξ_1, ξ_2), it turns the temperature actually equals U. In this case the function u is just an auxiliary function.

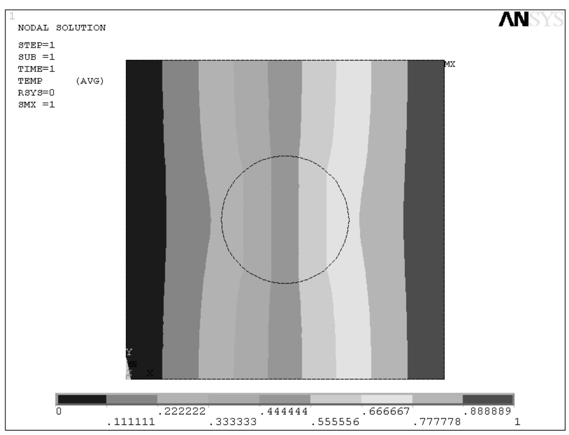

FIGURE 2. Plot of the computed temperature U in Y.

Since the harmonic conjugate of $x\xi_1 + y\xi_2$ is $y\xi_1 - x\xi_2$ we obtain from Theorem 5.3 that in a subdomain S_i the harmonic conjugate V of U is given by $V = -\lambda_i^{-1}\psi + (x(r - \xi_2) + y(s + \xi_1))$, where

$$\begin{bmatrix} s + \xi_1 \\ -(r - \xi_2) \end{bmatrix} = \frac{1}{\lambda_i} q\xi.$$

In other words,

$$V = -\lambda_i^{-1}\psi + (xr' + ys'), \tag{30}$$

where

$$\begin{bmatrix} s' \\ -r' \end{bmatrix} = \frac{1}{\lambda_i} q\xi.$$

For the interpretation of the harmonic conjugate V for various physical problems, see e.g. [13].

As an example we consider a stationary conduction problem consisting of two subdomains S_1, S_2, where $\xi_1 = 1$, $\xi_2 = 0$, $\lambda_1 = 0.5$, $\lambda_2 = 1$, $Y = (0,1)^2$ and $Y \cap S_1$ is a disk of radius $= 0.2$ centered at $(0.5, 0.5)$. This corresponds to a doubly periodic composite material of two materials consisting of fibres with conductivity λ_1 embedded in a connected material with conductivity λ_2. The information that $\xi_1 = 1$ and $\xi_2 = 0$ merely tells that the average temperature gradient $\langle \operatorname{grad} U \rangle$ taken over each period is equal to $(1,0)^T$. Due to symmetry it is possible to show that the off-diagonal elements of the effective conductivity matrix q vanish and that $q_{11} = q_{22}$. We first find the temperature U approximately by solving (5) numerically (see Figure 2). This is done by using the FE-program Ansys. The elementtype which is used is a 8-node thermal element called plane77. The number of nodes is 15653 and the number of elements is 5072. After computing the FE-solution $U_{\mathrm{FE}} \approx U$ we find that $q_{11} = q_{22} \approx 0.9196$ by calculating the following integral numerically

$$\frac{1}{|Y|} \int_Y \lambda(x,y) \frac{\partial U_{\mathrm{FE}}}{\partial x} \, dxdy \approx q_{11}.$$

Next, we find ψ by solving (27) numerically (see the plot of ψ in Figure 3), and finally we find the harmonic conjugate V directly from (30).

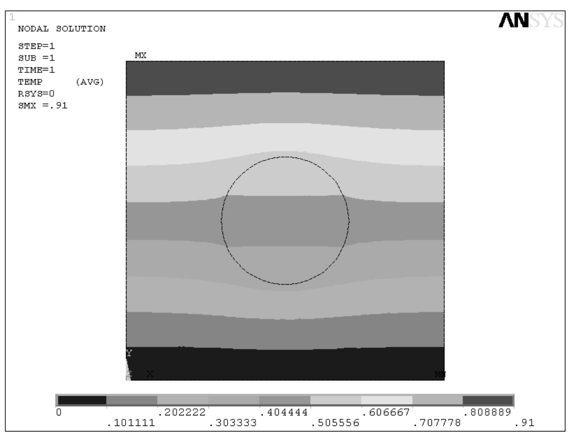

FIGURE 3. Plot of the computed function ψ of the dual problem in Y.

For details concerning the method of computing effective moduli in general, including more information on numerical and engineering aspects, see e.g. [4], [8], [9], [17] and [10].

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