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DIRECT INTEGRAL DECOMPOSITION FOR PERIODIC FUNCTION SPACES AND APPLICATION TO BLOCH WAVES

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Abstract. In this paper, we study a direct integral decomposition for the spaces $L^2(O)$ and $H^1(O)$ based on $(\xi, Y^*)-$periodic functions. Using this decomposition we can write the Green's operator (associated to the classical Stokes system in fluid mechanics) in terms of a family of self-adjoint compact operators which depend on the parameter ξ. As a consequence, we obtain the so-called Bloch waves associated to the Stokes system in the case of a periodic perforated domain.

1. **Introduction.** In this paper, we rigorously study a direct integral decomposition for the spaces $L^2(O)$ and $H^1(O)$ in $(\xi, Y^*)-$periodic function spaces, where the sets O and Y^* are defined in (3) and (2) respectively, in the sense given by J. Dixmier [8] (see also the book of G. W. Mackey [12]), and thus, a decomposition based on $(\xi, Y^*)-$periodic functions of the Green's operator for the Stokes system is obtained. As a consequence, we derive the so-called Bloch waves associated with this system.

The starting point of our study consists in introducing two basic reference cells denoted $Y =]0, 2\pi[^N$ and $Y^* = Y \setminus \overline{T}$, where T is a given star-shaped region included

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in Y. In this way, by means of Y and Y^*, we define the periodic perforated domain $O = \mathbb{R}^N \setminus \cup_{j\in\mathbb{Z}^N}\{\overline{T+2\pi j}\}$.

We are interested in the following spectral problem for the Stokes system:

$$\begin{cases} -\Delta \mathbf{u} + \nabla p &= \lambda \mathbf{u} \quad \text{in } O \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } O \\ \mathbf{u} &= 0 \quad \text{on } \partial T_j,\ j\in\mathbb{Z}^N, \end{cases} \tag{1}$$

which is a spectral boundary value problem in the unbounded domain O. The Green's function associated to (1), which is a bounded self-adjoint operator, is not necessarily compact. In order to obtain the spectral decomposition of system (1), we first need to decompose the corresponding Green's operator into a family of bounded compact self-adjoint operators. To this end, a direct integral decomposition will be used.

The method of Bloch wave decomposition (or Floquet decomposition) is well known for reducing the problem of solving the Schrödinger equation in an infinite periodic medium to a family of simpler Schrödinger equations posed in a single periodicity cell and parametrized by the so-called Bloch frequency (see [6] and the references therein). Originally, Bloch waves were introduced by F. Bloch [3] in 1928 in the context of solid state physics for the study of propagation of electrons in a crystal. More recently, Bloch waves have been used in the study of several issues in partial differential equations modeling heterogeneous media. Particularly, we refer to C. Conca and M. Vanninathan [7] (see also [5]) for the study of periodic homogenization, where the authors recover the classical results of convergence in the homogenization process, which were first obtained by F Murat and L. Tartar (see [13] and [20]), by using Bloch waves method. Bloch waves are also a useful tool in the study of asymptotic behavior of solutions to partial differential equations as times goes to infinity; in particular we refer to J.H. Ortega and E. Zuazua [15] for the study of a complete asymptotic expansion in time $t > 0$ of solutions for parabolic equations with periodic coefficients as time t goes to infinity.

In the case of the Stokes system, Bloch waves are obtained by solving the following spectral problem

$$\begin{cases} -\Delta \mathbf{u} + \nabla p &= \lambda \mathbf{u} & \text{in } Y^* \\ \nabla\cdot\mathbf{u} &= 0 & \text{in } Y^* \\ \mathbf{u} &= 0 & \text{on } \partial T \\ \mathbf{u}, p \text{ are } (\xi, Y^*) & - \text{ periodic functions.} \end{cases}$$

where $(\xi, Y^*)-$periodic functions are characterized by the following generalized periodic property $\psi(x+2\pi p) = e^{2\pi i\xi\cdot p}\psi(x)$ for a.e. $x \in \mathbb{R}^N$, and for all $p \in \mathbb{Z}^N$ (see [6] pp. 193 or [16] pp. 279 for more details). To prove existence of Bloch waves, we will first establish a new version of the classical De Rham's Theorem in the space of $(\xi, Y^*)-$periodic functions (see Proposition 3 below); for an alternative construction of these Bloch waves, the reader is referred to [1].

The content of this paper is the following. In Section 2 we study the direct integral decomposition of the spaces $\mathbf{L}^2(O)$ and $\mathbf{H}^1_0(O)$ in $(\xi, Y^*)-$periodic function spaces $\mathbf{L}^2_\#(\xi, Y^*)$, $\mathbf{H}^1_\#(\xi, Y^*)$ and $\mathbf{H}^1_{0,\#}(\xi, Y^*)$. Later, by using these decompositions, a new version of the classical De Rham's Theorem is established. Finally, in Section 3 we obtain the Bloch waves for the Stokes system rewriting its Green's operator as a family of self-adjoint compact operators which depend on the parameter ξ.

2. (ξ, Y^*)-periodic functions decomposition of $\mathbf{L}^2(O)$ and $\mathbf{H}^1_0(O)$. Let $\mathbf{Z}$ be a measurable space endowed with a positive measure ν. A family $(H(\xi))_{\xi\in\mathbf{Z}}$ is called a "Hilbert space field" if $H(\xi)$ is a Hilbert space for each $\xi \in \mathbf{Z}$. We shall denote by $||\cdot||$ the Hilbertian norm of $H(\xi)$. Define (see [8], [12] and [16]) the "direct integral":

$$\mathbb{H} = \int_Z^{\oplus} H(\xi) d\nu(\xi) \equiv \left\{ x \in \prod_{\xi\in Z} H(\xi) \;\middle|\; x \text{ is measurable and } \int_Z \|x(\xi)\|^2 d\nu(\xi) < \infty \right\}$$

An operator $T \in \mathcal{L}(\mathbb{H}, \mathbb{H})$ is decomposable in terms of operators acting on $H(\xi)$ if there exists a $\nu-$measurable linear bounded mapping field $(T(\xi))_{\xi\in Z}$, which is essentially bounded with respect to ξ, such that $Tx = (T(\xi)x(\xi))_{\xi\in Z}$; we write $T = \displaystyle\int_Z^{\oplus} T(\xi) d\nu(\xi)$.

2.1. **The spaces $\mathbf{L}^2_{\#}(\xi, Y^*)$, $\mathbf{H}^1_{\#}(\xi, Y^*)$ and $\mathbf{H}^1_{0,\#}(\xi, Y^*)$.** First of all, we state some basic notation following the book by C. Conca, J. Planchard and M. Vanninathan [6]. Set $Y =]0, 2\pi[^N$ and let $T \subseteq Y$ be a C^2, star-shaped domain with respect to $x_0 \in T$. We can thus define

$$Y^* = Y \setminus \overline{T} \tag{2}$$

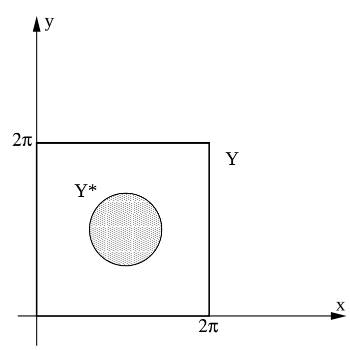

Figure 1. Basic cell Y^*

$$O = \mathbb{R}^N \setminus \cup_{j\in\mathbb{Z}^N} \{\overline{T_j}\}, \tag{3}$$

where

$$T_j = T + 2\pi j, \quad j \in \mathbb{Z}^N. \tag{4}$$

We say that $v \in L^2_{loc}(O)$ is (ξ, Y^*) − periodic if

$$v(x + 2\pi m) = e^{2\pi i m\cdot\xi} v(x) \;\; \text{a.e.} \;\; x \in O, \; \forall\, m \in \mathbb{Z}^N,$$

where $\xi \in \mathbb{R}^N$ is a parameter. Let us observe that if ξ is replaced by $(\xi + q)$ with $q \in \mathbb{Z}^N$, the above equality is unaltered and we can therefore restrict ξ to the cell $Y' = [0, 1[^N$. Let us define the space $L^2_{\#}(\xi, Y^*)$ by

$$L^2_{\#}(\xi, Y^*) = \{v \in L^2_{loc}(O) |\; v \text{ is } (\xi, Y^*) - \text{periodic}\}, \tag{5}$$

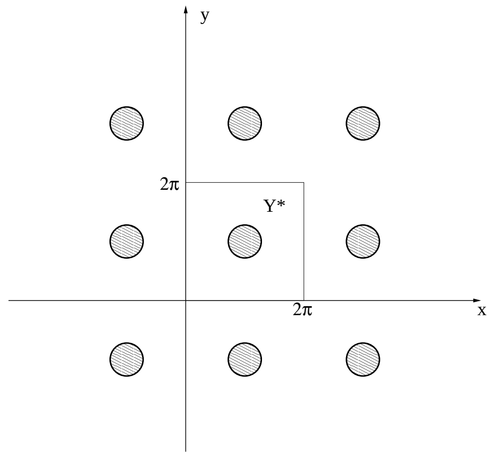

FIGURE 2. Periodic perforated domain O

Now, we can define the following spaces,

$$\begin{aligned} H^1_\#(\xi, Y^*) &= \left\{ v \in L^2_\#(\xi, Y^*) \mid \frac{\partial v}{\partial x_l} \in L^2_\#(\xi, Y^*),\ l = 1, \ldots, N \right\}, \\ H^1_{0,\#}(\xi, Y^*) &= \{ v \in H^1_\#(\xi, Y^*) \mid v = 0 \text{ on } \partial T \}, \end{aligned}$$

which are Hilbert spaces with respect to the inner product in $H^1(Y^*)$. The spaces labeled in bold face denote the cartesian product of the space with N-times itself.

Proposition 1. *([6])*

(1) The imbedding $H^1_\#(\xi, Y^*) \hookrightarrow L^2_\#(\xi, Y^*)$ is compact, so is $H^1_{0,\#}(\xi, Y^*) \hookrightarrow L^2_\#(\xi, Y^*)$.

(2) Define $H^1_{0,T}(Y^*) \equiv \{ v \in H^1(Y^*) \mid v = 0 \text{ on } \partial T \}$. Then $H^1_{0,\#}(\xi, Y^*)$ is a Hilbert subspace of $H^1_{0,T}(Y^*)$.

It is well known that there exists $C > 0$ such that

$$\int_{Y^*} |v|^2 dx \le C \int_{Y^*} |\nabla v|^2 dx \quad \text{for all } v \in H^1_{0,T}(Y^*).$$

Hence, from this and by Proposition 1, we have

Proposition 2. *There exists $C > 0$ such that for each $\xi \in Y'$*

$$\int_{Y^*} |u_\xi(x)|^2 dx \le C \int_{Y^*} |\nabla u_\xi(x)|^2 dx \quad \text{for all } u_\xi \in H^1_{0,\#}(\xi, Y^*),$$

in particular $C > 0$ does not depend on ξ.

Remark 1. For each $\xi \in Y' = [0, 1[^N$ we define:

$$\begin{array}{rcl} I(\xi) \ : \ L^2_\#(Y^*) & \to & L^2_\#(\xi, Y^*) \\ \phi & \mapsto & e^{ix\cdot\xi}\phi. \end{array} \tag{6}$$

This mapping is an isometry. Moreover, if $\phi \in H^1_\#(Y^*)$, $\Phi \in \mathbf{H}^1_\#(Y^*)$ we have

$$\nabla(e^{ix\cdot\xi}\phi) = e^{ix\cdot\xi} D(\xi)\phi, \quad \text{and} \quad \nabla \cdot (e^{ix\cdot\xi}\Phi) = e^{ix\cdot\xi} D(\xi) \cdot \Phi,$$

where $D(\xi) = \nabla + i\xi$.

It is well-known that if $u \in \mathcal{D}'_{\#}(Y)$, where $\mathcal{D}'_{\#}(Y)$ is the space of the $Y-$periodic distributions on $\mathbb{R}^N$ (see [18]), then

$$u = \sum_{p\in\mathbb{Z}^N} u_p e^{ip\cdot x}, \tag{7}$$

in the sense of distributions. Here u_p are complex numbers such that

$$|u_p| \leq C(1+|p|)^M \quad p \in \mathbb{Z}^N,$$

for some $C, M \geq 0$. In the same manner, if $v \in L^2_{\#}(Y)$ then,

$$v = \sum_{p\in\mathbb{Z}^N} \hat{v}(p) e^{ip\cdot x},$$

in the sense of $L^2(Y)$ and $\hat{v}(p)$ is defined by

$$\hat{v}(p) = \frac{1}{(2\pi)^N} \int_Y v(x) e^{-ip\cdot x} dx,$$

moreover,

$$\frac{1}{(2\pi)^N} \int_Y |v(x)|^2 dx = \sum_{p\in\mathbb{Z}^N} |\hat{v}(p)|^2, \tag{8}$$

i.e., $(\hat{v}(p))_{p\in\mathbb{Z}^N} \in l^2(\mathbb{Z}^N, \mathbb{C})$. Identity (8) allows us to identify $L^2_{\#}(Y)$ with $l^2(\mathbb{Z}^N, \mathbb{C})$.

According to the above facts, we can alternative define $H^1_{\#}(Y)$ as follows

$$H^1_{\#}(Y) = \{v \in \mathcal{D}'_{\#}(Y) \mid (v_p(1+|p|^2)^{\frac{1}{2}})_{p\in\mathbb{Z}^N} \in l^2(\mathbb{Z}^N, \mathbb{C})\},$$

where v_p, with $p \in \mathbb{Z}^N$, are the coefficients given in (7). Furthermore, for each $s > 0$ we can define

$$H^s_{\#}(Y) = \{v \in \mathcal{D}'_{\#}(Y) \mid (v_p(1+|p|^2)^{\frac{s}{2}})_{p\in\mathbb{Z}^N} \in l^2(\mathbb{Z}^N, \mathbb{C})\}.$$

This definition is similar to the one of the classical Sobolev spaces on $\mathbb{R}^N$ via Fourier series instead of Fourier transform. Moreover, as in the case of Sobolev spaces on $\mathbb{R}^N$, we have

$$\big(H^1_{\#}(Y)\big)' = H^{-1}_{\#}(Y).$$

Our next purpose is to obtain a generalized version of De Rham's Theorem for periodic functions. We start recalling the following definitions:

$$\chi(\Omega) \equiv \{u \in H^{-1}(\Omega) \mid \nabla u \in \mathbf{H}^{-1}(\Omega)\}.$$

By using Fourier transform it can be proved that $\chi(\mathbb{R}^N) = L^2(\mathbb{R}^N)$ (see [21]) and hence, by using a standard reflexion technique, $\chi(\mathbb{R}^N_+) = L^2(\mathbb{R}^N_+)$. We next define

$$\chi_{\#}(Y) := \{u \in H^{-1}_{\#}(Y) \mid \nabla u \in \mathbf{H}^{-1}_{\#}(Y)\}.$$

Lemma 2.1. *$\chi_{\#}(Y) = L^2_{\#}(Y)$.*

Proof. Let $u \in \chi_{\#}(Y)$ be given. Then $(u_p(1+|p|^2)^{-\frac{1}{2}})_{p\in\mathbb{Z}^N}$ and $(p_k u_p(1+|p|^2)^{-\frac{1}{2}})_{p\in\mathbb{Z}^N}$, $k = 1, \ldots, N$, are elements of $l^2(\mathbb{Z}^N, \mathbb{C})$, where u_p is given in (7). Therefore, $\big(\frac{1+|p|}{(1+|p|^2)^{\frac{1}{2}}} u_p\big)_{p\in\mathbb{Z}^N} \in l^2(\mathbb{Z}^N, \mathbb{C})$ and thus, $(u_p)_{p\in\mathbb{Z}^N} \in l^2(\mathbb{Z}^N, \mathbb{C})$. This implies $u \in L^2_{\#}(Y)$. □

Now, by using an appropiate partition of 1, we prove:

Lemma 2.2. *If $T \subseteq Y$ is an open set with smooth boundary and $Y^* = Y \setminus \overline{T}$. Then,*

$$\chi_{\#}(Y^*) = L^2_{\#}(Y^*).$$

We recall the following lemma which will be useful in our study.

Lemma 2.3. [10, pp.18] *Let B_1, B_2, B_3 be three Banach spaces, $S \in \mathcal{L}(B_1; B_2)$ and $R \in \mathcal{L}(B_1; B_3)$ be compact operators such that*

$$\|u\|_{B_1} \cong \|Su\|_{B_2} + \|Ru\|_{B_3}, \ \forall u \in B_1.$$

Then $Ker(S)$ is a finite dimensional space, the mapping S is an isomorphism from $B_1/Ker(S)$ onto $Im(S)$ and $Im(S)$ is a closed subspace of B_2.

Lemma 2.4. *The linear mapping*

$$\nabla \ : \ L^2_{\#}(Y^*) \rightarrow \quad \mathbf{H}^{-1}_{\#}(Y^*)$$

has closed range in $\mathbf{H}^{-1}_{\#}(Y^*)$.

Proof. To prove this result we will use Lemma 2.3. Let us consider the spaces $B_1 = L^2_{\#}(Y^*)$, $B_2 = \mathbf{H}^{-1}_{\#}(Y^*)$ and $B_3 = H^{-1}_{\#}(Y^*)$, moreover we consider the operators $S = \nabla$ and the injection R from $L^2_{\#}(Y^*)$ into $H^{-1}_{\#}(Y^*)$. We note that since the injection from $H^1_{0,\#}(Y^*)$ into $L^2_{\#}(Y^*)$ is compact, so the dual injection from $L^2_{\#}(Y^*)$ into $H^{-1}_{\#}(Y^*)$ is also compact. Now, we must show that $||u||_{L^2_{\#}(Y^*)} \cong ||\nabla u||_{\mathbf{H}^{-1}_{\#}(Y^*)} + ||u||_{H^{-1}_{\#}(Y^*)}$.
In fact, since $||\nabla u||_{\mathbf{H}^{-1}_{\#}(Y^*)} + ||u||_{H^{-1}_{\#}(Y^*)} \leq C||u||_{L^2_{\#}(Y^*)}$, it is only necessary to prove that

$$||\nabla u||_{\mathbf{H}^{-1}_{\#}(Y^*)} + ||u||_{H^{-1}_{\#}(Y^*)}$$

is a complete norm on $L^2_{\#}(Y^*)$.

Let $(u_n)_{n\geq 1}$ be a Cauchy sequence in the above norm. Then $(\nabla u_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbf{H}^{-1}_{\#}(Y^*)$ and $(u_n)_{n\geq 1}$ is also a Cauchy sequence in $H^{-1}_{\#}(Y^*)$. As $H^{-1}_{\#}(Y^*)$ is complete, it is easy to prove that there exists $u \in H^{-1}_{\#}(Y^*)$ such that

$$\begin{aligned} \lim_{n\to\infty} u_n &= u \ \text{ in } \ H^{-1}_{\#}(Y^*), \\ \lim_{n\to\infty} \nabla u_n &= \nabla u \ \text{ in } \ \mathbf{H}^{-1}_{\#}(Y^*), \end{aligned}$$

and thus $u \in \chi_{\#}(Y^*) = L^2_{\#}(Y^*)$, which completes the proof. □

In a similar way, we can prove that $\nabla \ : \ L^2_{\#}(\xi, Y^*) \rightarrow \mathbf{H}^{-1}_{\#}(\xi, Y^*)$ has a closed range for each $\xi \in Y'$. Now, if we define, for every $\xi \in Y'$, div $: \ \mathbf{H}^1_{0,\#}(\xi, Y^*) \rightarrow L^2_{\#}(\xi, Y^*)$ by div $\mathbf{u} = \nabla \cdot \mathbf{u}$, then it is not so difficult to prove that the dual operator of div is $-\nabla$. Thus, since ∇ has a closed range we can conclude that $Ker(\text{div}) = Im(\nabla)$ (see [4] pp 29 or [24] pp 205).

From the previous result we can obtain a new version of the classical De Rham's Theorem.

Proposition 3. *If $\xi \in Y'$ and $f \in \mathbf{H}^{-1}_{\#}(\xi, Y^*)$ are such that $< f, v >= 0$ for each $v \in \mathbf{H}^1_{0,\#}(\xi, Y^*)$, satisfying $\nabla \cdot v = 0$, then $f = \nabla p$, where $p \in L^2_{\#}(\xi, Y^*)$.*

In what follows we define

$$\mathbf{V}_{\#}(\xi, Y^*) = \{\mathbf{u} \in \mathbf{H}^1_{0,\#}(\xi, Y^*) \mid \nabla \cdot \mathbf{u} = 0\}. \tag{9}$$

In the same way, we consider

$$\mathbf{H}_{\#}(\xi, Y^*) = \overline{\mathbf{V}_{\#}(\xi, Y^*)}^{\mathbf{L}^2_{\#}(\xi, Y^*)}, \quad (\mathbf{V}_{\#}(\xi, Y^*) \text{ without topology}).$$

The following result give us a characterization of $\mathbf{H}_{\#}(\xi, Y^*)$.

Proposition 4. $\mathbf{H}_{\#}(\xi, Y^*) = \{\mathbf{u} \in \mathbf{L}^2_{\#}(\xi, Y^*) \mid \nabla \cdot \mathbf{u} = 0, \ \mathbf{u} \cdot \mathbf{n}|_{\partial T} = 0\}$.

2.2. **Direct decomposition of $L^2(O)$ and $H^1_0(O)$.** The following lemma give us a relationship between $L^2(O)$ and $L^2_{\#}(\xi, Y^*)$, for the proof the reader is referred to [2], [14] or [16].

Lemma 2.5. *Let $\phi, \psi \in L^2(O)$ then the function $\phi(\xi; \cdot)$ defined by*

$$\phi(\xi; x) = \sum_{p \in \mathbb{Z}^N} \phi(x + 2\pi p) e^{-2\pi i p \cdot \xi}, \tag{10}$$

satisfies

(i) $\phi(\xi; \cdot) \in L^2_{\#}(\xi, Y^*)$, *a.e.* $\xi \in Y'$.

(ii) $\phi(\cdot; \cdot) \in L^2(Y' \times Y^*)$, *furthermore* $\|\phi(\cdot; \cdot)\|_{L^2(Y' \times Y^*)} = \|\phi\|_{L^2(O)}$.

(iii) $\phi(x) = \int_{Y'} \phi(\xi; x) d\xi$.

(iv) $(\phi(\cdot + 2\pi m))(\xi; x) = \phi(\xi; x) e^{2\pi i m \cdot \xi}$, *for each* $\xi \in Y'$, $x \in Y^*$.

(v) $\int_O \psi(x) \overline{\phi}(x) dx = \int_{Y'} \int_{Y^*} \psi(\xi; x) \overline{\phi}(\xi; x) dx \, d\xi$.

Lemma 2.5 amounts to say that $L^2(O)$ can be written as a direct integral decomposition (see [8] and [9] for details), that is:

$$L^2(O) = \int_{Y'}^{\oplus} L^2_{\#}(\xi, Y^*) d\xi = \left\{ (f_\xi)_{\xi \in Y'} \in \prod_{\xi \in Y'} L^2_{\#}(\xi, Y^*) \mid \int_{Y'} \|f_\xi\|_{L^2(Y^*)} d\xi < \infty \right\}.$$

In the same manner, if $v \in H^1(O)$, then

$$v(\xi; x) = \sum_{p \in \mathbb{Z}^N} v(x + 2\pi p) e^{-2\pi i p \cdot \xi}, \tag{11}$$

belongs to $H^1_{\#}(\xi, Y^*)$ and satisfy (ii)-(v) of Lemma 2.5, moreover:

$$H^1(O) = \int_{Y'}^{\oplus} H^1_{\#}(\xi, Y^*) d\xi, \quad H^1_0(O) = \int_{Y'}^{\oplus} H^1_{0,\#}(\xi, Y^*) d\xi.$$

The following result provides a Poincaré type inequality for $H^1_0(O)$.

Proposition 5. *There is a constant $C > 0$ such that,*

$$\int_O |u(x)|^2 dx \leq C \int_O |\nabla u(x)|^2 dx \quad \forall \ u \in H^1_0(O).$$

Proof. By Proposition 2 there exists $C > 0$, which does not depend on ξ, so that

$$\int_{Y^*} |u(\xi; x)|^2 dx \leq C \int_{Y^*} |\nabla u(\xi; x)|^2 dx.$$

Therefore,

$$\begin{aligned}\int_O |u(x)|^2 dx &= \int_{Y'} \int_{Y^*} |u(\xi; x)|^2 dx d\xi \\ &\leq C \int_{Y'} \int_{Y^*} |\nabla u(\xi; x)|^2 dx d\xi \\ &= C \int_O |\nabla u(x)|^2 dx.\end{aligned}$$

□

The above result implies that $\|\nabla u\|_{\mathbf{L}^2(O)}$ is a norm on $H_0^1(O)$, which is equivalent to the norm of $H^1(O)$.

3. **Bloch waves for the Stokes system.** This section is devoted to the study of the Bloch waves for the Stokes system. To do this we study the Green's operator associated to this model.

3.1. **The Stokes system.** Firstly, we recall the classical De Rham's Theorem:

Proposition 6. [19] *Let Ω be an open subset of $\mathbb{R}^N$ and $f = (f_1, ..., f_N)$. $f_i \in \mathcal{D}'(\Omega)$, $i = 1, .., N$. A necessary and sufficient condition to have $f = \nabla p$ for some $p \in \mathcal{D}'(\Omega)$, is that $< f, v >= 0 \quad \forall v \in \mathbf{C}_c^\infty(\Omega)$ such that $\nabla \cdot v = 0$.*

Proposition 7. [22, pp. 14] *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$.*

(i) If a distribution p has all its first-order derivatives $D_i p$, $1 \leq i \leq N$, in $L^2(\Omega)$, then $p \in L^2(\Omega)$ and $\|p\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega) \|\nabla p\|_{L^2(\Omega)}$.

(ii) If a distribution p has all its first derivatives $D_i p$, $1 \leq i \leq N$, in $H^{-1}(\Omega)$, then $p \in L^2(\Omega)$ and $\|p\|_{L^2(\Omega)/\mathbb{R}} \leq C(\Omega) \|\nabla p\|_{H^{-1}(\Omega)}$.
In both cases if Ω is any open set in $\mathbb{R}^N$, $p \in L^2_{loc}(\Omega)$.

In this section we consider the following Stokes system

$$\left\{ \begin{array}{rcll} -\Delta \mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } O \\ \nabla \cdot \mathbf{u} &=& 0 & \text{in } O \\ \mathbf{u} &=& 0 & \text{on } \partial T_j, \quad j \in \mathbb{Z}^N. \end{array} \right. \tag{12}$$

where O is defined in (3) and T_j in (4).

In what follows we consider the spaces

$$\mathbf{V}(O) = \{\mathbf{u} \in \mathbf{H}_0^1(O) \mid \nabla \cdot \mathbf{u} = 0\}$$

and

$$\mathbf{H}(O) = \overline{\mathbf{V}(O)}^{\mathbf{L}^2(O)} = \{\mathbf{f} \in \mathbf{L}^2(O) \,|\nabla \cdot \mathbf{f} = 0,\ \mathbf{f} \cdot \mathbf{n} = 0 \text{ on } \partial T_j,\ j \in \mathbb{Z}^N\}.$$

It is not so hard to prove that

$$\mathbf{V}(O) = \int_{Y'}^{\oplus} \mathbf{V}_\#(\xi, Y^*) d\xi,\ \mathbf{H}(O) = \int_{Y'}^{\oplus} \mathbf{H}_\#(\xi, Y^*) d\xi$$

where $\mathbf{V}_\#(\xi, Y^*)$ and $\mathbf{H}_\#(\xi, Y^*)$ are defined in (9).

By taking $\mathbf{v} \in \mathbf{V}(O)$ and multiplying the first equation of (12) by $\mathbf{v}$ and integrating by parts, we obtain the following variational problem:

$$\left\{ \begin{array}{c} \text{Find } \mathbf{u} \in \mathbf{V}(O) \text{ such that} \\ \displaystyle\int_O \nabla \mathbf{u} \cdot \nabla \overline{\mathbf{v}} dx = \int_O \mathbf{f} \cdot \overline{\mathbf{v}} dx, \quad \text{for all } \mathbf{v} \in \mathbf{V}(O). \end{array} \right. \tag{13}$$

If we define
$$a(\mathbf{u},\mathbf{v}) = \int_O \nabla \mathbf{u} \cdot \nabla \overline{\mathbf{v}} dx,$$
then it is clear that a is a coercive bounded bilinear form in $\mathbf{V}(O)$, then from Lax-Milgram lemma, Proposition 7 and definition of $\mathbf{V}(O)$, we obtain

Theorem 3.1. *The problem (13) has a unique solution $\mathbf{u} \in \mathbf{V}(O)$. Furthermore, it satisfies (12) in the following weak sense: There exists a pressure $p \in L^2_{loc}(O)$ such that:*

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \text{ in the distribution sense in } O \\ \nabla \cdot \mathbf{u} &= 0 \text{ in the distribution sense in } O \\ \mathbf{u} &= 0 \text{ on } \partial T_j,\ j \in \mathbb{Z}^N, \text{ in the trace sense.} \end{aligned}$$

We define the Green's operator associated to the Stokes system
$$\begin{array}{cccc} \mathcal{G} : & \mathbf{H}(O) & \to & \mathbf{H}(O) \\ & \mathbf{f} & \mapsto & \mathbf{u}, \end{array} \tag{14}$$
where $\mathbf{u} \in \mathbf{V}(O)$ is the unique solution of (13).

3.2. **Bloch waves for the Stokes system.** Let ξ be a parameter in $Y' = [0,1[^N$ and $\mathbf{f}_\xi \in \mathbf{L}^2_\#(\xi, Y^*)$ be a given function. Let us consider the following variational problem:
$$\left\{ \begin{array}{c} \text{Find } \mathbf{u}_\xi \in \mathbf{V}_\#(\xi, Y^*) \text{ such that,} \\ \displaystyle\int_{Y^*} \nabla \mathbf{u}_\xi(x) \cdot \nabla \overline{\mathbf{v}}_\xi(x) dx = \int_{Y^*} \mathbf{f}_\xi(x) \cdot \overline{\mathbf{v}}_\xi(x) dx, \quad \forall \mathbf{v}_\xi \in \mathbf{V}_\#(\xi, Y^*), \end{array} \right. \tag{15}$$
where $\mathbf{V}_\#(\xi, Y^*) = \{\mathbf{v}_\xi \in \mathbf{H}^1_{0,\#}(\xi, Y^*) \mid \nabla \cdot \mathbf{v}_\xi = 0\}$. As in the above subsection, by using Lax-Milgram Lemma, we can prove that problem (15) has a unique solution. Moreover, by Proposition 3 we establish:

Theorem 3.2. *For every $\xi \in Y'$, the problem (15) has a unique solution $\mathbf{u}_\xi \in \mathbf{V}_\#(\xi, Y^*)$. Furthermore, there exists $p_\xi \in L^2_\#(\xi, Y^*)$ such that*
$$\left\{ \begin{aligned} -\Delta \mathbf{u}_\xi + \nabla p_\xi &= \mathbf{f}_\xi \quad \text{in the sense of } \mathbf{H}^{-1}_\#(\xi, Y^*) \\ \nabla \cdot \mathbf{u}_\xi &= 0 \quad \text{in the sense of } \mathbf{L}^2_\#(\xi, Y^*) \\ \mathbf{u}_\xi &= 0 \quad \text{in the trace sense on } \partial T. \end{aligned} \right. \tag{16}$$

Now, we define the Green's operator associated to (16)
$$\begin{array}{cccc} G(\xi) : & \mathbf{H}_\#(\xi, Y^*) & \to & \mathbf{H}_\#(\xi, Y^*) \\ & \mathbf{f}_\xi & \mapsto & \mathbf{u}_\xi, \end{array}$$
where $\mathbf{u}_\xi \in \mathbf{V}_\#(\xi, Y^*)$ is the unique solution of (15). Here $\mathbf{H}_\#(\xi, Y^*) = \{\mathbf{f} \in \mathbf{L}^2_\#(\xi, Y^*) \mid \nabla \cdot \mathbf{f} = 0 \text{ and } \mathbf{f} \cdot \mathbf{n} = 0 \text{ on } \partial T\}$. Again, as in the case of operators $\mathcal{G}$ in the above subsection, for any $\xi \in Y'$, $G(\xi)$ is bounded compact self-adjoint operator. Moreover, we have
$$\int_{Y^*} G(\xi)\mathbf{f}_\xi \cdot \mathbf{f}_\xi dx > 0, \quad \mathbf{f}_\xi \neq 0.$$
Thefore, by using the standard theory of compact self-adjoint operators, we obtain

Theorem 3.3. *Let $\xi \in Y'$ be fixed. Then there exists a subsequence of eigenvalues $(\mu_n(\xi))_{n\geq 1}$ for the operator $G(\xi)$ each of which is of finite multiplicity. As usual, repeating each value as many times as its multiplicity*

$$\mu_1(\xi) \geq \mu_2(\xi) \geq \ . \ . \ . \ \geq \mu_n(\xi) \geq \ . \ . > 0.$$

The corresponding eigenfunctions denoted by $(\Psi_n(\xi,\cdot))_{n\geq 1}$ form an orthonormal basis in $\mathbf{H}_\#(\xi, Y^*)$.

Definition 3.4. The eigenfunctions $\{\Psi_n(\xi,\cdot) \ : \ \xi \in Y', \ n \geq 1\}$ given by Theorem 3.3 are so-called Bloch waves for the Stokes system.

Remark 2. a) If we define $\lambda_n(\xi) = \dfrac{1}{\mu_n(\xi)}$, $n \in \mathbb{N}$, then Bloch waves $\Psi_n(\xi,\cdot)$ and Bloch eigenvalues $\lambda_n(\xi)$ are solution of

$$\left\{\begin{array}{rcll} -\Delta\Psi + \nabla p & = & \lambda\Psi & \text{in } Y^* \\ \nabla\cdot\Psi & = & 0 & \text{in } Y^* \\ \Psi & = & 0 & \text{on } \partial T \\ \Psi, p \text{ are } (\xi, Y^*) & - & \text{periodic functions.} & \end{array}\right. \tag{17}$$

b) From Lemma 2.5, by translating the test function $\mathbf{v}$ in $2\pi m$, $m \in \mathbb{Z}^N$, problem (13) can be written

$$\int_{Y'}\int_{Y^*} e^{2\pi i m\cdot\xi}\nabla\mathbf{u}(\xi;x)\cdot\nabla\mathbf{v}(\xi;x)dxd\xi = \int_{Y'}\int_{Y^*} e^{2\pi i m\cdot\xi}\mathbf{f}(\xi;x)\cdot\mathbf{v}(\xi;x)dxd\xi,$$

$\forall\mathbf{v} \in \mathbf{V}(O)$, $\forall m \in \mathbb{Z}^N$, therefore,

$$\int_{Y^*}\nabla\mathbf{u}(\xi;x)\cdot\nabla\mathbf{v}(\xi;x)dx = \int_{Y^*}\mathbf{f}(\xi;x)\cdot\mathbf{v}(\xi;x)dx,$$

a.e. $\xi \in Y'$, for all $\mathbf{v} \in \mathbf{V}(O)$. In other words, the Green's operators are related by:

$$\mathcal{G}\mathbf{f} = \int_{Y'} G(\xi)\mathbf{f}(\xi;\cdot)d\xi, \quad \forall\mathbf{f} \in \mathbf{H}(O).$$

Theorem 3.5. *Let $\mathbf{v} \in \mathbf{H}(O)$ be given. Then the following integrals:*

$$\hat{v}_n(\xi) = \int_O \mathbf{v}(x)\cdot\overline{\Psi}_n(\xi,x)dx \tag{18}$$

exists a.e. $\xi \in Y'$, for $n \geq 1$, where $(\Psi_n(\xi,\cdot))_{n\geq 1}$ are the Bloch waves, and they satisfy

$$\|\mathbf{v}\|^2_{\mathbf{L}^2(O)} = \int_{Y'}(\sum_{n=1}^{\infty}|\hat{v}_n(\xi)|^2)d\xi. \tag{19}$$

Moreover, we have the following inversion formula

$$\mathbf{v}(x) = \int_{Y'}\sum_{n=1}^{\infty}\hat{v}_n(\xi)\Psi_n(\xi,x)d\xi. \tag{20}$$

Remark 3. If we put $\Phi = e^{-ix\cdot\xi}\Psi$ and $q = e^{-ix\cdot\xi}p$, then problem (17) becomes:

$$\left\{\begin{array}{rcll} -D(\xi)\cdot(D(\xi)\Phi)+D(\xi)q & = & \lambda\Phi & \text{in } Y^* \\ D(\xi)\cdot\Phi & = & 0 & \text{in } Y^* \\ \Phi & = & 0 & \text{on } \partial T \\ \Phi, q \text{ are } Y^* & - & \text{periodic functions,} & \end{array}\right.$$

where $D(\xi) = \nabla + i\xi$. This is the spectral problem considered in [1] to obtain the Bloch waves for the Stokes system in a perforated domain.

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