RANDOM HOMOGENIZATION OF FRACTIONAL OBSTACLE PROBLEMS

Luis Caffarelli

Department of Mathematics, University of Texas at Austin
Austin, TX 78712, USA

Antoine Mellet

Department of Mathematics, University of British Columbia
Vancouver, BC V6T 1Z2, Canada

Abstract. We use a characterization of the fractional Laplacian as a Dirichlet to Neumann operator for an appropriate differential equation to study its obstacle problem in perforated domains.

1. **Introduction.** Given a smooth function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}^n$ and a subset T_ε of $\mathbb{R}^n$, we consider $v^\varepsilon(x)$ solution of the following obstacle problem:

$$\begin{cases} v^\varepsilon(x) \geq \varphi(x) & \text{for } x \in T_\varepsilon \\ (-\Delta)^s v^\varepsilon \geq 0 & \text{for } x \in \mathbb{R}^n \\ (-\Delta)^s v^\varepsilon = 0 & \text{for } x \in \mathbb{R}^n \setminus T_\varepsilon \text{ and for } x \in T_\varepsilon \text{ if } v^\varepsilon(x) > \varphi(x). \end{cases} \quad (1)$$

The operator $(-\Delta)^s$ denotes the fractional Laplace operator of order s, where s is a real number between 0 and 1. It can be defined using Fourier transform, by $\mathcal{F}((-\Delta)^s f)(\xi) = |\xi|^{2s} \widehat{f}(\xi)$. In particular, (1) can be seen as the Euler-Lagrange equation for the minimization of the $\dot{H}^s$ norm $||f||_{\dot{H}^s} = ||\widehat{f}(\xi)|\xi|^s||_{L^2}$ with the constrain that $f \geq \varphi$ on T_ε. We will see that this system of equations can also be stated as a boundary obstacle problem for elliptic degenerate equations.

In (1), the domain $\mathbb{R}^n$ is perforated and the obstacle $\varphi(x)$ is viewed by $v^\varepsilon(x)$ only on the subset T_ε. A typical example of T_ε is given by:

$$T_\varepsilon = \bigcup_{k \in \mathbb{Z}^n} B_{a^\varepsilon}(\varepsilon k), \quad (2)$$

with $a^\varepsilon \ll \varepsilon$. The goal of this paper is to study the asymptotic behavior of v^ε as $\varepsilon \to 0$. When T_ε is given by (2), the effective equation satisfied by the limit of v^ε strongly depends on the radius a^ε: If a^ε is large enough, the limit turns out to be an obstacle problem with obstacle $\varphi(x)$. On the other hand, if a^ε is small then the limiting problem is a simple elliptic equation without any obstacle condition. It is well known in the case of the regular Laplace operator $(s = 1)$ that there is a critical size for a^ε for which interesting behavior arises.

In the case of the regular Laplace operator, this problem was first studied for periodic T_ε by L. Carbone and F. Colombini [2] and then in a more general framework by E. De Giorgi, G. Dal Maso and P. Longo [9] and G. Dal Maso and P. Longo [11], G. Dal Maso [10]. Our main reference will be the papers of D. Cioranescu and F. Murat [3, 4], in which the case of a periodic distribution of balls is studied. More precisely, they prove that when $s = 1$ and if T_ε is given by (2) with $a^\varepsilon = r_0\,\varepsilon^{\frac{n}{n-2}}$, then the function $v = \lim_{\varepsilon\to 0} v^\varepsilon$ solves

$$-\Delta v - \mu(v - \varphi)_- = 0,$$

where μ is a real number (depending on r_0) and $w_- = \max(-w, 0)$. The obstacle condition thus disappears when ε goes to zero, but it gives rise to a new term $\mu(v - \varphi)_-$ in the equation.

In [5], we generalize this result (still with $s = 1$) to sets T_ε that are the union of small sets $S_\varepsilon(k) \subset \mathbb{R}^n$ still periodically distributed, but with random sizes and shapes. More precisely, we introduce a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and we assume that for every $\omega \in \Omega$ and every $\varepsilon > 0$ we are given some subsets $S_\varepsilon(k, \omega)$ such that

$$S_\varepsilon(k, \omega) \subset B_\varepsilon(\varepsilon k).$$

We then consider

$$T_\varepsilon(\omega) = \bigcup_{k\in\mathbb{Z}^n} S_\varepsilon(k, \omega).$$

The only assumptions necessary to generalize the result of D. Cioranescu and F. Murat [3]-[4] is that each set $S_\varepsilon(k, \omega)$ is of capacity of order ε^n: $\mathrm{cap}(S_\varepsilon(k, \omega)) = \varepsilon^n \gamma(k, \omega)$ (this is where the critical exponent $\varepsilon^{\frac{n}{n-2}}$ comes from) and that the $\gamma(k, \omega)$ have some averaging properties (stationary ergodicity).

In the present paper, we extend the result of [5] to the case of fractional Laplace operators $s \in (0, 1)$. We will show that under appropriate assumptions on the size of the sets $S_\varepsilon(k, \omega)$, the function $v(x) = \lim_{\varepsilon\to 0} v^\varepsilon(x)$ solves

$$(-\Delta)^s v - \mu(v - \varphi)_- = 0.$$

In the particular case of sets T_ε of the form (2), the critical size is now given by

$$a^\varepsilon = r_0\,\varepsilon^{\frac{n}{n-2s}}$$

(the critical exponent $\frac{n}{n-2s}$ is related to the s-capacity of the sets $S_\varepsilon(k, \omega)$).

In the remainder of this section, we briefly motivate the problem and we introduce the extension problem for the fractional Laplace operators, which allows us to rewrite (1) as a boundary obstacle problem for a local (degenerate) elliptic operator. The precise hypothesis on $T_\varepsilon(\omega)$ will be detailed in the following section in which the precise statement of the main theorem is also given. The remainder of the paper is devoted to the proof of our main statement.

1.1. **A semipermeable membrane problem.** When $s = 1/2$, (1) naturally arises as a boundary obstacle problem for the regular Laplace operator (also know as Signorini problem): We consider the following problem set in the upper-half space

$\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \, ; \, y \geq 0\}$:

$$\begin{cases} -\Delta u(x,y) = 0 & \text{for } (x,y) \in \mathbb{R}^{n+1}_+ \\ u(x,0) \geq \varphi(x) & \text{for } x \in T_\varepsilon \\ \partial_y u(x,0) \leq 0 & \text{for } x \in \mathbb{R}^n \\ \partial_y u(x,0) = 0 & \text{for } x \in \mathbb{R}^n \setminus T_\varepsilon \text{ and for } x \in T_\varepsilon \text{ if } u(x,0) > \varphi(x) \end{cases} \tag{3}$$

with the boundary condition

$$\lim_{y \to \infty} u(x,y) = 0.$$

It is then well-known that $v(x) = u(x,0)$ is solution of (1) with $s = 1/2$ (see [14] and [7] for details).

It can be of interest to state equation (3) in a bounded domain $D \subset \mathbb{R}^{n+1}_+$: Introducing

$$\Sigma = D \cap \{y = 0\} \quad \text{and} \quad \Gamma = \partial D \cap \{y > 0\},$$

we can consider the following boundary obstacle problem:

$$\begin{cases} -\Delta u(x,y) = 0 & \text{for } (x,y) \in D \\ u(x,0) \geq \varphi(x) & \text{for } x \in \Sigma \cap T_\varepsilon \\ \partial_y u(x,0) \leq 0 & \text{for } x \in \Sigma \\ \partial_y u(x,0) = 0 & \text{for } x \in \Sigma \setminus T_\varepsilon \text{ and for } x \in T_\varepsilon \text{ if } u(x,0) > \varphi(x) \end{cases} \tag{4}$$

with the boundary condition

$$u(x,y) = g(x,y) \text{ for } (x,y) \in \Gamma.$$

Equation (4) arises, for instance, in the modeling of diffusion through semi-permeable membranes (such as the membrane of a cell): The membrane is modeled by the surface Σ. The outside concentration of molecules is given by $\varphi(x)$, and the transport of molecules through the membrane and in the direction of the concentration gradient is possible only across some given channels (represented by the set T_ε) and only from the outside of the cell ($\{y < 0\}$) toward the inside of the cell D. At equilibrium, the concentration inside the cell is then given by the solution $u(x,y)$ of (4).

1.2. **An extension problem for fractional obstacle problems.** Following L. Caffarelli, S. Salsa and L. Silvestre [7], we can actually rewrite (1) as a boundary obstacle problem for all fractional powers $s \in (0,1)$. We rely for this on the following extension formula established by L. Caffarelli and L. Silvestre [6]: For a given function $f(x)$ defined in $\mathbb{R}^n$, and for $a \in (-1,1)$, if we define $u(x,y)$ by

$$\begin{cases} -\text{div}\,(y^a \nabla u) = 0 & \text{for } (x,y) \in \mathbb{R}^{n+1}_+ \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{5}$$

then

$$(-\Delta)^s f(x) = \lim_{y \to 0} y^a \partial_y u(x,y)$$

with

$$s = (1-a)/2.$$

We can thus rewrite the fractional obstacle problem (1) as follows:

$$\begin{cases} -\mathrm{div}\,(y^a \nabla u^\varepsilon) = 0 & \text{for } (x,y) \in \mathbb{R}^{n+1}_+ \\ u^\varepsilon(x,0) \geq \varphi(x) & \text{for } x \in T_\varepsilon \\ \lim_{y \to 0} y^a \partial_y u^\varepsilon(x,y) \leq 0 & \text{for } x \in \mathbb{R}^n \\ \lim_{y \to 0} y^a \partial_y u^\varepsilon(x,y) = 0 & \text{for } x \in \mathbb{R}^n \setminus T_\varepsilon \text{ and } x \in T_\varepsilon \cap \{u^\varepsilon > \varphi\} \end{cases} \tag{6}$$

where $a = 1 - 2s$ (note that $a \in (-1,1)$). Our main result will concern problems such as (6) with possibly bounded domain D instead of $\mathbb{R}^{n+1}_+$.

In the sequel, the theory of degenerate elliptic equations in weighted Sobolev spaces will play an important role. We refer to [13] for many results that will be used in this paper.

1.3. **Variational formulation.** The system of equations (6) can also be written as a minimization problem. For a given open subset D of $\mathbb{R}^{n+1}_+$, we denote by $L^2(D, |y|^a)$ the weighted L^2 space with weight $|y|^a$ and by $W^{1,2}(D, |y|^a)$ the corresponding Sobolev's space. We have

$$||u||^2_{W^{1,2}(D,|y|^a)} = \int_D |y|^a |u|^2 \, dx \, dy + \int_D |y|^a |\nabla u|^2 \, dx \, dy.$$

We then introduce the energy functional:

$$\mathscr{I}(u) = \int_D \frac{1}{2} |y|^a |\nabla u|^2 \, dx \, dy$$

and the set

$$K_\varepsilon = \left\{ v \in W^{1,2}(D, |y|^a) \,;\, v(x,0) \geq \varphi(x) \text{ for } x \in T_\varepsilon(\omega)\,,\, v = g \text{ on } \Gamma \right\}.$$

It is readily seen that (6) is the Euler-Lagrange equation associated to the minimization problem:

$$\mathscr{I}(u^\varepsilon) = \inf_{v \in K_\varepsilon} \mathscr{I}(v), \qquad u^\varepsilon \in K_\varepsilon. \tag{7}$$

(Note that since K_ε is closed, convex and not empty, (7) has a unique solution $u^\varepsilon \in K_\varepsilon$).

Finally, we notice (see [6]) that if $u(x,y)$ is the extension of a function $f(x)$ as in (5), then

$$\int_{\mathbb{R}^{n+1}_+} |y|^a |\nabla u|^2 \, dx \, dy = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi = ||f||^2_{\dot{H}^s(\mathbb{R}^n)}.$$

In particular, the minimization problem (7) is equivalent to the variational formulation of problem (1).

In this paper, we study the asymptotic behavior of the solutions of (7) for any open subset D of $\mathbb{R}^{n+1}_+$. The assumptions and the main result are made precise in the next section. The proof of the main theorem, which is detailed in Section 3, relies on the construction of an appropriate corrector. This construction is detailed in Sections 4 and 5.

2. Assumptions and main result.

2.1. The set T_ε. We consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. For all $\omega \in \Omega$, the set $T_\varepsilon(\omega)$ is given by:

$$T_\varepsilon(\omega) = \bigcup_{k \in \mathbb{Z}^n} S_\varepsilon(k, \omega),$$

where the sets $S_\varepsilon(k, \omega) \subset \mathbb{R}^n$ satisfy the following assumptions:

Assumption 1: *For all $k \in \mathbb{Z}^n$ and $\omega \in \Omega$, there exists $\gamma(k, \omega)$ (independent of ε) such that*

$$\mathrm{cap}_s(S_\varepsilon(k, \omega)) = \varepsilon^n \, \gamma(k, \omega),$$

where $\mathrm{cap}_s(A)$ denotes the s-capacity of subset A of $\mathbb{R}^{n+1}$ (defined below).
   Moreover, we assume that

$$S_\varepsilon(k, \omega) \subset B_{M\varepsilon^{\frac{n}{n-2s}}}(\varepsilon k) \text{ for all } k \in \mathbb{Z}^n \text{ a.e. } \omega \in \Omega, \tag{8}$$

for some large constant M, and that there exists a constant $\overline{\gamma} > 0$ such that

$$\gamma(k, \omega) \leq \overline{\gamma} \quad \text{ for all } k \in \mathbb{Z}^n \text{ and a.e. } \omega \in \Omega. \tag{9}$$

This first assumption defines the critical size of the set T_ε. It will guarantee that $\mathrm{cap}_s(T_\varepsilon)$ remains finite as ε goes to zero. A natural definition for the s-capacity of a subset A of $\mathbb{R}^n$ is the following:

$$\mathrm{cap}_s(A) = \inf \left\{ \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \, ; \, f \in H_0^s(\mathbb{R}^n), \, f(x) \geq 1 \text{ for } x \in A \right\}$$

(note that the case $s = 1$ corresponds to the "classical" capacity involving the H_0^1 norm). Using the extension problem for the fractional Laplace operator (see [6] for details), an equivalent definition (up to a multiplicative constant) is given by

$$\mathrm{cap}_s(A) = \inf \left\{ \int_{\mathbb{R}^{n+1}} y^a |\nabla h|^2 \, dx \, dy \, ; \, h \in W_0^{1,2}(\mathbb{R}_+^{n+1}, |y|^a), \, h(x, 0) \geq 1, \, x \in A \right\}.$$

We will use this second definition in this paper. If B_r^n is a n-dimensional ball, then its s-capacity in $\mathbb{R}^{n+1}$ is given by

$$\mathrm{cap}_s(B_r^n) = c_{n+1-a} r^{n-1+a} = c_{n+2s} r^{n-2s}$$

for some constant c_k. Assumption **1** is thus satisfied in particular if the sets $S_\varepsilon(k, \omega)$ are balls centered on $\varepsilon \mathbb{Z}^n$ with radius $r(k, \omega) \varepsilon^{\frac{n}{n-2s}}$.

Assumption 2: *The process $\gamma : \mathbb{Z}^n \times \Omega \mapsto [0, \infty)$ is stationary ergodic: There exists a family of measure-preserving transformations $\tau_k : \Omega \to \Omega$ satisfying*

$$\gamma(k + k', \omega) = \gamma(k, \tau_{k'} \omega) \quad \text{ for all } k, k' \in \mathbb{Z}^n \text{ and } \omega \in \Omega,$$

and such that if $A \subset \Omega$ and $\tau_k A = A$ for all $k \in \mathbb{Z}^n$, then $P(A) = 0$ or $P(A) = 1$ (the only invariant set of positive measure is the whole set).

This second assumption is necessary to ensure that some averaging process occurs as ε goes to zero (the hypothesis of stationarity is the most general extension of the notions of periodicity and almost periodicity for a function to have some self-averaging behavior).

2.2. **Main result.** We are now ready to state our main result:

Theorem 2.1. *Let D be an open subset of $\mathbb{R}^{n+1}_+$ ($n \geq 2$), denote*

$$\Sigma = D \cap \{y = 0\}, \qquad \Gamma = \partial D \cap \{y > 0\}$$

*and let $T_\varepsilon(\omega)$ be a subset of Σ satisfying Assumptions **1** and **2** above.*

There exists a constant $\alpha_0 \geq 0$ such that for any $\varphi(x,y) \in \mathcal{C}^{1,1}(D)$ the solution $u^\varepsilon(x,y,\omega)$ of

$$\min\left\{\frac{1}{2}\int_D y^a |\nabla v|^2 \, dx \, dy \, ; \, v \in W_0^{1,2}(D, |y|^a), \, v(x,0) \geq \varphi(x,0) \text{ for } x \in T_\varepsilon(\omega)\right\},$$

converges $W^{1,2}(D, |y|^a)$-weak and almost surely $\omega \in \Omega$ to a function $\overline{u}(x,y)$ solution of the following minimization problem

$$\min\left\{\frac{1}{2}\int_D y^a |\nabla v|^2 \, dx \, dy + \frac{1}{2}\int_\Sigma \alpha_0 (v - \varphi)^2_-(x,0) \, dx \, ; \, v \in W_0^{1,2}(D, |y|^a)\right\},$$

where $w_- = \max(0, -w)$.

If, moreover, there exists $\underline{\gamma} > 0$ such that $\gamma(k, \omega) \geq \underline{\gamma}$ for all $k \in \mathbb{Z}^n$ and a.e. $\omega \in \Omega$, then $\alpha_0 > 0$.

In particular the function $\overline{u}(x,y)$ solves

$$\begin{cases} -\operatorname{div}(y^a \nabla \overline{u}) = 0 & \text{for } (x,y) \in D \\ \lim_{y \to 0} y^a \partial_y \overline{u}(x,y) = \alpha_0 (\overline{u} - \varphi)_-(x,0) & \text{for } x \in \Sigma \\ \overline{u}(x,y) = 0 & \text{for } (x,y) \in \Gamma \end{cases}$$

Remark 2.2. When D is a bounded subset of $\mathbb{R}^{n+1}_+$, the condition $u \in W_0^{1,2}(D, |y|^a)$ could easily be replaced by

$$u \in W^{1,2}(D, |y|^a), \qquad u(x,y) = g(x,y) \text{ for } x \in \partial D \cap \{y > 0\}$$

for some function $g(x,y) \in L^\infty(\partial D \cap \{y > 0\})$.

We stated Theorem 2.1 in its most general form. It contains the semipermeable membrane problem, as well as our original problem (1) with the fractional operator. More precisely, if we have $D = \mathbb{R}^{n+1}_+$ and if we consider the trace $v(x) = \overline{u}(x,0)$ in Theorem 2.1 we get:

Corollary 2.3. *Let T_ε be a subset of $\mathbb{R}^n$ ($n \geq 2$) satisfying Assumptions **1** and **2** above. There exists $\alpha_0 \geq 0$ such that for any $\varphi(x) \in C^{1,1}(\mathbb{R}^n)$, the solution $v^\varepsilon(x,\omega)$ of (1) converges, as $\varepsilon \to 0$, $H^s(D)$-weak and almost surely to a function $\overline{v}(x)$ solution of*

$$(-\Delta)^s v - \alpha_0 (v - \varphi)_- = 0.$$

As in Cioranescu - Murat [3, 4] and Caffarelli-Mellet [5], the proof of Theorem 2.1 relies on the construction of an appropriate corrector. More precisely, we use the following result:

Proposition 2.4. *Let $T_\varepsilon(\omega)$ be a subset of $\mathbb{R}^n$ satisfying Assumptions **1** and **2** above. There exists a non-negative constant α_0 such that for every bounded subset*

D of $\mathbb{R}^{n+1}_+$, there is a function $w^\varepsilon_0(x, y, \omega)$ defined in D and satisfying

$$w^\varepsilon(x, 0) = 1 \quad for\ x \in T_\varepsilon(\omega) \cap (D \cap \{y = 0\}) \tag{10}$$

$$\|w^\varepsilon\|_{L^\infty(D)} \leq C \tag{11}$$

$$w^\varepsilon \longrightarrow 0 \qquad W^{1,2}(D, |y|^a)\text{-weak a.s. } \omega \in \Omega \tag{12}$$

and

$$\begin{cases} \textit{For all sequences } v^\varepsilon(x, y, \omega) \textit{ satisfying:} \\ \quad \begin{cases} v^\varepsilon(x, 0) \geq 0 \quad \textit{for } x \in T_\varepsilon(\omega) \cap \Sigma \\ \|v^\varepsilon\|_{L^\infty(D)} \leq C \\ v^\varepsilon \longrightarrow v \quad \textit{in } W^{1,2}(D, |y|^a) - \textit{weak, a.s.} \end{cases} \\ \textit{and for any } \phi \in \mathcal{D}(D) \textit{ such that } \phi \geq 0, \textit{ we have:} \\ \quad \lim_{\varepsilon \to 0} \int_D y^a \nabla w^\varepsilon \cdot \nabla v^\varepsilon \phi\, dx\, dy \geq -\int_\Sigma \alpha_0\, \phi\, dx \\ \textit{with equality if } v^\varepsilon(x, 0) = 0 \textit{ for } x \in T_\varepsilon \cap \Sigma. \end{cases} \tag{13}$$

The proof of Proposition 2.4 will occupy most of this paper. We stress the fact that Assumptions **1** and **2** are sufficient but by no mean necessary to the proof of this Proposition. Any set $T_\varepsilon(\omega)$ such that Proposition 2.4 holds would be admissible for Theorem 2.1.

The condition (13) may seem rather obscure and the next Lemma will suggest a nicer (but stronger) condition to replace it. However (13) is the condition that appears naturally in the proof of Theorem 2.1.

Lemma 2.5. *Let D be a bounded subset of $\mathbb{R}^{n+1}_+$, and assume that w^ε satisfies*

$$\begin{cases} -\operatorname{div}(y^a \nabla w^\varepsilon) = 0 & \textit{for } (x, y) \in D \\ w^\varepsilon(x, 0) = 1 & \textit{for } x \in T_\varepsilon(\omega) \cap \Sigma \\ \lim_{y \to 0} y^a \partial_y w^\varepsilon(x, y) = \alpha_0 & \textit{for } x \in \Sigma_\varepsilon \cap \Sigma \\ \lim_{y \to 0} y^a \partial_y w^\varepsilon(x, y) \leq 0 & \textit{for } x \in T_\varepsilon \cap \Sigma \end{cases} \tag{14}$$

together with (11) and (12). Then (13) holds.

This lemma also gives an indication of how to construct $w^\varepsilon(x, y, \omega)$: We will look for a constant α_0 such that the solution of (14) converges to zero in $W^{1,2}(D, |y|^a)$-weak.

Proof. Let $v^\varepsilon \in L^\infty(D) \cap W^{1,2}(D, |y|^a)$ be such that $v^\varepsilon(x, 0) \geq 0$ on $T_\varepsilon \cap \Sigma$ and let ϕ be a smooth test function with compact support in D. Then, we have:

$$\begin{aligned} 0 &= \int_D \operatorname{div}(y^a \nabla w^\varepsilon) \phi\, v^\varepsilon\, dx\, dy \\ &= -\int_D y^a \phi \nabla w^\varepsilon \cdot \nabla v^\varepsilon\, dx\, dy - \int_D y^a \nabla \phi \cdot \nabla w^\varepsilon v^\varepsilon\, dx\, dy \\ &\quad - \int_{\Sigma \setminus T_\varepsilon} \lim_{y \to 0} (y^a \partial_y w^\varepsilon)\, v^\varepsilon\, \phi\, dx - \int_{T_\varepsilon} \lim_{y \to 0} (y^a \partial_y w^\varepsilon)\, v^\varepsilon\, \phi\, dx. \end{aligned}$$

Since $\lim_{y \to 0} y^a \partial_y w^\varepsilon(x, y) \leq 0$ and $v^\varepsilon(x, 0) \geq 0$ for $x \in T_\varepsilon$, we deduce:

$$\int_D y^a \phi \nabla w^\varepsilon \cdot \nabla v^\varepsilon \, dx \, dy \; \geq \; -\int_D y^a \nabla \phi \nabla w^\varepsilon v^\varepsilon \, dx \, dy - \int_{\Sigma_\varepsilon} \alpha_0 \, v^\varepsilon \, \phi \, dx.$$

$$\geq \; -\int_D y^a \nabla \phi \nabla w^\varepsilon v^\varepsilon \, dx \, dy - \int_\Sigma \alpha_0 \, v^\varepsilon \, \phi \, dx. \qquad (15)$$

with equality if $v^\varepsilon(x, 0) = 0$ for $x \in T_\varepsilon$. In order to pass to the limit in (15), we note that we have the following convergences:

$$w^\varepsilon \longrightarrow 0 \qquad W^{1,2}(D, |y|^a)\text{-weak} \qquad \text{a.s. } \omega \in \Omega,$$

and

$$v^\varepsilon \longrightarrow v \qquad L^2(D, |y|^a)\text{-strong} \qquad \text{a.s. } \omega \in \Omega.$$

Hence the first term in the right hand side of (15) goes to zero. Moreover we have

$$v^\varepsilon(\cdot, 0) \longrightarrow v(\cdot, 0) \qquad H^s(\Sigma)\text{-weak and } L^2(\Sigma)\text{-strong} \qquad \text{a.s. } \omega \in \Omega,$$

so (15) gives

$$\lim_{\varepsilon \to 0} \int_D y^a \phi \nabla w^\varepsilon \cdot \nabla v^\varepsilon \, dx \, dy \geq -\int_\Sigma \alpha_0 \, v \, \phi \, dx.$$

with equality if $v^\varepsilon(x, 0) = 0$ for $x \in T_\varepsilon$.   □

2.3. Related problems.

Before turning to the proof of Theorem 2.1, we briefly mention other results that follow from Proposition 2.4: If we consider energy functionals of the form

$$\mathscr{J}(v) = \frac{1}{2} \int_D y^a |\nabla u|^2 \, dx \, dy + \int_\Sigma u \, h \, dx$$

for some $h \in L^\infty(\Sigma)$, then a proof similar to that of Theorem 2.1 shows that the homogenization of the following equation

$$\begin{cases} v^\varepsilon(x) \geq \varphi(x) & \text{for } x \in T_\varepsilon \\ (-\Delta)^s v^\varepsilon \geq h(x) & \text{for } x \in \mathbb{R}^n \\ (-\Delta)^s v^\varepsilon = h(x) & \text{for } x \in \mathbb{R}^n \setminus T_\varepsilon \text{ and on } T_\varepsilon \text{ if } v^\varepsilon > \varphi \end{cases}$$

leads to

$$(-\Delta)^s v - \alpha_0 (v - \varphi)_- = h \qquad \text{in } \mathbb{R}^n.$$

More interestingly, we can replace the constrain $v^\varepsilon \geq \varphi$ on T_ε by a Dirichlet condition of the form $v^\varepsilon = 0$ on T_ε. This amounts to minimizing $\mathscr{J}(v)$ in the convex set

$$K_\varepsilon = \{ v \in W^{1,2}(\mathbb{R}_+^{n+1}, |y|^a) \, ; \, v(x, 0) = 0 \text{ for } x \in T_\varepsilon(\omega) \}.$$

The corresponding Euler equation is

$$\begin{cases} (-\Delta)^s v^\varepsilon(x) = h(x) & \text{for } x \in \mathbb{R}^n \setminus T_\varepsilon \\ v^\varepsilon(x) = 0 & \text{for } x \in \partial T_\varepsilon. \end{cases}$$

We can then show that the solution $v^\varepsilon(x)$ converges to a function $v(x)$ solution of

$$(-\Delta)^s v - \alpha_0 v = h \qquad \text{in } \mathbb{R}^n.$$

3. **Proof of Theorem 2.1.** In this section, we prove that Theorem 2.1 follows from Proposition 2.4. For the sake of simplicity, we assume that D is a bounded domain in $\mathbb{R}^{n+1}_+$. This allows us to take the corrector $w^\varepsilon(x, y, \omega)$ given by Proposition 2.4 and corresponding to the domain D. When D is unbounded, we note that every integral involving w^ε is computed with a compactly supported test function ϕ. We can thus use, for each of them, the corrector w^ε corresponding to the domain supp ϕ. The final result is of course independent of w^ε.

The maximum principle and the natural energy estimate easily give that u^ε is bounded in $L^\infty(D) \cap W^{1,2}_0(D, |y|^a)$ almost surely. In particular, there exists a function $\overline{u}(x, y, \omega)$ such that

$$u^\varepsilon \longrightarrow \overline{u} \qquad W^{1,2}_0(D, |y|^a) - \text{weak} \quad \text{a.e. } \omega \in \Omega.$$

In order to prove Theorem 2.1, we have to show that

$$\mathscr{J}_\alpha(\overline{u}) = \inf_{v \in W^{1,2}_0(D, |y|^a)} \mathscr{J}_\alpha(v) \quad \text{a.e. } \omega \in \Omega \tag{16}$$

where $\mathscr{J}_\alpha$ is the energy associated to the limiting problem, given by:

$$\mathscr{J}_\alpha(v) = \frac{1}{2} \int_D y^a |\nabla v|^2 \, dx \, dy + \frac{1}{2} \int_\Sigma \alpha_0 (u - \varphi)^2_- \, dx.$$

Equality (16) will be a consequence of the following lemmas:

Lemma 3.1. *For any test function $\phi \in \mathcal{D}(D)$, we have*

$$\lim_{\varepsilon \to 0} \int_D y^a |\nabla w^\varepsilon|^2 \phi \, dx \, dy = \int_\Sigma \alpha_0 \phi \, dx.$$

Lemma 3.2. *Let u^ε be a bounded sequence in $W^{1,2}(D, |y|^a) \cap L^\infty(D)$. If*

$$u^\varepsilon \rightharpoonup \overline{u} \quad \text{in } W^{1,2}(D, |y|^a)\text{-weak,}$$

then

$$\liminf_{\varepsilon \to 0} \mathscr{J}(u^\varepsilon) \geq \mathscr{J}_\alpha(\overline{u}).$$

Proof of Theorem 2.1. For any $v \in \mathcal{D}(D)$, we consider the function $v + (v - \varphi)_- w^\varepsilon$ (note that this function satisfies the obstacle constrain). Its energy is given by:

$$\mathscr{J}(v + (v - \varphi)_- w^\varepsilon)$$
$$= \frac{1}{2} \int_D y^a \Big[|\nabla v|^2 + |\nabla(v - \varphi)_-|^2 w^{\varepsilon 2} + |(v - \varphi)_-|^2 |\nabla w^\varepsilon|^2 \Big] \, dx \, dy$$
$$+ \int_D y^a \Big[(v - \varphi)_- \nabla(v - \varphi)_- w^\varepsilon \nabla w^\varepsilon + \nabla v \nabla(v - \varphi)_- w^\varepsilon$$
$$+ \nabla v (v - \varphi)_- \nabla w^\varepsilon \Big] \, dx \, dy.$$

Lemma 3.1 and the weak convergence of w^ε to 0 in $W^{1,2}(D, |y|^a)$ thus implies

$$\lim_{\varepsilon \to 0} \mathscr{J}(v + (v - \varphi)_- w^\varepsilon) = \mathscr{J}_\alpha(v).$$

Morever, it is readily seen that the function $v + (v - \varphi)_- w^\varepsilon$ belongs to K_ε. Since u^ε minimizes $\mathscr{J}$ on K_ε, we deduce

$$\mathscr{J}(v + (v - \varphi)_- w^\varepsilon) \geq \mathscr{J}(u^\varepsilon),$$

and therefore

$$\mathscr{J}_\alpha(v) \geq \limsup_{\varepsilon \to 0} \mathscr{J}(u^\varepsilon) \qquad \text{for all } v \in \mathcal{D}(D).$$

On the other hand, Lemma 3.2 gives

$$\liminf_{\varepsilon \to 0} \mathscr{J}(u^\varepsilon) \geq \mathscr{J}_\alpha(\overline{u}).$$

and so

$$\mathscr{J}_\alpha(\overline{u}) \leq \mathscr{J}_\alpha(v) \qquad \text{for all } v \in \mathcal{D}(D).$$

Equality (16) follows by a density argument. $\qquad\qquad\square$

Proof of Lemma 3.1. This first lemma is a straightforward consequence of (13): If we take $v^\varepsilon = 1 - w^\varepsilon$, we have $v^\varepsilon(x, 0) = 0$ for $x \in T_\varepsilon$, $v^\varepsilon(x, y)$ bounded in $L^\infty(D)$ and $v^\varepsilon(x, y)$ converges to 1 in $W^{1,2}(D, |y|^a)$-weak, $L^2(D, |y|^a)$-strong, and almost surely $\omega \in \Omega$. We can thus use (13), which implies

$$-\int_D y^a \phi \nabla w^\varepsilon \cdot \nabla(1 - w^\varepsilon) \, dx \, dy \longrightarrow \int_\Sigma \alpha_0 \, \phi \, dx,$$

and so

$$\int_D y^a \phi |\nabla w^\varepsilon|^2 \, dx \, dy \longrightarrow \int_\Sigma \alpha_0 \, \phi \, dx$$

for all $\phi \in \mathcal{D}(D)$. $\qquad\qquad\square$

Proof of Lemma 3.2. Following Cioranescu-Murat (see [4], Proposition 3.1), we evaluate the quantity

$$\int_D |y|^a |\nabla(u^\varepsilon - (z + (z - \varphi)_- w^\varepsilon))|^2 \, dx \, dy$$

for some test function z with compact support in D and then take the limit as ε goes to zero.

Using (12), we obtain:

$$\liminf_{\varepsilon \to 0} \int_D y^a |\nabla u^\varepsilon|^2 \, dx \, dy \;\geq\; 2 \int_D y^a \nabla \overline{u} \cdot \nabla z \, dx \, dy - \int_D y^a |\nabla z|^2 \, dx \, dy$$

$$+ 2 \lim_{\varepsilon \to 0} \int_D y^a (z - \varphi)_- \nabla u^\varepsilon \cdot \nabla w^\varepsilon \, dx \, dy$$

$$- \lim_{\varepsilon \to 0} \int_D y^a (z - \varphi)_-^2 |\nabla w^\varepsilon|^2 \, dx \, dy.$$

Lemma 3.1 yields

$$\lim_{\varepsilon \to 0} \int_D y^a (z - \varphi)_-^2 |\nabla w^\varepsilon|^2 \, dx \, dy = \int_\Sigma \alpha_0 (z - \varphi)_-^2 \, dx.$$

Property (13), together with the facts that $u^\varepsilon \in L^\infty(D)$ and $(u^\varepsilon - \varphi)(x,0) \geq 0$ for $x \in T_\varepsilon$, implies

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \int_D y^a(z-\varphi)_- \nabla u^\varepsilon \cdot \nabla w^\varepsilon &= \lim_{\varepsilon \to 0} \int_D y^a(z-\varphi)_- \nabla(u^\varepsilon - \varphi) \cdot \nabla w^\varepsilon \, dx \, dy \\
&\quad + \lim_{\varepsilon \to 0} \int_D y^a(z-\varphi)_- \nabla\varphi \cdot \nabla w^\varepsilon \, dx \, dy \\
&\geq - \int_\Sigma \alpha_0(\overline{u}-\varphi)(z-\varphi)_- \, dx.
\end{aligned}
$$

It follows that for any test function $z \in \mathcal{D}(D)$ we have:

$$
\begin{aligned}
\liminf_{\varepsilon \to 0} \int_D y^a|\nabla u^\varepsilon|^2 \, dx \, dy &\geq 2\int_D y^a \nabla\overline{u} \cdot \nabla z \, dx \, dy - \int_D y^a|\nabla z|^2 \, dx \, dy \\
&\quad -2\int_\Sigma \alpha_0(\overline{u}-\varphi)(z-\varphi)_- \, dx \\
&\quad - \int_\Sigma \alpha_0(z-\varphi)_-^2 \, dx.
\end{aligned}
$$

We can now take a sequence z_n that converges to $\overline{u}$ strongly in $W^{1,2}(D, |y|^a)$ and such that $z_n(\cdot, 0)$ converges to $\overline{u}(\cdot, 0)$ strongly in $L^2(\Sigma, |y|^a)$. Using the fact that $(\overline{u}-\varphi)(\overline{u}-\varphi)_- = -(\overline{u}-\varphi)_-^2$, we get

$$
\liminf_{\varepsilon \to 0} \int_D y^a|\nabla u^\varepsilon|^2 \, dx \, dy \geq \int_D y^a|\nabla \overline{u}|^2 \, dx \, dy + \int_\Sigma \alpha_0(\overline{u}-\varphi)_-^2 \, dx.
$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. The auxiliary corrrector.

4.1. Notations and scheme of the proof. We recall that

$$
\mathbb{R}_+^{n+1} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \, ; \, y \geq 0\},
$$

and we fix a bounded domain $D \subset \mathbb{R}_+^{n+1}$. For any $x_0 \in \mathbb{R}^n$ and $y_0 > 0$, we introduce the following notation for the Euclidian balls:

$$
B_r(x_0, y_0) = \left\{(x,y) \in \mathbb{R}^{n+1} \, ; \, \left(|x-x_0|^2 + |y-y_0|^2\right)^{1/2} \leq r\right\},
$$

$$
B_r^+(x_0, 0) = B_r(x_0, 0) \cap \{y > 0\},
$$

$$
B_r^n(x_0) = \{x \in \mathbb{R}^n \, ; \, |x - x_0| \leq r\}.
$$

4.1.1. *The fundamental solution.* We recall (see [7] for details) that the function

$$
h(x,y) = \frac{\nu_{n+1+a}}{|x^2 + y^2|^{\frac{n-1+a}{2}}} \qquad \text{with} \qquad \nu_k = \frac{\pi^{\frac{k}{2}}\Gamma(\frac{k-1}{2})}{4},
$$

solves

$$
\begin{cases}
-\mathrm{div}\,(y^a \nabla h)(x,y) = 0 & \text{for } y > 0 \\
\lim_{y \to 0} y^a \partial_y h(x,y) \longrightarrow -\delta(x),
\end{cases}
$$

where $\delta(x)$ denotes the Dirac distribution centered at 0 in $\mathbb{R}^n$. We also have

$$
\mathrm{div}\,(y^a \nabla h) = -\mu_{n,a}\delta(x,y) \qquad \text{in } \mathbb{R}^{n+1}
$$

where $\delta(x,y)$ denotes the Dirac distribution centered at 0 in $\mathbb{R}^{n+1}$ and for some constant $\mu_{n,a}$.

4.1.2. *An auxiliary corrector.* One of the key points in the proof of Proposition 2.4 is to see that away from εk, the set $S_\varepsilon(k, \omega)$ is equivalent to a (n+1)-dimensional ball. More precisely, we introduce the capacitary potential $\varphi_k^\varepsilon(x, y, \omega)$ associated to the set $S_\varepsilon(k, \omega)$. It is defined by the following minimization problem:

$$\inf\left\{\int_{\mathbb{R}^{n+1}} y^a |\nabla\varphi|^2 \, dx \, dy \, ; \, \varphi \in W^{1,2}(\mathbb{R}^{n+1}_+, |y|^a), \, \varphi(x, 0) \geq 1 \, \forall x \in S_\varepsilon(k, \omega)\right\}.$$

It is readily seen that, almost surely in ω, $\varphi_k^\varepsilon(x, y, \omega)$ satisfies

$$\begin{cases} -\text{div}\,(y^a \nabla\varphi_k^\varepsilon) = 0 & \text{for } (x, y) \in \mathbb{R}^{n+1}_+ \\ \varphi_k^\varepsilon(x, 0) = 1 & \text{for } x \in S_\varepsilon(k, \omega) \\ \lim_{y \to 0} y^a \partial_y \varphi_k^\varepsilon(x, y) = 0 & \text{for } x \notin S_\varepsilon(k, \omega) \end{cases} \tag{17}$$

and by definition of the capacity as seen in the introduction, Assumption **1** yields

$$\int_{\mathbb{R}^{n+1}} y^a |\nabla\varphi_k^\varepsilon|^2 \, dx \, dy = \varepsilon^n \gamma(k, \omega). \tag{18}$$

Moreover, we have the following lemma (the proof of which is presented in Appendix A):

Lemma 4.1. *For any $\delta > 0$, there exists R_δ such that*

$$\left|\varphi_k^\varepsilon(x, y, \omega) - \varepsilon^n \gamma(k, \omega)\frac{2}{\mu_{n,a}}h(x - \varepsilon k, y)\right| \leq \delta\varepsilon^n \gamma(k, \omega)\frac{2}{\mu_{n,a}}h(x - \varepsilon k, y)$$

for all (x, y) such that $|(x - \varepsilon k, y)| \geq \varepsilon^{\frac{n}{n-1+a}} R_\delta$ and for all $\varepsilon > 0$.

*Moreover, R_δ depends only on the constant M appearing in Assumption **1** (in particular, R_δ is independent of k and ω).*

This Lemma will play a fundamental role in the proof of Proposition 2.4 (see Section 5). It suggests that at distance $\varepsilon^{\frac{n}{n-1+a}} R$ away from εk, the corrector w^ε should behave like the function

$$h_k^\varepsilon(x, y, \omega) := \varepsilon^n \gamma(k, \omega)\frac{2}{\mu_{n,a}}h(x - \varepsilon k, y).$$

For later use, we introduce the notation

$$a^\varepsilon = \varepsilon^{\frac{n}{n-1+a}}.$$

The first step in the proof, and the main goal of this section is to construct a function $\widetilde{w}^\varepsilon$ that would be a good approximation of w^ε away from εk and that behaves like h_k^ε at distance $a^\varepsilon R$ from εk

For that purpose, we introduce

$$\widetilde{D}_\varepsilon = D \setminus \bigcup_{k \in \mathbb{Z}^n} B^+_{r(k,\omega)a^\varepsilon}(\varepsilon k), \quad \text{and} \quad \widetilde{\Sigma}_\varepsilon = \Sigma \setminus B^n_{r(k,\omega)a^\varepsilon}(\varepsilon k),$$

where $r(k, \omega)$ is chosen in such a way that $h_k^\varepsilon(x, y) = 1$ on $\partial B^+_{r(k,\omega)a^\varepsilon}(\varepsilon k)$, i.e.

$$r(k, \omega) = \left(\frac{2\nu_{n+1+a}}{\mu_{n,a}}\gamma(k, \omega)\right)^{1/(n-1+a)}. \tag{19}$$

We will prove the following proposition:

Proposition 4.2. *There exist a non-negative real number α_0 (independent of the choice of D) and a function $\widetilde{w}^\varepsilon(x, y, \omega)$ satisfying*

$$\begin{cases} -div\,(y^a \nabla \widetilde{w}^\varepsilon) = 0 & for\ (x, y) \in \widetilde{D}_\varepsilon \\[2mm] \lim_{y \to 0} y^a \partial_y \widetilde{w}^\varepsilon(x, y) = \alpha_0 & for\ x \in \widetilde{\Sigma}_\varepsilon \end{cases} \tag{20}$$

for almost all $\omega \in \Omega$, such that

$$\widetilde{w}^\varepsilon(x, y) \ = \ h_k^\varepsilon(x, y) + o(1) \quad for\ (x, y) \in B_{\varepsilon/2}^+(\varepsilon k) \cap \widetilde{D}_\varepsilon\ a.s.\ \omega \in \Omega \tag{21}$$

Moreover, we have:
(i) $\|\widetilde{w}^\varepsilon\|_{L^\infty(\widetilde{D}_\varepsilon)} \leq C$
(ii) $\|\widetilde{w}^\varepsilon\|_{L^2(\widetilde{D}_\varepsilon)} \longrightarrow 0$ as $\varepsilon \to 0$.
(iii) $\|\nabla \widetilde{w}^\varepsilon\|_{L^2(\widetilde{D}_\varepsilon)} \leq C$

The goal of this section is to establish Proposition 4.2. The main advantage of $\widetilde{w}^\varepsilon$ over w^ε is that the former only depends on the capacity of $S_\varepsilon(k, \omega)$. This explains why no assumptions are needed on the shape of $S_\varepsilon(k, \omega)$. In the last section of the paper (Section 5), we will see how to use both the functions φ_k^ε (near εk) and the corrector $\widetilde{w}^\varepsilon$ (at distance $a^\varepsilon R$ of εk) in order to prove Proposition 2.4.

4.1.3. *Effective equation.* The main idea to prove Proposition 4.2 (and in particular (21)) makes use of the fact that $h_k^\varepsilon(x, y, \omega)$ solves:

$$\begin{cases} -\mathrm{div}\,(y^a \nabla h_k^\varepsilon)(x, y) = 0 & for\ (x, y) \in \mathbb{R}_+^{n+1} \\[2mm] \lim_{y \to 0} y^a \partial_y h_k^\varepsilon(x, y) = -\varepsilon^n \widetilde{\gamma}(k, \omega) \delta(x - \varepsilon k) & for\ x \in \mathbb{R}^n \end{cases}$$

with

$$\widetilde{\gamma}(k, \omega) = \gamma(k, \omega) \frac{2}{\mu_{n,a}}.$$

Proposition 4.2 will thus be a consequence of the following proposition:

Proposition 4.3. *There exists $\alpha_0 \geq 0$ such that the solution $w_0^\varepsilon(x, y, \omega)$ of*

$$\begin{cases} -div\,(y^a \nabla w_0^\varepsilon) = 0 & for\ (x, y) \in \mathbb{R}_+^{n+1} \\[2mm] \lim_{y \to 0} y^a \partial_y w_0^\varepsilon = \alpha_0 - \sum_{k \in \mathbb{Z}^n \cap \Sigma} \varepsilon^n \widetilde{\gamma}(k, \omega) \delta(x - \varepsilon k) & for\ x \in \Sigma \\[2mm] w_0^\varepsilon(x, 0) = 0 & for\ x \in \mathbb{R}^n \setminus \Sigma \end{cases} \tag{22}$$

satisfies:

$$w_0^\varepsilon(x, y) \ = \ h_k^\varepsilon(x, y) + o(1) \quad for\ (x, y) \in B_{\varepsilon/2}^+(\varepsilon k) \cap D\ a.s.\ \omega \in \Omega \tag{23}$$

This proposition is the main step in the proof of Proposition 4.2 and its proof will occupy most of section.

4.2. **Proof of Proposition 4.3.** In order to prove Proposition 4.3, it is more convenient to work with the rescaled function

$$v_0^\varepsilon(x, y, \omega) = \varepsilon^{-1+a} w_0^\varepsilon(\varepsilon x, \varepsilon y, \omega). \tag{24}$$

Equation (22) then becomes:

$$\begin{cases} -\mathrm{div}\,(y^a \nabla v_0^\varepsilon) = 0 & \text{for } (x,y) \in \mathbb{R}_+^{n+1} \\ \displaystyle\lim_{y \to 0} y^a \partial_y v_0^\varepsilon(x,y) = \alpha_0 - \sum_{k \in \mathbb{Z}^n \cap D} \widetilde{\gamma}(k,\omega) \delta(x-k) & \text{for } x \in \varepsilon^{-1}\Sigma \\ v_0^\varepsilon(x,0) = 0 & \text{for } x \in \mathbb{R}^n \setminus \varepsilon^{-1}\Sigma, \end{cases} \quad (25)$$

and (23) is equivalent to

$$v_0^\varepsilon(x,y,\omega) = h_k(x,y,\omega) + o(\varepsilon^{-1+a}) \quad \text{for } (x,y) \in B_{1/2}^+(\varepsilon k) \cap \varepsilon^{-1}D \text{ a.s. } \omega \in \Omega$$

where

$$h_k(x,y) := \widetilde{\gamma}(k,\omega)\, h(x-k,y) = \frac{r(k,\omega)^{n-1+a}}{|(x-k)^2 + y^2|^{(n-1+a)/2}}.$$

Note that $h_k = \varepsilon^{-1+a}$ on $\partial B_{\overline{a}^\varepsilon r(k,\omega)}$ with $\overline{a}^\varepsilon = \varepsilon^{\frac{1-a}{n-1+a}}$.

In order to find the critical α_0 for which the solution v_0^ε has the appropriate behavior near the lattice points $k \in \mathbb{Z}^n$, we follow the method developed by Caffarelli-Souganidis-Wang in [8] and which was already the corner stone in [5]: We introduce the following obstacle problem, for every open set $A \subset \mathbb{R}^n$ and for every real number $\alpha \in \mathbb{R}$:

$$\begin{cases} v(x,0) \geq 0 & \text{for } x \in \mathbb{R}^n \\ \displaystyle\lim_{y \to \infty} v(x,y) = 0 & \text{for } x \in \mathbb{R}^n \\ -\mathrm{div}\,(y^a \nabla v^\varepsilon) \geq 0 & \text{for } (x,y) \in \mathbb{R}_+^{n+1} \\ \displaystyle\lim_{y \to 0} y^a \partial_y v(x,y) \leq \alpha - \sum_{k \in \mathbb{Z}^n \cap D} \widetilde{\gamma}(k,\omega) \delta(x-k) & \text{for } x \in A. \end{cases} \quad (26)$$

We then define the smallest super-solution of the obstacle problem:

$$\overline{v}_{\alpha,A}(x,y,\omega) = \inf\big\{v(x,y)\,;\, v \text{ solution of } (26)\big\}. \quad (27)$$

It is readily seen that the function $\overline{v}_{\alpha,A}$ satisfies

$$\begin{cases} -\mathrm{div}\,(y^a \nabla \overline{v}_{\alpha,A}) = 0 & \text{for } (x,y) \in \mathbb{R}_+^{n+1} \\ \displaystyle\lim_{y \to 0} y^a \partial_y \overline{v}_{\alpha,A}(x,y) = \alpha - \sum_{k \in \mathbb{Z}^n \cap A} \widetilde{\gamma}(k,\omega) \delta(x-k) & \text{for } x \in A \cap \{\overline{v}_{\alpha,A} > 0\} \end{cases} \quad (28)$$

and

$$\lim_{y \to 0} y^a \partial_y \overline{v}_{\alpha,A}(x,y) \geq 0 \qquad \text{for } x \in A \cap \{\overline{v}_{\alpha,A} = 0\}. \quad (29)$$

Remark 4.4. The function

$$h_{k,\alpha}(x,y) = h_k(x,y) - \alpha \int_{B_1^n(k)} \frac{\nu_{n+1+a}}{(|x-x'|^2 + y^2)^{\frac{n-1+a}{2}}}\, dx' \quad (30)$$

satisfies

$$\begin{cases} -\mathrm{div}\,(y^a \nabla h_{k,\alpha}) = 0 & \text{for } x \in \mathbb{R}_+^{n+1} \\ \displaystyle\lim_{y \to 0} y^a \partial_y h_{k,\alpha}(x,y) \longrightarrow \alpha - \widetilde{\gamma}(k,\omega) \delta(x-k) & \text{for } x \in B_1^n(k). \end{cases}$$

It is radially symmetric around k and $\sup_{|x|=1,\, y>0} h_{\alpha,k}(x,y) \leq r^{n-1+a}$. In particular, the maximum principle and (28) implies that if $B_1^n(k) \subset A$, then:

$$\overline{v}_{\alpha,A}(x,y,\omega) \geq h_{\alpha,k}(x,y,\omega) - r^{n-1+a} \qquad \text{for } (x,y) \in B_1^+(k), \text{ a.s.} \quad (31)$$

We now want to show that there exists a critical α_0 such that the followings hold:

1. The solution of the obstacle problem $\overline{v}_{\alpha,A}(x,y,\omega)$ behaves like $h_{\alpha,k}(x,y,\omega)$ near any point $k \in A \cap \mathbb{Z}^n$.
2. The solution of (25) is not far from $\overline{v}_{\alpha,A}$.

For that purpose, we introduce the following quantity, which measures the size of the contact set along the boundary $\{y = 0\}$:

$$\overline{m}_\alpha(A,\omega) = |\{x \in A \, ; \, \overline{v}_{\alpha,A}(x,0,\omega) = 0\}|$$

where $|A|$ denotes the Lebesgue measure of a set A in $\mathbb{R}^n$.

The starting point of the proof is the following lemma:

Lemma 4.5. *The random variable $\overline{m}_\alpha$ is subadditive, and the process*

$$T_k m(A,\omega) = m(k + A, \omega)$$

has the same distribution for all $k \in \mathbb{Z}^n$.

Proof of Lemma 4.5. Assume that the finite family of sets $(A_i)_{i \in I}$ is such that

$$\begin{aligned}
A_i &\subset A & \text{for all } i \in I \\
A_i \cap A_j &= \emptyset & \text{for all } i \neq j \\
|A - \cup_{i \in I} A_i| &= 0 &
\end{aligned}$$

then $\overline{v}_{\alpha,A}$ is admissible for each A_i, and so $\overline{v}_{\alpha,A_i} \leq u_{\alpha,A}$. It follows that

$$\{\overline{v}_{\alpha,A}(\cdot,0,\omega) = 0\} \cap A_i \subset \{\overline{v}_{\alpha,A_i}(\cdot,0,\omega) = 0\}$$

and so

$$\begin{aligned}
\overline{m}_\alpha(A,\omega) &= \sum_{i \in I} |\{\overline{v}_{\alpha,A}(\cdot,0,\omega) = 0\} \cap A_i| \\
&\leq \sum_{i \in I} |\{\overline{v}_{\alpha,A_i}(\cdot,0,\omega) = 0\}| = \sum_{i \in I} \overline{m}_\alpha(A_i,\omega),
\end{aligned}$$

which gives the subadditive property. Assumption **2** then yields

$$T_k m(A,\omega) = m(A, \tau_k \omega)$$

which gives the last assertion of the lemma. $\qquad\square$

Since $\overline{m}_\alpha(A,\omega) \leq |A|$, and thanks to the ergodicity of the transformations τ_k, it follows from the subadditive ergodic theorem (see Akcoglu, Krengel [1] and Dal Maso, Modica [12]) that for each α, there exists a constant $\overline{\ell}(\alpha)$ such that

$$\lim_{t \to \infty} \frac{\overline{m}_\alpha(B_t(0), \omega)}{|B_t(0)|} = \overline{\ell}(\alpha) \quad \text{a.s.,}$$

where $B_t(0)$ denotes the ball centered at the origin with radius t. Note that the limit exists and is the same if instead of $B_t(0)$, we use cubes or balls centered at tx_0 for some x_0.

If we scale back and consider the function

$$\overline{w}_\alpha^\varepsilon(x,y,\omega) = \varepsilon^{1-a} \, \overline{v}_{\alpha, B_{\varepsilon^{-1}}(\varepsilon^{-1}x_0)}(x/\varepsilon, y/\varepsilon, \omega), \qquad \text{in } B_1(x_0),$$

we deduce

$$\lim_{\varepsilon \to 0} \frac{|\{x \, ; \, \overline{w}_\alpha^\varepsilon(x,0,\omega) = 0\}|}{|B_1|} = \overline{\ell}(\alpha) \quad \text{a.s.}$$

The next lemma summarizes the properties of $\overline{\ell}(\alpha)$:

Lemma 4.6. (i) $\overline{\ell}(\alpha)$ *is a nondecreasing functions of α.*

(ii) *If $\alpha < 0$, then $\overline{\ell}(\alpha) = 0$. Moreover, if the $\gamma(k, \omega)$ are bounded from below, then $\overline{\ell}(\alpha) = 0$ for α positive small enough $(0 < \alpha < C(\underline{\gamma}))$.*

(iii) *If α is large enough $(\alpha \geq C(\overline{\gamma}))$, then $\overline{\ell}(\alpha) > 0$.*

The proof of this Lemma is rather technical and of little interest. It is presented in full details in Appendix B. Using Lemma 4.6, we can define

$$\alpha_0 = \sup\{\alpha \,;\, \overline{\ell}(\alpha) = 0\}.$$

We observe that α_0 is finite (Lemma 4.6 (iii)) and that α_0 is non negative (Lemma 4.6 (ii)). Moreover, α_0 is strictly positive if the $\gamma(k, \omega)$ are bounded from below almost surely by a positive constant.

We now fix a bounded subset A of $\mathbb{R}^n$ and we denote by

$$\overline{v}_\alpha^\varepsilon(x, y, \omega) = \overline{v}_{\alpha, \varepsilon^{-1}A}(x, y, \omega) \tag{32}$$

the solutions of (27) corresponding to $\varepsilon^{-1}A$. We also introduce the rescaled function

$$\overline{w}_\alpha^\varepsilon(x, y, \omega) = \varepsilon^{1-a} \, \overline{v}_\alpha^\varepsilon(x/\varepsilon, y/\varepsilon, \omega).$$

In order to complete the proof of Proposition 4.3, we are first going to prove that $\overline{w}_\alpha^\varepsilon$ satisfies inequality (23), and then that the solution w_0^ε of (22) behaves like $\overline{w}_\alpha^\varepsilon$.

We recall the definition of $h_{\alpha,k}$:

$$h_{\alpha,k}(x, y) = \frac{r(k)^{n-1+a}}{(|x - k|^2 + y^2)^{\frac{n-1+a}{2}}} - \alpha \int_{B_1^n(k)} \frac{\nu_{n+1+a}}{(|x - x'|^2 + y^2)^{\frac{n-1+a}{2}}} \, dx',$$

and we introduce the scaled function

$$h_{\alpha,k}^\varepsilon(x, y) := \varepsilon^{1-a} h_{\alpha,k}(x/\varepsilon, y/\varepsilon).$$

Note that when $(x, y) \in \partial B_{\overline{a}^\varepsilon r(k,\omega)}^+(k)$, then

$$h_{\alpha,k}(x, y) = \varepsilon^{-1+a} - \alpha \int_{B_1^n(0)} \frac{\nu_{n+1+a}}{(|x - x'|^2 + y^2)^{\frac{n-1+a}{2}}} \, dx'$$

(we recall that $\overline{a}^\varepsilon = \varepsilon^{\frac{1-a}{n-1+a}}$).

We then have the following lemma:

Lemma 4.7. (i) *For every α and for every $k \in \mathbb{Z}^n \cap A$, we have*

$$\overline{v}_\alpha^\varepsilon(x, y) \geq h_{\alpha,k}(x, y) - r^{n-1+a} \quad \text{for } (x, y) \in B_1^+(k) \text{ a. s.}$$

(ii) *For every $\alpha > \alpha_0$ and every $k \in \mathbb{Z}^n \cap A$, we have*

$$\overline{v}_\alpha^\varepsilon(x, y) \leq h_{\alpha,k}(x, y) + o(\varepsilon^{-1+a}) \quad \text{for } (x, y) \in B_{1/2}^+(k) \text{ a. s.}$$

We deduce:

Corollary 4.8. (i) *For every α and every $k \in \mathbb{Z}^n \cap A$ such that $r(k, \omega) > 0$, we have*

$$\overline{v}_\alpha^\varepsilon(x, y) \geq \varepsilon^{-1+a} + o(1) \quad \text{for } (x, y) \in \partial B_{\overline{r}(k,\omega)\overline{a}^\varepsilon}^+(k) \quad a.e. \ \omega \in \Omega$$

and so

$$\overline{w}_\alpha^\varepsilon(x,y) \geq 1 + o(\varepsilon^{1-a}) \quad for \ (x,y) \in \partial B_{r(k,\omega)a^\varepsilon}^+(k) \quad a.e. \ \omega \in \Omega$$

for all α.

(ii) *For every $\alpha > \alpha_0$ and every $k \in \mathbb{Z}^n \cap A$, we have*

$$\overline{v}_\alpha^\varepsilon(x,y) \leq \varepsilon^{-1+a} + o(\varepsilon^{-1+a}) \quad for \ (x,y) \in \partial B_{r(k,\omega)\overline{a}^\varepsilon}^+(k) \quad a.e. \ \omega \in \Omega$$

and so

$$\overline{w}_\alpha^\varepsilon(x,y) \leq 1 + o(1) \ for \ (x,y) \in \partial B_{r(k,\omega)a^\varepsilon}^+(k) \quad a.e. \ \omega \in \Omega$$

Proof of Lemma 4.7. (i) This is an immediate consequence of (31).

(ii) The proof of (ii) is more delicate and is split in several steps.
Preliminary: First of all since A is bounded, we have $A \subset B_R^n(x_0)$ for some R. Without loss of generality, we can always assume that $B_R^n(x_0) = B_1^n(0)$. If we consider

$$v_\alpha^\varepsilon(x,y,\omega) = \overline{v}_{\alpha,\varepsilon^{-1}B_1^n}(x,y,\omega),$$

the solution of (27) corresponding to $A = B_{\varepsilon^{-1}}^n(0)$, it is readily seen that

$$\overline{v}_\alpha^\varepsilon(x,y,\omega) \leq v_\alpha^\varepsilon(x,y,\omega) \quad for \ all \ (x,y) \in \mathbb{R}_+^{n+1} \ a.e. \ \omega \in \Omega.$$

It is thus enough to prove (ii) for v_α^ε.

In the sequel, we will need the following consequence of Lemma 4.5 (see [8] for the proof):

Lemma 4.9. *For any ball $B_r^n(x_0) \in B_1^n(0)$, the following limit holds, a.s. in ω:*

$$\lim_{\varepsilon \to 0} \frac{|\{v_\alpha^\varepsilon(x,0,\omega) = 0\} \cap B_{\varepsilon^{-1}r}^n(\varepsilon^{-1}x_1)|}{|B_{\varepsilon^{-1}r}^n|} = \overline{\ell}(\alpha)$$

Step 1: We now start the proof: For any $\delta > 0$, we can cover $B_{\varepsilon^{-1}}^n$ by a finite number N ($\leq C\delta^{-n}$) of balls $B_i^n = B_{\delta\varepsilon^{-1}}^n(\varepsilon^{-1}x_i)$ with radius $\delta\varepsilon^{-1}$ and center $\varepsilon^{-1}x_i$. Since $\alpha > \alpha_0$, we have $\overline{\ell}(\alpha) > 0$. By Lemma 4.9, we deduce that for every i, there exists ε_i such that if $\varepsilon \leq \varepsilon_i$, then

$$|\{v_\alpha^\varepsilon(x,0,\omega) = 0\} \cap B_i^n| > 0 \quad a.s. \ \omega.$$

In particular, if $\varepsilon \leq \inf \varepsilon_i$, then $v_\alpha^\varepsilon(x_i',0) = 0$ for some x_i' in B_i^n a.s. $\omega \in \Omega$.
Introducing $B_i = B_{\delta\varepsilon^{-1}}(\varepsilon^{-1}x_i)$ the $n+1$ dimensional ball with same radius and same center as B_i^n, we now have to show that v_α^ε remains small in each B_i^+ as long as we stay away from the lattice points $k \in \mathbb{Z}^n$. More precisely, we want to show that

$$\sup_{\cup_{k \in \mathbb{Z}^n} B_1^+(k) \setminus B_{1/4}^+(k)} v_\alpha^\varepsilon(x,y) \leq C\delta^{1-a}\varepsilon^{-1+a}.$$

Step 2: Let $\eta(x)$ be a nonnegative function defined in $\mathbb{R}^n$ such that $0 \leq \eta(x) \leq 1$ for all x, $\eta(x) = 1$ in $B_{1/8}$ and $\eta = 0$ in $\mathbb{R}^n \setminus B_{1/4}$. We then consider the function

$$u = v_\alpha^\varepsilon \star_x \eta$$

where $\star_x$ indicates the convolution in $\mathbb{R}^n$ with respect to the x-variable. The function $u(x, y)$ is nonnegative on $2B_i^+$ and satisfies

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{for } (x, y) \in 2B_i^+ \\ -C \leq \lim_{y \to 0} y^a \partial_y u(x, y) \leq C & \text{for } x \in 2B_i^n \end{cases} \tag{33}$$

where C is a universal constant depending only on n, $\overline{r}$ and α. We deduce:

Lemma 4.10. *There exists a universal constant C such that*

$$\sup_{B_i} u \leq C \inf_{B_i} u + C\delta^{1-a}\varepsilon^{-1+a}.$$

Proof. We write $u = u_1 + u_2$ where u_1 and u_2 are two functions solution of $\operatorname{div}(y^a \nabla u_i) = 0$ in $2B_i^+$ and satisfying

$$\begin{cases} \lim_{y \to 0} y^a \partial_y u_1(x, y) = \lim_{y \to 0} y^a \partial_y u(x, y) & \text{for } x \in 2B_i^n, \\ u_1(x, y) = 0 & \text{for } (x, y) \in \partial(2B_i^+) \cap \{y > 0\} \end{cases}$$

and

$$\begin{cases} \lim_{y \to 0} y^a \partial_y u_2(x, y) = 0 & \text{for } x \in 2B_i^n, \\ u_2(x, y) = u(x, y) & \text{for } (x, y) \in \partial(2B_i^+) \cap \{y > 0\}. \end{cases}$$

The maximum principle and the fact that B_i has radius $\delta\varepsilon^{-1}$ yield:

$$\begin{aligned} |u_1(x, y)| &\leq C((2\delta\varepsilon^{-1})^{1-a} - y^{1-a}) \\ &\leq C(\delta\varepsilon^{-1})^{1-a} \end{aligned}$$

for all $(x, y) \in 2B_i^+$. On the other hand, boundary Harnack inequality for degenerate elliptic equation (see [13]) implies

$$\sup_{B_i} u_2 \leq C \inf_{B_i} u_2.$$

The Lemma follows easily. $\square$

For the next step, we will need the following lemma:

Lemma 4.11. *If v satisfies*

$$div(y^a \nabla v) = 0 \quad in \ B_r^+(x_0, 0)$$

and

$$\lim_{y \to 0} y^a \partial_y v(x, y) \leq \alpha \quad for \ x \in B_r^n(x_0),$$

then

$$\frac{2}{\omega_{n+a}r^{n+a}} \int_{B_r^+(x_0,0)} |y|^a v(x, y)\, dx\, dy \leq v(x_0, 0) + \alpha C(n) r^{1-a}$$

where $C(n)$ is a universal constant and $\omega_{n+a} = \int_{B_1(x_0,0)} |y|^a\, dx\, dy$.

Proof. The function $w(x, y) = v(x, y) + \alpha \int_{B_r^n(x_0)} \frac{C_{n+1+a}}{(|x-x'|^2+y^2)^{\frac{n-1+a}{2}}}\, dx'$ satisfies

$$\operatorname{div}(y^a \nabla w) = 0 \quad \text{and} \quad \lim_{y \to 0} y^a \partial_y w \leq 0.$$

Proceeding as in [6], we now reflect w about the plane $\{y = 0\}$. The function

$$\overline{w}(x, y) = \begin{cases} w(x, y) & \text{if } y > 0 \\ w(x, -y) & \text{if } y < 0 \end{cases}$$

is now defined in the whole space $\mathbb{R}^{n+1}$ and it satisfies

$$\operatorname{div}\left(|y|^a \nabla \overline{w}\right) \leq 0 \quad \text{in } B_r(x_0, 0).$$

We can thus use the mean value formula (see [7]):

$$\frac{1}{\omega_{n+a} r^{n+a}} \int_{B_r(x_0, 0)} |y|^a \overline{w}(x, y)\, dx\, dy$$
$$\leq \overline{w}(x_0, 0)$$
$$\leq w(x_0, 0)$$
$$\leq v(x_0, 0) + \alpha \int_{B_r^n(x_0)} \frac{C_{n+1+a}}{|x_0 - x'|^{n-1+a}}\, dx'.$$

Since $\alpha \geq 0$, we see that $v \leq w$ and so

$$\frac{2}{\omega_{n+a} r^{n+a}} \int_{B_r^+(x_0, 0)} y^a v(x, y)\, dx\, dy \quad \leq \quad \frac{1}{\omega_{n+a} r^{n+a}} \int_{B_r(x_0, 0)} |y|^a \overline{w}(x, y)\, dx\, dy$$

Moreover, we have

$$\int_{B_r^n(x_0)} \frac{C_{n+1+a}}{|x_0 - x'|^{n-1+a}}\, dx' = \int_{B_r^n(0)} \frac{C_{n+1+a}}{|z|^{n-1+a}}\, dz = C(n+a) r^{1-a},$$

hence the lemma. $\qquad\qquad\square$

Step 3: We have $v_\alpha^\varepsilon(x_i', 0) = 0$ and $\lim_{y \to 0} y^a \partial_y v_\alpha^\varepsilon(x, y) \leq \alpha$ for $x \in B_{1/2}(x_i')$. Lemma 4.11 thus applies and yields:

$$\int_{B_{1/2}^+(x_i', 0)} |y|^a v_\alpha^\varepsilon(x, y)\, dx\, dy \leq C(v_\alpha^\varepsilon(x_i', 0) + \alpha) \leq C(\alpha, n+a). \tag{34}$$

We want to deduce an upper bound on u in B_i. Since $u \geq 0$, we note that

$$\int_0^{1/4} \tau^a u(x, \tau)\, d\tau \geq \left(\inf_{\tau \in [0, 1/4]} u \right) \int_0^{1/4} \tau^a\, d\tau.$$

Then, using the definition of u (and the fact that $\eta(x) = 0$ outside $B_{1/4}^n(x)$), we deduce:

$$\inf_{B_{1/4}^+(x_i', 0)} u \quad \leq \quad C \inf_x \int_0^{1/4} \tau^a u(x, \tau)\, d\tau$$

$$\leq \quad C \inf_x \int_0^{1/4} \int_{B_{1/4}^n(x)} \tau^a v_\alpha^\varepsilon(\xi, \tau)\, d\xi\, d\tau$$

$$\leq \quad C \int_{B_{1/2}(x_i', 0)} \tau^a v_\alpha^\varepsilon(\xi, \tau)\, d\xi\, d\tau,$$

which, together with (34) yields:

$$\inf_{B_{1/4}^+(x_i', 0)} u \leq C(\alpha, n). \tag{35}$$

Using Lemma 4.10 we see that for every δ and for ε small enough, we have:

$$\sup_{B_i} u \leq C \inf_{B_i} u + C \delta^{1-a} \varepsilon^{-1+a} \leq C(\alpha, n) + C \delta^{1-a} \varepsilon^{-1+a} \leq C \delta^{1-a} \varepsilon^{-1+a}. \tag{36}$$

Step 4: We now want to use (36) to get an upper bound on v_α^ε. For that purpose, we note that $\lim_{y \to 0} y^a \partial_y v_\alpha^\varepsilon \geq 0$ in $B_i \setminus \cap_{k \in \mathbb{Z}^n} \{k\}$, and so a proof similar to that of Lemma 4.11 yields

$$v_\alpha^\varepsilon(x, y) \leq C_{n+a} \int_{B_{1/8}^+(x,y)} |\tau|^a v_\alpha^\varepsilon(\xi, \tau) \, d\xi \, d\tau \tag{37}$$

for all $(x, y) \in B_i \setminus \cap_{k \in \mathbb{Z}^n} \overline{B}_{1/4}(k)$.

Inequality (37) and the definition of $u(x, y)$ yield that for all (x, y) in $B_i \setminus \cap_{k \in \mathbb{Z}^n} B_{1/4}(k)$, we have:

$$\begin{aligned}
v_\alpha^\varepsilon(x, y) &\leq C_{n+a} \int_{y-1/8}^{y+1/8} \int_{B_{1/8}^n(x)} |\tau|^a v_\alpha^\varepsilon(\xi, \tau) \, d\xi \, d\tau \\
&\leq C_{n+a} \int_{y-1/8}^{y+1/8} |\tau|^a u(x, \tau) \, d\tau \\
&\leq C(n+a)|y|^{1+a} \sup_{B_i} u.
\end{aligned}$$

Inequality (36) therefore implies

$$\sup_{(x,y) \in \cup_{k \in \mathbb{Z}^n} B_1^+(k) \setminus B_{1/4}^+(k)} v_\alpha^\varepsilon(x, y) \leq C \delta^{1-a} \varepsilon^{-1+a}. \tag{38}$$

Step 5: In order to complete the proof of the lemma, we only have to notice that since $\inf_{\partial B_{1/2}} h_{\alpha,k}(x, y) \geq -C\alpha$, (38) and the definition of v_α^ε imply

$$v_\alpha^\varepsilon(x, y) \leq h_{\alpha,k}(x, y) + C \delta^{1-a} \varepsilon^{-1+a} \qquad \text{in } B_{1/2}(k)$$

for all $k \in \mathbb{Z}^n$. $\qquad \square$

This conclude the proof of Lemma 4.7, and we are now in position to complete the proof of Propositions 4.3. $\qquad \square$

Proof of Proposition 4.3. For every α, we denote by $\overline{v}_\alpha^\varepsilon$ the solution of the obstacle problem (26) corresponding to $A = \varepsilon^{-1}\Sigma$:

$$\overline{v}_\alpha^\varepsilon(x, y, \omega) = \overline{v}_{\alpha, \varepsilon^{-1}\Sigma}(x, y, \omega),$$

and by $\overline{w}_\alpha^\varepsilon$ the rescaled function:

$$\overline{w}_\alpha^\varepsilon(x, y, \omega) = \varepsilon^{1-a} \, \overline{v}_{\alpha, \varepsilon^{-1}\Sigma}(x/\varepsilon, y/\varepsilon, \omega).$$

We recall that w_0^ε is solution of

$$\begin{cases}
-\operatorname{div}(y^a \nabla w_0^\varepsilon) = 0 & \text{for } (x, y) \in \mathbb{R}_+^{n+1} \\
\lim_{y \to 0} y^a \partial_y w_0^\varepsilon(x, y) = \alpha_0 - \sum_{k \in \mathbb{Z}^n \cap D} \widetilde{\gamma}(k, \omega) \delta(x - \varepsilon k) & \text{for } x \in \Sigma \\
w_0^\varepsilon(x, 0) = 0 & \text{for } x \in \mathbb{R}^n \setminus \Sigma
\end{cases}$$

In order to prove Proposition 4.3, we have to establish (23). This is done in two steps using the properties of the function $\overline{w}_\alpha^\varepsilon$:

1. For every $\alpha > \alpha_0$, we have $\operatorname{div}(y^a \nabla(w_0^\varepsilon - w_\alpha^\varepsilon)) = 0$ for $(x, y) \in \mathbb{R}_+^{n+1}$, $\lim_{y \to 0} y^a \partial_y(w_0^\varepsilon - w_\alpha^\varepsilon) \geq \alpha_0 - \alpha$ on Σ and $(w_0^\varepsilon - w_\alpha^\varepsilon)(x, 0) = 0$ on $\mathbb{R}^n \setminus \Sigma$. We deduce

$$w_0^\varepsilon(x_0, y_0) - w_\alpha^\varepsilon(x_0, y_0) \leq \int_\Sigma \frac{\alpha_0 - \alpha}{|(x_0 - x)^2 + y_0^2|^{\frac{n-1+a}{2}}} \, dx,$$

   and therefore

$$\sup_{(x,y) \in \mathbb{R}_+^{N+1}} (w_0^\varepsilon(x, y) - w_\alpha^\varepsilon(x, y)) \leq C |\Sigma|^{\frac{1-a}{n+1}} \rho_\Sigma^{\frac{1-a}{n+1}} |\alpha - \alpha_0|$$

   with

$$\rho_\Sigma = \inf\{\rho \, ; \, \Sigma \subset B_\rho\}.$$

   In particular, we thus have

$$w_0^\varepsilon(x, y) \leq w_\alpha^\varepsilon(x, y) + O(\alpha - \alpha_0) \qquad \text{for } (x, y) \in \mathbb{R}_+^{n+1},$$

   and Lemma 4.7 (ii) (since $\alpha > \alpha_0$) yields:

$$w_0^\varepsilon(x, y) \leq h_{\alpha,k}^\varepsilon(x, y) + O(\alpha - \alpha_0) + o(1) \quad \text{for } (x, y) \in B_{\varepsilon/2}(\varepsilon k) \text{ a.s.}$$

   (Note that this argument shows the continuity of w_α^ε with respect to α).

2. Similarly, we observe that for $\alpha \leq \alpha_0$, we have $\operatorname{div}(y^a \nabla(w_\alpha^\varepsilon - w_0^\varepsilon)) = 0$ for $(x, y) \in \mathbb{R}_+^{n+1}$, $(w_\alpha^\varepsilon - w_0^\varepsilon)(x, 0) = 0$ for $x \in \mathbb{R}^n \setminus \Sigma$ and

$$\lim_{y \to 0} y^a \partial_y(w_\alpha^\varepsilon - w_0^\varepsilon)(x, y) \geq \alpha - \alpha_0 - \alpha 1_{\{w_\alpha^\varepsilon = 0\} \cap \Sigma} \qquad \text{for } x \in \Sigma.$$

   Proceeding as before, we deduce:

$$\sup_{\mathbb{R}_+^{N+1}} (w_\alpha^\varepsilon - w_0^\varepsilon) \leq C \rho_\Sigma^{\frac{1-a}{n+1}} \left[|\Sigma|^{\frac{1-a}{n+1}} (\alpha_0 - \alpha) \right.$$

$$\left. + C\alpha |\{w_\alpha^\varepsilon(x, 0) = 0\} \cap \Sigma|^{\frac{1-a}{n+1}} \right].$$

   So Lemma 4.7 (i) yields

$$w_0^\varepsilon(x, y) \geq h_{\alpha,k}^\varepsilon(x, y) - o(\varepsilon) - O(\alpha_0 - \alpha) - C\alpha |\{w_\alpha^\varepsilon(x, 0) = 0\} \cap \Sigma|^{\frac{1-a}{n+1}}$$

   for all $(x, y) \in B_{\varepsilon/2}(\varepsilon k)$. Finally, using the fact that

$$\lim_{\varepsilon \to 0} |\{w_\alpha^\varepsilon(x, 0) = 0\} \cap \Sigma| = \overline{\ell}(\alpha)|\Sigma| = 0$$

   for all $\alpha \leq \alpha_0$ we easily deduce the first inequality in (23).

$\square$

4.3. **Proof of Proposition 4.2.** In order to complete the proof of Proposition 4.2, we construct a corrector $\widetilde{w}^\varepsilon$ which is equal to 1 on the (n+1)-dimensional balls $B_{r(k,\omega)a^\varepsilon}^+(\varepsilon k)$. More precisely, we recall that D is a bounded subset of $\mathbb{R}_+^{n+1}$, and we introduce

$$\widetilde{T}_\varepsilon = D \cap \bigcup_{k \in \mathbb{Z}^n \cap \Sigma} B_{r(k,\omega)a^\varepsilon}^+(\varepsilon k)$$

and

$$\widetilde{\Sigma}_\varepsilon = \Sigma \setminus \bigcup_{k \in \mathbb{Z}^n \cap \Sigma} B_{r(k,\omega)a^\varepsilon}^n(\varepsilon k).$$

We then define a corrector $\widetilde{w}^\varepsilon(x, y, \omega)$ which will satisfy all the conditions of Proposition 2.4, with the set $\widetilde{T}_\varepsilon$ instead of T_ε. In particular, we will prove that $\widetilde{w}^\varepsilon$ behaves like h_k^ε near the $B_{r(k,\omega)a^\varepsilon}^+(\varepsilon k)$.

We consider the following obstacle problem:

$$
\begin{cases}
\operatorname{div}\left(y^a \nabla w\right) \le 0 & \text{for } (x, y) \in \mathbb{R}_+^{n+1} \setminus \widetilde{T}_\varepsilon \\
\lim_{y \to 0} y^a \partial_y w(x, y) \le \alpha_0 & \text{for } x \in \widetilde{\Sigma}_\varepsilon \\
w(x, y) \ge 1 & \text{for } (x, y) \in \widetilde{T}_\varepsilon \\
w(x, 0) = 0 & \text{for } x \in \mathbb{R}^n \setminus \Sigma,
\end{cases}
\tag{39}
$$

and we define:

$$\widetilde{w}^\varepsilon(x, y, \omega) = \inf\left\{ w(x, y, \omega) \, ; \, w \text{ solution of } (39) \right\}.$$

It is readily seen that $\widetilde{w}^\varepsilon$ satisfies (20). So in order to complete the proof of Proposition 4.2, we only have to show that $\widetilde{w}^\varepsilon$ is bounded uniformly in $L^\infty(D)$ and that $\widetilde{w}^\varepsilon \longrightarrow 0$ in $W_{\text{loc}}^{1,2}(D, |y|^a)$-weak as ε goes to zero.

Strong convergence in $L^2(D, |y|^a)$:
First of all, since $\widetilde{w}^\varepsilon = 1 = h_{\alpha,k}^\varepsilon(x, y) + o(1)$ on $\widetilde{T}_\varepsilon$, (23) implies

$$w_0^\varepsilon(x, y) - o(1) \le \widetilde{w}^\varepsilon(x, y, \omega) \le w_0^\varepsilon(x, y) + o(1) \quad \text{in } D \quad \text{a.e. } \omega \in \Omega,$$

which in turn implies (using Proposition 4.3 again):

$$h_{\alpha,k}^\varepsilon(x, y) - o(1) \le \widetilde{w}^\varepsilon(x, \omega) \le h_{\alpha,k}^\varepsilon(x, y) + o(1) \quad \forall (x, y) \in B_{\varepsilon/2}^+(\varepsilon k). \tag{40}$$

In particular, we get:

$$\|\widetilde{w}^\varepsilon\|_{L^\infty(\mathbb{R}_+^{n+1})} \le C.$$

Moreover, a simple computation shows that

$$\int_{B_\varepsilon(\varepsilon k) \setminus B_{a^\varepsilon}(\varepsilon k)} y^a |h_{\alpha,k}^\varepsilon|^2 \, dx \, dy \le C\varepsilon^{n+1}$$

and it is readily seen that (40) implies

$$|w_0^\varepsilon(x, y)| \le C\varepsilon^{1-a} + o(1) = o(1) \qquad \forall (x, y) \in \bigcup_{k \in \mathbb{Z}^n} \partial B_{\varepsilon/2}(\varepsilon k).$$

We deduce:

$$\|\widetilde{w}^\varepsilon\|_{L^2(D, |y|^a)}^2 \le \sum_{k \in \{\mathbb{Z}^n \cap \varepsilon^{-1}\Sigma\}} \int_{B_\varepsilon \setminus B_{a^\varepsilon}} y^a |h_{\alpha,k}^\varepsilon|^2 \, dx \, dy + o(1) \int_D |y|^a \, dx \, dy$$

and since $\#\{\mathbb{Z}^n \cap \varepsilon^{-1}\Sigma\} \le C\varepsilon^{-n}$ for all n, we have:

$$\|\widetilde{w}^\varepsilon\|_{L^2(D, |y|^a)}^2 \le \varepsilon + o(1) = o(1). \tag{41}$$

In particular

$$\widetilde{w}^\varepsilon \longrightarrow 0 \quad \text{in } L^2(D, |y|^a) - \text{strong}.$$

as ε goes to zero.

Bound in $W^{1,2}(D, |y|^a)$:
Using the definition if $\widetilde{w}^\varepsilon$ and an integration by parts, we get:

$$
\begin{aligned}
\int_{\mathbb{R}^{n+1}_+ \setminus \widetilde{T}_\varepsilon} y^a |\nabla \widetilde{w}^\varepsilon|^2 \, dx \, dy &= \int_{\mathbb{R}^{n+1}_+ \setminus \widetilde{T}_\varepsilon} y^a \nabla \widetilde{w}^\varepsilon \cdot \nabla (\widetilde{w}^\varepsilon - 1) \, dx \, dy \\
&= -\int_{\partial \widetilde{T}_\varepsilon \cup \widetilde{\Sigma}_\varepsilon} [\lim_{y \to 0} y^a \widetilde{w}^\varepsilon_y (x, y)] (\widetilde{w}^\varepsilon (x, y) - 1) \, d\sigma(x, y) \\
&= -\alpha_0 \int_{\widetilde{\Sigma}_\varepsilon} (\widetilde{w}^\varepsilon (x, 0) - 1) \, dx
\end{aligned}
$$

The L^∞ bound thus yieds

$$
\int_{\mathbb{R}^{n+1}_+ \setminus \widetilde{T}_\varepsilon} y^a |\nabla \widetilde{w}^\varepsilon|^2 \, dx \, dy \leq C\alpha_0 |\widetilde{\Sigma}_\varepsilon| (\|\widetilde{w}^\varepsilon\|_{L^\infty} + 1) \leq C,
$$

which completes the proof. $\square$

5. **Proof of proposition 2.4.** This section is devoted to the proof of the main proposition. We recall that the sets $S_\varepsilon(k, \omega)$ are subsets of $\mathbb{R}^n$ with unspecified shapes and they satisfy

$$
\mathrm{cap}_s (S_\varepsilon(k, \omega)) = \varepsilon^n \gamma(k, \omega).
$$

Lemma 4.1 gives the existence of a function $\varphi^\varepsilon_k (x, y, \omega)$ such that

$$
\begin{cases}
\mathrm{div}\, (y^a \nabla \varphi) = 0 & \text{for } (x, y) \in \mathbb{R}^{n+1}_+ \\
\varphi(x, 0) = 1 & \text{for } x \in S_\varepsilon(k, \omega) \\
\lim_{y \to 0} y^a \partial_y \varphi(x, y) = 0 & \text{for } x \notin S_\varepsilon(k, \omega)
\end{cases}
$$

and we let α_0 and $\widetilde{w}^\varepsilon (x, y, \omega)$ be given by Proposition 4.2.
We then have:

1. For a given $\delta > 0$, Lemma 4.1 implies that for every $k \in \mathbb{Z}^n$ and $\omega \in \Omega$ there exists a constant $R_\delta(k, \omega)$ such that

$$
|\varphi^\varepsilon_k (x, \omega) - h^\varepsilon_k (x, y, \omega)| \leq \delta \, h^\varepsilon_k (x, y, \omega) \leq \delta \frac{\widetilde{\gamma}(k, \omega)}{R_\delta^{n-1+a}} \tag{42}
$$

in $B^+_{2a^\varepsilon R_\delta} \setminus B^+_{a^\varepsilon R_\delta}(\varepsilon k)$ and for all $\varepsilon > 0$. It is readily seen that for any R there exists $\varepsilon_1(R)$ such that

$$
a^\varepsilon R \leq \varepsilon^\sigma / 4 \quad \text{for all } \varepsilon \leq \varepsilon_1 \tag{43}
$$

for some $\sigma > 1$.

2. Inequality (21) in Proposition 4.2 implies that for given δ and R, there exists $\varepsilon_2(\delta, R) < \varepsilon_1(R)$ such that for all $\varepsilon \leq \varepsilon_2(\delta, R)$, we have

$$
|\widetilde{w}^\varepsilon (x) - h^\varepsilon_k (x, y, \omega)| \leq \frac{\delta}{R^{n-1+a}} \quad \text{in } B^+_{\varepsilon/2}(\varepsilon k). \tag{44}
$$

Thanks to (43), Inequality (44) holds in particular in $B^+_{2a^\varepsilon R} \setminus B^+_{a^\varepsilon R}(\varepsilon k)$.

The corrector will be constructed by gluing together the functions φ^ε_k (near the sets $S_\varepsilon(k)$) and the function $\widetilde{w}^\varepsilon$ (away from the sets $S_\varepsilon(k)$). The gluing has to be done very carefully so that the corrector satisfies all the properties listed in Proposition 2.4: For a given ε, we define δ_ε to be the smallest positive number such

that (43) and (44) hold with $\delta = \delta_\varepsilon$ and $R = R_{\delta_\varepsilon}$. From the remarks above, we see that δ_ε is well defined as soon as ε is small enough (say smaller than $\varepsilon_2(1, R_1)$). Moreover, for any $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_2(\delta, R_\delta)$ such that $\delta_\varepsilon \leq \delta$ for all $\varepsilon \leq \varepsilon_0$. In particular

$$\lim_{\varepsilon \to 0} \delta_\varepsilon = 0.$$

From now on, we write

$$R_\varepsilon = R_{\delta_\varepsilon}.$$

In order to define w^ε, we introduce the cut-off function $\eta_\varepsilon(x, y)$ defined on D and such that

$$\eta_\varepsilon(x, y) = 1 \quad \text{for } (x, y) \in D \setminus \bigcup_{k \in \mathbb{Z}^n} B^+_{2a^\varepsilon R_\varepsilon}(\varepsilon k)$$

$$\eta_\varepsilon(x, y) = 0 \quad \text{for } (x, y) \in \bigcup_{k \in \mathbb{Z}^n} B^+_{a^\varepsilon R_\varepsilon}(\varepsilon k).$$

We can always choose η in such a way that

$$|\nabla \eta_\varepsilon| \leq C(a^\varepsilon R_\varepsilon)^{-1} \quad \text{and} \quad |\Delta \eta_\varepsilon| \leq C(a^\varepsilon R_\varepsilon)^{-2}$$

for $(x, y) \in B^+_{2a^\varepsilon R_\varepsilon}(\varepsilon k) \setminus B^+_{a^\varepsilon R_\varepsilon}(\varepsilon k)$. We now set:

$$w^\varepsilon(x, y) = \eta_\varepsilon(x, y)\widetilde{w}^\varepsilon(x, y) + (1 - \eta_\varepsilon(x, y)) \sum_{k \in \mathbb{Z}^n \cap D} \varphi^\varepsilon_k(x, y) \mathbf{1}_{B^+_{\varepsilon/2}(\varepsilon k)}(x, y).$$

It satisfies

$$w^\varepsilon(x, y, \omega) = \begin{cases} \varphi^\varepsilon_k(x, y) & \text{for } (x, y) \in D \cap B^+_{a^\varepsilon R_\varepsilon}(\varepsilon k) \quad \forall k \in \mathbb{Z}^n \\[2mm] \widetilde{w}^\varepsilon(x, y) & \text{for } (x, y) \in D \setminus \bigcup_{k \in \mathbb{Z}^n} B^+_{2a^\varepsilon R_\varepsilon}(\varepsilon k). \end{cases}$$

To simplify the notations in the sequel, we denote

$$\varphi^\varepsilon(x, y) := \sum_{k \in \mathbb{Z}^n \cap D} \varphi^\varepsilon_k(x, y, \omega) \mathbf{1}_{B^+_{\varepsilon/2}(\varepsilon k)}(x, y).$$

The properties of w^ε are summarized in the following lemma, which implies Proposition 2.4:

Lemma 5.1. *The function w^ε satisfies the following properties:*
  (i) $w^\varepsilon(x, 0) = 1$ *for $x \in S_\varepsilon$ and $||w^\varepsilon||_{L^\infty(D)} \leq C$.*
  (ii) w^ε *converges to zero in $L^2(D, |y|^a)$-strong as ε goes to zero.*
  (iii) w^ε *is bounded in $W^{1,2}(D, |y|^a)$.*
  (iv) w^ε *satisfies (13).*

Proof. (i) Immediate consequence of the definition of w^ε since $\varphi^\varepsilon_k = 1$ on $S_\varepsilon(k, \omega)$ and $\widetilde{w}^\varepsilon$ and φ^ε_k are bounded in L^∞.
  (ii) Since $S_\varepsilon(k, \omega) \subset B^n_{a^\varepsilon M}(\varepsilon k)$, we have:

$$\varphi^\varepsilon_k(x, y, \omega) \leq C\varepsilon^n \frac{M^{n-1+a}}{\nu_{n-1+a}} h(x - \varepsilon k, y)$$

for all (x, y) such that $|(x - \varepsilon k, y)| \geq a^\varepsilon M$. Since $\varphi_k^\varepsilon \leq 1$ in $B_{a^\varepsilon M}(\varepsilon k)$, we get

$$
\int_D |y|^a \, |(1 - \eta_\varepsilon)\varphi^\varepsilon|^2 \, dx \, dy
$$

$$
\leq \sum_{k \in \mathbb{Z}^n \cap \varepsilon^{-1}\Sigma} \int_{B_{2a^\varepsilon R}(\varepsilon k)} |y|^a \, |\varphi_k^\varepsilon|^2 \, dx \, dy
$$

$$
\leq \sum_{k \in \mathbb{Z}^n \cap \varepsilon^{-1}\Sigma} \int_{B_{a^\varepsilon M}(\varepsilon k)} |y|^a \, dx \, dy
$$

$$
+ C \sum_{k \in \mathbb{Z}^n \cap \varepsilon^{-1}\Sigma} \int_{B_{2a^\varepsilon(R_\varepsilon)(\varepsilon k)} \setminus B_{a^\varepsilon M}(\varepsilon k)} |y|^a \left(\varepsilon^n \frac{M^{n-1+a}}{\nu_{n-1+a}} h(x - \varepsilon k) \right)^2 dx
$$

$$
\leq \sum_{k \in \mathbb{Z}^n \cap \varepsilon^{-1}\Sigma} (a^\varepsilon M)^{n+1+a}
$$

$$
+ C \sum_{k \in \mathbb{Z}^n \cap \varepsilon^{-1}\Sigma} \varepsilon^{2n} (a^\varepsilon M)^{n+1-2(n-1+a)} M^{n-1+a}.
$$

Using (43) and the definition of a^ε, we deduce:

$$
\|(1 - \eta_\varepsilon)\varphi^\varepsilon\|^2_{L^2(D, |y^a|)} \leq C(M) \varepsilon^{\frac{2n - an}{n-1+a}}.
$$

Estimate (41) thus implies

$$
\|w^\varepsilon\|_{L^2(D, |y^a|)} \leq \|\widetilde{w}^\varepsilon\|_{L^2(D, |y^a|)} + \|(1 - \eta_\varepsilon)\varphi^\varepsilon\|_{L^2(D, |y^a|)} = o(1).
$$

and therefore

$$
w^\varepsilon \longrightarrow 0 \qquad L^2(D, |y|^a)\text{-strong.}
$$

(iii) Next, we want to show that w^ε is bounded in $W^{1,2}(D, |y|^a)$. First, we note that outside $\cup_{k \in \mathbb{Z}^n} B_{\varepsilon/2}(\varepsilon k)$ we have $\nabla w^\varepsilon = \nabla \widetilde{w}^\varepsilon$ which is bounded in $W^{1,2}(D, |y|^a)$. Next, we see that in $B_{\varepsilon/2}(\varepsilon k)$, we have:

$$
\nabla w^\varepsilon = \nabla \eta_\varepsilon (\widetilde{w}^\varepsilon - \varphi_k^\varepsilon) + \eta_\varepsilon \nabla \widetilde{w}^\varepsilon + (1 - \eta_\varepsilon) \nabla \varphi_k^\varepsilon \tag{45}
$$

Since $\widetilde{w}^\varepsilon$ and φ^ε are both bounded in $W^{1,2}(D, |y|^a)$ (thanks to (18)), we see that in order to show that ∇w^ε is bounded in $L^2(D, |y|^a)$, we only have to show that

$$
\int_D y^a |\nabla \eta_\varepsilon (\widetilde{w}^\varepsilon - \varphi^\varepsilon)|^2 \, dx \, dy \leq C.
$$

For that purpose, we notice that (42) and (44) yield

$$
|\widetilde{w}^\varepsilon - \varphi_k^\varepsilon| \leq C \frac{\delta_\varepsilon}{R_\varepsilon^{n-1+a}} \quad \text{in } B_{2R_\varepsilon a^\varepsilon}(\varepsilon k) \setminus B_{R_\varepsilon a^\varepsilon}(\varepsilon k),
$$

and so, using the definition of $\eta_\varepsilon(x,y)$, we deduce:

$$\int_D y^a |\nabla \eta_\varepsilon(\widetilde{w}^\varepsilon - \varphi^\varepsilon)|^2 \, dx \, dy$$

$$\leq \sum_{k \in \varepsilon \mathbb{Z}^n \cap \Sigma} \int_{B_{2R_\varepsilon a^\varepsilon}(\varepsilon k) \setminus B_{R_\varepsilon a^\varepsilon}(\varepsilon k)} y^a |\nabla \eta_\varepsilon(\widetilde{w}^\varepsilon - \varphi^\varepsilon_k)|^2 \, dx$$

$$\leq \sum_{k \in \varepsilon \mathbb{Z}^n \cap \Sigma} (R_\varepsilon a^\varepsilon)^{n+1+a} (R_\varepsilon a^\varepsilon)^{-2} \frac{\delta_\varepsilon^2}{R_\varepsilon^{2(n-1+a)}}$$

$$\leq \sum_{k \in \varepsilon \mathbb{Z}^n \cap \Sigma} R_\varepsilon^{-(n-1+a)} \varepsilon^n \delta_\varepsilon^2$$

$$\leq C\varepsilon^{-n} \varepsilon^n \delta_\varepsilon = C\delta_\varepsilon,$$

where we used the fact that we can always assume that $\delta_\varepsilon < 1$ and $R_\varepsilon \geq 1$. For latter use, we note that we actually proved

$$\int_D y^a |\nabla \eta_\varepsilon(\widetilde{w}^\varepsilon - \varphi^\varepsilon)|^2 \, dx \, dy \longrightarrow 0 \text{ when } \varepsilon \to 0. \tag{46}$$

(iv) It remains to show that (13) holds. We only show the inequality (the equality follows easily). Let v^ε be a sequence of functions satisfying:

$$\begin{cases} v^\varepsilon(x,0) \geq 0 & \text{for } x \in T_\varepsilon \\ \|v^\varepsilon\|_{L^\infty(D)} \leq C \\ v^\varepsilon \longrightarrow v & \text{in } W^{1,2}(D, |y|^a) - \text{weak}. \end{cases}$$

Then for any $\phi \in \mathcal{D}(D)$, we have:

$$-\int_D y^a \nabla w^\varepsilon \cdot \nabla v^\varepsilon \phi \, dx \, dy$$

$$= -\int_D y^a \nabla \eta_\varepsilon \cdot \nabla v^\varepsilon (\widetilde{w}^\varepsilon - \varphi^\varepsilon) \phi \, dx \, dy - \int_D y^a \nabla \widetilde{w}^\varepsilon \cdot \nabla v^\varepsilon \phi \, \eta_\varepsilon \, dx \, dy$$

$$- \int_D y^a \nabla \varphi^\varepsilon \cdot \nabla v^\varepsilon \phi \, (1 - \eta_\varepsilon) \, dx \, dy$$

$$= -\int_D y^a \nabla \eta_\varepsilon \cdot \nabla v^\varepsilon (\widetilde{w}^\varepsilon - \varphi^\varepsilon) \phi \, dx \, dy$$

$$+ \int_\Sigma (\lim_{y \to 0} y^a \partial_y \widetilde{w}^\varepsilon) v^\varepsilon \phi \, \eta_\varepsilon \, dx + \int_\Sigma (\lim_{y \to 0} y^a \partial_y \varphi^\varepsilon) v^\varepsilon \phi \, (1 - \eta_\varepsilon) \, dx$$

$$+ \int_D y^a \nabla \widetilde{w}^\varepsilon \cdot \nabla(\phi \eta_\varepsilon) v^\varepsilon \, dx \, dy + \int_D y^a \nabla \varphi^\varepsilon \cdot \nabla(\phi(1 - \eta_\varepsilon)) v^\varepsilon \, dx \, dy$$

where we used the fact that $\text{div}\,(y^a \nabla \widetilde{w}^\varepsilon) = 0$ on supp η^ε and $\text{div}\,(y^a \nabla \varphi^\varepsilon) = 0$ on supp $(1 - \eta^\varepsilon)$. The first term goes to zero thanks to (46) and the weak convergence of ∇v^ε in $L^2(D, |y|^a)$, and the boundary terms satisfy

$$\lim_{\varepsilon \to 0} \int_\Sigma (\lim_{y \to 0} y^a \partial_y \widetilde{w}^\varepsilon) v^\varepsilon \phi \eta_\varepsilon \, dx = \lim_{\varepsilon \to 0} \int_\Sigma \alpha_0 v^\varepsilon \phi \eta_\varepsilon \, dx$$

$$= \lim_{\varepsilon \to 0} \int_\Sigma \alpha_0 v \phi \eta \, dx$$

and

$$\lim_{\varepsilon \to 0} \int_{\Sigma} (\lim_{y \to 0} y^a \partial_y \varphi^{\varepsilon}) v^{\varepsilon} \phi (1 - \eta_{\varepsilon}) \, dx \;\; = \;\; \lim_{\varepsilon \to 0} \int_{T_{\varepsilon}} (\lim_{y \to 0} y^a \partial_y \varphi^{\varepsilon}) v^{\varepsilon} \phi (1 - \eta_{\varepsilon}) \, dx$$
$$\leq \;\; 0.$$

Finally, the last two terms can be rewritten as:

$$\int_D y^a \nabla \widetilde{w}^{\varepsilon} \cdot \nabla (\phi \eta_{\varepsilon}) v^{\varepsilon} \, dx \, dy + \int_D y^a \nabla \varphi^{\varepsilon} \cdot \nabla (\phi (1 - \eta_{\varepsilon})) v^{\varepsilon} \, dx \, dy$$
$$= \int_D y^a \nabla (\widetilde{w}^{\varepsilon} - \varphi^{\varepsilon}) \cdot (\nabla \eta_{\varepsilon}) \, v^{\varepsilon} \phi \, dx \, dy$$
$$+ \int_D y^a \, v^{\varepsilon} \eta_{\varepsilon} \, \nabla \widetilde{w}^{\varepsilon} \cdot \nabla \phi \, dx \, dy + \int_D y^a \, v^{\varepsilon} (1 - \eta_{\varepsilon}) \, \nabla \varphi^{\varepsilon} \cdot \nabla \phi \, dx \, dy$$

Using the weak convergence of $\nabla \widetilde{w}^{\varepsilon}$ and $\nabla \varphi^{\varepsilon}$ to zero, we see that in order to prove (13), it only remains to prove that

$$\int_D y^a \nabla (\widetilde{w}^{\varepsilon} - \varphi^{\varepsilon}) \cdot (\nabla \eta_{\varepsilon}) \, v^{\varepsilon} \phi \, dx \, dy \longrightarrow 0 \text{ when } \varepsilon \to 0.$$

Since v^{ε} is bounded in L^{∞}, it is enough to show that

$$\int_D |y|^a |\nabla (\widetilde{w}^{\varepsilon} - \varphi^{\varepsilon})| \, |\nabla \eta_{\varepsilon}| \, dx \, dy \longrightarrow 0 \text{ when } \varepsilon \to 0.$$

For that purpose, we recall that

$$|\widetilde{w}^{\varepsilon} - \varphi_k^{\varepsilon}| \leq \frac{\delta_{\varepsilon}}{R_{\varepsilon}^{n-1+a}} \qquad \text{in } B_{2a^{\varepsilon} R_{\varepsilon}}^+ \setminus B_{a^{\varepsilon} R_{\varepsilon}}^+,$$

and

$$\begin{cases} \operatorname{div} (y^a \nabla (\widetilde{w}^{\varepsilon} - \varphi_k^{\varepsilon})) = 0 & \text{for } (x, y) \in B_{4a^{\varepsilon} R}^+ \setminus B_{a^{\varepsilon} R_{\varepsilon}/2}^+ \\ \lim_{y \to 0} y^a \partial_y (\widetilde{w}^{\varepsilon} - \varphi_k^{\varepsilon})(x, y) = \alpha_0 & \text{for } x \in B_{4a^{\varepsilon} R}^n \setminus B_{a^{\varepsilon} R_{\varepsilon}/2}^n. \end{cases}$$

In particular, interior gradient estimates (see [7]) imply

$$|\nabla (\widetilde{w}^{\varepsilon} - \varphi_k^{\varepsilon})| \leq \frac{\delta_{\varepsilon}}{R_{\varepsilon}^{n-1+a}} (a^{\varepsilon} R_{\varepsilon})^{-1} + C(a^{\varepsilon} R_{\varepsilon})^{-a}$$

in $B^+_{2a^\varepsilon R_\varepsilon} \setminus B^+_{a^\varepsilon R_\varepsilon}$. We deduce:

$$
\int_D |y|^a |\nabla(\widetilde{w}^\varepsilon - \varphi^\varepsilon)| \, |\nabla \eta_\varepsilon| \, dx \, dy
$$

$$
\leq \sum_{k \in \varepsilon \mathbb{Z}^n \cap \Sigma} \int_{B^+_{2a^\varepsilon R_\varepsilon} \setminus B^+_{a^\varepsilon R_\varepsilon}} |y|^a |\nabla(\widetilde{w}^\varepsilon - \varphi^\varepsilon)| \, |\nabla \eta_\varepsilon| \, dx \, dy
$$

$$
\leq \sum_{k \in \varepsilon \mathbb{Z}^n \cap \Sigma} \frac{C\delta_\varepsilon}{R_\varepsilon^{n-1+a}} (a^\varepsilon R_\varepsilon)^{-2} \int_{B^+_{2a^\varepsilon R_\varepsilon} \setminus B^+_{a^\varepsilon R_\varepsilon}} |y|^a \, dx \, dy
$$

$$
+ \sum_{k \in \varepsilon \mathbb{Z}^n \cap \Sigma} C(a^\varepsilon R_\varepsilon)^{-1-a} \int_{B^+_{2a^\varepsilon R_\varepsilon} \setminus B^+_{a^\varepsilon R_\varepsilon}} |y|^a \, dx \, dy
$$

$$
\leq \sum_{k \in \varepsilon \mathbb{Z}^n \cap \Sigma} \frac{\delta_\varepsilon}{R_\varepsilon^{n-1+a}} (a^\varepsilon R_\varepsilon)^{-2} (a^\varepsilon R_\varepsilon)^{n+1+a}
$$

$$
+ \sum_{k \in \varepsilon \mathbb{Z}^n \cap \Sigma} C(a^\varepsilon R_\varepsilon)^{-1-a} (a^\varepsilon R_\varepsilon)^{n+1+a}
$$

$$
\leq \frac{C\delta_\varepsilon}{R_\varepsilon^{n-1+a}} \varepsilon^{-n} (a^\varepsilon R_\varepsilon)^{n-1+a} + C\varepsilon^{-n} (a^\varepsilon R_\varepsilon)^n.
$$

Using (43) and the definition of a^ε, we deduce:

$$
\int_D |y|^a |\nabla(\widetilde{w}^\varepsilon - \varphi^\varepsilon)| \, |\nabla \eta_\varepsilon| \, dx \, dy \leq C\delta_\varepsilon + C\varepsilon^{n(\sigma-1)}.
$$

which concludes the proof since $\sigma > 1$ and $\lim_{\varepsilon \to 0} \delta_\varepsilon = 0$.

$\square$

Appendix A. **Proof of Lemma 4.1.** We now turn to the proof of Lemma 4.1. We take $k = 0$ and we recall that φ^ε_0 is the capacity potential associated to $S_\varepsilon(0)$. It satisfies (17) and (18). We then introduce the function

$$
G(x, \xi, y, \tau) = h(x - \xi, y - \tau) + h(x - \xi, y + \tau)
$$

which satisfies

$$
\mathrm{div}_{\xi,\tau}(|\tau|^a \nabla_{\xi,\tau} G) = -\mu_{n,a}\delta(x - \xi, y - \tau) - \mu_{n,a}\delta(x - \xi, y + \tau)
$$

and

$$
\lim_{\tau \to 0} \tau^a \partial_\tau G(x, \xi, y, \tau) = 0
$$

for all x, ξ and y. If $y > 0$, we deduce that for any function $\varphi(x, y)$, we have:

$$
\int_{\tau > 0} \tau^a \nabla_{\xi,\tau} G(x, \xi, y, \tau) \nabla_{\xi,\tau} \varphi(\xi, \tau) \, d\xi \, d\tau.
$$

$$
= -\int_{\tau > 0} \mathrm{div}\,(\tau^a \nabla_{\xi,\tau} G(x, \xi, y, \tau)) \varphi(\xi, \tau) \, d\xi \, d\tau
$$

$$
- \int_{\mathbb{R}^n} \lim_{\tau \to 0} \tau^a \partial_\tau G(x, \xi, y, \tau) \varphi(\xi, 0) \, d\xi
$$

$$
= \mu_{n,a}\varphi(x, y).
$$

Moreover, if $\varphi_0^\varepsilon(x,y)$ is the capacity potential associated to $S_\varepsilon(0)$, then (17) yields

$$\int_{\tau>0} \tau^a \nabla_{\xi,\tau} G(x,\xi,y,\tau) \nabla_{\xi,\tau} \varphi_0^\varepsilon(\xi,\tau) \, d\xi \, d\tau.$$

$$= -\int_{\tau>0} G(x,\xi,y,\tau) \mathrm{div}\left(\tau^a \nabla_{\xi,\tau}\varphi_0^\varepsilon(\xi,\tau)\right) d\xi \, d\tau$$

$$- \int_{\mathbb{R}^n} G(x,\xi,y,0) \lim_{\tau\to0} \tau^a \partial_\tau \varphi_0^\varepsilon(\xi,\tau) \, d\xi$$

$$= -2\int_{\mathbb{R}^n} h(x-\xi,y) \lim_{\tau\to0} \tau^a \partial_\tau \varphi_0^\varepsilon(\xi,\tau) \, d\xi.$$

Combining those two equalities, we get:

$$\mu_{n,a}\varphi_k^\varepsilon(x,y) = -2\int_{S_\varepsilon(0)} h(x-\xi,y) \lim_{\tau\to0} \tau^a \partial_\tau \varphi_0^\varepsilon(\xi,\tau) \, d\xi.$$

Next, we note that (18) yields, after integration by parts and using (17):

$$\varepsilon^n\gamma(0) = \int_{\mathbb{R}^n} \tau^a |\nabla\varphi_0^\varepsilon(\xi,\tau)|^2 \, d\xi = -\int_{S_\varepsilon(0)} \lim_{\tau\to0} \tau^a \partial_\tau \varphi_0^\varepsilon(\xi,\tau) \, d\xi,$$

and therefore

$$\varphi_0^\varepsilon(x,y) - \frac{2}{\mu_{n,a}}\varepsilon^n\gamma(0)h(x,y)$$

$$= -\frac{2}{\mu_{n,a}}\int_{S_\varepsilon(0)} [h(x-\xi,y) - h(x,y)] \lim_{\tau\to0} \tau^a \partial_\tau \varphi_0^\varepsilon(\xi,\tau) \, d\xi.$$

In order to conclude, we recall that $S_\varepsilon(0) \subset B_{Ma^\varepsilon}(0)$ and so we have $|\xi| \le Ma^\varepsilon$ in the previous integral. If (x,y) is such that $|(x,y)| \ge Ra^\varepsilon$ with $R \ge 8M$, we deduce that for all $\xi \in S_\varepsilon(0)$, we have:

$$
\begin{aligned}
|h(x-\xi,y) - h(x,y)| &\le \sup_{\xi^*\in B_{Ma^\varepsilon}(0)} |\nabla_{x,y}h(x-\xi^*,y)||\xi| \\
&\le \sup_{\xi^*\in B_{Ma^\varepsilon}(0)} \frac{|\xi|}{((x-\xi^*)^2+y^2)^{\frac{n-a}{2}}} \\
&\le \frac{C|\xi|}{(x^2+y^2)^{\frac{n-a}{2}}} \\
&\le \frac{C|\xi|}{(x^2+y^2)^{\frac{1}{2}}}h(x,y) \\
&\le \frac{CM}{R}h(x,y).
\end{aligned}
$$

We can thus write

$$\left|\varphi_0^\varepsilon(x,y) - \frac{2}{\mu_{n,a}}\varepsilon^n\gamma(0)h(x,y)\right|$$

$$\le \frac{CM}{R}\frac{2}{\mu_{n,a}}h(x,y)\int_{S_\varepsilon(0)} \left|\lim_{\tau\to0} \tau^a \partial_\tau \varphi_0^\varepsilon(\xi,\tau)\right| d\xi$$

$$\le \frac{CM}{R}\frac{2}{\mu_{n,a}}\varepsilon^n\gamma(0)h(x,y),$$

where the right hand side is bounded by $\delta\frac{2}{\mu_{n,a}}\varepsilon^n\gamma(0)h(x,y)$ if R is large enough. $\square$

Appendix B. **Proof of Lemma 4.6.** (i) For a given set A, it is readily seen from the definition of $\overline{v}_{\alpha,A}$ that if $\alpha' \leq \alpha$, then $\overline{v}_{\alpha',A}$ is admissible for the obstacle problem with α: It follows that

$$\overline{v}_{\alpha,A} \leq \overline{v}_{\alpha',A} \qquad \text{for any } \alpha, \alpha' \text{ such that } \alpha' \leq \alpha$$

and so $\alpha \mapsto m_\alpha(A,\omega)$ is nondecreasing. The result follows from the definition of $\overline{\ell}(\alpha)$.

(ii) If α is negative, then we have

$$\lim_{y \to 0} y^a \partial_y \overline{v}_{\alpha,tB}(x,y) < 0 \quad \text{for } x \in \mathbb{R}^n.$$

Since $\overline{v}_{\alpha,tB}(x,y) \geq 0$ for $(x,y) \in \mathbb{R}^{n+1}_+$, we deduce

$$\overline{v}_{\alpha,tB}(x,0) > 0 \quad \text{for } x \in \mathbb{R}^n.$$

It follows that $m_\alpha(tB,\omega) = 0$ for all $t > 0$, and so $\overline{\ell}(\alpha) = 0$ for all $\alpha < 0$.

If $r(k,\omega)$ is bounded from below:

$$r(k,\omega) \geq \underline{r} > 0 \text{ for all } k \in \mathbb{Z}^n, \text{ a.e. } \omega \in \Omega,$$

then, we define

$$\varphi(x,y) = \frac{\underline{r}^{n-1+a}}{(|x|^2 + y^2)^{\frac{n-1+a}{2}}} - \alpha \int_{B_1^n(0)} \frac{\nu_{n+1+a}}{(|x-x'|^2 + y^2)^{\frac{n-1+a}{2}}} \, dx' - C_0$$

with

$$C_0 = \underline{r}^{n-1+a} - \alpha \int_{B_1^n(e)} \frac{\nu_{n+1+a}}{|z|^{n-1+a}} \, dz$$

where e denotes any unit vector in $\mathbb{R}^n$. In particular, we have

$$\varphi(x,0) = 0 \text{ if } |x| = 1,$$

and, if α is small enough

$$\varphi(x,0) > 0 \text{ if } |x| < 1,$$

and

$$\varphi(x,y) < 0 \text{ if } |x| = 1, \quad y > 0$$

(we note that φ is the sum of a term which is decreasing with respect to $|x|$ and one which is increasing). Since φ satisfies

$$\lim_{y \to 0} y^a \partial_y \varphi(x,y) = \alpha - \underline{\gamma}\delta(x) \geq \alpha - \widetilde{\gamma}(0,\omega)\delta(x),$$

for all $x \in B_1^n(0)$, we deduce

$$\overline{v}_{\alpha,tB}(x,0) > \varphi(x,0) > 0 \qquad \text{in } B_1^n(0).$$

Since we can do this in any ball $B_1^n(k)$, we must have $m_\alpha(tB^n,\omega) = 0$ for all $t > 0$, and so $\overline{\ell}(\alpha) = 0$ for all α small enough.

(iii) We consider the function

$$\psi(x,y) = \frac{\overline{r}^{n-1+a}}{(|x|^2 + y^2)^{\frac{n-1+a}{2}}} - \alpha \int_{B_1^n(0)} \frac{\nu_{n+1+a}}{(|x-x'|^2 + y^2)^{\frac{n-1+a}{2}}} \, dx' + C,$$

where the constant C will be chosen later. It satisfies

$$\lim_{y \to 0} y^a \partial_y \psi(x,y) = \alpha - \overline{\gamma}\delta(x) \leq \alpha - \widetilde{\gamma}(0,\omega)\delta(x) \qquad \forall x \in B_1^n(0),$$

$$\psi(x,y) \longrightarrow C \qquad \text{when } |x|^2 + y^2 \to \infty$$

and we note that $\psi(x,0)$ is radially symmetric. Moreover, when α is such that

$$\alpha \int_{B_1(0)} \frac{\nu_{n+1+a}}{|e_1 - x'|^{n-1+a}} \, dx' \geq \overline{r}^{n-1+a}$$

then

$$\psi(x,0) < C \qquad \text{when } |x| = 1.$$

Since $\operatorname{div}(y^a \nabla \psi) = 0$ for $y > 0$ and $\lim_{y \to 0} y^a \partial_y \psi(x,y) = 0$ for $x \notin B_1^n(0)$, the strong maximum principle and Hopf Lemma yield that the minimum of $\psi(x,y)$ is reached for $y = 0$ and $x \in B_1^n(0)$, and with an appropriate choice of the constant C, we can always assume that this minimum is 0:

$$\inf_{\mathbb{R}_+^{n+1}} \psi(x,y) = \inf_{B_1^n(0)} \psi(x,0) = 0.$$

Finally, if α is such that

$$\alpha \int_{B_1^n(0)} \left[\frac{1}{|\frac{e_1}{2} - x'|^{n-1+a}} - \frac{1}{|e_1 - x'|^{n-1+a}} \right] dx' \geq \overline{r}^{n-1+a}(4^{n+1} - 1)$$

then $\psi(x,0)$ reaches its minimum when $|x| = R_\alpha$ with $R_\alpha < 1/4$.

We now consider the function $\varphi(x,y)$ defined by:

$$\varphi(x,y) = \begin{cases} \psi(x-k,y) & \text{for } (x,y) \in B_{1/4}^+(k) \\ \inf_{k'} \psi(x-k',y) & \text{for } (x,y) \in \mathbb{R}_+^{n+1} \setminus \cup_{k'} B_{1/4}^+(k') \end{cases}$$

We clearly have

$$\lim_{y \to 0} y^a \partial_y \varphi(x,y) \leq \alpha_0 - \gamma(k,\omega)\delta(x-k) \qquad \text{for } x \in B_{1/4}^n(k)$$

and

$$\lim_{y \to 0} y^a \partial_y \varphi(x,y) \leq \alpha_0 \qquad \text{for } x \in \mathbb{R}^n \setminus \cup_{k'} B_{1/4}^n(k').$$

In order to prove that φ is a supersolution for the obstacle problem, we only have to check that

$$\psi(x-k,y) = \inf_{k'} \psi(x-k',y) \qquad \text{for } (x,y) \in \partial B_{1/4}^+(k)$$

or equivalently

$$\psi(x,y) = \inf_{k'} \psi(x-k',y) \qquad \text{for } (x,y) \in \partial B_{1/4}^+(0).$$

It is readily seen that this amounts to showing that

$$\alpha \nu_{n+1+a} \int_{B_1^n(0)} \frac{1}{(|x-x'|^2 + y^2)^{\frac{n-1+a}{2}}} - \frac{1}{(|x-k-x'|^2 + y^2)^{\frac{n-1+a}{2}}} \, dx' \geq 4^{n-1} \overline{r}^{n-1+a}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ and all $(x,y) \in \partial B_{1/4}^+(0)$. This inequality is obviously satisfied if α is large enough provided we can prove that

$$\int_{B_1^n(0)} \frac{1}{(|x-x'|^2 + y^2)^{\frac{n-1+a}{2}}} - \frac{1}{(|x-k-x'|^2 + y^2)^{\frac{n-1+a}{2}}} \, dx' > 0$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ and all $(x,y) \in \partial B_{1/4}^+(0)$. This is equivalent to

$$\int_{B_1^n(x)} \frac{1}{(|x'|^2 + y^2)^{\frac{n-1+a}{2}}} \, dx' > \int_{B_1^n(x-k)} \frac{1}{(|x'|^2 + y^2)^{\frac{n-1+a}{2}}} \, dx' > 0$$

whick holds for all $(x,y) \in \partial B_{1/4}^+(0)$ since $|k| \geq 1$.

By definition of $\overline{v}_{\alpha,tB}$, we deduce that

$$\overline{v}_{\alpha,tB}(x) \leq \varphi(x,y) \quad \text{in } tB^n \text{ a.s.}$$

In particular, this implies that $\overline{v}_{\alpha,tB^n}$ vanishes in $tB^n \setminus \cup_{k \in \mathbb{Z}^n} B_{1/2}(k)$, and so

$$\frac{\overline{m}_{\alpha}(tB^n,\omega)}{|tB^n|} \geq \left(\frac{|C_1| - |B_{1/2}^n|}{|C_1|} \right) = 1 - \frac{\omega_n}{2^n} \quad \text{a.s.}$$

We conclude

$$\overline{\ell}(\alpha) \geq 1 - \frac{\omega_n}{2^n} > 0.$$

$\square$

REFERENCES

[1] M. A. Akcoglu and U. Krengel, *Ergodic theorems for superadditive processes*, J. Reine Angew. Math., **323** (1981), 53–67.

[2] L. Carbone and F. Colombini, *On convergence of functionals with unilateral constraints*, J. Math. Pures Appl., **59** (1980), 465–500.

[3] D. Cioranescu and F. Murat, *Un terme étrange venu d'ailleurs*, in "Nonlinear Partial Differential Equations and Their Applications," Collège de France Seminar, Vol. II (Paris, 1979/1980). Pitman. Res. Notes in Math., **60** (1982), 98–138, 389–390.

[4] D. Cioranescu and F. Murat, *Un terme étrange venu d'ailleurs II*, in "Nonlinear Partial Differential Equations and Their Applications," Collège de France Seminar, Vol. III (Paris, 1979/1980). Pitman. Res. Notes in Math., **70** (1982), 154–178, 425–426.

[5] L. Caffarelli and A. Mellet, *Random homogenization of an obstacle problem*, Annales de l'IHP, Analyse non lineaire, to appear.

[6] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Preprint (2006).

[7] L. Caffarelli, S. Salsa and L. Silvestre, *Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian*, Invent. Math., **171** (2008), 425–461.

[8] L. Caffarelli, P. E. Souganidis and L. Wang *Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media*, Comm. Pure Appl. Math., **58** (2005), 319–361.

[9] E. De Giorgi, G. Dal Maso and P. Longo, Γ-*limits of obstacles*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), **68** (1980), 481–487.

[10] G. Dal Maso, *Asymptotic behaviour of minimum problems with bilateral obstacles*, Ann. Mat. Pura Appl. (4), **129** (1981), 327–366.

[11] G. Dal Maso and P. Longo, Γ-*limits of obstacles* , Ann. Mat. Pura Appl. (4), **128** (1981), 1–50.

[12] G. Dal Maso and L. Modica, *Nonlinear stochastic homogenization and ergodic theory*, J. Reine Angew. Math., **368** (1986), 28–42.

[13] E. Fabes, C. Kenig and R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations, **7** (1982), 77–116.

[14] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math., **60** (2007), 67–112.