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TWO-PARAMETER HOMOGENIZATION FOR A GINZBURG-LANDAU PROBLEM IN A PERFORATED DOMAIN

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Abstract. Let A be an annular type domain in $\mathbb{R}^2$. Let A_δ be a perforated domain obtained by punching periodic holes of size δ in A; here, δ is sufficiently small. Suppose that $\mathcal{J}$ is the class of complex-valued maps in A_δ, of modulus 1 on ∂A_δ and of degrees 1 on the components of ∂A, respectively 0 on the boundaries of the holes.

We consider the existence of a minimizer of the Ginzburg-Landau energy

$$E_\lambda(u) = \frac{1}{2}\int_{A_\delta} (|\nabla u|^2 + \frac{\lambda}{2}(1-|u|^2)^2)$$

among all maps in $u \in \mathcal{J}$.

It turns out that, under appropriate assumptions on $\lambda = \lambda(\delta)$, existence is governed by the asymptotic behavior of the H^1-capacity of A_δ.
When the limit of the capacities is $> \pi$, we show that minimizers exist and that they are, when $\delta \to 0$, equivalent to minimizers of the same problem in the subclass of $\mathcal{J}$ formed by the $\mathbb{S}^1$-valued maps. This result parallels the one obtained, for a fixed domain, in [3], and reduces homogenization of the Ginzburg-Landau functional to the one of harmonic maps, already known from [2].

When the limit is $< \pi$, we prove that, for small δ, the minimum is not attained, and that minimizing sequences develop vortices. In the case of a fixed domain, this was proved in [1].

1. Introduction.

Let Ω_o and Ω_i be two smooth bounded simply connected domains in $\mathbb{R}^2$ such that $\bar{\Omega}_i \subset \Omega_o$. Consider the annular type domain $A = \Omega_o \setminus \bar{\Omega}_i$. Set $\Gamma_o = \partial\Omega_o$, $\Gamma_i = \partial\Omega_i$, so that $\partial A = \Gamma_o \cup \Gamma_i$.

We define a perforated domain A_δ obtained by "punching" holes of size δ in A. To this end we first introduce a unit periodicity cell V. Let $\mathbf{a}$ and $\mathbf{b}$ be two linearly independent vectors in $\mathbb{R}^2$ and set $P = \{s\mathbf{a} + t\mathbf{b}; s, t \in (0,1)\}$. Let U be a smooth simply connected domain such that $\bar{U} \subset P$. The unit cell is defined as $V = P \setminus \bar{U}$. Set $\Gamma = \partial U$.

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Consider, for a small number $\delta > 0$, a point $x_\delta \in \mathbb{R}^2$ and define $Z_\delta = \{m \in \mathbb{Z}^2 \; ; \; \delta m + x_\delta + \delta P \subset A\}$. Then the perforated domain is defined as follows:

$$A_\delta = A \setminus \cup_{m \in Z_\delta} (\delta m + x_\delta + \delta \bar{U}), \tag{1}$$

with the boundary

$$\partial A_\delta = \Gamma_o \cup \Gamma_i \cup \cup_{m \in Z_\delta} (\delta m + x_\delta + \delta \Gamma). \tag{2}$$

Our goal is to study asymptotic behavior as $\delta \to 0$ and $\lambda \to \infty$ of solutions (minimizers) of the following minimization problem:

$$m_\delta := \operatorname{Inf}\left\{ E_\lambda(u) = \frac{1}{2} \int_{A_\delta} |\nabla u|^2 + \frac{\lambda}{4} \int_{A_\delta} (1 - |u|^2)^2 \; ; \; u \in \mathcal{J} \right\}. \tag{3}$$

Here, $\lambda = \lambda(\delta) \to \infty$ is a Ginzburg-Landau (GL) parameter (the exact relation between λ and δ will be specified later), E_λ is a GL energy functional. The class $\mathcal{J}$ of testing maps is

$$\begin{aligned} \mathcal{J} = \Big\{ & u \in H^1(A_\delta \; ; \mathbb{C}); \text{ tr } u \in H^{1/2}(\partial A_\delta \; ; \; \mathbb{S}^1), \; \deg(u, \Gamma_o) \\ & = \; \deg(u, \Gamma_i) = 1, \; \deg(u, \delta m + x_\delta + \delta \Gamma) = 0, \text{for any } m \in Z_\delta \Big\}. \end{aligned} \tag{4}$$

The degrees here are computed with respect to the direct orientation of the connected components of ∂A_δ.

To begin with, we note that the definition of $\mathcal{J}$ is meaningful. Indeed, if γ is a simple closed curve and $u \in H^{1/2}(\gamma \; ; \; \mathbb{S}^1)$, then u has a degree on γ (since $u \in VMO(\gamma \; ; \; \mathbb{S}^1)$ and such maps have a degree [10]). If u is a minimizer of (3)-(4), then actually $u \in C^\infty(\bar{A}_\delta)$ [4], thus for a minimizer, the degree is the classical winding number.

Recall that, for fixed δ and large λ, the minimizers of the problem (3)-(4) exist (subcritical domain) or may not exist (supercritical domain) depending on the H^1-capacity of the domain A_δ [3, 1]. It turns out that asymptotic behavior of minimizers of (3)-(4) for subcritical domains can be understood by establishing asymptotic equivalence (see Theorem 1 below) between these minimizers and the minimizers of the following problem

$$\begin{aligned} M_\delta := \operatorname{Inf}\Big\{ & \frac{1}{2} \int_A |\nabla u|^2; \; u \in H^1(A_\delta \; ; \mathbb{S}^1), \; \deg(u, \Gamma_o) = \deg(u, \Gamma_i) = 1, \\ & \deg(u, \delta m + x_\delta + \delta \Gamma) = 0, \text{ for any } m \in Z_\delta \Big\}. \end{aligned} \tag{5}$$

The infimum is always attained in (5) [6], Chapter 1. Note that while the variational problem (3)-(4) is nonlinear, the latter problem (5) has an underlying linear problem for the phase of the corresponding harmonic maps and therefore is much easier to analyze both asymptotically and numerically. Indeed homogenization for the problem (5) as $\delta \to 0$ has been established in [2], and therefore the above mentioned asymptotic equivalence provides the homogenization result for (3)-(4).

While the main effort in this work is on establishing the asymptotic equivalence in the subcritical domains, we also find the asymptotic behavior of the minimizing sequences for supercritical domains Theorems 2-3.

In order to state our main result on asymptotic equivalence, we begin with the following

Proposition 1. *There exists* $\lim_{\delta\to 0} M_\delta = M \in (0, +\infty)$.

Proof. Recall the following "torsion" problem:

$$N_\delta = \operatorname{Inf}\left\{\frac{1}{2}\int_A |\nabla\phi|^2;\ \phi \in H^1(A_\delta\ ;\mathbb{R}),\ \phi_{|\Gamma_o} = 1,\ \phi_{|\Gamma_i} = 0,\right. \\ \left. \phi_{|\delta m + x_\delta + \delta\Gamma} = \text{unknown constant, for any } m \in Z_\delta\right\}. \tag{6}$$

Let N_δ be the infimum (actually minimum) in (6). Recall [13] that there exists

$$\lim_{\delta\to 0} N_\delta = N \in (0, +\infty).$$

Problems (5) and (6) are related as follows: we have $M_\delta = \frac{1}{2}\int_A |\nabla\Phi|^2$, where Φ solves

$$\begin{cases} \Delta\Phi = 0, & \text{in } A_\delta \\ \Phi = 0 & \text{on } \Gamma_i \\ \Phi = \text{unknown constant} & \text{on } \Gamma_o \\ \Phi = \text{unknown constant} & \text{on each } \delta m + x_\delta + \delta\Gamma \\ \displaystyle\int_{\Gamma_o} \frac{\partial\Phi}{\partial\nu} = 2\pi \\ \displaystyle\int_{\delta m + x_\delta + \delta\Gamma} \frac{\partial\Phi}{\partial\nu} = 0, & \text{for any } \ m \in Z_\delta \end{cases} \tag{7}$$

(see. e.g, Chapter 1 in [6], where it is shown that, if u is a minimizer of (5) and we write locally $u = exp(i\psi)$, with ψ smooth, then ψ is harmonic. The harmonic conjugate Φ of ψ, *a priori* only locally defined, turns out to be globally defined and satisfies (7)). On the other hand, if ϕ is a minimizer of (6), then it is easy to see that ϕ satisfies:

$$\begin{cases} \Delta\phi = 0, & \text{in } A_\delta \\ \phi = 0 & \text{on } \Gamma_i \\ \phi = 1 & \text{on } \Gamma_o \\ \phi = \text{unknown constant} & \text{on each } \delta m + x_\delta + \delta\Gamma \\ \displaystyle\int_{\delta m + x_\delta + \delta\Gamma} \frac{\partial\phi}{\partial\nu} = 0, & \text{for any } \ m \in Z_\delta \end{cases}. \tag{8}$$

Thus $\phi = C\Phi$ for some constant C. Since

$$N_\delta = \frac{1}{2}\int_{A_\delta} |\nabla\phi|^2 = \frac{1}{2}\int_{\Gamma_o} \frac{\partial\phi}{\partial\nu} = \frac{1}{2}\int_{\Gamma_o} C\frac{\partial\Phi}{\partial\nu} = \frac{1}{2}C2\pi,$$

we find that $\Phi = \frac{\pi}{N_\delta}\phi$. Thus $M_\delta = \frac{\pi^2}{N_\delta}$, and the conclusion of the proposition follows. $\square$

Remark 1. N_δ is the (generalized) H^1-capacity of A_δ. Therefore, N may be viewed as a homogenized H^1-capacity.

We now show that the existence of minimizers of (3) is governed by the value of M (thus by the value of N).

Proposition 2. *Assume that $M < 2\pi$. Then, for sufficiently small $\delta > 0$, the infimum m_δ in (3) is attained.*

Proof. For small δ, we have $M_\delta < 2\pi$. Since test functions for (5) are also test functions for (3), we find that $m_\delta \le M_\delta < 2\pi$. It remains to recall that Proposition 5 in [3] states that if $m_\delta < 2\pi$, then m_δ is attained. □

Theorem 1. *Assume that $M < 2\pi$ and that*

$$(H1) \lim_{\delta\to 0} \lambda(\delta)\delta^2 = \infty.$$

Then:
a) for sufficiently small $\delta > 0$, minimizers u_δ of (3) are **unique** *up to a rotation, i.e., if u'_δ and u_δ are minimizers, then $u'_\delta = \alpha u_\delta$ for some $\alpha \in \mathbb{S}^1$.*
b) if u^δ is a minimizer of (5)(a harmonic map), then there is some $\alpha_\delta \in \mathbb{S}^1$ such that, as $\delta \to 0$, $u_\delta - \alpha_\delta u^\delta \to 0$ both in $H^1(A_\delta)$ and uniformly in $\bar{A}_\delta$.
In particular, $|u_\delta| \to 1$ uniformly in $\bar{A}_\delta$, so that u_δ is **vortexless** *for sufficiently small δ.*

Corollary 1. *The map $\bar{\alpha}_\delta u_\delta / u^\delta$, initially defined in the perforated domain A_δ, has an extension w_δ defined in A such that $w_\delta \to 1$ in $H^1(A)$.*

Remark 2. Clearly, if u minimizes either (3) or (5), then so does $\alpha\, u$, $\forall\, \alpha \in \mathbb{S}^1$, and therefore one can not hope to "get rid" of the constant α_δ in the above statements.

Remark 3. With more standard notations, we have $\lambda = \kappa^2 = 1/\varepsilon^2$, where κ is the usual GL parameter and $\varepsilon = 1/\kappa$ behaves like a length (penetration depth). With these units, (H1) becomes $\varepsilon \ll \delta$, that is the penetration length is much smaller then the size of the periodicity cells.

Concerning the case $M > 2\pi$, we have the following result, where, presumably only for technical reasons, we had to replace the rather natural condition $(H1)$ by a slightly stronger one.

Theorem 2. *Assume that $M > 2\pi$ and $(H2) \lim_{\delta\to 0} \sqrt{\lambda}\delta/\ln\lambda = \infty$. Then there is some δ_0 such that, for $\delta < \delta_0$, m_δ is not attained.*

Remark 4. In terms of ε, assumption $(H2)$ reads $\varepsilon|\ln\varepsilon| \ll \delta$. Thus, $(H2)$ is satisfied if, e. g., $\varepsilon \le C\delta^{1+a}$ for some $a > 0$.

The next result asserts that, if $M > 2\pi$ and δ is sufficiently small, then minimizing sequences "develop exactly two vortices, one near Γ_o, the other one near Γ_i". Stated in this form, the result is not true, since the testing maps are merely H^1, and there is no good notion of zero set in this case. In order to have a rigorous result, we proceed as in [3]. We first regularize a testing map: given $v \in \mathcal{J}$, let u equal v on ∂A_δ and minimize the GL energy with respect to its boundary value. Then u is smooth in A_δ; thus its zero set is well-defined, unlike the one v. We call u a quasi-minimizer. A quasi-minimizing sequence is a sequence $\{u_n\}$ of quasi-minimizers such that $E_\lambda(u_n) \to m_\delta$.

Theorem 3. *Assume that $M > 2\pi$ and $(H1) \lim_{\delta\to 0} \lambda\delta^2 = \infty$. Let δ be sufficiently small and let $\{u_n\}$ be a quasi-minimizing sequence. Then, for large n, u_n has exactly two zeroes, one ζ_n, of degree 1 and such that $\zeta_n \to \Gamma_i$, the other one ξ_n, of degree -1 and such that $\xi_n \to \Gamma_i$.*

Here, the degree is the degree of u_n computed on a small circle around ζ_n, respectively ξ_n.

Remark 5. Unlike the case of a fixed domain [3], we do not know what happens when $M = 2\pi$. The answer seems to depend on whether M_δ converges to M from above or from below.

A word about the proofs. The only part where periodicity comes into the picture is Proposition 1, which is needed to define the value M. Otherwise, the proofs could deal with non periodic holes of size δ, at distance $\geq C\delta$ from ∂A, of mutual distance $\geq C\delta$ and of "uniform geometry" (what this means, it will be clear from Section 2).

2. **Proof of Theorem 1 and of Corollary 1. Outline of the proof.** The main part consists in proving that $|u_\delta| \to 1$ uniformly in $\overline{A}_\delta$ as $\delta \to 0$. To this end, we first prove that such convergence holds "far away" from ∂A_δ (Step 2); this is an easy consequence of an estimate taken from [14]. More delicate is convergence "near" ∂A_δ; this is the core of the proof. In a slightly different context, a similar situation is considered in [12]. We present below a different approach (Step 3). Step 1 provides preliminary estimates in Step 3. Once the uniform convergence is known, H^1 convergence of u_δ/u^δ (modulo $\mathbb{S}^1$) is straightforward. Better estimates are obtained in Step 4, using an idea from [15]. These estimates are required in Step 5 (uniform convergence of u_δ/u^δ). The key ingredients in this part are the fact that the phase of u_δ/u^δ satisfies a jacobian type equation (idea borrowed from [9]) together with some estimate for such equations [7]. In Step 6, we prove the fundamental estimate $|\nabla u_\delta| = o(1/\sqrt{\lambda})$; the proof is obtained via the analysis of a Gagliardo-Nirenberg type estimate obtained in [5]. Once this estimate is obtained, uniqueness of u_δ modulo $\mathbb{S}^1$ is well-known (Step 7).

Throughout this section, we assume that the hypotheses of Theorem 1 hold: that is, $M < 2\pi$ and $\lim\limits_{\delta\to 0} \lambda\delta^2 = \infty$. In addition, we assume δ sufficiently small, in order to have existence of u_δ.

Step 1. Comparison results for M_δ

With $\Gamma = \partial U$, let $d(x)$ denote the signed distance of a point x to Γ ($d(x)$ is positive outside Γ, negative inside it). Set $\Gamma_t = \{x; d(x) = t\}$, so that $\Gamma_0 = \Gamma$. In what follows, we suppose that $|t|$ is sufficiently small.

Let U_t be the interior of Γ_t and let $V_t = P\backslash \bar{U}_t$. Then U_t is simply connected and it makes sense to consider:
(i) $A_{\delta,t} = A \setminus \cup_{m\in Z_\delta} (\delta m + x_\delta + \delta \bar{U}_t)$;

(ii) the minimization problem

$$(P2_t) \qquad M_{\delta,t} := \inf\left\{\frac{1}{2}\int\limits_{A_{\delta,t}} |\nabla u|^2; u \in H^1(A_{\delta,t}\,; \mathbb{S}^1), \deg(u, \Gamma_o)\right.$$
$$\left. = \deg(u,\Gamma_i) = 1, \deg(u, \delta m + x_\delta + \delta\Gamma_t) = 0, \text{ for any } \ m \in Z_\delta\right\}$$

Lemma 1. *We have $|M_{\delta,t} - M_\delta| \leq C|t|$ for some C independent of t and δ.*

Proof. We start by noting that $M_{\delta,t} \le C$ for some C independent of δ and t. Indeed, fix a map $u \in C^\infty(\bar{A};\mathbb{S}^1)$ such that $\deg(u,\Gamma_o) = \deg(u,\Gamma_i) = 1$. Then $u_{|A_{\delta,t}}$ is a test function for $(P2_t)$. Therefore,

$$M_{\delta,t} \le \frac{1}{2}\int_{A_{\delta,t}} |\nabla u|^2 \le \frac{1}{2}\int_{A} |\nabla u|^2 = \text{const}.$$

Returning to the proof, we consider only the case $t > 0$; the proof of the case $t < 0$ is similar.

Let, for $x \in \Gamma, \vec{\nu}(x)$ be the inner normal to Γ. If $\varepsilon > 0$ is sufficiently small, then each y such that $|d(y)| \le \varepsilon$ may be uniquely written as $y = x - d(y)\vec{\nu}(x)$ for some $x \in \Gamma$. We set, for such a $y, \vec{\nu}(y) = \vec{\nu}(x)$. Then $y \mapsto \vec{\nu}(y)$ is smooth. Assume that $|t| < \varepsilon/2$. Define $\Phi_t : \overline{V_t} \to \overline{V}$ through the formula

$$\Phi_t(x) = \begin{cases} x, & \text{if } d(x) \ge \varepsilon \\ x + \dfrac{t(\varepsilon - d(x))}{\varepsilon - t}\,\vec{\nu}(x), & \text{if } t \le d(x) < \varepsilon \end{cases}.$$

Clearly, Φ_t^{-1} is given by

$$\Phi_t^{-1}(x) = \begin{cases} x, & \text{if } d(x) \ge \varepsilon \\ x - \dfrac{t(\varepsilon - d(x))}{\varepsilon}\,\vec{\nu}(x), & \text{if } 0 \le d(x) < \varepsilon \end{cases}.$$

It is obvious from these two formulae that Φ_t has the following properties:
(i) $\Phi_t = \text{id}$ near ∂P;
(ii) Φ_t is an orientation preserving diffeomorphism of Γ_t into Γ;
(iii) $\|D\Phi_t - \text{id}\| \le Ct$, $\|D\Phi_t^{-1} - \text{id}\| \le Ct$, for some C independent of t.

With the help of Φ_t, we may construct a diffeomorphism $\Phi_{\delta,t}$ of $\bar{A}_{\delta,t}$ into $\bar{A}_\delta$ by setting

$$\Phi_{\delta,t}(x) = \begin{cases} x, & \text{if } x \notin \bigcup_{m\in Z_\delta}(\delta m + x_\delta + \delta\overline{V_t}) \\ \delta m + x_\delta + \delta\Phi_t(\dfrac{1}{\delta}(x - x_\delta) - m), & \text{if } x \in \delta m + x_\delta + \delta\overline{V_t} \end{cases}.$$

Thus
(i) $\Phi_{\delta,t}$ is a Lipschitz diffeomorphism, with inverse

$$\Phi_{\delta,t}^{-1}(x) = \begin{cases} x, & \text{if } x \notin \bigcup_{m\in Z_\delta}(\delta m + x_\delta + \delta\overline{V}) \\ \delta m + x_\delta + \delta\Phi_t^{-1}(\dfrac{1}{\delta}(x - x_\delta) - m), & \text{if } x \in \delta m + x_\delta + \delta\overline{V} \end{cases};$$

(ii) $\Phi_{\delta,t}$ is an orientation preserving diffeomorphism of $\delta m + x_\delta + \delta\Gamma_t$ into $\delta m + x_\delta + \delta\Gamma$. In addition, $\Phi_{\delta,t}$ restricted to $\Gamma_o \cup \Gamma_i$ is the identity map.
(iii) $\|D\Phi_{\delta,t} - \text{id}\| \le Ct$, $\|D\Phi_{\delta,t}^{-1} - \text{id}\| \le Ct$, for some $C > 0$ independent of t.

If u is a test function for $(P2)$, then $u \circ \Phi_{\delta,t}$ is a test function for $(P2_t)$. Thus

$$M_{\delta,t} \le \frac{1}{2}\int_{A_{\delta,t}} |\nabla(u \circ \Phi_{\delta,t})|^2 \le \frac{1}{2}(1 + Ct)\int_{A_\delta} |\nabla u|^2,$$

so that

$$M_{\delta,t} \le (1 + Ct)M_\delta \le M_\delta + C't.$$

Similarly, $M_\delta \le M_{\delta,t} + C''t$. □

We will need below a version of Lemma 1. Set $\Gamma_o^\varepsilon = \{x \in A \ ; \ \mathrm{dist}(x, \Gamma_o) = \varepsilon\}$, $\Gamma_i^\varepsilon = \{x \in A \ ; \ \mathrm{dist}(x, \Gamma_i) = \varepsilon\}$. Set also $A^\varepsilon = \{x \in A \ ; \ \mathrm{dist}\ (x, \partial A) > \varepsilon\}$ and denote $A'_{\delta,t} = A^{|t|\delta} \cap A_{\delta,t}$. We note that, if $|t|$ is sufficiently small, then we have

$$\partial A'_{\delta,t} = \Gamma_o^{|t|\delta} \cup \Gamma_i^{|t|\delta} \cup \bigcup_{m \in Z_\delta} (\delta m + x_\delta + \delta\Gamma_t).$$

Lemma 2. *Let*

$$(P2'_t)\ M'_{\delta,t} := \inf \left\{ \frac{1}{2} \int_{A'_{\delta,t}} |\nabla u|^2 ; u \in H^1(A'_{\delta,t}\ ; \mathbb{S}^1), \deg(u, \Gamma_o^{|t|\delta}) = \deg(u, \Gamma_i^{|t|\delta}) = 1, \deg(u, \delta m + x_\delta + \delta\Gamma_t) = 0,\ \textit{for any } m \in Z_\delta \right\}.$$

Then $|M'_{\delta,t} - M_\delta| \leq C|t|$ for some C independent of small t and δ.

The proof, very similar to the one of Lemma 1, is left to the reader.

Step 2. For small δ, $|u_\delta|$ is close to 1 "far away" from ∂A_δ

We recall the following estimates obtained in [14].

Lemma 3. *Let $B > 0$ be a fixed constant. Then, with constants $C = C_{\ell,B}$ depending only on B and on $\ell \in \mathbb{N}$, a solution u of the Ginzburg–Landau equation $-\Delta u = \lambda u(1 - |u|^2)$ in $B(0,R) \subset \mathbb{R}^2$ satisfying $|u| \leq 1$ and the energy estimate*

$$\frac{1}{2} \int_{B(0,R)} |\nabla u|^2 + \frac{\lambda}{4} \int_{B(0,R)} (1 - |u|^2)^2 \leq B$$

satisfies the inequalities

$$|D^\ell u(0)| \leq \frac{C}{R^\ell},$$
$$|D^\ell (1 - |u|^2)(0)| \leq \frac{C}{\lambda R^{\ell+2}}.$$

Actually, Lemma 3 was proved in [14] for $R = 1$; the general case follows by scaling.

Lemma 4. *Let $t > 0$ be sufficiently small and fixed. Let $\mu \in (0,1)$. Then, for sufficiently small δ, we have $|u_\delta| \geq \mu$ in $A'_{\delta,t}$, provided that the hypotheses of Theorem 1 hold.*

Proof. If $x \in A_\delta$ and $R = \mathrm{dist}(x, \partial A_\delta)$, then

$$\frac{1}{2} \int_{B(x,R)} |\nabla u_\delta|^2 + \frac{\lambda}{4} \int_{B(x,R)} (1 - |u_\delta|^2)^2 \leq m_\delta \leq M_\delta \leq B,$$

for some B independent of x or δ. In addition, minimizers of $(P1)$ satisfy $|u_\delta| \leq 1$ and the Ginzburg–Landau equation [3]. We are in position to apply Lemma 3, which yields

$$1 - |u_\delta(x)|^2 \leq \frac{C}{\lambda R^2}.$$

If $x \in A'_{\delta,t}$, then $R \geq t\delta$. Therefore,

$$1 - |u_\delta(x)|^2 \leq \frac{C'}{\lambda \delta^2}.$$

The conclusion of the lemma follows from assumption $(H1)$. □

Step 3. For small δ, $|u_\delta|$ is close to 1 in $\overline{A}_\delta$

Lemma 5. *Let C be a smooth annular domain with outer (inner) boundary γ_o (γ_i). Let $u \in C^1(\bar{C};\mathbb{C})$ be such that $|u| \le 1$ in C and $\mu \le |u| \le 1$ on ∂C. Here, $0 < \mu < 1$. Let $d_o = \deg(u/|u|, \gamma_o), d_i = \deg(u/|u|, \gamma_i)$. Then*

$$\frac{1}{2}\int_C |\nabla u|^2 \ge \pi\mu^2 |d_o - d_i|.$$

Proof. Set $v = f(u)$, where $f(z) = \begin{cases} z/|z|, & \text{if } |z| \ge \mu \\ z/\mu, & \text{if } |z| < \mu \end{cases}$. Then $|\nabla u| \ge \mu |\nabla v|$, $|v| = 1$ on ∂C, $\deg(v, \gamma_o) = d_o$, $\deg(v, \gamma_i) = d_i$. It suffices to prove that $\frac{1}{2}\int_C |\nabla v|^2 \ge \pi|d_o - d_i|$. This follows from

$$\frac{1}{2}\int_C |\nabla v|^2 \ge \left|\int_C \text{Jac } v\right| = \frac{1}{2}\left|\int_{\partial C} v \times \frac{\partial v}{\partial \tau}\right| = \pi|d_o - d_i|. \tag{9}$$

We used here the degree formula

$$\deg(v, \gamma) = \frac{1}{2\pi}\int_\gamma v \times \frac{\partial v}{\partial \tau},$$

where γ is positively oriented and $v : \gamma \to \mathbb{S}^1$. For further use, we note that the equality

$$\int_C \text{Jac } v = \pi(\deg(v, \gamma_o) - \deg(v, \gamma_i))$$

(and thus the conclusion of Lemma 5) holds if we merely suppose $u \in H^1$; see [4] for details. □

Lemma 6. *For sufficiently small δ and fixed $t > 0$, we have $\deg(u_\delta/|u_\delta|, \Gamma_o^{t\delta}) = \deg(u_\delta/|u_\delta|, \Gamma_i^{t\delta}) = 1$ and $\deg(u_\delta/|u_\delta|, \delta m + x_\delta + \delta\Gamma_t) = 0$, $\forall\, m \in Z_\delta$.*
In other words, $u_\delta/|u_\delta|$ is a test function for $(P2'_t)$.

Proof. In view of Lemma 4, we may assume the $|u_\delta| \ge \mu$ on $\partial A'_{\delta,t}$; here, $\mu \in (0,1)$ is to be chosen later. The set $A_\delta \setminus \overline{A'_{\delta,t}}$ is a union of disjoint smooth annular domains:

$$\begin{aligned} A_\delta \setminus \overline{A'_{\delta,t}} =& \{x \in A\ ;\ 0 < \text{dist}(x, \Gamma_o) < \delta t\} \\ &\cup \{x \in A\ ;\ 0 < \text{ dist}(x, \Gamma_i) < \delta t\} \cup \bigcup_{m \in Z_\delta} (\delta m + x_\delta + \delta(U_t \setminus \bar{U})). \end{aligned}$$

Applying Lemma 5 to each of these domains, we find that, with $v = v_\delta = u_\delta/|u_\delta|$, we have

$$\begin{aligned} M_\delta \ge m_\delta \ge \frac{1}{2}\int_{A_\delta \setminus \overline{A'_{\delta,t}}} |\nabla u_\delta|^2 \ge& \pi\mu^2 \Big(\sum_{m \in Z_\delta} |\deg(v, \delta m + x_\delta + \delta\Gamma_t)| \\ &+ |\deg(v, \Gamma_o^{\delta t}) - 1| + |\deg(v, \Gamma_i^{\delta t}) - 1| \Big) \end{aligned}. \tag{10}$$

Since v_δ is smooth and of modulus 1 in $A'_{\delta,t}$, we have

$$\deg(v_\delta, \Gamma_o^{\delta t}) = \deg(v_\delta, \Gamma_i^{\delta t}) + \sum_{m \in Z_\delta} \deg(v_\delta, \delta m + x_\delta + \delta\Gamma_t). \tag{11}$$

Argue by contradiction: assume that one of the equalities stated in the lemma is false. By (11), there has to be a second equality violated among the ones stated. Therefore, the right-hand side of (10) is at least $2\pi\mu^2$. We find that $M_\delta \geq 2\pi\mu^2$. If we pick δ sufficiently small and μ sufficiently close to 1, this inequality contradicts the fact that $M < 2\pi$. □

Lemma 7. *We have*

$$\lim_{\delta\to 0} \lambda \int_{A_\delta} (1 - |u_\delta|^2)^2 = 0 \tag{12}$$

and

$$\lim_{t\to 0} \limsup_{\delta\to 0} \int_{A_\delta \setminus A'_{\delta,t}} |\nabla u_\delta|^2 = 0. \tag{13}$$

Proof. Fix $\mu \in (0,1)$. For small δ, we have $|u_\delta| \geq \mu$ in $A'_{\delta,t}$. Set $v_\delta = u_\delta/|u_\delta|$. Then $|\nabla u_\delta| \geq \mu |\nabla v_\delta|$ and $\frac{1}{2}\int_{A'_{\delta,t}} |\nabla v_\delta|^2 \geq M'_{\delta,t}$. Thus

$$\begin{aligned} M_\delta \geq m_\delta &= \frac{1}{2}\int_{A'_{\delta,t}} |\nabla u_\delta|^2 + \frac{1}{2}\int_{A_\delta \setminus A'_{\delta,t}} |\nabla u_\delta|^2 + \frac{\lambda}{4}\int_{A_\delta} (1-|u_\delta|^2)^2 \\ &\geq \mu^2 M'_{\delta,t} + \frac{1}{2}\int_{A_\delta \setminus A'_{\delta,t}} |\nabla u_\delta|^2 + \frac{\lambda}{4}\int_{A_\delta} (1-|u_\delta|^2)^2. \end{aligned}$$

The conclusion follows then immediately from Lemma 2. □

Lemma 8. *Let C be a chord in the unit disk, C different from a diameter. Let S be the smallest of the two closed regions delimited by C inside the closed unit disk $\bar{\mathbb{D}}$.*

Let $\mathcal{O}$ be a smooth bounded domain and let $g \in C^\infty(\partial\mathcal{O}; S)$.

If u minimizes the Ginzburg–Landau energy $\frac{1}{2}\int_{\mathcal{O}} |\nabla u|^2 + \frac{\lambda}{4}\int_{\mathcal{O}} (1-|u|^2)^2$ among all the functions that equal g on $\partial\mathcal{O}$, then $u(\mathcal{O}) \subset S$.

Proof. We may assume that, for some $\mu \in (0,1)$, we have $C = \{z \in \bar{\mathbb{D}};\ \mathrm{Re}\ z = \mu\}$ and $S = \{z \in \bar{\mathbb{D}};\ \mathrm{Re}\ z \geq \mu\}$.

We first claim that $\mathrm{Re}\ u \geq 0$. Indeed, the map $v = |\mathrm{Re}\ u| + i\ \mathrm{Im}\ u$ equals g on $\partial\mathcal{O}$ and has same energy as u. Thus both u and v satisfy the Ginzburg–Landau equation. It follows that $\mathrm{Re}\ u$ and $|\mathrm{Re}\ u|$ are (real) analytical; therefore, so is $\mathrm{Re}\ u_-$. Since $\mathrm{Re}\ u_-$ vanishes near $\partial\mathcal{O}$, we find that $\mathrm{Re}\ u_- = 0$, i. e. $\mathrm{Re}\ u \geq 0$.

Let P be the orthogonal projection on S. When $z \in \bar{\mathbb{D}} \cap \{\mathrm{Re}\ z \geq 0\}$, we have

$$P(z) = \begin{cases} z, & \text{if } \ \mathrm{Re}\ z \geq \mu \\ \mu + i\ \mathrm{Im}\ z, & \text{if } |\mathrm{Im}\ z| \leq \sqrt{1-\mu^2} \text{ and } \mathrm{Re}\ z < \mu \\ \mu + i(\mathrm{sgn}\ \ \mathrm{Im}\ z)\sqrt{1-\mu^2}, & \text{if } |\mathrm{Im}\ z| > \sqrt{1-\mu^2} \text{ and } \mathrm{Re}\ z < \mu \end{cases}.$$

Set $w = P \circ u$, which equals g on $\partial\mathcal{O}$. Since P is 1-Lipschitz, we have $|\nabla w| \le |\nabla u|$. One may easily check that, for $z \in \bar{\mathbb{D}} \cap \{\mathrm{Re}\ z \ge 0\}$, we have $|z| \le |P(z)| \le 1$. Consequently, the GL energy of w is at most the one of u. By minimality of u, this implies that $|u| = |P \circ u|$ everywhere, that is, $u(\mathcal{O}) \subset S \cup \{z\ ;\ |z| = 1$ and $0 \le \mathrm{Re}\ z < \mu\}$. If there is some point $Q \in \bar{\mathcal{O}}$ such that $|u(Q)| = 1$ and $0 \le \mathrm{Re}\ u(Q) < \mu$, then $Q \in \mathcal{O}$. Thus Q is an interior maximum point for $|u|$, which yields $|u| \equiv 1$ (this is easily seen by applying the maximum principle to the equation $-\Delta|u|^2 = 2\lambda|u|^2(1 - |u|^2) - 2|\nabla u|^2$ satisfied by $|u|^2$). The Ginzburg–Landau equation implies that u is constant. This contradicts the existence of Q. In conclusion, $u(\mathcal{O}) \subset S$, as stated in the lemma. □

Lemma 9. *Let $0 < \mu < \nu < 1$. Let U be a smooth bounded simply connected domain and let $g \in C^\infty(\partial U; \mathbb{C})$ be such that $\nu \le |g| \le 1$. Let u be a minimizer of the Ginzburg–Landau energy among all maps that equal g on ∂U.*

There is some $\varepsilon > 0$, depending on μ and ν, but not on U, g or u, such that, if $\int_U |\nabla u|^2 < \varepsilon$, then $|u| \ge \mu$ in $\bar{U}$.

A variant of Lemma 9, with U a circular annulus, appears in [12]. In our case, U is supposed simply connected, but otherwise its geometry is arbitrary.

Proof. Let $m = \min_{\bar{U}} |u|$. Assume that $m < \mu$, for otherwise we are done. Let $m < t < \nu$ be a regular value of $|u|$. Then at least one of the connected components of the level set $\{|u| = t\}$, say γ, encloses a minimum point for $|u|$. Let $\mathcal{O}$ be the interior of γ. Thus $\mathcal{O}$ is a smooth set with boundary γ and $\min_{\bar{\mathcal{O}}} |u| = m$. By Lemma 8, $u(\gamma)$ is not contained in any zone S delimited by a chord at distance $> m$ from the origin. Given a point $P_1 \in \gamma$, let C be the chord orthogonal to the segment $\overline{0\ u(P_1)}$, that crosses this segment and is at distance m from the origin. Then there is some point $P_2 \in \gamma$ such that $u(P_2)$ and $u(P_1)$ are separated by C. Since $|u(P_1)| = |u(P_2)| = t$, this implies that $|u(P_1) - u(P_2)| \ge \sqrt{2t(t-m)}$. Let φ be a simple arc on γ connecting P_1 to P_2. Then $|u(P_1) - u(P_2)| = \left|\int_\varphi \frac{\partial u}{\partial \tau}\, d\ell\right| \le \int_\gamma |\nabla u| \le \int_{|u|=t} |\nabla u|$.

The co-area formula yields

$$\begin{aligned}\varepsilon \ge \int_U |\nabla u|^2 \ge \int_U |\nabla u|\,|\nabla|u|\,| &= \int (\int_{|u|=t} |\nabla u| d\ell)\ \mathrm{dt} \\ &\ge \int_\mu^\nu (\int_{|u|=t} |\nabla u| d\ell)\ \mathrm{dt}\ \ge \int_\mu^\nu \sqrt{2t(t-m)}\ \mathrm{dt}\end{aligned}.$$

We have a contradiction if $\varepsilon < \int_\mu^\nu \sqrt{2t(t-m)}\ \mathrm{dt}$. □

We note for further use that, in the above lemma, U need not be smooth. It suffices to know that $|u| > \frac{\mu + \nu}{2}$ near ∂U.

Lemma 10. *Let $0 < \mu < \nu < 1$ and $B > 0$. Let $\mathcal{V}$ be a smooth annular domain of Newtonian capacity $\geq B$ (this is equivalent to saying that $\mathcal{V}$ is conformally equivalent to $\{z; 1 < |z| < R\}$ for some R such that $R \leq e^{\pi/B}$).*

Let $g : \partial\mathcal{V} \to \mathbb{C}$ be a smooth function such that $|g| \geq \nu$ and let u minimize the Ginzburg–Landau energy among all the maps that equal g on $\partial\mathcal{V}$. Then there is some $\varepsilon > 0$, depending only on μ, ν and B, but not on g or u, such that $|u| \geq \mu$ whenever $\int_{\mathcal{V}} |\nabla u|^2 < \varepsilon$.

Before proceeding to the proof of the lemma, let us note that the following condition is sufficient in order to have the capacity of $\mathcal{V} \geq B$: there are two concentric disks of radii R_1 and $e^{\pi/B}R_1$, such that $\mathcal{V}$ is contained in the annulus $\mathcal{A}$ determined by the two disks. Indeed, capacity decreases as the domain increases, and the capacity of $\mathcal{A}$ is B.

In particular, we may apply, for some B independent of sufficiently small $t > 0$ and δ, the above lemma to each connected component of $A_\delta \setminus \overline{A'_{\delta,t}}$. Consequently, Lemma 10 combined with (13) in Lemma 7 and Lemma 4 yields immediately the following

Lemma 11. *We have $|u_\delta| \to 1$ uniformly in $\overline{A_\delta}$ as $\delta \to 0$.*

Proof. Let Φ^{-1} be a conformal representation of $\mathcal{V}$ into $\{z; 1 < |z| < R\}$, with $R \leq N = e^{\pi/B}$. Then

$$\begin{aligned}\int_0^{2\pi} \Big(\int_1^R |\nabla(u\circ\Phi)\left(re^{i\theta}\right)| dr\Big)^2 d\theta &\leq \int_0^{2\pi}\int_1^R r|\nabla u \circ \Phi(re^{i\theta})|^2 dr\, d\theta \, \ln R \\ &= \ln R \int_{\mathcal{V}} |\nabla u|^2 < \varepsilon \ln R \leq \frac{\varepsilon\pi}{B}\end{aligned}.$$

Therefore, there is some θ such that

$$\int_1^R |\nabla(u\circ\Phi)(re^{i\theta})|dr \leq \sqrt{\frac{\varepsilon}{2B}}.$$

We find that

$$|u(\Phi(re^{i\theta})) - u(\Phi(se^{i\theta}))| \leq \sqrt{\frac{\varepsilon}{2B}}, \quad \forall\, r, s \in [1, R].$$

Since $|u(\Phi(Re^{i\theta}))| \geq \nu$, we obtain that $|u(\Phi(re^{i\theta}))| \geq \dfrac{\mu+\nu}{2}$, $\forall\, r, s \in [1, R]$, provided ε is sufficiently small (depending only on μ, ν, B).

Let now $U = \mathcal{V} \setminus \{\Phi(re^{i\theta}); r \in (1, R)\}$. Then U is simply connected, since $U = \Phi(\mathcal{B})$, with $\mathcal{B} = \{z;\ 1 < |z| < R\} \setminus \{re^{i\theta}; r \in (1, R)\}$, which is simply connected. Since clearly $|u| \geq \dfrac{2\mu+\nu}{3}$ near ∂U, we are in position to apply Lemma 9 in order to conclude. □

Step 4. Proof of Theorem 1 b) (the H^1 part) and of Corollary 1

To summarize, up to now we know that $|u_\delta| \to 1$ uniformly on $\bar{A}_\delta$.

We may write locally u^δ, which is smooth and of modulus 1, as $u^\delta = e^{i\varphi^\delta}$, with φ^δ smooth. φ^δ is not globally defined; however, its gradient is, since $\nabla\varphi^\delta = u^\delta \times \nabla u^\delta$. The fact that u^δ is a minimizer for $(P2)$ reads [6], Chapter 1

$$\begin{cases} \text{div } (\nabla\varphi^\delta) = 0 & \text{in } A_\delta \\ \nabla\varphi^\delta \cdot \nu = 0 & \text{on } \partial A_\delta \\ \displaystyle\int_{\Gamma_o} \nabla\varphi^\delta \cdot \tau = 2\pi & \\ \displaystyle\int_{\Gamma_i} \nabla\varphi^\delta \cdot \tau = 2\pi & \\ \displaystyle\int_{\delta m + x_\delta + \delta\Gamma} \nabla\varphi^\delta \cdot \tau = 0 & \text{for any } \ m \in Z_\delta \end{cases}. \tag{14}$$

For small δ, the map $v_\delta = u_\delta/u^\delta$ does not vanish, has modulus 1 and degree 0 on each component of ∂A_δ. Thus we may write globally $u_\delta = u^\delta \rho_\delta e^{i\psi_\delta}$, where $0 < \rho_\delta < 1$ and ψ_δ is smooth. The fact that (u_δ) is a minimizer for $(P1)$ translates into [4]

$$\begin{cases} \text{div } (\rho_\delta^2(\nabla\varphi^\delta + \nabla\psi_\delta)) = 0 & \text{in } A_\delta \\ \dfrac{\partial\psi_\delta}{\partial\nu} = 0 & \text{on } \partial A_\delta \\ -\Delta\rho_\delta = \lambda\rho_\delta(1-\rho_\delta^2) - \rho_\delta|\nabla\varphi^\delta + \nabla\psi_\delta|^2 & \text{in } A_\delta \\ \rho_\delta = 1 & \text{on } \partial A_\delta \end{cases}. \tag{15}$$

The following two results are not optimal, but suffice to our purposes.

Lemma 12. *We have* $|\nabla u^\delta| \le \dfrac{C}{\delta}$ *for some* C *independent of small* δ.

Proof. With the notations in the proof of Proposition 1, we have $|\nabla\Phi| = |\nabla\varphi^\delta| = |\nabla u^\delta|$, so that the lemma amounts to $|\nabla\Phi| \le \dfrac{C}{\delta}$.

Given any small number $\varepsilon > 0$ and any integer M, we may cover $\overline{A_\delta}$ with a collection of disks $(D_i)_{i\in I}$ such that:
(i) for each i, either $D_i \subset A_\delta$, or D_i is centered on ∂A_δ;
(ii) the radius r_i of the disk D_i is bounded from below by $c_1\delta$ and from above by $c_2\delta$, where $c_1, c_2 > 0$ do not depend on small δ;
(iii) the disks D_i^*, concentric with the D_i's and twice smaller, cover A_δ;
(iv) for each i such that D_i is centered on ∂A_δ, $\overline{D_i \cap A_\delta}$ is diffeomorphic to the half unit disk $\overline{\mathbb{D}} \cap \{\text{Im } z \ge 0\}$ through a diffeomorphism Ψ_i mapping $\overline{D_i} \cap \partial A_\delta$ onto $\overline{\mathbb{D}} \cap \mathbb{R}$ and such that $\|D^k(\Psi_i - r_i^{-1}R_i)\|_{L^\infty} \le C\varepsilon\delta^{-k}$. Here, R_i is an appropriate isometry and C does not depend on $0 \le k \le M$.

Assume first that $D_i \subset A_\delta$. By standard estimates for harmonic functions, we have $\|\nabla\Phi\|_{L^\infty(D_i^*)} \le \dfrac{C}{r_i}\|\nabla\Phi\|_{L^2(D_i)}$, so that $|\nabla\Phi| \le \dfrac{C}{\delta}$ on D_i^* (here, we use (ii) and the uniform bound $\|\nabla\Phi\|^2_{L^2(A_\delta)} = 2M_\delta \le C$).

Assume next that the center of D_i is on ∂A_δ. Provided that, in (iv), ε is sufficiently small and M is sufficiently large, we have the following estimate [11], Chapter 9: $\|\nabla\psi\|_{L^\infty(D_i^*\cap A_\delta)} \le \dfrac{C}{r_i}\|\nabla\psi\|_{L^2(D_i\cap A_\delta)}$ for each harmonic function ψ that vanishes on $D_i \cap \partial A_\delta$. Noting that Φ is constant on each component of A_δ ([6], Chapter 1), we may apply the preceding estimate to $\psi = \Phi - \Phi(x_i)$, where x_i is the center of D_i, and find as above $|\nabla\Phi| \le \dfrac{C}{\delta}$ on $D_i^* \cap A_\delta$.

Finally, (iii) implies that $|\nabla\Phi| \le \dfrac{C}{\delta}$ in A_δ. □

Corollary 2. *We have* $\displaystyle\int |\nabla u^\delta|^4 \le \frac{C}{\delta^2}$.

Proof. We have $\displaystyle\int |\nabla u^\delta|^4 \le \|\nabla u^\delta\|^2_{L^\infty} \int |\nabla u^\delta|^2 \le \frac{C}{\delta^2}$. □

Before stating the next result, let us recall that, for small δ, we may write $u_\delta = \rho_\delta u^\delta e^{i\psi_\delta}$.

Lemma 13. *We have, for small* δ, $\displaystyle\int_{A_\delta} (|\nabla\rho_\delta|^2 + |\nabla\psi_\delta|^2 + \lambda(1-\rho_\delta^2)^2) \le \frac{C}{\lambda\delta^2}$.

Proof. We have $\displaystyle E_\lambda(u_\delta) = \frac{1}{2}\int_{A_\delta} (|\nabla\rho_\delta|^2 + \rho_\delta^2|\nabla\varphi^\delta + \nabla\psi_\delta|^2 + \frac{\lambda}{2}(1-\rho_\delta^2)^2)$ and $E_\lambda(u^\delta) = \displaystyle\frac{1}{2}\int_{A_\delta} |\nabla\varphi^\delta|^2$. The inequality $E_\lambda(u_\delta) \le E_\lambda(u^\delta)$ reads, after some algebraic manipulations:

$$\int_{A_\delta} (\rho_\delta^2|\nabla\psi_\delta|^2 + |\nabla\rho_\delta|^2 + \frac{\lambda}{2}(1-\rho_\delta^2)^2) + 2\int_{A_\delta} \rho_\delta^2 \nabla\varphi^\delta \cdot \nabla\psi_\delta \le \int_{A_\delta} (1-\rho_\delta^2)|\nabla\varphi^\delta|^2. \quad (16)$$

Multiplying by ψ_δ the equation (14) satisfied by $\nabla\varphi^\delta$, we find that $\displaystyle\int_{A_\delta} \nabla\varphi^\delta \cdot \nabla\psi_\delta = 0$.

Using Cauchy-Schwarz and the fact that $\rho_\delta \to 1$ uniformly in $\overline{A_\delta}$ as $\delta \to 0$, this yields, for small δ:

$$\begin{aligned} \left|2\int_{A_\delta} \rho_\delta^2 \nabla\varphi^\delta \cdot \nabla\psi_\delta\right| &= \left|2\int_{A_\delta} (\rho_\delta^2 - 1)\nabla\varphi^\delta \cdot \nabla\psi_\delta\right| \\ &\le \frac{1}{2}\int_{A_\delta} \rho_\delta^2|\nabla\psi_\delta|^2 + \frac{1}{2}\int_{A_\delta} (1-\rho_\delta^2)|\nabla\varphi^\delta|^2 \end{aligned} \quad (17)$$

Inserting (17) into (16) and using again the fact that $\rho_\delta \to 1$, we find, for small δ, that

$$\int_{A_\delta} (|\nabla\rho_\delta|^2 + |\nabla\psi_\delta|^2 + \lambda(1-\rho_\delta^2)^2) \le C\int_{A_\delta} (1-\rho_\delta^2)|\nabla\varphi^\delta|^2. \quad (18)$$

Corollary 2 combined with Cauchy-Schwarz and (18) implies that

$$\int_{A_\delta} (|\nabla\rho_\delta|^2 + |\nabla\psi_\delta|^2 + \frac{\lambda}{2}(1-\rho_\delta^2)^2) \le \frac{C}{\lambda}\int_{A_\delta} |\nabla\varphi^\delta|^4 \le \frac{C}{\lambda\delta^2}. \quad (19)$$

□

We may now complete the proof of Theorem 1 b) (the H^1 part) and Corollary 1: assumption $(H1)$ and the preceding lemma imply that $\displaystyle\int_{A_\delta} (|\nabla\rho_\delta|^2 + |\nabla\psi_\delta|^2) \to 0$.

The extension ζ_δ of ρ_δ to A with the value 1 satisfies $\zeta_\delta - 1 \to 0$ in $H^1(A)$ (recall that $\rho_\delta \to 1$ uniformly in $\overline{A_\delta}$). Using a standard extension result, we may extend ψ_δ to a map η_δ in A such that $\nabla\eta_\delta \to 0$ in $L^2(A)$. Let a_δ be the average of η_δ on A. Then $\eta_\delta - a_\delta \to 0$ in $H^1(A)$. Setting $\alpha_\delta = e^{ia_\delta}$ and $w_\delta = \overline{\alpha_\delta}\zeta_\delta e^{i\eta_\delta}$, we then clearly have $w_\delta \to 1$ in $H^1(A)$ and, in A_δ, $w_\delta = \overline{\alpha}_\delta \dfrac{u_\delta}{u^\delta}$. This implies both Theorem 1 b) and Corollary 1. □

Step 5. Proof of Theorem 1 b) (uniform convergence part)

By (15), the smooth vector field $X = \rho_\delta^2(\nabla\varphi^\delta + \nabla\psi_\delta)$ satisfies div $X = 0$. Thus, we may write (at least locally) $X = (\partial H/\partial y, -\partial H/\partial x)$ for some smooth $H = H_\delta$. The condition $X \cdot \nu = 0$ on ∂A_δ implies ([6], Chapter 1) that H is single-valued and constant on each component of ∂A_δ.
On the other hand, the fact that $\rho_\delta \to 1$ uniformly in $\overline{A_\delta}$ implies that, for small δ, we have $\dfrac{1}{2}|\nabla\varphi^\delta + \nabla\psi_\delta| \le |\nabla H| \le 2|\nabla\varphi^\delta + \nabla\psi_\delta|$. Using the identity

$$|\nabla u_\delta|^2 = |\nabla\rho_\delta|^2 + \rho_\delta^2|\nabla\varphi^\delta + \nabla\psi_\delta|^2,$$

we find that

$$\int_{A_\delta} |\nabla H|^2 \le C. \tag{20}$$

We may rewrite the equation div $X = 0$ as

$$\Delta\psi_\delta = \Delta\psi_\delta + \text{div } \nabla\varphi^\delta = -2/\rho_\delta\nabla\rho_\delta \cdot (\nabla\psi_\delta + \nabla\varphi^\delta).$$

In terms of H, this may be reformulated as

$$\Delta\psi_\delta = \nabla\left(\frac{1}{\rho_\delta^2}\right) \times \nabla H. \tag{21}$$

This equation is complemented with the Neumann condition $\partial\psi_\delta/\partial\nu = 0$ on ∂A_δ.
We next recall the following result (due to Choné and quoted in [7])

Lemma 14. *Let φ solve* $\begin{cases} \Delta\varphi = \nabla u \times \nabla v & \text{in } \Omega \\ \partial\varphi/\partial\nu = 0 & \text{on } \partial\Omega \end{cases}$. *Here,*
(i) Ω is a smooth bounded domain in $\mathbb{R}^2$;
(ii) $u, v \in H^1$;
(iii) u is constant on each component of $\partial\Omega$.
Then, with some constant $C > 0$ **independent of** *Ω and for some $c \in \mathbb{R}$ (depending on φ), we have*

$$\|\varphi - c\|_{L^\infty(\Omega)} \le C(\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}). \tag{22}$$

Before going further, let us note that one may transform the additive estimate (22) into a multiplicative one. Indeed, if we replace u, v by tu, $t^{-1}v$, for arbitrary $t > 0$, this will not affect the equation satisfied by φ. If we write (22) for tu, $t^{-1}v$, then minimize over t, we find

$$\|\varphi - c\|_{L^\infty(\Omega)} \le 2C\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}. \tag{23}$$

Applying this estimate to the equation (21) (note that *(iii)* in the above lemma comes from the fact that $\dfrac{1}{\rho_\delta^2} = 1$ on ∂A_δ) and using Lemma 13 and (20), we find, with $c = c_\delta$, that $\|\psi_\delta - c_\delta\|_{L^\infty(A_\delta)} \to 0$. On the other hand, Corollary 1 implies that

$\|\psi_\delta - a_\delta\|_{L^2(A_\delta)} \to 0$. We easily obtain that $\|\psi_\delta - a_\delta\|_{L^\infty(A_\delta)} \to 0$, which implies Theorem 1 b) (uniform convergence part). □

Step 6. Pointwise estimates for ∇u_δ

We rely on the following Gagliardo-Nirenberg type estimate established in the Appendix of [5]

Lemma 15. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. If $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, then $\|\nabla u\|_{L^\infty(\Omega)} \leq C_\Omega \|u\|^{1/2}_{L^\infty(\Omega)} \|\Delta u\|^{1/2}_{L^\infty(\Omega)}$. Here, C_Ω depends on Ω, but not on u.*

Of especial interest to us is the dependence of C_Ω on Ω. An inspection of the proof of the above lemma in [5] shows that C_Ω depends only on the geometry of Ω. More specifically, let ε be sufficiently small, $M \in \mathbb{N}$ be sufficiently large. Then there is some $r = r_\Omega$ such that we may cover $\overline{\Omega}$ with disks D_i of radius r and:
(i) for each i, either $D_i \subset \Omega$, or D_i is centered on $\partial\Omega$;
(ii) the disks D_i^*, concentric with the D_i's and twice smaller, cover Ω;
(iii) for each i such that D_i is centered on $\partial\Omega$, $\overline{D_i \cap \Omega}$ is diffeomorphic to the half unit disk $\overline{\mathbb{D}} \cap \{\text{Im } z \geq 0\}$ through a diffeomorphism Ψ_i mapping $\overline{D_i} \cap \partial\Omega$ onto $\overline{\mathbb{D}} \cap \mathbb{R}$ and such that $\|D^k(\Psi_i - r^{-1}R_i)\|_{L^\infty} \leq C\varepsilon r^{-k}$. Here, R_i is an appropriate isometry and C does not depend on $0 \leq k \leq M$.

Then C_Ω depends only on r. If we consider the scaled domains $\Omega = \Omega_\delta = \delta^{-1}A_\delta$, we may clearly pick an r_Ω satisfying (i)-(iii) and independent of small δ. Thus, we may choose a constant C independent of small δ such that, for any $u \in C^2(\overline{\delta^{-1}A_\delta})$ such that $u = 0$ on $\partial(\delta^{-1}A_\delta)$, we have

$$\|\nabla u\|_{L^\infty(\delta^{-1}A_\delta)} \leq C \|u\|^{1/2}_{L^\infty(\delta^{-1}A_\delta)} \|\Delta u\|^{1/2}_{L^\infty(\delta^{-1}A_\delta)}. \tag{24}$$

On the other hand, the estimate in Lemma 15 is scale invariant, that is, $C_{t\Omega} = C_\Omega$, $t > 0$. Thus, with C independent of small δ, we have

$$\|\nabla u\|_{L^\infty(A_\delta)} \leq C \|u\|^{1/2}_{L^\infty(A_\delta)} \|\Delta u\|^{1/2}_{L^\infty(A_\delta)}, \quad u \in C^2(\overline{A_\delta}),\ u = 0 \text{ on } \partial A_\delta. \tag{25}$$

Recalling the equation (15) satisfied by ρ_δ and Lemma 12, we find, with the help of (25) applied to $u = 1 - \rho_\delta$, that

$$\|\nabla \rho_\delta\|_{L^\infty(A_\delta)} = o\left(\frac{1}{\sqrt{\lambda}} + \|\nabla \psi_\delta\|_{L^\infty(A_\delta)}\right) \quad \text{as } \delta \to 0. \tag{26}$$

Lemma 16. *We have $\|\nabla \psi_\delta\|_{L^\infty(A_\delta)} \leq C\left(\frac{1}{\delta} + \|\nabla \rho_\delta\|_{L^\infty(A_\delta)}\right)$, with C independent of small δ.*

Proof. The idea is to estimate rather ∇H than $\nabla \psi_\delta$. It suffices to prove, with C independent of small δ, the following inequality:

$$\|\nabla H\|_{L^\infty(A_\delta)} \leq C\left(\frac{1}{\delta} + \|\nabla \rho_\delta\|_{L^\infty(A_\delta)}\right). \tag{27}$$

Recall that $\nabla \varphi^\delta + \nabla \psi_\delta = \dfrac{1}{\rho_\delta^2}(\partial H/\partial y, -\partial H/\partial x)$. This implies that H satisfies the equation $\text{div }(1/\rho_\delta^2 \nabla H) = 0$. On the other hand, recall that H is constant on each component of ∂A_δ. The degree conditions on u_δ on ∂A_δ together with the preceding discussion imply that H is solution of

$$\begin{cases} \Delta H = 2\rho_\delta^{-1}\nabla\rho_\delta\cdot\nabla H & \text{in } A_\delta \\ H = \text{const.} & \text{on each component of } \partial A_\delta \\ \displaystyle\int_{\Gamma_o}\frac{\partial H}{\partial\nu} = 2\pi & \\ \displaystyle\int_{\Gamma_i}\frac{\partial H}{\partial\nu} = 2\pi & \\ \displaystyle\int_{\delta m + x_\delta + m\Gamma}\frac{\partial H}{\partial\nu} = 0 & \text{for each } m\in Z_\delta \end{cases}.$$

Note that H achieves its minimum only on Γ_i and its maximum only on Γ_o; this is a consequence of the equation $\operatorname{div}\left(\frac{1}{\rho_\delta^2}\nabla H\right) = 0$, complemented by Neumann conditions, satisfied by H. By adding an appropriate constant to H, we may further assume that $\min H = H_{|\Gamma_i} = 0$.
We first claim that H is bounded independently of small δ. Indeed, by (20), we have

$$\max H = H_{|\Gamma_o} = \frac{1}{2\pi}\int_{\Gamma_o} H\frac{\partial H}{\partial\nu} = \frac{1}{2\pi}\int_{A_\delta}\frac{1}{\rho_\delta^2}|\nabla H|^2 \le C.$$

We next split $H = H_1 + H_2$, where H_1 is harmonic and agrees with H on ∂A_δ. We first note that, by (20), we have

$$\int_{A_\delta}|\nabla H_1|^2 \le \int_{A_\delta}|\nabla H|^2 \le C.$$

The proof of Lemma 12 implies that

$$\|\nabla H_1\| \le \frac{C}{\delta}. \tag{28}$$

On the other hand, $|H_2| \le |H_1| + H \le 2\|H\|_{L^\infty(A_\delta)} \le C$. Therefore, estimate (25) applied to H_2 yields

$$\begin{aligned}\|\nabla H_2\|_{L^\infty(A_\delta)} &\le C\|\nabla H\|^{1/2}_{L^\infty(A_\delta)}\|\nabla\rho_\delta|^{1/2}_{L^\infty(A_\delta)} \\ &\le C\left(\frac{1}{\delta^{1/2}} + \|\nabla H_2\|^{1/2}_{L^\infty(A_\delta)}\right)\|\nabla\rho_\delta\|^{1/2}_{L^\infty(A_\delta)},\end{aligned}$$

so that

$$\|\nabla H_2\|_{L^\infty(A_\delta)} \le C\left(\frac{1}{\delta} + \|\nabla\rho_\delta\|_{L^\infty(A_\delta)}\right). \tag{29}$$

We conclude by combining (28) to (29). □

Lemma 12, Lemma 16 and (26) imply immediately the following pointwise estimate:

$$\|\nabla u_\delta\|_{L^\infty(A_\delta)} = o\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{as } \delta\to 0. \tag{30}$$

Step 7. Uniqueness (modulo $\mathbb{S}^1$) of u_δ for small δ

The proof in [17], which yields uniqueness for the Dirichlet problem (see also [16], Chapter 8 and [15]), adapts to minimizers of (3). The key ingredients are the inequality $\|\nabla u_\delta\|_{L^\infty(A_\delta)} = o(1/\sqrt{\lambda})$ and the fact that $\rho_\delta \to 1$ uniformly in $\overline{A}_\delta$ as $\delta \to 0$. As proved in [4], unde these two hypotheses, if δ is sufficiently small and u_δ, u'_δ minimize (3), then there is some $\alpha \in \mathbb{S}^1$ such that $u'_\delta = \alpha u_\delta$. □

3. **Proof of Theorem 3.** We essentialy follow [3]. The main step consists in proving that the energy and the zeroes of u_n concentrate "near" ∂A.
We start by recalling the following upper bound for m_δ [3]

Lemma 17. *We have $m_\delta \le 2\pi$.*

Without loss in generality, we may assume that a quasi-minimizing sequence $\{u_n\}$ satisfies $E_\lambda(u_n) < 2\pi + e^{-\lambda}$. This energy bound, together with Lemma 3 implies, as in the proof of Lemma 4, that, for each $\mu \in (0,1)$ and sufficiently small $t > 0$, we have

$$|u_n| \ge \mu \quad \text{in } A'_{\delta,t},\ n \in \mathbb{N},\ \delta \text{ small.} \tag{31}$$

Lemma 18. *For sufficiently small δ and t, we have*

$$\int\limits_{\{x\in A_\delta;\ \delta t<\operatorname{dist}(x,\delta m+x_\delta+\delta\Gamma)<2\delta t\}} |\nabla(u_n/|u_n|)|^2 \ge Ct \deg^2(u_n/|u_n|\,,\,\delta m + x_\delta + \delta\Gamma_t),$$

with C independent of δ, t and $m \in Z_\delta$.

Proof. By scale invariance, we may assume that $\delta = 1$, $x_\delta = 0$, $m = 0$. Let v be the $\mathbb{S}^1$-valued map obtained by rescaling $u_n/|u_n|$. The map $x \mapsto f(x) :=\operatorname{dist}(x,\Gamma)$ is, near Γ, smooth and has gradient of modulus 1. The co-area formula implies that

$$\int\limits_{\{t<f(x)<2t\}} |\nabla v|^2 = \int_t^{2t}\int\limits_{\Gamma_s} |\nabla v|^2 d\ell\, ds. \tag{32}$$

On the other hand, we have, for each $s \in (t, 2t)$,

$$\deg\,(v\,,\,\Gamma_s) = \deg\,(u_n/|u_n|\,,\,\delta m + x_\delta + \delta\Gamma_t) := d.$$

Since

$$d^2 = \frac{1}{4\pi^2}\left(\int\limits_{\Gamma_s} v \times \frac{\partial v}{\partial \tau}\right)^2 \le \frac{|\Gamma_s|}{4\pi^2}\int\limits_{\Gamma_s} |\nabla v|^2 d\,\ell,$$

(32) implies that

$$\int\limits_{\{t<f(x)<2t\}} |\nabla v|^2 \ge \frac{Ct}{\min\{|\Gamma_s|\ ;\ t<s<2t\}} d^2 \ge Ctd^2.$$

□

The analog of Lemma 6 is

Lemma 19. *For sufficiently small δ and $t > 0$ and for sufficiently large n, the map $u_n/|u_n|$ has degree 0 on each component of $\partial A'_{\delta,t}$.*

Proof. In view of (31), we may assume the $|u_n| \geq \mu$ on $\partial A'_{\delta,t}$; here, $\mu \in (0,1)$ is to be chosen later. As in the proof of Lemma 6, Lemma 5 implies that, with $v = v_n = u_n/|u_n|$, we have

$$\begin{aligned} 2\pi + e^{-\lambda} \geq & \pi\mu^2 \Bigg(\sum_{m \in Z_\delta} |\deg(v, \delta m + x_\delta + \delta\Gamma_t)| + |\deg(v, \Gamma_o^{\delta t}) - 1| \\ & + |\deg(v, \Gamma_i^{\delta t}) - 1|\Bigg) + \frac{\lambda}{4}\int_{A_\delta} (1 - |u_n|^2)^2 + \frac{\mu^2}{2} \int_{A'_{\delta,t}} |\nabla v|^2 \Bigg). \end{aligned} \tag{33}$$

If all the terms containing degrees vanish, then v is a test function for $(P2'_t)$. Thus (by Lemma 2) the last integral in (33) is larger than $2\pi + e^{-\lambda}$, provided μ is sufficiently close to 1 and δ, t are sufficiently small. Therefore, there has to be a term containing a degree that does not vanish (and therefore a second one, by the proof of Lemma 6). We easily find that, for small δ and t, exactly two of these terms equal 1, all the others vanish. It follows, in addition, that

$$\lim_{\delta \to 0} \left(\lambda \int_{A_\delta} (1 - |u_n|^2)^2 + \int_{A'_{\delta,t}} |\nabla(u_n/|u_n|)|^2 \right) = 0 \quad \text{uniformly in } n. \tag{34}$$

In view of the preceding discussion, we are done if we prove that, for small δ and t, we have $\deg(u_n/|u_n|, \Gamma_o^{\delta t}) = \deg(u_n/|u_n|, \Gamma_i^{\delta t}) = 0$.
We start by noting that, for small δ and t and for each $m \in Z_\delta$, we have

$$\deg\,(u_n/|u_n|\,,\,\delta m + x_\delta + \delta\Gamma_t) = 0.$$

This follows immediately by combining Lemma 18 to (34).
In view of the balancing condition (11), this leaves us with two possibilities, for small δ and t and possibly after passing to a subsequence in δ: either the degrees on $\Gamma_o^{\delta t}$ and $\Gamma_i^{\delta t}$ equal 0, or they equal 2. If we rule out the second possibility, then we are done. Argue by contradiction and assume that the degrees are 2. Then, in $A'_{\delta,t}$, $u_n/|u_n|$ is a test function for a problem (P''_t), similar to $(P2'_t)$, but this time with degrees 2 instead of 1 on $\Gamma_o^{t\delta}$ and $\Gamma_i^{t\delta}$. It turns out that the energy of this problem is **four** times the one of $(P2'_t)$. (Indeed, it is easy to see that, if v minimizes $(P2'_t)$, then v^2 minimizes (P''_t). On the other hand, we have $|\nabla v^2| = 4|\nabla v|^2$). In view of Lemma 2, we find that

$$\liminf \frac{1}{2} \int_{A'_{\delta,t}} |\nabla(u_n/|u_n|)|^2 \geq 4M > 0.$$

This contradicts (34). ∎

Lemma 20. *For small t, we have:*

a) $\displaystyle\lim_{\delta \to 0} \int_{\{x \in A\ ;\ \mathrm{dist}(x,\Gamma_o) < \delta t\}} |\nabla u_n|^2 = \lim_{\delta \to 0} \int_{\{x \in A\ ;\ \mathrm{dist}(x,\Gamma_i) < \delta t\}} |\nabla u_n|^2 = \pi$*;*

b) $\displaystyle\lim_{\delta \to 0} \int_{A^{\delta t}} |\nabla u_n|^2 = 0$*;*

c) $\displaystyle\lim_{\delta \to 0} \lambda \int_A (1 - |u_n|^2)^2 = 0$*;*

d) $|u_n| \to 1$ *uniformly, as* $\delta \to 0$*, in* $\overline{A^{\delta t}}$.

Proof. By Lemma 5 and Lemma 19, we have

$$\liminf_{\delta\to 0} \int_{\{x\in A\ ;\ \mathrm{dist}(x,\Gamma_o)<\delta t\}} |\nabla u_n|^2 \geq \pi,$$

and a similar estimate holds for the second integral in a). The upper bound $E_\lambda(u_n) < 2\pi + e^{-\lambda}$ implies a), b) and c). d) is a consequence of b) and of Lemma 10. □

As explained in [3], the information contained in Lemma 20 yield the conclusion of Theorem 3. In [3], the domain considered is fixed, but the proof there applies with no changes to our situation. □

4. **Proof of Theorem 2. Outline of the proof.** We argue by contradiction and assume that, for small δ, the minimum is attained in (3). We consider, for such δ, a minimizer u. Recalling the upper bound $m_\delta \leq 2\pi$, we have $E_\lambda(u) \leq 2\pi$. In Step 3, we prove that this upper bound implies that "far away" from ∂A, u is "almost" constant. In the case of a fixed domain, this was proved in [4]; here, we use an alternative approach. Steps 1 and 2 provide preliminary estimates needed in Step 3. In Step 5, we prove that the energy of a map which is almost constant far away from ∂A is, for small δ, strictly larger than 2π. Thus, the minimum is not attained in (3). The method comes from [1]. The technical part needed in the proof is adapted to our situation in Step 4.

We will use $(H2)$ in the following equivalent form: for each $C, K, a > 0$, we have $Ke^{-C\delta\sqrt{\lambda}} \leq \lambda^{-a}$ for sufficiently small δ.

Step 1. Estimates for small energy solutions of the GL equation

To start with, we recall the following result [5]

Lemma 21. *Let $\lambda \geq 1$ and let ρ satisfy $-\Delta\rho = \lambda\rho(1-\rho^2) - f$ in $\mathbb{D}(0,1/3)$, with $0 \leq \rho \leq 1$ and $f \geq 0$. Then $1-\rho^2(x) \leq \dfrac{C}{\lambda}\|f\|_{L^\infty(\mathbb{D})}$, $|x| \leq 1/4$, where C does not depend on f or ρ.*

We will need the following quantitative version of Lemma 3

Lemma 22. *Let u be a solution of the GL equation $-\Delta u = \lambda u(1-|u|^2)$ in $\mathbb{D}(0,R)$ satisfying:*
(i) $\lambda R^2 \geq 1$;
(ii) the energy bound $E_\lambda(u) = K^2 \leq 1/\sqrt{\lambda R^2}$.
Then there is some ε, independent of u, K, λ or R, such that, if u satisfies in addition
(iii) $1-\varepsilon \leq |u| \leq 1$,
then $|\nabla u(0)| \leq \dfrac{CK}{R}$ and $1-|u(0)|^2 \leq \dfrac{CK^2}{\lambda R^2}$.
Here, C does not depend on u, K or λ.

Proof. We may assume $R = 1$; the general case follows by scaling. Throughout the proof, C, C' will denote universal constants.

Let $\varepsilon > 0$ to be fixed later. We write, in $\mathbb{D}$, $u = \rho e^{i\varphi}$, with $1-\varepsilon \le \rho \le 1$ and φ smooth.

We start by exploiting the equation satisfied by φ. By Fubini, there is some $r \in (3/4, 1)$ such that the restriction v of u to $C(0,r)$ satisfies $\|\nabla v\|_{L^2(C(0,r))} \le CK$. Then the restriction ψ of φ to $C(0,r)$ satisfies $\|\nabla \varphi\|_{L^2(C(0,r))} \le C'K$. Let Φ be the harmonic extension of ψ to $\mathbb{D}(0,r)$. Since φ satisfies $\operatorname{div}(\rho^2 \nabla\varphi) = 0$, we find that $\zeta := \varphi - \Phi$ is solution of $\begin{cases} \Delta\zeta = \operatorname{div}\ ((1-\rho^2)\nabla\varphi) & \text{in } \mathbb{D}(0,r) \\ \zeta = 0 & \text{on } C(0,r) \end{cases}$. Thus

$$\|\nabla\zeta\|_{L^4(\mathbb{D}(0,r))} \le C\|(1-\rho^2)\nabla\varphi\|_{L^4(\mathbb{D}(0,r))} \le 2C\varepsilon(\|\nabla\zeta\|_{L^4(\mathbb{D}(0,r))} + \|\nabla\Phi\|_{L^4(\mathbb{D}(0,r))}).$$

Here, C is independent of r (this follows from the scale invariance of the preceding estimate). If ε is sufficiently small, we find that $\|\nabla\zeta\|_{L^4(\mathbb{D}(0,r))} \le C'\|\nabla\Phi\|_{L^4(\mathbb{D}(0,r))}$. On the other hand, we have

$$\|\nabla\Phi\|_{L^4(\mathbb{D}(0,r))} \le C\left\|\Phi - \frac{1}{\pi r^2}\int_{\mathbb{D}(0,r)} \Phi\right\|_{H^{3/2}(\mathbb{D}(0,r))} \le C'\|\psi\|_{H^1(C(0,r))};$$

here, C, C' are independent of $r \in (3/4, 1)$, by scale invariance. By choice of r, we find that

$$\|\nabla\varphi\|_{L^4(\mathbb{D}(0,3/4))} \le CK. \tag{35}$$

We now turn to the equation $-\Delta\rho = \lambda\rho(1-\rho)^2 - \rho|\nabla\varphi|^2$ satisfied by ρ. By Fubini, there is some $r \in (2/3, 3/4)$ such that $\int_{C(0,r)} (|\nabla\rho|^2 + \lambda(1-\rho^2)^2) \le CK^2$. For such r, we have

$$\left|\int_{C(0,r)} (1-\rho)\frac{\partial\rho}{\partial\nu}\right| \le \left(\int_{C(0,r)} (1-\rho)^2\right)^{1/2}\left(\int_{C(0,r)} |\nabla\rho|^2\right)^{1/2} \le \frac{CK^2}{\sqrt{\lambda}}. \tag{36}$$

We multiply by $1-\rho$ the equation of ρ and integrate it over $\mathbb{D}(0,r)$. Using Cauchy-Schwarz and assuming ε sufficiently small, we find, with the help of (35) and (36), that

$$\begin{aligned} \int_{\mathbb{D}(0,r)} (\lambda(1-\rho^2)^2 + |\nabla\rho|^2) &\le -\int_{C(0,r)} (1-\rho)\frac{\partial\rho}{\partial\nu} + \int_{\mathbb{D}(0,r)} (1-\rho)|\nabla\varphi|^2 \\ &\le \frac{CK^2}{\sqrt{\lambda}} + \frac{\lambda}{2}\int_{\mathbb{D}(0,r)} (1-\rho^2)^2 + \frac{CK^4}{\lambda} \end{aligned}. \tag{37}$$

In view of the energy bound $K^2 \le 1/\sqrt{\lambda} \le 1$, we find that $\frac{K^4}{\lambda} \le \frac{K^2}{\sqrt{\lambda}}$, so that

$$\|\Delta\rho\|^2_{L^2(\mathbb{D}(0,r))} \le \lambda^2 \int_{\mathbb{D}(0,r)} (1-\rho^2)^2 + \|\nabla\varphi\|^4_{L^4(\mathbb{D}(0,r))} \le CK^2\sqrt{\lambda} + CK^4 \le C'K^2\sqrt{\lambda}. \tag{38}$$

Returning to (37), we obtain also

$$\int_{\mathbb{D}(0,r)} |\nabla\rho|^2 \le \frac{CK^2}{\sqrt{\lambda}}. \tag{39}$$

The standard estimate

$$\|\nabla\rho\|_{L^p(\mathbb{D}(0,1/2))} \le C_p(\|\nabla\rho\|_{L^2(\mathbb{D}(0,2/3))} + \|\Delta\rho\|_{L^2(\mathbb{D}(0,2/3))}), \quad 1 < p < \infty,$$

yields, with the help of (38), (39) and of the fact that $\lambda \ge 1$,

$$\|\nabla\rho\|_{L^p(\mathbb{D}(0,1/2))} \le C_p K \sqrt[4]{\lambda}, \quad 1 < p < \infty. \tag{40}$$

We next return to the equation $\Delta\varphi = -2\rho^{-1}\nabla\rho\cdot\nabla\varphi$ satisfied by φ. In view of (35) and (40), we have $\|\Delta\varphi\|_{L^q(\mathbb{D}(0,1/2))} \le C_q K^2 \sqrt[4]{\lambda}$, $q < 4$. The estimate

$$\|\nabla\varphi\|_{L^\infty(\mathbb{D}(0,1/3))} \le C_q(\|\nabla\varphi\|_{L^2(\mathbb{D}(0,1/2))} + \|\Delta\varphi\|_{L^q(\mathbb{D}(0,1/2))}), \quad q > 2,$$

implies that

$$\|\nabla\varphi\|_{L^\infty(\mathbb{D}(0,1/3))} \le CK. \tag{41}$$

We are now in position to apply Lemma 21 and infer, with the help of (41) and of the equation satisfied by ρ, that

$$1 - \rho^2 \le \frac{CK^2}{\lambda} \quad \text{in } \mathbb{D}(0, 1/4). \tag{42}$$

Thus $\|\Delta\rho\|_{L^\infty(\mathbb{D}(0,1/4))} \le CK^2$. Finally, the estimate

$$\|\nabla\rho\|_{L^\infty(\mathbb{D}(0,1/8))} \le C(\|\nabla\rho\|_{L^2(\mathbb{D}(0,1/2))} + \|\Delta\rho\|_{L^\infty(\mathbb{D}(0,1/4))})$$

implies that

$$\|\nabla\rho\|_{L^\infty(\mathbb{D}(0,1/8))} \le CK. \tag{43}$$

We conclude by combining (41), (42) and (43). □

Step 2. Concentration of the energy near ∂A

In the remaining part of the proof, u is a minimizer of (3).
Recall that $A^t = \{x \in A \; ; \; \text{dist}(x, \partial A) > t\}$.

Lemma 23. *There is some sufficiently large C such that, for small δ, we have:*
(i) $|u| \ge 1/2$ *in* $A^{C/\sqrt{\lambda}}$*;*
(ii) $\deg(u/|u|, \Gamma_o^{C/\sqrt{\lambda}}) = \deg(u/|u|, \Gamma_i^{C/\sqrt{\lambda}}) = 0$.

Proof. (i) is a consequence of Lemma 3 and of Lemma 20 d). In order to obtain (ii), we rely on Lemma 19. By homotopy invariance of the degree, we have, for small $t > 0$ and δ, that $\deg(u/|u|, \Gamma_o^{C/\sqrt{\lambda}}) = \deg(u/|u|, \Gamma_o^{t\delta}) = 0$, and a similar equality holds for the inner boundary Γ_i. □

Recall that we set $A'_{\delta,t} = \{x \in A_\delta \; ; \; \text{dist}(x, \partial A) > t\delta\}$.

Lemma 24. *Let $t > 0$ be sufficiently small and let $a > 0$. Then, for small δ, we have*

$$\int_{A'_{\delta,t}} (|\nabla u|^2 + \lambda(1 - |u|^2)^2) \le \lambda^{-a}.$$

Proof. Throughout the proof, C_j will denote a constant independent of δ or u. Let $f(s) = f(s, \delta)$ be the energy of u in $A^s \cap A_\delta$, that is

$$f(s) = \frac{1}{2} \int\limits_{A^s \cap A_\delta} |\nabla u|^2 + \frac{\lambda}{4} \int\limits_{A^s \cap A_\delta} (1 - |u|^2)^2.$$

We will consider only $s < t_0\delta$, with sufficiently small t_0. Since, near ∂A, the distance to ∂A is smooth and has gradient of modulus 1, we have

$$f'(s) = -\frac{1}{2} \int\limits_{\Gamma_o^s \cup \Gamma_i^s} |\nabla u|^2 - \frac{\lambda}{4} \int\limits_{\Gamma_o^s \cup \Gamma_i^s} (1 - |u|^2)^2$$

(this follows from the co-area formula). Let C be as in the preceding lemma and let $C/\sqrt{\lambda} \le s \le t_0\delta$. By (9), we have

$$\int\limits_{A_\delta \setminus A^s} |\nabla u|^2 \ge 2 \int\limits_{A_\delta \setminus A^s} |\text{Jac } u| \ge \left| \int\limits_{\Gamma_o} u \times \frac{\partial u}{\partial \tau} - \int\limits_{\Gamma_o^s} u \times \frac{\partial u}{\partial \tau} \right| + \left| \int\limits_{\Gamma_i} u \times \frac{\partial u}{\partial \tau} - \int\limits_{\Gamma_i^s} u \times \frac{\partial u}{\partial \tau} \right|.$$

Since $u \in \mathcal{J}$, we have

$$\int\limits_{\Gamma_o} u \times \frac{\partial u}{\partial \tau} = \int\limits_{\Gamma_i} u \times \frac{\partial u}{\partial \tau} = 2\pi.$$

On the other hand, since $|u| \ge 1/2$ in $A^s \cap A_\delta$ and the degrees of $u/|u|$ on each component of $\partial(A^s \cap A_\delta)$ equal zero, we may write, globally in $A^s \cap A_\delta$, $u = \rho e^{i\varphi}$, with $1/2 \le \rho \le 1$, ρ and φ smooth. Thus

$$\int\limits_{\Gamma_o^s} u \times \frac{\partial u}{\partial \tau} = \int\limits_{\Gamma_o^s} \rho^2 \frac{\partial \varphi}{\partial \tau} = \int\limits_{\Gamma_o^s} (\rho^2 - 1) \frac{\partial \varphi}{\partial \tau}.$$

We find that

$$\left| \int\limits_{\Gamma_o^s} u \times \frac{\partial u}{\partial \tau} \right| \le \left(\int\limits_{\Gamma_o^s} (\rho^2 - 1)^2 \right)^{1/2} \left(\int\limits_{\Gamma_o^s} |\nabla \varphi|^2 \right)^{1/2} \le C_1 \left(\int\limits_{\Gamma_o^s} (\rho^2 - 1)^2 \right)^{1/2} \left(\int\limits_{\Gamma_o^s} |\nabla u|^2 \right)^{1/2};$$

a similar estimate holds for Γ_i^s. We obtain

$$\int\limits_{A_\delta \setminus A^s} |\nabla u|^2 \ge 4\pi - C_1 \left(\int\limits_{\Gamma_o^s \cup \Gamma_i^s} (\rho^2 - 1)^2 \right)^{1/2} \left(\int\limits_{\Gamma_o^s \cup \Gamma_i^s} |\nabla u|^2 \right)^{1/2} \ge 4\pi + \frac{C_2 f'(s)}{\sqrt{\lambda}}. \tag{44}$$

On the other hand, we have

$$2\pi \ge E_\lambda(u) = \frac{1}{2} \int\limits_{A_\delta \setminus A^s} |\nabla u|^2 + \frac{1}{2} \int\limits_{A^s \cap A_\delta} |\nabla u|^2 + \frac{\lambda}{4} \int\limits_{A_\delta} (1 - |u|^2)^2. \tag{45}$$

By combining (44) to (45), we find that $f(s) + \dfrac{C_2 f'(s)}{\sqrt{\lambda}} \le 0$. Equivalently, the map $s \mapsto e^{s\sqrt{\lambda}/C_2} f(s)$ is non-increasing in $[C/\sqrt{\lambda}, t_0\delta]$. Since $f(s) \le E_\lambda(u) \le 2\pi$, we find that

$$f(s) \le f(C/\sqrt{\lambda}) e^{(C/\sqrt{\lambda})\sqrt{\lambda}/C_2} e^{-s\sqrt{\lambda}/C_2} \le C_3 e^{-s\sqrt{\lambda}/C_2}, \quad \text{if } C/\sqrt{\lambda} \le s \le t_0\delta. \tag{46}$$

The conclusion of the lemma follows by taking, in (46), $s = t\delta$ and using $(H2)$. $\square$

Step 3. u is almost constant far away from ∂A_δ

Lemma 25. *Let $a>0$. Then, for small t and δ and for any $x\in A'_{\delta,t}$, we have $|\nabla u(x)|\le\lambda^{-a}$ and $1-|u(x)|^2\le\lambda^{-a}$.*

Proof. We apply Lemma 22 in $B(x,t\delta/2)$. The upper bound in Lemma 24 (with t replaced by $t/2$) yields immediately the desired conclusion. □

Recall that we may write, for small δ and t, $u=\rho e^{i\varphi}$ in $A^{t\delta}$, with $1/2\le\rho\le 1$ and smooth ρ and φ.

Lemma 26. *There is some $b=b(u)$ such that, for small δ and t and fixed $a>0$, we have $|\varphi-b|\le\lambda^{-a}$ in $A'_{\delta,t}$.*

Proof. There is some C independent of δ, t such that two points x,y in $A'_{\delta,t}$ can be connected by a path γ contained in $A'_{\delta,t}$ and of length $\le C$. Since $|\varphi(x)-\varphi(y)|\le C\,\mathrm{length}(\gamma)\sup_\gamma|\nabla u|$, the conclusion follows then from the bound $|\nabla u|\le\lambda^{-a}$ in $A'_{\delta,t}$; it suffices to take $b=\varphi(x)$, where x is any point in $A'_{\delta,t}$. □

Setting $\alpha=\alpha(u)=e^{i\,b}$, we find immediately from Lemma 25 and Lemma 26 that

$$\|u-\alpha\|_{L^\infty(A'_{\delta,t})}\le\lambda^{-a}\qquad\text{for small }\delta. \tag{47}$$

Lemma 27. *Let $t>0$ be sufficiently small and let $a>0$. Then, for small δ, we have $\int_{A_\delta}(1-|u|^2)^2\le\lambda^{-a}$.*

Proof. It suffices to combine (45) to (44). The desired conclusion follows with the help of Lemma 25. □

Step 4. An auxiliary linear problem

We adapt here the main idea in [1] to our situation. We fix $c>0$ and let $r_j=r_j(\delta)=j\,c\,\delta$, $j=1,2$; here, δ is sufficiently small. Let $g\in H^{1/2}((0,2\pi);\mathbb{S}^1)$ be 2π-periodic. We may identify this map with an $H^{1/2}$-map on $\mathbb{S}^1$ and compute its degree. We assume in what follows that $\deg(g,(0,2\pi))=1$ and $\mathrm{Im}\left(\int_0^{2\pi}g(\theta)d\theta\right)=0$ (in this section, such a g will be called admissible).

We consider the following class $\mathcal{L}=\mathcal{L}_g$ of test maps w, consisting of complex-valued H^1-maps defined in $(0,r_2)\times(0,2\pi)$ and defined by the constraints

$$\mathcal{L}=\{w\ ;\ w(r_2,\theta)=g(\theta)\text{ a. e.}, w\text{ is }2\pi-\text{periodic in }\theta\}.$$

For $w\in\mathcal{L}$, consider the energy

$$F_\mu(w)=\int_0^{r_2}dr\int_0^{2\pi}d\theta\,|\nabla w|^2+\int_0^{r_1}dr\int_0^{2\pi}d\theta\left(\frac{\mu}{2}(\mathrm{Re}\ w-1)^2-\frac{1}{2\mu}(\mathrm{Im}\ w)^2\right),$$

and the associated minimization problem

$$(P_\mu)\ R_\mu:=\min\{F_\mu(w)\ ;\ w\in\mathcal{L}\}. \tag{48}$$

Lemma 28. *Assume that* $(H3)$ $\lim\limits_{\delta\to 0}\sqrt{\mu}\delta/\ln\mu=\infty$. *Then there is some* δ_0 *independent of the admissible map* g *such that, for* $\delta<\delta_0$, *we have* $R_\mu>\pi$.

Proof. Let $w\in\mathcal{L}$. We may extend w by symmetry with respect to r to $(-r_2,0)\times(0,2\pi)$. The new map, still denoted w, satisfies

$$w(\pm r_2,\theta)=g(\theta)\text{ a. e.},\quad w\text{ is }2\pi-\text{periodic in }\theta. \tag{49}$$

We have $F_\mu(w)=\dfrac{1}{2}G_\mu(w)$, where

$$G_\mu(w)=\int_{-r_2}^{r_2}dr\int_0^{2\pi}d\theta|\nabla w|^2+\int_{-r_1}^{r_1}dr\int_0^{2\pi}d\theta\Big(\frac{\mu}{2}(\text{Re }w-1)^2-\frac{1}{2\mu}(\text{Im }w)^2\Big).$$

Thus $R_\mu\geq\dfrac{1}{2}S_\mu$, where S_μ is the minimum of G_μ over all the maps w defined in $(-r_2,r_2)\times(0,2\pi)$ satisfying (49). We are bound to prove that $S_\mu>2\pi$ for small δ. The value of S_μ is explicitely computed in [1]; in particular, S_μ (thus R_μ) is attained if $\mu<1$. As explained there, for $\mu<1$ we have $S_\mu>2\pi$ for each admissible g provided that the following condition (which is independent of g) is satisfied:

$$\alpha_n^\mu<\beta_n^\mu,\quad\forall\, n\in\mathbb{N}^*, \tag{50}$$

with

$$\alpha_n^\mu=\frac{1-\sqrt{1-\mu^{-1}n^{-2}}\,\tanh(r_1\sqrt{n^2-\mu^{-1}})}{1+\sqrt{1-\mu^{-1}n^{-2}}\,\tanh(r_1\sqrt{n^2-\mu^{-1}})}$$

and

$$\beta_n^\mu=\frac{\sqrt{1+\mu n^{-2}}\,\tanh(r_1\sqrt{n^2+\mu})-1}{\sqrt{1+\mu n^{-2}}\,\tanh(r_1\sqrt{n^2+\mu})+1}.$$

Inequality (50) is proved in [1] for fixed r_1 and large μ. It still holds under our assumptions, but the argument is slightly more involved. Set

$$\begin{aligned}x=x_n^\mu&:=\sqrt{1-\mu^{-1}n^{-2}}\,\tanh(r_1\sqrt{n^2-\mu^{-1}})\\ y=y_n^\mu&:=\sqrt{1+\mu n^{-2}}\,\tanh(r_1\sqrt{n^2+\mu})\end{aligned}.$$

We have to prove that $\dfrac{1-x}{1+x}<\dfrac{y-1}{y+1}$, which amounts to $xy>1$.

Noting that $n\geq 1$, we easily see that, for $\mu>4$ (and thus, for small δ) we have $\sqrt{1-\mu^{-1}n^{-2}}\sqrt{1+\mu n^{-2}}>\sqrt{1+\mu n^{-2}/2}$, and this for each $n\in\mathbb{N}^*$.

On the other hand, the map tanh is increasing. Since, on the one hand, we have $\sqrt{n^2+\mu}>\dfrac{n+\sqrt{\mu}}{2}$ and, on the other hand, for $\mu>4/3$ and $n\geq 1$ we have $\sqrt{n^2-\mu^{-1}}>\dfrac{n}{2}$, we obtain, for large μ, the inequality

$$xy>z=z_n^\mu:=\sqrt{1+\mu n^{-2}/2}\,\tanh(r_1n/2)\,\tanh(r_1(n+\sqrt{\mu})/2).$$

It suffices thus to prove that $z>1$ for small δ.

Recalling that $r_1=c\delta$ and assumption $(H3)$, we may write $\dfrac{r_1}{2}=\dfrac{a\ln\mu}{\sqrt{\mu}}$, where $a=a(r_1)\to\infty$ as $r_1\to 0$. On the other hand, set $b=b(n,\mu)=\dfrac{n}{\sqrt{\mu}}$, so that

$n = b\sqrt{\mu}$. Substituting $r_1/2$ and n into the expression of z, we see that we have to prove that

$$X = X^\mu := \sqrt{1 + \frac{1}{2b^2}}\, \tanh(a\, b\, \ln \mu)\, \tanh(a\, b\, \ln \mu + a\, \ln \mu) > 1 \qquad (51)$$

provided μ and a are sufficiently large.

We distinguish three cases: (i) $b < 1/\sqrt{2}$ and $a\, b\, \ln \mu < 1$, (ii) $b < 1/\sqrt{2}$ and $a\, b\, \ln \mu \geq 1$ and (iii) $b \geq 1/\sqrt{2}$. In what follows, C_j denotes a universal constant.

In case (i), we have:

- $\sqrt{1 + \frac{1}{2b^2}} \geq \frac{C_1}{b}$;
- $\tanh(a\, b\, \ln \mu) \geq C_2 a\, b\, \ln \mu$;
- $\tanh(a\, b\, \ln \mu + a\, \ln \mu) >\ \tanh(a\, \ln \mu)$.

Thus $X > C_1 C_2\, a\, \ln \mu\, \tanh(a\, \ln \mu) > 1$ for large a and μ.

In case (ii), we rely on:

- $\sqrt{1 + \frac{1}{2b^2}} > \sqrt{2}$;
- $\tanh(a\, b\, \ln \mu) \geq \tanh 1$;
- $\tanh(a\, b\, \ln \mu + a\, \ln \mu) >\ \tanh(a\, \ln \mu)$.

Thus $X > \sqrt{2}\, \tanh 1\, \tanh(a\, \ln \mu) > 1$ for large a and μ (here, we use $\sqrt{2}\, \tanh 1 > 1$).

In case (iii), we use the inequality

$$\tanh x > 1 - \frac{2}{e^x}, \quad x > 0$$

together with:

- $\sqrt{1 + \frac{1}{2b^2}} \geq 1 + \frac{C_3}{b^2}$
- $\tanh(a\, b\, \ln \mu + a\, \ln \mu) >\ \tanh(a\, b\, \ln \mu)$.

We obtain

$$\begin{aligned} X &> \left(1 + \frac{C_3}{b^2}\right)\left(1 - \frac{2}{e^{a\, b\, \ln \mu}}\right)^2 > \left(1 + \frac{C_3}{b^2}\right)\left(1 - \frac{4}{e^{a\, b\, \ln \mu}}\right) \\ &> 1 + \frac{C_3}{b^2} - \frac{4}{e^{a\, b\, \ln \mu}} - \frac{4C_3}{b^2 e^{a\, b\, \ln \mu}} \end{aligned}.$$

Setting $c = a\, \ln \mu$, we have $e^{a\, b\, \ln \mu} > \frac{c^2 b^2}{2}$. Thus, clearly, $X > 1$ for large c (and therefore, for large a and μ). □

Step 5. $E_\lambda(u) > 2\pi$ **for large** λ

We follow closely [1]. Let R be such that A is conformally equivalent to $\mathcal{A} := \mathbb{D}(0, R) \setminus \overline{\mathbb{D}}(0, 1/R)$ and let $\Phi : \overline{A} \to \overline{\mathcal{A}}$ be a conformal representation such that $\Phi(\Gamma_o) = C(0, R)$ and $\Phi(\Gamma_i) = C(0, 1/R)$. Set $v = u \circ \Phi^{-1} : \Phi(\overline{A_\delta}) \to \mathbb{C}$. In $\Phi(A_\delta)$, v satisfies $-\Delta v = \lambda g(x) v(1 - |v|^2)$, where $g = \text{Jac}\ \Phi^{-1}$ is bounded from above and below (note that g does not depend on δ).

Let α be the constant in (47). We multiply by $\ln\frac{|z|}{R}$ the equation of v and integrate over $\{z\ ;\ R-c_1\delta<|z|<R\}$. Here, $c_1>0$ is fixed sufficiently small in order to have $\Phi^{-1}(C(0,R-c_1\delta))\subset A'_{\delta,t}$ for some small fixed t. As in [1], we find, with the help of Lemma 25, Lemma 27 and (47), that, for each $a>0$, we have

$$\left|\int_{C(0,R)} v d\ell - 2\pi R\alpha\right| \le \lambda^{-a} \quad \text{for small } \delta; \tag{52}$$

a similar estimate holds for $\displaystyle\int_{C(0,1/R)} v d\ell$. As explained in [1], (52) together with Lemma 25 imply that we may find for some appropriate $\beta\in\mathbb{S}^1$ such that

$$\operatorname{Im}\left(\int_{C(0,R)} \beta v d\ell\right)=0 \tag{53}$$

and, for small fixed $c_2, c_3>0$,

$$|1-\beta v(z)|\le\lambda^{-a},\quad |\nabla(\beta v)|\le\lambda^{-a} \qquad \text{for } R-c_2\delta/<|z|<R-c_3\delta \text{ and small } \delta. \tag{54}$$

Similarly, one may find $\gamma\in\mathbb{S}^1$ such that γv satisfies similar estimates near $C(0,1/R)$. We assume henceforth that $\beta=1$. This does not affect the generality, since, if u minimizes (3), then so does βu.

We set $w(r,\theta)=v(\exp(r+\ln R-2c\delta+i\theta))$, $0\le r\le 2c\delta$, $0\le\theta\le 2\pi$; here, $c>0$ is to be fixed later. As explained in [1], if c is fixed sufficiently small, then (54) implies that, with small fixed $c_4, c_5>0$, we have

$$\frac{1}{2}\int_{\Phi^{-1}(\{z\ ;\ R-c_4\delta<|z|<R\})}\left(|\nabla u|^2+\frac{\lambda}{2}(1-|u|^2)^2\right)\ge F_{c_5\lambda}(w); \tag{55}$$

here, F_μ is the energy considered in Step 4.

Clearly, w is 2π-periodic in θ and, if we set $g(\theta)=v(Re^{i\theta})$, then g has degree 1. On the other hand, if λ satisfies $(H2)$, then $\mu:=c_5\lambda$ satisfies $(H3)$. In view of Lemma 28 and of (55), we find that

$$\frac{1}{2}\int_{\Phi^{-1}(\{z\ ;\ R-c_4\delta<|z|<R\})}\left(|\nabla u|^2+\frac{\lambda}{2}(1-|u|^2)^2\right)>\pi$$

for small δ. Similarly, we have

$$\frac{1}{2}\int_{\Phi^{-1}(\{z\ ;\ 1/R<|z|<1/R+c_6\delta\})}\left(|\nabla u|^2+\frac{\lambda}{2}(1-|u|^2)^2\right)>\pi.$$

Provided we choose c_4, c_6 sufficiently small, we have

$$\Phi^{-1}(\{z\ ;\ R-c_4\delta<|z|<R\})\cap\Phi^{-1}(\{z\ ;\ 1/R<|z|<1/R+c_6\delta\})=\emptyset.$$

We find that, for small δ, we have $E_\lambda(u)>2\pi$. This completes the proof of Theorem 2. □

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