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DISTRIBUTION OF MINIMUM VALUES OF STOCHASTIC FUNCTIONALS

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Abstract. Some mathematical problems of mechanics and physics have a form of the following variational problem. There is a functional, I, which is a sum of some quadratic positive functional and a linear functional. The quadratic functional is deterministic. The linear functional is a sum of a large number, N, of statistically independent linear functionals. The minimum value of the functional, I, is random. One needs to know the probability distribution of the minimum values for large N. The probability distribution was found in [2] in terms of solution of some deterministic variational problem. It was clear from the derivation that the class of quadratic and linear functionals for which this probability distribution can be used is not empty. It was not clear though how wide this class is. This paper aims to give some sufficient conditions for validity of the results of [2].

1. Introduction.

Let $\mathcal{H}$ be a Hilbert space of functions u. The scalar product of two elements u, v of $\mathcal{H}$ is denoted by (u, v). Consider a linear functional (l_0, u) on $\mathcal{H}$. The linear functional is random. To emphasize this in our notation, we write for the linear functional $(l_0(r), u)$ where r is an event, an element of a set with some prescribed probabilistic measure. We define an empirical "average" of the linear functional l_0, as

$$(l, u) = \frac{1}{N}\sum_{a=1}^{N}(l_0(r_a), u)\,, \tag{1}$$

where $r_1, ..., r_N$ are independent identically distributed random variables.

Consider also a linear operator A acting on elements u of $\mathcal{H}$. The operator A is assumed to be positive, i.e. for all u the quadratic form (Au, u) is nonnegative. The operator A is deterministic. We set a minimization problem for the functional

$$I(u) = \frac{1}{2}(Au, u) - (l, u). \tag{2}$$

The minimum is sought either on all space $\mathcal{H}$ or on its linear subspace $\mathcal{H}'$.

We assume that the minimum value of the functional $I(u)$ is achieved at a unique element, $\check{u}$. This element obeys the Euler equation

$$A\check{u} = l. \tag{3}$$

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Since, in accordance with (3), $(A\check{u},\check{u}) = (l,\check{u})$, the minimum value is given by Clapeyron's formula

$$\min_u I(u) = \frac{1}{2}(A\check{u},\check{u}) - (l,\check{u}) = -\frac{1}{2}(A\check{u},\check{u}). \tag{4}$$

The minimum is negative due to positiveness of the quadratic form (Au,u). In physical problems, the functional $\frac{1}{2}(Au,u)$ has usually the sense of energy. Therefore, we call $-\min I(u)$ energy and denote it by E. Energy is random. The question under consideration is: What is the probability distribution of energy for large N? Such a question arises in various branches of physics and mechanics : statistical mechanics of point vortices and statistical mechanics of vortex lines [3-5,17], homogenization problems [6], Kosterlitz-Thouless phase transition [7], plasticity of micro-samples [8], random excitation of elastic bodies [2], dynamic Saint-Venant's principle [11]. To formulate the answer let us first rectify the question.

We may expect that the linear functional (1) converges, in some sense, to the averaged functional

$$(\bar{l},u) = M\left(l_0(r),u\right). \tag{5}$$

Here M stands for mathematical expectation. The right hand side of (5) is an integral of a function of events, $(l_0(r),u)$, over the probabilistic measure.

Denote by $\bar{E}$ energy, corresponding to $(\bar{l},u)$:

$$-\bar{E} = \min\left(\frac{1}{2}(Au,u) - (\bar{l},u)\right). \tag{6}$$

One may expect that, as in the central limit theorem, energy fluctuates around the average value $\bar{E}$ with the magnitude of fluctuations in the order of $N^{-1/2}$. We are interested to find the probability of large fluctuations, when E differs from $\bar{E}$ for a finite amount independent on N.

The probability density was found in [2] in the following terms. Consider a functional of three variables, parameter E, number z and element u of $\mathcal{H}$,

$$S(E,z,u) = Ez + \frac{z}{2}(Au,u) + \ln Me^{-z(l_0,u)}. \tag{7}$$

Here, as in (5), $Me^{-z(l_0,u)}$ means the integral over the probabilistic measure of the function of r, $e^{-z(l_0(r),u)}$.

Consider the stationary points of the functional $S(E,z,u)$ with respect to z and u. Denote the maximum stationary value of functional (7) by $S(E)$. In physical problems, function $S(E)$ has usually the sense of entropy. We use this term for $S(E)$ in what follows, and call $S(E,z,u)$ the entropy functional.

To emphasize that functional $I(u)$ and probability density of energy, $f(E)$, depend on N we write $I_N(u)$ and $f_N(E)$, respectively. Note that the entropy functional (7) and entropy $S(E)$ do not depend on N. Probability density of energy is given by the relation:

$$f_N(E) = \text{const}\, e^{NS(E)}. \tag{8}$$

The value of the prefactor was also found (see [2] and section VII below).

Entropy is always negative and achieves its maximum at the point $E = \bar{E}$. This is the most probable value of energy. In vicinity of this point,

$$S(E) = \frac{1}{2}\,\frac{d^2S}{dE^2}\,\Big|_{E=\bar{E}}(E-\bar{E})^2.$$

The second derivative is negative, denote it by $-1/\sigma^2$. In vicinity of $\bar{E}$ the probability density of energy is

$$\text{const}\; e^{-N\frac{(E-\bar{E})^2}{2\sigma^2}} = \text{const}\; e^{-\frac{(E-\bar{E})^2}{2(\sigma/\sqrt{N})^2}}.$$

We obtain Gaussian distribution with the small variance, $\sigma/\sqrt{N}$. In the limit $N\to\infty$ the probability density converges to δ-function with the support at the point $E=\bar{E}$.

Probability that energy deviates from $\bar{E}$ on a finite amount is exponentially small in the order $e^{NS(E)}$ (remind that N is large and $S(E)$ is negative).

Derivative $dS(E)/dE=\beta$ has the sense of inverse temperature. Temperature is positive for $E<\bar{E}$ and negative for $E>\bar{E}$. If $\bar{E}$ decreases and goes to zero the branch of positive temperatures disappears.

Energy $\bar{E}$ is equal to zero in the limit $N\to\infty$ if the linear functional $(l_0(r),u)$ has zero mean value. Indeed, let us write down energy in terms of the inverse operator, A^{-1}, explicitly. From (3),

$$\check{u} = A^{-1}l. \tag{9}$$

Plugging(9) and (1) in the energy expression, we have

$$E = \frac{1}{2}(A\check{u},\check{u}) = \frac{1}{2}(l,A^{-1}l) = \frac{1}{2N^2}\sum_{a,b}\left(A^{-1}l_0(r_a),\,l_0(r_b)\right) \tag{10}$$

Averaged energy is obtained by applying math expectation M to (10):

$$\begin{aligned} M\,E &= \frac{1}{2N^2}M\sum_{a,b}\left(A^{-1}l_0(r_a),l_0(r_b)\right) \\ &= \frac{1}{2N^2}M\left(\sum_{a=1}^{N}\left(A^{-1}l_0(r_a),l_0(r_a)\right)+\sum_{a\neq b}\left(A^{-1}l_0(r_a),l_0(r_b)\right)\right) \\ &= \frac{1}{2N^2}\left(NM(A^{-1}l_0,l_0)+N(N-1)(A^{-1}\bar{l}_0,\bar{l}_0)\right). \end{aligned} \tag{11}$$

If $\bar{l}_0\neq 0$, the last term in (11) dominates, and, in the limit $N\to\infty$, averaged energy ME coincides with $\bar{E}$

$$ME = \bar{E} = \frac{1}{2}(A^{-1}\bar{l}_0,\bar{l}_0)$$

If $\bar{l}_0=0$, then the last term in (11) vanishes, and

$$ME = \frac{1}{2N}M(A^{-1}l_0,l_0). \tag{12}$$

We see that averaged energy, as it should be, is never zero, but, if l_0 has zero mean value, the averaged energy becomes small, in the order of N^{-1}, and tends to zero as N tends to infinity. This yields disappearance of the range of positive temperatures in the limit $N\to\infty$.

The range of positive temperatures is quite important in some applications. In order to study this range, we have to rescale energy, multiplying it, as follows from (12), by N. Therefore, we have to study the minimization problem for the functional

$$N\left[\frac{1}{2}(Au,u)-\frac{1}{N}\sum_{a=1}^{N}(l_0(r_a),u)\right]=\frac{N}{2}(Au,u)-\sum_{a=1}^{N}(l_0(r_a,u) \tag{13}$$

Note that minimization of the functional (13) is carried out over all elements of a linear space. Making a change $u\to u/\sqrt{N}$ we do not change the set of admissible elements. The functional I, after this change, takes the form (2) where

$$(l,u)=\frac{1}{\sqrt{N}}\sum_{a=1}^{N}(l_0(r_a),u)\,.. \tag{14}$$

Emphasize that the linear functional $(l_0(r),u)$in (14) is supposed to have zero mean value.

The cases (1) and (14) were called in [2] small and large excitation cases, respectively.

Probability density of energy of large excitations can be also obtained explicitly [2]. To write it down we introduce a positive operator B by the formula,

$$(Bu,u)=M(l_0(r),u)^2, \tag{15}$$

and consider the eigenvalue problem:

$$A\varphi=\mu B\varphi. \tag{16}$$

Let the eigen-functions of this problem form a basis in $\mathcal{H}$. Quadratic forms (Au,u) and (Bu,u) are diagonal in this basis:

$$(Au,u)=\sum_{k=1}^{\infty}\lambda_k u_k^2 \quad , \quad (Bu,u)=\sum_{k=1}^{\infty}b_k u_k^2. \tag{17}$$

We define an analytic function of complex variable z,

$$\Phi(z)=\prod_{i=1}^{\infty}\left(1+\frac{zb_k}{\lambda_k}\right). \tag{18}$$

Here it is assumed that $0\le b_1/\lambda_1\le b_2/\lambda_2\le\cdots$ and

$$\sum_{k=1}^{\infty}\frac{b_k}{\lambda_k}<\infty \tag{19}$$

Denote by $\hat{f}_N(z)$ Laplace's transform of probability density of energy:

$$f_N(E)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{Ez}\hat{f}_N(z)\,dz.$$

Then

$$\lim_{N\to\infty}\hat{f}_N(z)=\frac{1}{\sqrt{\Phi(z)}}, \tag{20}$$

where $\Phi(z)$ is the function (18).

A heuristic derivation of (7), (8) and (20) in [2] contains a number of steps which need a justification. In particular, it was far from obvious that these steps are legitimate for singular excitations like point vortices, electric charges or dislocations, for which the linear functional $(l_0(r), u)$ is not continuous in the norm of the operator A.

This paper aims to prove the above-mentioned formulas for probability density of energy for a wide class of functionals. First we consider the case of large excitation and give some sufficient conditions for validity of (20) which can be effectively checked. The case of small excitation is more difficult. Formula (8) is not valid, in general, because probability density $f_N(E)$ may have δ-type contributions. We prove a version of (8) for probability distribution,

$$F_N(E) = \int_0^E f_N(E')\, dE,$$

namely,

$$F_N(E) = R_N e^{NS(E)}, \tag{21}$$

and determine the asymptotics of the prefactor R_N. The key point of the proof is the identity discussed in section VI which reduces computing of the prefactor to studying of a large excitation problem with some auxiliary measure. The auxiliary measure contains an information on the stationary point of the entropy functional. This makes the sufficient conditions in case of small excitations less explicit.

The paper is organized as follows. First, for the reader's convenience, we repeat a heuristic derivation of the formulas for probability density. Then, in section III, the statements of the paper are formulated. The proof is given in sections V and VII. The proof is preceded by a discussion of peculiarities of integration in infinite-dimensional spaces (section IV), and derivation of an identity for probability distribution (section VI). The paper is concluded by checking the sufficient conditions for a case of large singular excitations (section VIII).

2. **Heuristic derivation.** To find probability density of energy we make first a finite dimensional truncation, $\mathcal{H}_m$, of space $\mathcal{H}$. Then u and l_0 become m-dimensional vectors, A is $m \times m$ positive matrix, $(Au, u)_m$ and $(l_0, u)_m$ are m-dimensional quadratic and linear functions, respectively. We are seeking for the probability density of the minimum values of the quadratic function

$$I_{m,N}(u) = \frac{1}{2}(Au, u)_m - (l, u)_m \ , \qquad (l, u)_m \equiv \frac{1}{N}\sum_{a=1}^{N}(l_0(r_a), u)_m, \tag{22}$$

where l_0 is a random vector. We put indices m, N to indicate the dependence on the space dimension m and the number of independent random variables, N. We assume that minimum is taken over the entire space $\mathcal{H}_m$ (actually, this is a generic case because $\mathcal{H}_m$ can be always taken as a subspace of $\mathcal{H}'$).

Probability density is sought in the limit $m \to \infty$, $N \to \infty$. By definition, the probability density of energy is

$$f_{m,N}(E) = M\delta(E + \min_u I_{m,N}), \tag{23}$$

where $\delta(E)$ is the δ-function. Following to Lyapunov's idea, it is useful to consider Fourier (or Laplace) transform of probability density if one deals with independent random variables. Thus, we present δ-function in (23) as an integral over imaginary axis in a plane of complex variable z:

$$\delta(E) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{Ez} dz. \tag{24}$$

We have

$$f(E) = \frac{1}{2\pi i} M \int_{-i\infty}^{i\infty} e^{Ez + z \min_u I_{m,N}} dz. \tag{25}$$

Suppose we may change the order of integration and mathematical expectation in (25). Then

$$f(E) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{Ez} M e^{z \min_u I_{m,N}} dz. \tag{26}$$

Our goal is to change, in a sense, the order of minimization and mathematical expectation in (26): if we find first mathematical expectation of $\exp(z\, I_{m,N})$ and do minimization after that, then computing of the probability density is reduced to solving deterministic problems only. A key to perform such "a change of order" is the following observation:

There is a well known relation,

$$\int e^{-\frac{1}{2}(Au,u)_m + (l,u)_m} d^m u = e^{\frac{1}{2}(A^{-1}l,l)_m} \sqrt{\frac{(2\pi)^m}{\det A_m}}, \tag{27}$$

where A^{-1} is the inverse matrix and $\det A_m$ is the determinant of the matrix A. It is remarkable that this relation can be written also in the form:

$$e^{\min_u \left[\frac{1}{2}(Au,u)_m - (l,u)_m\right]} = \sqrt{\frac{\det A_m}{(2\pi)^m}} \int e^{-\left[\frac{1}{2}(Au,u)_m - (l,u)_m\right]} d^m u. \tag{28}$$

The exponent of the minimum value is presented in the form of integral. Applying mathematical expectation to (28) we may change the order of integration over u and M-operation which is an integration over the event space. Then the whole problem is reduced to computing integrals: minimization has gone.

To get to the final result faster it is more convenient to take (28) in the form

$$e^{z \min_u \left[\frac{1}{2}(Au,u)_m - (l,u)_m\right]} = \sqrt{\frac{z^m \det A_m}{(2\pi)^m}} \int_{\alpha_1 - i\infty}^{\alpha_1 + i\infty} \cdots \int_{\alpha_m - i\infty}^{\alpha_m + i\infty} e^{z\left[\frac{1}{2}(A\zeta,\zeta)_m - (l,\zeta)_m\right]} \frac{d^m \zeta}{i^m}. \tag{29}$$

Here $\zeta_1, ..., \zeta_m$ are complex variables running over lines $[\alpha_1 - i\infty,\ \alpha_1 + i\infty], ... [\alpha_m - i\infty,\ \alpha_m + i\infty]$, vector $\alpha = (\alpha_1, ..., \alpha_m)$ is arbitrary, $\mathrm{Re}\, z \geq 0$. Dealing with complex valued vectors ζ, we use the scalar product without complex conjugation, i.e., for real v, $\quad (l_0, iv) = i(l_0, v)$.

Applying M-operation to (29) we have

$$Me^{-z\min\limits_u\left[\frac{1}{2}(Au,u)_m-(l,u)_m\right]} = \sqrt{\frac{z^m \det A_m}{(2\pi)^m}} \int\limits_{\alpha-i\infty}^{\alpha+i\infty} e^{\frac{z}{2}(A\zeta,\zeta)_m} Me^{-z(l,\zeta)_m} \frac{d^m\zeta}{i^m}. \quad (30)$$

The second term in the integral can be written in terms of the characteristic function of l_0:

$$\chi_m(u) = M\, e^{i(l_0,u)_m}.$$

Indeed, using the statistical independence of variables $r_1, ..., r_N$ we have

$$M\, e^{-z(l,\zeta)_m} = M\, e^{\frac{z}{N}\sum\limits_{a=1}^{N}(l_0(r_a),\zeta)_m} = \left(M\, e^{-\frac{z}{N}(l_0(r),\zeta)_m}\right)^N = \left[\chi_m\left(\frac{iz\zeta}{N}\right)\right]^N. \quad (31)$$

We will usually write (31) without introducing the characteristic function explicitly:

$$M\, e^{-z(l,\zeta)_m} = e^{N\ln M\, e^{-\frac{z}{N}(l_0(r),\zeta)_m}}. \quad (32)$$

Combining (26), (30) and (32) we obtain

$$f(E) = \int\limits_{-i\infty}^{i\infty}\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \sqrt{\frac{z^m \det A_m}{(2\pi)^m}}\, e^{Ez+\frac{z}{2}(A\zeta,\zeta)_m+N\ln M\, e^{-\frac{z}{N}(l_0(r),\zeta)_m}} d^m\zeta\, \frac{dz}{2\pi i^{m+1}}.$$

Let us make a change of variable $z \to zN$. Then we finally have

$$f(E) = \frac{N}{2\pi i}\int\limits_{-i\infty}^{i\infty}\int\limits_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{i^m}\sqrt{\left(\frac{Nz}{2\pi}\right)^m \det A_m}\; e^{NS_m(E,z,\zeta)} d^m\zeta\, dz, \quad (33)$$

$$S_m(E,z,\zeta) = Ez + \frac{z}{2}(A\zeta,\zeta)_m + \ln Me^{-z(l_0,\zeta)_m}.$$

One may expect that in the limit $m \to \infty$ the function $S_m(E,z,\zeta)$ transforms to the functional (7) while the asymptotics of the integral (33) as $N \to \infty$ is given by formula (8).

The case of large excitations is treated similarly:

$$f(E) = M\delta\left(E + \min_u I(u)\right) = \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} e^{Ez} Me^{z\min\limits_u I(u)}\, dz \; =$$

$$= \lim_{m\to\infty}\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} e^{Ez}\sqrt{\frac{\det A_m}{(2\pi)^m}}\, M\int e^{-\frac{1}{2}(Au,u)_m+i\sqrt{z}(l,u)_m}\, d^m u\, dz \; =$$

$$= \lim_{m\to\infty}\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\int e^{Ez}\sqrt{\frac{\det A_m}{(2\pi)^m}}\, e^{-\frac{1}{2}(Au,u)_m+N\ln Me^{i\sqrt{\frac{z}{N}}(l_0,u)_m}}\, d^m u.$$

Changing the function,

$$N\ln M\, e^{i\sqrt{\frac{z}{N}}(l_0,u)},$$

by its limit value as $N \to \infty$,

$$-\frac{z}{2}(Bu, u) \equiv -\frac{z}{2}M(l_0, u)^2,$$

we find

$$f(E) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{Ez} \lim_{m\to\infty} \sqrt{\frac{\det A_m}{\det(A + zB)_m}}\, dz,$$

which transforms to (20) if all changes of limit procedures made in this derivation are legitimate.

Formula (8) looks like the leading term in an integral asymptotics obtained by the steepest descent method. Note that the steepest descent method cannot be applied to the integral (33) directly because this method assumes that $N \to \infty$ while m is fixed. In our case $m \gg N$, and we first tend m to infinity.

The limit of m-dimensional integrals like (33) as $m \to \infty$ may be considered as an integral in infinite-dimensional space. Such integrals appear in various issues of physics and mathematics [9,10,12-15,18-21]. A peculiarity of the problem under consideration is that the integral (33) does not converge absolutely and uniformly with respect to m: if one substitute the integrand by its absolute value, then the integral tends to infinity as $m \to \infty$. Similar difficulty is encountered if one attempts to give a probabilistic interpretation to Feynman's path integrals.

For real valued integrals, the Laplace method in infinite-dimensional spaces has been developed by Ellis and Rosen [9] (see also a review paper [18]). Note also that the problem under consideration is similar in nature to the problems considered in theory of large deviations by S.Varahdan [21], M.Freidlin and A.Wentzel [13].

3. **Statements.** We formulate in this section three statements the proof of which is the subject of the rest of the paper.

1. Let the following properties are satisfied:

A. The spectrum of the eigenvalue problem,

$$A\varphi = \mu B\varphi ,$$

where the operator B is defined by (15) is discrete. The corresponding eigen-functions $\varphi_1, \varphi_2, ...$ form a basis in $\mathcal{H}$. The quadratic forms (Au, u) and (Bu, u) in this basis are diagonal:

$$(Au, u) = \sum_{k=1}^{\infty} \lambda_k u_k^2 \quad , \quad (Bu, u) = \sum_{k=1}^{\infty} b_k u_k^2.$$

We number the eigen-functions in such a way that the corresponding λ_k increases $\lambda_1 \le \lambda_2 \le ...$ Operator A is positive definite, i.e.

$$\lambda_1 > 0.$$

Operator B is positive, and $b_k \ge 0$.

B.The series of inverse eigenvalues converges:

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty.$$

C. Denote $(\overset{\circ}{l}(r), \varphi_k)$ by $\overset{\circ}{l}_k(r)$. We assume that $\overset{\circ}{l}_k(r)$ are uniformly bounded for all r and k:

$$\left|\overset{\circ}{l}_k(r)\right| \le h.$$

D. There exist a positive operator C such that for any u

a. $\max_r (l_0(r), u)^2 \le (Cu, u)$,

b. operator C is diagonal in the basis $\{\varphi_k\}$:

$$(Cu, u) = \sum_{k=1}^{\infty} c_k u_k^2,$$

c. for some positive number α $\quad A - \alpha C > 0$, i.e.

$$\alpha c_k < \lambda_k \qquad \text{for all } k,$$

d. the series $\sum\limits_{k=1}^{\infty} \frac{c_k}{\lambda_k}$ converges.

Under these conditions, if $\lim\limits_{N\to\infty} \hat{f}_N(z)$ is an analytic function of z in some vicinity of the point $z = 0$, then (20) holds true.

2. If, in addition to the above conditions,

E. $\ln\left|M\, e^{(l_0(r),u)}\right| \le (Du, \bar{u})$,

for some positive operator D , u is complex-valued, bar means complex conjugate;

F. for α from condition D and some $\alpha' > 0$,

$$A - \alpha' D - \alpha C > 0,$$

and

$$\lim_{m\to\infty} \sqrt{\det A_m / \det(A - \alpha' D - \alpha C)_m} \quad \text{exists},$$

then $\lim\limits_{N\to\infty} \hat{f}_N(z)$ is an analytic function in some vicinity of the point $z = 0$.

A. Beliaev proved that condition E follows from condition D.a with $D = 4C$ [1].

3. Let $(\beta, \check{u})$ be the stationary point of the entropy functional. Define an auxiliary probabilistic measure with mathematical expectation $\tilde{M}$ by the formula: for any function $\varphi(r)$

$$\tilde{M}\varphi(r) \; = \; M\left[e^{-\beta(l_0(r),\check{u})}\varphi(r)\right] \Big/ M\, e^{-\beta(l_0(r),\check{u})}. \tag{34}$$

Suppose that all assumptions of the statement 1 are fulfilled for the functional l_0 substituted by $l_0' = l_0 - \tilde{M} l_0$ and the mathematical expectation M substituted by $\tilde{M}$. Consider the function,

$$\hat{f}_N(z, \zeta) \; = \; \tilde{M}\, e^{z \min I(u) - \zeta(l,u)}, \tag{35}$$

and assume that $\lim\limits_{N\to\infty} \hat{f}_N(z, \zeta)$ is an analytical function in vicinity of the point $z = 0, \quad \zeta = 0$. Assume also that $\beta > 0$ and for $\xi = \frac{1}{\sqrt{N}} \sum_{a=1}^{N} l_0'(r_a)$ the integral,

$$\int_{-\infty}^{+\infty} \left| \tilde{M}\, e^{-\frac{\beta + i\eta}{2}(A^{-1}\xi,\xi) + i\lambda\eta(\xi,\check{u})} \right| d\eta, \tag{36}$$

exists and does not exceed $\text{const}\big/\lambda^{1+\kappa}$, $\kappa > 0$. Then (7), (21) hold true.

4. **Asymptotics of some integrals in functional spaces.** Consider an integral,

$$J_m(R) = \int\limits_{(Cu,u)_m \geq R^2} e^{-\frac{1}{2}(Au,u)_m} \sqrt{\frac{\det A_m}{(2\pi)^m}}\, d^m u, \tag{37}$$

where $(Au,u)_m$ and $(Cu,u)_m$ are positive quadratic forms in m-dimensional space. The factor $\sqrt{\det A_m/(2\pi)^m}$ is included to make the integral over the entire space equal to unity:

$$J_m(0) = 1.$$

For brevity, we use the notation,

$$\mathcal{D}^m u = \sqrt{\frac{\det A_m}{(2\pi)^m}}\, d^m u,$$

and write $J_m(R)$ in the form

$$J_m(R) = \int\limits_{(Cu,u)_m \geq R^2} e^{-\frac{1}{2}(Au,u)_m} \mathcal{D}^m u.$$

For the limit value of the integral $J_m(R)$ as $m \to \infty$ we use a symbolic notation

$$J(R) = \int\limits_{(Cu,u) \geq R^2} e^{-\frac{1}{2}(Au,u)} \mathcal{D} u.$$

Note that $J(0) = 1$.

We are interested in asymptotics of the integral $J_m(R)$ as $R \to \infty$ and $m \to \infty$. The multidimensional integrals behave quite differently from the low dimensional ones. This is caused by a peculiar behavior of the volume in the multidimensional spaces: most of the volume of a sphere is concentrated near its surface; in a sense, spheres in multidimensional spaces are practically empty. This can be seen from the following estimate. Since the volume of a sphere of radius R is $V = c_m R^m, \quad c_m$ is a constant depending on the space dimension m, the volume $\triangle V$ of the spherical layer of the thickness, $\triangle$, which is adjacent to the sphere boundary is: $\triangle V/V = [R^m - (R-\triangle)^m]/R^m$, or

$$\frac{\triangle V}{V} = 1 - \left(1 - \frac{\triangle}{R}\right)^m. \tag{38}$$

Let the spherical layer contains 99% of the volume, i.e. $\triangle V/V = 0.99\,V$. Then, solving (38) with respect to $\triangle/R$, we find: $\triangle = 0.99\,R$ for $m = 1, \quad \triangle \sim 3/4\,R$ for $m = 3, \quad \triangle \sim 1/3\,R$ for $m = 10, \quad \triangle \sim 0.05\,R$ for $m = 1000$.

Including the "mass density" $e^{-\frac{1}{2}(u,u)}$ exponentially decaying away from the origin does not change the situation. Indeed, let us put in (37) $A = C = I, \quad I$ is the identity matrix, and consider the integral,

$$\tilde{J}_m(R) = \frac{1}{(2\pi)^{m/2}} \int\limits_{(u,u) \geq R^2} e^{-\frac{1}{2}(u,u)} d^m u. \tag{39}$$

If m is fixed and R tends to infinity, integral (39) goes to zero. If R fixed and m tends to infinity then the integral (40) goes to the unity, as if $R = 0$. To make this obvious, let us write (39) in "spherical coordinates", denoting $(u,u)=r^2$. Then

$$\tilde{J}_m(R)=\frac{1}{(2\pi)^{m/2}}\sigma(m)\int\limits_R^\infty e^{-\frac{1}{2}r^2}r^{m-1}dr, \tag{40}$$

where $\sigma(m)$ is the area of the unit sphere,

$$\sigma(m)=\frac{2\pi^{m/2}}{\Gamma(m/2)}.$$

Asymptotics of integral (40) can be found by Laplace's method. We change the variable $r=\sqrt{m}\,\xi$ and write the integral in the form

$$\int\limits_R^\infty e^{-\frac{1}{2}r^2+(m-1)\ln r}dr=m^{m/2}\int\limits_{\frac{R}{\sqrt{m}}}^\infty e^{m[-\frac{1}{2}\xi^2+\ln\xi]}\,\frac{d\xi}{\xi}.$$

The stationary point of the function $-\frac{1}{2}\xi^2+\ln\xi$ is $\xi=1$, and, in the first approximation, the integral does not depend on R:

$$\int\limits_R^\infty e^{-\frac{1}{2}r^2+(m-1)\ln r}dr=m^{m/2}e^{-\frac{1}{2}m}\int\limits_{-\infty}^{+\infty} e^{-m\xi^2}\,d\xi=\frac{m^{m/2}}{\sqrt{2m}}e^{-\frac{1}{2}m}2\pi \quad ,$$

and, using the asymptotics of Γ-function for large argument, we obtain $\tilde{J}_m(R)=\tilde{J}_m(0)$.

After these preliminary comments we return to evaluation of the integral (37). It is clear that the region $(Cu,u)_m\geq R^2$ must be much bigger then the region $(Au,u)_m\geq R^2$ to let the integral feel the value of the constant R for large m. That means that (Cu,u) is, in a sense, much smaller then (Au,u). A precise meaning of "smallness" is given by condition D.d. Let us show that, if (Cu,u) satisfies condition D.d, $\sum c_k/\lambda_k<\infty$, then a simple estimate of $J_m(R)$ holds:

$$J_m(R)\leq\kappa e^{-\frac{1}{2}\alpha R^2}\ , \tag{41}$$

the constant κ is specified later. First, let us note that, due to condition D.d, the product $\prod\limits_{k=1}^m\left(1+\frac{zc_k}{\lambda_k}\right)$ converges to an analytical function $\exp[2\psi(z)]$,

$$\psi(z)=\frac{1}{2}\sum_{k=1}^\infty\ln\left(1+\frac{zc_k}{\lambda_k}\right). \tag{42}$$

This function is analytic in the half plane $\mathrm{Re}z>-c_0$, $\quad 0<c_0=\min\limits_k\{\lambda_k/c_k\}$. It is equal to unity at $z=0$. Choosing $\alpha<c_0$ we have for all m

$$\left(\prod_{k=1}^m\left(1-\frac{\alpha c_k}{\lambda_k}\right)\right)^{-1/2}\leq\ e^{-\psi(-\alpha)}, \tag{43}$$

because $\prod\limits_{k=m+1}^\infty\left(1-\frac{\alpha c_k}{\lambda_k}\right)\ <\ 1.$

For that α we have

$$J_m(R)=e^{-\alpha R^2}\int\limits_{(Cu,u)_m\geq\quad R^2} e^{-\frac{1}{2}(Au,u)_m+\alpha R^2}\mathcal{D}^m u$$

$$\leq e^{-\alpha R^2} \int\limits_{(Cu,u)_m \geq R^2} e^{-\frac{1}{2}(Au,u)_m + \alpha(Cu,u)} \mathcal{D}^m u$$

$$\leq e^{-\alpha R^2} \int e^{-\frac{1}{2}((A-\alpha C)u,u)}\, \mathcal{D}^m u = e^{-\alpha R^2} \left(\prod_{k=1}^{m} \left(1 - \frac{\alpha c_k}{\lambda_k} \right) \right)^{-1} \leq \kappa e^{-\alpha R^2},$$

where we put $\kappa = \exp[-\psi(-\alpha)]$.

Hence,

$$J(R) \leq \kappa e^{-\alpha R^2} \tag{44}$$

Estimate (44) corresponds to Lemma 4.3 of paper [9].

5. **Large excitations: Proof of the statements.** We begin with some simple corollaries of the assumptions made.

Corollaries:

1.Coefficients of the form (Bu, u) are uniformly bounded:

$$b_k \leq h^2. \tag{45}$$

Indeed, for any sequence $\{u_k\}$,

$$(Bu, u) = M \left(l_0(r), \sum_k u_k \varphi_k \right)^2 = \sum_{k,s} M \left(l_0(r), \varphi_k \right) \left(l_0(r), \varphi_s \right)\, u_k u_s$$
$$= \sum_k b_k u_k^2.$$

Therefore,

$$M \left(l_0(r), \varphi_k \right)\, \left(l_0(r), \varphi_s \right) = 0 \quad \text{for} \quad k \neq s \quad \text{and} \quad b_k = M \left(l_0(r), \varphi_k \right)^2. \tag{46}$$

Estimate (45) follows from (46) and condition C.

2.The quadratic functional (Bu, u) does not exceed (Cu, u),

$$(Bu, u) \leq (Cu, u).$$

Indeed,

$$b_k = M \left(l_0(r), \varphi_k \right)^2 \leq \max_r (l_0(r), \varphi_k)^2 \leq c_k.$$

The latter inequality follows from D.a if one puts in D.a $u = u_k \varphi_k$.

3.The series $\sum\limits_{k=1}^{\infty} \frac{b_k}{\lambda_k}$ converges:

$$\sum_{k=1}^{\infty} \frac{b_k}{\lambda_k} \leq h^2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty. \tag{47}$$

4.Function $\Phi(z) = \prod\limits_{1}^{\infty} \left(1 + \frac{z b_k}{\lambda_k} \right)$ is an entire analytic function in the half plane Re $z > -a$, $\quad a = \min\{\lambda_k / b_k\}$. This follows from (47).

5.Function $\Phi(z)$ does not have a zero at any finite point of the half plane Re $z > -a$ (see, for example, [16]).

6. Energy of “an elementary excitation”, $\overset{\circ}{E}(r)$, defined by the relation

$$-\overset{\circ}{E}(r) \equiv \min_u \left[\frac{1}{2}(Au, u) - (l_0(r), u) \right] \ ,$$

is finite and uniformly bounded.

Indeed,

$$\overset{\circ}{E}(r) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{\overset{\circ}{l}{}_k^2(r)}{\lambda_k} \ \leq \ \frac{h^2}{2}\sum_{k=1}^{\infty}\frac{1}{\lambda_k} \ < +\infty$$

Proposition 1. *Let $\hat{f}_N(z)$ be Laplace's transform of probability density of energy,*

$$\hat{f}_N(z) = M\, e^{z \min\limits_u I_N(u)}$$

Denote by $I_{N,m}$ the m-dimensional truncation of the functional I_N:

$$I_{N,m}(u) = \frac{1}{2}(Au,u)_m - \frac{1}{\sqrt{N}}\sum_{a=1}^{N}(l_0(r_a),u)_m$$

$$(Au,u)_m = \sum_{k=1}^{m}\frac{1}{2}\lambda_k u_k^2 \quad , \quad (l_0(r),u)_m = \sum_{k=1}^{m}\overset{\circ}{l}_k(r)u_k$$

and by $\hat{f}_{N,m}(z)$ the Laplace transform of the probability density of truncated energy,

$$\hat{f}_{N,m}(z) = M\, e^{z \min\limits_u I_{N,m}(u)}.$$

Then

$$\hat{f}_N(z) = \lim_{m\to\infty}\hat{f}_{N,m}(z) = \lim_{m\to\infty}\sqrt{\frac{\det A_m}{(2\pi)^m}}\int e^{-\frac{1}{2}(Au,u)_m + N\ln M e^{-i\sqrt{\frac{z}{N}}(l_0,u)_m}}d^m u. \tag{48}$$

Proof. The sum,

$$-\min_u I_{N,m}(u) = \frac{1}{2}\sum_{k=1}^{m}\frac{1}{\lambda_k}\left(\frac{1}{\sqrt{N}}\sum_{a=1}^{N}\overset{\circ}{l}_k(r_a)\right)^2, \tag{49}$$

converges uniformly as $m \to \infty$ because the members of the sum are bounded by the converging series,

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\lambda_k}(\sqrt{N}h)^2.$$

Therefore,

$$M\lim_{m\to\infty} e^{z\min\limits_u I_{N,m}} = \lim_{m\to\infty} M e^{z\min\limits_u I_{N,m}}. \tag{50}$$

The sum (49) converges to

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\left(\frac{1}{\sqrt{N}}\sum_{a=1}^{N}\overset{\circ}{l}_k(r)\right)^2 = -\min_u I_N(u,r).$$

Thus, due to (50),

$$M\lim_{m\to\infty} e^{z\min\limits_u I_{N,m}(u,r)} = Me^{z\min\limits_u I_N(u,r)} = \hat{f}_N(z) = \lim_{m\to\infty} Me^{z\min\limits_u I_{N,m}(u,r)} \tag{51}$$

$$= \lim_{m\to\infty}\hat{f}_{N,m}(z).$$

Now we recall the identity for any positive quadratic form $(Au,u)_m$, any linear form $(l,u)_m$ and a complex number z,

$$\sqrt{\frac{\det A_m}{(2\pi)^m}} \int e^{-\frac{1}{2}(Au,u)_m - i\sqrt{z}(l,u)_m} d^m u = e^{-\frac{1}{2}z(A_m^{-1}l,l)} = e^{z\min[\frac{1}{2}(Au,u)_m-(l,u)_m]}. \tag{52}$$

Using (52) we can write:

$$Me^{z\min_u I_{N,m}(u,r)} = M\sqrt{\frac{\det A_m}{(2\pi)^m}} \int e^{-\frac{1}{2}(Au,u)_m - i\sqrt{\frac{z}{N}}\sum_{a=1}^{N}(l_0(r),u)_m} d^m u. \tag{53}$$

The integral in (53) converges absolutely, therefore we may compute integral and mathematical expectation in any order. Computing first the mathematical expectation,

$$Me^{-i\sqrt{\frac{z}{N}}\sum_{a=1}^{N}(l_0(r),u)_m} = \left(Me^{-i\sqrt{\frac{z}{N}}(l_0(r),u)_m}\right)^N = e^{N\ln Me^{-i\sqrt{\frac{z}{N}}(l_0(r),u)_m}},$$

we arrive at (48). □

Proposition 2. *The limit value of the characteristic function on the real positive axis in a small vicinity of the origin, $z \le \delta$, is given by the relation,*

$$\hat{f}(z) = \lim_{N\to\infty} \hat{f}_N(z) = \frac{1}{\sqrt{\Phi(z)}}. \tag{54}$$

Proof. Let us split the region of integration in (48) into two subregions, $(Cu,u) \le R^2$ and $(Cu,u) \ge R^2$. For the integral I_1 over the first region we have

$$\begin{aligned} I_1 &= \sqrt{\frac{\det A_m}{(2\pi)^m}} \int\limits_{(Cu,u)_m \le R^2} e^{-\frac{1}{2}(Au,u)_m + N\ln Me^{-i\sqrt{\frac{z}{N}}(l_0(r),u)_m}} d^m u = \\ &= \sqrt{\frac{\det A_m}{(2\pi)^m}} \int\limits_{(Cu,u)_m \le R^2} e^{-\frac{1}{2}((A+zB)u,u)_m} d^m u \\ &+ \sqrt{\frac{\det A_m}{(2\pi)^m}} \int\limits_{(Cu,u)_m \le R^2} e^{-\frac{1}{2}(Au,u)_m} \triangle d^m u, \end{aligned} \tag{55}$$

where

$$\triangle = \left[Me^{-i\sqrt{\frac{z}{N}}(l_0(r),u)_m}\right]^N - e^{-\frac{z}{2}(Bu,u)_m}.$$

The first integral in the right hand side of (54) can be presented as the difference of integrals over the entire space and over the region $(Cu,u)_m \ge R^2$:

$$\sqrt{\frac{\det A_m}{(2\pi)^m}} \int e^{-\frac{1}{2}((A+zB)u,u)} d^m u - \sqrt{\frac{\det A_m}{(2\pi)^m}} \int\limits_{(Cu,u)_m \ge R^2} e^{-\frac{1}{2}((A+zB)u,u)} d^m u. \tag{56}$$

The first integral over the entire space can be found exactly. It is equal to $1\Big/\sqrt{\Phi_m(z)}$,

$$\Phi_m(z) = \prod_{k=1}^{m} \left(1 + \frac{zb_k}{\lambda_k}\right).$$

The second integral over the region $(Cu,u)_m \geq R^2$ may be estimated by means of (41). In the estimate of I_1 we assume that z is any complex number from a small vicinity of the origin, $|z| \leq \delta$. We have

$$\left| \sqrt{\frac{\det A_m}{(2\pi)^m}} \int\limits_{(Cu,u)_m \geq R^2} e^{-\frac{1}{2}((A+zB)u,u)_m}\, d^m u \right| \leq$$

$$\leq \sqrt{\frac{\det A_m}{(2\pi)^m}} \int\limits_{(Cu,u)_m \geq R^2} e^{-\frac{1}{2}((A-\delta B)u,u)_m}\, d^m u \leq$$

$$\leq \sqrt{\frac{\det A_m}{(2\pi)^m}} \int\limits_{(Cu,u)_m \geq R^2} e^{-\frac{1}{2}((A-\delta C)u,u)_m}\, d^m u \leq$$

$$\sqrt{\frac{\det A_m}{(2\pi)^m}}\, e^{-\alpha' R^2} \int\limits_{(Cu,u)_m \geq R^2} e^{-\frac{1}{2}((A-\delta C-\alpha' C)u,u)_m}\, d^m u \leq$$

$$\leq e^{-\alpha' R^2} \frac{1}{\sqrt{\prod\limits_{k=1}^{m} \left(1 - \frac{(\delta+\alpha')c_k}{\lambda_k}\right)}} \leq \kappa e^{-\alpha' R^2}. \tag{57}$$

It is assumed in the last inequality that δ and α' are small enough to satisfy the condition

$$\delta + \alpha' \leq \alpha.$$

To estimate the second integral in (55) we have to estimate $\triangle$. We may write $\triangle$ in the form

$$\triangle = e^{-\frac{z}{2}(Bu,u)_m} \left(e^{\frac{z}{2}(Bu,u)_m} \left(1 - \frac{z}{2N}(Bu,u)_m + \frac{\rho}{N}\right)^N - 1 \right), \tag{58}$$

where

$$\frac{\rho}{N} = M \left(e^{-i\sqrt{\frac{z}{N}}(l_0(r),u)_m} - 1 + i\sqrt{\frac{z}{N}}(l_0(r),u)_m + \frac{z}{2N}(l_0(r),u)_m \right)..$$

First we bound ρ by means of inequality,

$$\left| e^w - 1 - w - \frac{w^2}{2} \right| \leq \frac{|w|^3}{3!} e^{|w|}.$$

Let us put in this inequality

$$w = i\sqrt{\frac{z}{N}}(l_0(r),u)_m\ ,$$

and note that

$$|w| \leq \sqrt{\frac{|z|}{N}} \sqrt{(Cu,u)_m} \leq \sqrt{\frac{\delta}{N}} R.$$

Imposing a constraint on R:

$$\sqrt{\frac{\delta}{N}}R < 1/2,$$

we find a bound for ρ

$$\frac{|\rho|}{N} \ \leq \ \frac{1}{2}\left(\sqrt{\frac{\delta}{N}}R\right)^3,$$

or

$$|\rho| \ \leq \ \frac{1}{2}\ \frac{\delta^{3/2}R^3}{N^{1/2}}.$$

Denote, for brevity, $-\frac{z}{2}(Bu,u)$ by w in (58):

$$\triangle = e^w\left(e^{N\ln\left(1+\frac{w+\rho}{N}\right)-w}-1\right)$$

Using inequality $|e^z-1| \ \leq \ |z|\,e^{|z|}$ we have

$$|\triangle| \ \leq \ e^{\operatorname{Re}w}\left|N\ln\left(1+\frac{w+\rho}{N}\right)-w\right|\,e^{\left|N\ln\left(1+\frac{w+\rho}{N}\right)-w\right|}. \tag{59}$$

Function $N\ln\left(1+\frac{w+\rho}{N}\right)-w$ may be estimated as follows. Let us introduce a function of t,

$$\varphi(t) = N\ln\left(1+\frac{(w+\rho)t}{N}\right)-wt.$$

We need to estimate $\varphi(1)$. Derivative of φ has a simple form

$$\frac{d\varphi}{dt} = \frac{w+\rho}{1+\frac{(w+\rho)t}{N}}-w = -\frac{(w+\rho)^2}{N}\ \frac{t}{1+\frac{(w+\rho)t}{N}}+\rho.$$

Thus,

$$\varphi(1) = \int_0^1 \frac{d\varphi}{dt}\,dt = \ -\frac{(w+\rho)^2}{N}\int_0^1 \frac{tdt}{1+\frac{(w+\rho)t}{N}}+\rho. \tag{60}$$

Since

$$\frac{\delta}{N}R^2 \ < \ \frac{1}{4}\ , \qquad \frac{|\rho|}{N} \ \leq \ \frac{1}{2}\left(\sqrt{\frac{\delta}{N}}R\right)^3 \ < \ \frac{1}{4},$$

we have

$$\frac{|w+\rho|}{N} \ \leq \ \frac{\left|\frac{z}{2}(Bu,u)_m\right|}{N}+\frac{|\rho|}{N} \ < \ \frac{\frac{\delta}{2}R^2}{N}+\frac{|\rho|}{N} \ < \ \frac{1}{2}. \tag{61}$$

Combining (60) and (61) we obtain

$$\left|N\ln\left(1+\frac{w+\rho}{N}\right)-w\right| \ = \ |\varphi(1)| \ \leq \tag{62}$$

$$\leq \ \frac{|w+\rho|^2}{N}\int_0^1 \frac{tdt}{1-\frac{1}{2}}+|\rho| = \frac{|w+\rho|^2}{N}+|\rho| \ \leq \ 2\frac{|w|^2+|\rho|^2}{N}+|\rho|.$$

We add one more constraint on R to have

$$\left|N\ln\left(1+\frac{w+\rho}{N}\right)-w\right| \ < \ 1 \tag{63}$$

Requiring that the right hand side of (62) be less then 1 we have

$$\frac{2}{N}\left(\left|\frac{z}{2}(Bu,u)_m\right|^2+\frac{1}{4}\,\frac{\delta^3R^6}{N}\right)\;+\;\frac{1}{2}\,\frac{\delta^{3/2}R^3}{N^{1/2}}\;<\;\triangle_1\;<\;1\,, \tag{64}$$

$$\triangle_1\;\equiv\;\frac{2}{N}\left(\frac{\delta^2}{4}R^4+\frac{1}{4}\,\frac{\delta^3R^6}{N}\right)\;+\;\frac{1}{2}\frac{\delta^{3/2}R^3}{N^{1/2}}.$$

We see that the inequality (63) is satisfied if

$$\frac{R}{N^{1/6}}\qquad\text{is sufficiently small} \tag{65}$$

In accordance with (62) and (63) the estimate for $\triangle$ (59) takes the form

$$|\triangle|\;\leq\;3\triangle_1\,e^{\frac{\delta}{2}(Bu,u)_m}. \tag{66}$$

From (66)

$$\sqrt{\frac{\det A_m}{(2\pi)^m}}\int\limits_{(Cu,u)_m\leq R^2}e^{-\frac{1}{2}(Au,u)_m}|\triangle|\,d^mu\;\leq \tag{67}$$

$$\leq\sqrt{\frac{\det A_m}{(2\pi)^m}}\int e^{-\frac{1}{2}(Au,u)_m+\frac{1}{2}\delta(Bu,u)_m}\;d^mu\;3\triangle_1=\frac{3\triangle_1}{\sqrt{\Phi_m(-\delta)}}.$$

Combining (67) and (57) we obtain

$$\left|I_1-\frac{1}{\sqrt{\Phi_m(z)}}\right|\;\leq\;\kappa e^{-\alpha' R^2}+\frac{3\triangle_1}{\sqrt{\Phi_m(-\delta)}}. \tag{68}$$

Let us estimate now the integral over the second subregion,

$$I_2=\sqrt{\frac{\det A_m}{(2\pi)^m}}\int\limits_{(Cu,u)_m\geq R^2}e^{-\frac{1}{2}(Au,u)_m+N\ln Me^{-i\sqrt{\frac{z}{N}}(l_0(r),u)_m}}\;d^mu. \tag{69}$$

We choose in (69) z to lie on the real positive semi-axis. Then $\mathrm{Re}\ln Me^{-i\sqrt{\frac{z}{N}}(l_0(r),u)_m}\;\leq\;0$, and, due to (41), integral I_2 admits an estimate

$$|I_2|\;\leq\;\sqrt{\frac{\det A_m}{(2\pi)^m}}\int\limits_{(Cu,u)_m\geq R^2}e^{-\frac{1}{2}(Au,u)_m}d^mu\;\leq\;\kappa e^{-\alpha R^2}.$$

Tending R to infinity as, for example, $N^{1/7}$, we see that I converges to $1/\sqrt{\Phi(z)}$ uniformly in m and N. □

Proposition 3. *If conditions E and F are fulfilled then* $\lim\limits_{N\to\infty}\hat{f}_N(z)$ *is an analytic function of* z.

Proof. Indeed, in this case we may estimate integral I_2 (69) for all complex z from some vicinity of the point $z=0\,,\quad |z|\leq\delta$

$$I_2\;\leq\;\sqrt{\frac{\det A_m}{(2\pi)^m}}\int\limits_{(Cu,u)_m\leq R^2}e^{-\frac{1}{2}(Au,u)_m+N\ln|Me^{-i\sqrt{\frac{z}{N}}l_0(r),u)}|}\;d^mu$$

$$\leq\;\sqrt{\frac{\det A_m}{(2\pi)^m}}\int\limits_{(Cu,u)_m\leq R^2}e^{-\frac{1}{2}(Au,u)_m+\delta(Du,u)_m}\;d^mu$$

$$\leq \; e^{-\alpha' R^2} \sqrt{\frac{\det A_m}{\det(A - \delta D - \alpha' C)_m}}.$$

The estimate (68) was derived for all complex z in a vicinity of the origin. Choosing again $R \sim N^{1/7}$, we have a sequence of analytic functions $\hat{f}_{N,m}(z)$ converging uniformly in the circle $|z| \leq \delta$ as $N \to \infty, \quad m \to \infty$. Thus the limit is an analytic function. It is equal to $1/\sqrt{\Phi(z)}$ in the circle and can be continued outside of the circle as $1/\sqrt{\Phi(z)}$. □

6. **Small excitations: An identity.** Denote by $F_N(E)$ the probability distribution of energy,

$$F_N(E) = \int_0^E f_N(E')dE',$$

and by $(\beta, \check{u})$ any stationary point of the entropy functional. Let us show that, for the case of small excitations, the following identity holds:

$$F_N(E) = R_N e^{NS(E)}, \tag{70}$$

$$R_N = \tilde{M}\left[e^{N\beta(l',\check{u})}\theta\left(-(l',\check{u}) - \frac{1}{2}(A^{-1}l',l')\right)\right], \tag{71}$$

where $\tilde{M}$ is the mathematical expectation for the auxiliary probability measure (34), $l' = l - \tilde{M}l_0$, and $\theta(E)$ is the step function ($\theta(E) = 1$ if $E > 0$ and $\theta(E) = 0$ if $E < 0$). In [2] this identity was obtained for probability density.

Since,

$$\theta(E) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{1}{z}e^{Ez}dz = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{1}{z}e^{NEz}dz, \tag{72}$$

we have

$$F(E) = M\theta\left(E - \frac{1}{2}(A^{-1}l,l)\right) = M\,\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{1}{z}e^{NEz - \frac{Nz}{2}(A^{-1}l,l)}dz =$$

$$= M\left[e^{-\beta\sum_{a=1}^{N}(l_0(r_a),\check{u})}\int_{-i\infty}^{+i\infty} e^{NEz+\beta\sum_{a=1}^{N}(l_0(r_a),\check{u}) - \frac{Nz}{2}(A^{-1}(\tilde{M}l_0+l'),\,\tilde{M}l_0+l')}\frac{dz}{2\pi iz}\right] =$$

$$= \left(Me^{-\beta(l_0,\check{u})}\right)^N \times \tag{73}$$

$$\times\tilde{M}\int_{-i\infty}^{+i\infty} e^{NEz+\beta\sum_{a=1}^{N}(l_0(r_a),\check{u}) - \frac{Nz}{2}(A^{-1}\tilde{M}l_0,\tilde{M}l_0) - Nz(A^{-1}\tilde{M}l_0,l') - \frac{Nz}{2}(A^{-1}l',l')}\frac{dz}{2\pi iz}.$$

Note that the equations for the stationary point of the entropy functional (7) reads

$$A\check{u} = \frac{Ml_0e^{-\beta(l_0,\check{u})}}{Me^{-\beta(l_0,\check{u})}} = \tilde{M}l_0,$$
$$\frac{1}{2}(A\check{u},\check{u}) = E.$$

Thus

$$(\tilde{M}l_0, \breve{u}) = 2E\ , \qquad (A^{-1}M\tilde{l}_0, M\tilde{l}_0) = (\breve{u}, M\tilde{l}_0) = 2E,$$
$$(A^{-1}M\tilde{l}_0, l') = (\breve{u}, l'),$$

and

$$S(E) = E\beta + \frac{\beta}{2}(A\breve{u}, \breve{u}) + \ln M \int e^{-\beta(l_0,\breve{u})} = 2E\beta + \ln M \int e^{-\beta(l_0,\breve{u})}.$$

Using these relations we simplify (73):

$$F(E) = e^{NS(E)}\tilde{M}\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}\frac{1}{z}e^{N\,\triangle S}dz,$$
$$\triangle S = \beta(l', \breve{u}) - z(l', \breve{u}) - \frac{z}{2}(A^{-1}l', l'). \tag{74}$$

Identity (70) follows from (74) and (72). Note that the identity holds for any stationary point of the entropy functional, including the points with negative temperature.

The formula (70) means that the prefactor is an integral over the region

$$-(l', \breve{u}) - \frac{1}{2}(A^{-1}l', l')\ \geq\ 0$$

This region is a sphere in the space of linear functionals l'. The sphere passes through the origin. The equation of the sphere can be written in the form

$$\frac{1}{2}(A^{-1}(l' + A\breve{u}), l' + A\breve{u}) = \frac{1}{2}(A\breve{u}, \breve{u})$$

or, since $A\breve{u} = \tilde{M}l_0$ and $(A\breve{u}, \breve{u}) = 2E$,

$$\frac{1}{2}(A^{-1}(l' + \tilde{M}l_0), l' + \tilde{M}l_0) = E$$

That means that the sphere has the radius $\sqrt{2E}$ and the center is at the point $-\tilde{M}l_0$. If, in the limit $N \to \infty$, the linear functional

$$\sqrt{N}l' = \frac{1}{\sqrt{N}}\sum_{a=1}^{N} l_0'(r_a)\ , \qquad l_0'(r) = l_0(r) - \tilde{M}l_0, \tag{75}$$

has a Gaussian distribution, the prefactor can be computed [2]. Here we describe a similar way in which the necessary assumptions can be seen more clear.

Denote the linear functional (75) by ξ. Then equation (71) takes the form

$$R_N = \tilde{M}\left[e^{\sqrt{N}\beta(\xi,\breve{u})}\theta\left(-\frac{1}{\sqrt{N}}(\xi, \breve{u}) - \frac{1}{2N}(A^{-1}\xi, \xi)\right)\right] \tag{76}$$
$$= \tilde{M}\left[e^{\sqrt{N}\beta(\xi,\breve{u})}\theta\left(-\sqrt{N}(\xi, \breve{u}) - \frac{1}{2}(A^{-1}\xi, \xi)\right)\right].$$

Here $\breve{u}$, as a solution of deterministic problem, does not depend on "an event". Averaging is conducted with respect to a random functional ξ.

The functional $\frac{1}{2}(A^{-1}\xi, \xi)$ is the energy in the variational problem with large excitations:

$$-\frac{1}{2}(A^{-1}\xi, \xi) = \min_u\left[\frac{1}{2}(Au, u) - (\xi, u)\right].$$

Thus, if we know the joint probability of energy and linear functional $(\xi, \breve{u})$,

$$f_N(E,t) = \tilde{M}\left[\delta\left(E - \frac{1}{2}(A^{-1}\xi,\xi)\right)\delta\left(t - (\xi,\check{u})\right)\right],$$

we can determine R_N from (76):

$$R_N = \int e^{\sqrt{N}\beta t}\theta(-E - \sqrt{N}\,t)\, f_N(E,t)\, dE\, dt. \tag{77}$$

Computing the probability density $f_N(E,t)$ is similar to computing $f_N(E)$. It is more convenient, as before, to deal with Laplace's transform of the probability density,

$$f_N(E,t) = \frac{1}{(2\pi i)^2}\int\limits_{c-i\infty}^{c+i\infty}\int\limits_{-i\infty}^{+i\infty} e^{Ez+t\zeta}\hat{f}(z,\zeta)\, dz\, d\zeta,$$

$$\hat{f}(z,\zeta) = \tilde{M}\, e^{z\min I(u)-\zeta(\xi,\check{u})}.$$

We have

$$R_N = \int e^{\sqrt{N}\beta t}\theta(-E - \sqrt{N}\,t)\frac{1}{(2\pi i)^2}\, e^{Ez+t\zeta}\hat{f}_N(z,\zeta)\, dE\, dt\, dz\, d\zeta =$$

$$= -\frac{1}{(2\pi i)^2\sqrt{N}}\int\int \hat{f}_N(z,\zeta)\frac{dz\, d\zeta}{\left(z - \beta - \frac{\zeta}{\sqrt{N}}\right)\left(\beta + \frac{\zeta}{\sqrt{N}}\right)}. \tag{78}$$

Here we assumed that

$$\mathrm{Re}\,(\zeta + \sqrt{N}\beta) \;>\; 0, \qquad \mathrm{Re}\left(z - \beta - \frac{\zeta}{\sqrt{N}}\right) \;>\; 0.$$

Let $\hat{f}_N(z,\zeta)$ be an analytical function of z for each ζ on imaginary axis, Re $z > 0\,,\quad \hat{f}_N(z,\zeta) \to 0$ as Re $z \to \infty$ so that

$$\int\limits_{c-i\infty}^{c+i\infty} \hat{f}_N(z,\zeta)\frac{dz}{z - \beta - \frac{\zeta}{\sqrt{N}}} \;\to 0 \qquad \text{as} \quad c \to \infty.$$

Then, the only contribution to (78) is given by the pole at $z = \beta + \zeta/\sqrt{N}$, and the prefactor is given by the formula:

$$R_N = \frac{1}{2\pi i\sqrt{N}}\int\limits_{-i\infty}^{+i\infty} \hat{f}_N\left(\beta + \frac{\zeta}{\sqrt{N}},\zeta\right)\frac{d\zeta}{\beta + \frac{\zeta}{\sqrt{N}}}. \tag{79}$$

In conclusion of this section, note that an identity similar to (70) can be obtained for the antiderivative of $F_N(E)$,

$$\underset{\bullet}{F}_N(E) = \int\limits_{-\infty}^{E} F_N(E')\, dE'\,, \qquad \frac{d}{dE}\underset{\bullet}{F}_N(E) = F_N(E)$$

Denote by $\underset{\bullet}{\theta}(E)$ the antiderivative of $\theta(E)$:

$$\underset{\bullet}{\theta}(E) = 0 \quad \text{if} \quad E < 0\,, \qquad \underset{\bullet}{\theta}(E) = E \quad \text{if} \quad E > 0$$

Then

$$F_{\bullet N}(E) = \tilde{R}_N\, e^{NS(E)}, \tag{80}$$

$$\tilde{R}_N = \tilde{M}\left[e^{N\beta(l',\breve{u})}\underset{\bullet}{\theta}\left(-(l',\breve{u}) - \frac{1}{2}(A^{-1}l',l')\right)\right].$$

Identity (80) can be obtained by repeating the derivation of (70) with the change of (72) by the relation,

$$\underset{\bullet}{\theta}(E) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{1}{z^2}\,e^{Ez}\,dz = \frac{1}{2\pi i N}\int_{-i\infty}^{+i\infty}\frac{1}{z^2}\,e^{NEz}\,dz.$$

7. **Prefactor.** In this section we find the limit value of the prefactor.

Consider function (35). For each fixed z and ζ the limit of this function as $N \to \infty$ can be computed in the same way as in section II

$$\begin{aligned}
\hat{f}(z,\zeta) &= \lim_{N\to\infty}\hat{f}_N(z,\zeta) = \lim_{N\to\infty}\tilde{M}\,e^{-z\frac{1}{2}(A^{-1}l,l)-\zeta(l,\breve{u})}\\
&= \tilde{M}\lim_{\substack{m\to\infty\\ N\to\infty}}\sqrt{\frac{\det A_m}{(2\pi)^m}}\int e^{-\frac{1}{2}(Au,u)_m+i\sqrt{z}(l,u)-\zeta(l,\breve{u})}\,d^m u\\
&= \lim_{\substack{m\to\infty\\ N\to\infty}}\sqrt{\frac{\det A_m}{(2\pi)^m}}\int e^{-\frac{1}{2}(Au,u)_m+N\ln\tilde{M}e^{i\frac{1}{\sqrt{N}}(l_0',\sqrt{z}\,u-i\zeta\breve{u})_m}}\,d^m u\\
&= \lim_{m\to\infty}\sqrt{\frac{\det A_m}{(2\pi)^m}}\int e^{-\frac{1}{2}(Au,u)_m-\frac{1}{2}\left(B(\sqrt{z}\,u-i\zeta\breve{u}),\sqrt{z}\,u-i\zeta\breve{u}\right)_m}\,d^m u\\
&= \lim_{m\to\infty}\sqrt{\frac{\det A_m}{(2\pi)^m}}\int e^{-\frac{1}{2}((A+zB)u,u)_m+\sqrt{z}\zeta i(Bu,\breve{u})_m+\frac{\zeta^2}{2}(B\breve{u},\breve{u})}\,d^m u\\
&= e^{\frac{\zeta^2}{2}\left((B\breve{u},\breve{u})-z((A+zB)^{-1}B\breve{u},B\breve{u})\right)}\sqrt{\frac{\det A}{\det(A+zB)}}.
\end{aligned}\tag{81}$$

Validity of all equalities in (81) can be established in the same way as in section V. If we drop in (79) the term $\zeta/\sqrt{N}$ and change $\hat{f}_N$ by $\hat{f}$ we find

$$\begin{aligned}
\sqrt{N}R_N &\simeq \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{\eta^2}{2}\left((B\breve{u},\breve{u})-\beta\left((A+\beta B)^{-1}B\breve{u},B\breve{u}\right)\right)}\sqrt{\frac{\det A}{\det(A+\beta B)}}\,d\eta =\\
&= \sqrt{\frac{\det A}{2\pi\det(A+\beta B)\left((B\breve{u},\breve{u})-\beta\left((A+\beta B)^{-1}B\breve{u},B\breve{u}\right)\right)}} \equiv \sqrt{N}R_\infty.
\end{aligned}\tag{82}$$

Let us check if (82) is indeed the limit value of $\sqrt{N}R$. We split the integral (79) into a sum of two integrals

$$\begin{aligned}
\sqrt{N}R_N = &\frac{1}{2\pi}\int_{|\eta|\le c}\hat{f}_N\left(\beta+\frac{i\eta}{\sqrt{N}},i\eta\right)\frac{d\eta}{\beta+\frac{i\eta}{\sqrt{N}}}\\
&+\frac{1}{2\pi}\int_{|\eta|\ge c}\hat{f}_N\left(\beta+\frac{i\eta}{\sqrt{N}},i\eta\right)\frac{d\eta}{\beta+\frac{i\eta}{\sqrt{N}}}.
\end{aligned}$$

Choose $c = N^{\alpha}$, $0 < \alpha < 1/2$. Then the first integral converges to (82) as $N \to \infty$. The second integral is estimated by means of (36).

$$\left| \frac{1}{2\pi} \int\limits_{|\eta| \geq N^{\alpha}} \hat{f}_N \left(\beta + \frac{i\eta}{\sqrt{N}}, i\eta \right) \frac{d\eta}{\beta + \frac{i\eta}{\sqrt{N}}} \right|$$
$$= \left| \frac{\sqrt{N}}{2\pi} \int\limits_{|\eta| \geq N^{\alpha - 1/2}} \hat{f}_N(\beta + i\eta, \sqrt{N} i\eta) \frac{d\eta}{\beta + i\eta} \right|$$
$$= \frac{\sqrt{N}}{2\pi} \int \left| \tilde{M} e^{-\frac{\beta + i\eta}{2}(A^{-1}l, l) + i\sqrt{N}\eta(l, \breve{u})} \right| \frac{d\eta}{\beta} \leq \text{const} \frac{1}{N^{\kappa/2}}.$$

So,

$$\sqrt{N}\,(R_N - R_\infty) \to 0 \quad \text{as} \quad N \to \infty.$$

8. **Large singular excitations.** Consider the minimization problem for the functional,

$$I_N(u) = \frac{1}{2} \int\limits_V \left(\bigtriangledown u^2 + h^{4s-2} (\bigtriangledown \bigtriangledown u)^{2s} \right) \, d^2x - \frac{1}{\sqrt{N}} \sum_{a=1}^{N} \left(u\left(r_a^+\right) - u\left(r_a^-\right) \right). \quad (83)$$

Here V is a two-dimensional square of the size 2π, $r_a (a = 1, .., N)$ is a couple of two points r_a^+ and r_a^- in region V, all vectors r_a^+, r_a^- are distributed homogeneously and independently in V. Minimum is sought over all periodic functions on the square V. Such a problem appears in the theory of Kosterlitz-Thouless phase transition [7] and theory of point vortices [17].

The functional is invariant under a shift of function u on a constant. To specify the minimizing element uniquely we put an additional constraint on the set of admissible functions,

$$< u > = \ 0, \quad (84)$$

where we use the notation

$$< \cdot > \equiv \ \frac{1}{|V|} \int\limits_V \cdot \, d^2x.$$

Let us check that formula (20) can be used in this case.

Indeed,

$$(l_0(r), u) = u\left(r^+\right) - u\left(r^-\right)$$

and for operator B we have

$$(Bu, u) = M(l_0, u)^2 = \frac{1}{|V|^2} \int \left(u\left(r^+\right) - u\left(r^-\right) \right)^2 \, d^2r^+ \, d^2r^-$$
$$= \frac{2}{|V|} \int\limits_V u^2(x) \, d^2x.$$

Here we used condition (83). The eigenvalue problem (16) has the eigen-functions $e^{i(kx)}$ where k is a node vector of the lattice with unit spacing. The point $k = 0$ is excluded due to condition (84). The eigenvalues are $\lambda_k = |k|^2 + h^{4s-2}|k|^{4s}$ while $b_k \sim 1$. Let us check the conditions of statement 1. The series $\sum \lambda_k^{-1}$ converges if

$s > 0$. Functions $\overset{\circ}{l}_k(r) = e^{ikr}$ are all bounded. Operator C can be chosen in the following way. Let μ_k be any series such that $\sum \frac{1}{\mu_k} = a < +\infty$. We put $c_k = a\mu_k$. Then

$$\max_r (l_0(r), u)^2 \;\le\; \left(\sum |u_k|\right)^2 \;\le\; \sum \frac{1}{\mu_k} \sum \mu_k u_k^2 \;=\; (Cu, u),$$

or

$$c_k = \mu_k \sum_{s=1}^{\infty} \frac{1}{\mu_s}.$$

To have

$$\alpha c_k \;<\; \lambda_k \;=\; |k|^2 + h^{4s-2}|k|^{4s}$$

we may put $\mu_k = |k|^2 + \varepsilon^{4\sigma-2}|k|^{4\sigma}$ and choose $\sigma < s, \quad \varepsilon < h, \quad \alpha = 1/2a$. For convergence of the series $\sum c_k/\lambda_k$ we have to put additionally,

$$4s - (4\sigma + 1) \;>\; 1,$$

or $s - \sigma \;>\; 1/2$. The minimum integer value of s is $s = 1$.

The term with higher derivatives in (83) can be considered as a means to smear out the δ-type singularity of the excitation. The parameter h has the meaning of a finite core of a singularity. One cannot put $h = 0$ in (83) because the variational problem becomes ill-posed: the functional $I_N(u)$ is not bounded below if $h = 0$. Probability density of minimum values of functional (83) depends on two parameters, N and h; denote it by $f_{N,h}(E)$. Formula (20) allows one to find the limit $\lim_{N\to\infty} f_{N,h}(E)$ for a fixed value of h. One may consider then the limit $\lim_{h\to 0} \lim_{N\to\infty} f_{N,h}(E)$. It turns out that this limit, after some "infinite shift" of energy, exists and can be found explicitly [17]. It remains unclear though whether one can change the order of the limits because the integral (69) does not seem converging uniformly.

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