

## HOMOGENIZATION OF SPECTRAL PROBLEMS IN BOUNDED DOMAINS WITH DOUBLY HIGH CONTRASTS

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**ABSTRACT.** Homogenization of a spectral problem in a bounded domain with a high contrast in both stiffness and density is considered. For a special critical scaling, two-scale asymptotic expansions for eigenvalues and eigenfunctions are constructed. Two-scale limit equations are derived and relate to certain non-standard self-adjoint operators. In particular they explicitly display the first two terms in the asymptotic expansion for the eigenvalues, with a surprising bound for the error of order  $\varepsilon^{5/4}$  proved.

**1. Introduction.** Homogenization for problems with physical properties which are not only highly oscillatory but also highly heterogeneous has long been documented to display unusual effects, such as the memory effects observed by E. Ya. Khruslov [17, 21, 22] and other nonlocal effects, e.g. [27, 3, 11, 12, 1]. In models of particular interest, often referred to as double porosity models, e.g. [4, 10], the parameter of high-contrast  $\delta$  is critically scaled against the periodicity size  $\varepsilon$ ,  $\delta \sim \varepsilon^2$ . Among other approaches, those have been treated by a high-contrast version of the classical method of asymptotic expansions, e.g. [24, 26, 15, 20], and using the techniques of two-scale convergence, e.g. [29, 30, 14, 13]. In particular, for spectral problems in bounded [29] and unbounded [30] periodic domains V.V. Zhikov studied the spectral convergence, introduced two-scale limit operator, developed the techniques of two-scale resolvent convergence and two-scale compactness. In [20] the spectral convergence of eigenvalues in the gaps of Floquet-Bloch spectrum due to defects in double-porosity type media were studied, and [13] supplemented this by the analysis of eigenfunction convergence based on an analysis of a two-scale uniform exponential decay.

In this work we study spectral problems of double-porosity type in a bounded domain  $\Omega$  where the high contrast might occur not only in the “stiffness” coefficient but also in the “density”, and argue that this leads to some interesting new effects. Namely, referring to the next section for precise technical formulations, for the spectral problem

$$-\operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon) = \lambda^\varepsilon \rho_\varepsilon(x) u_\varepsilon, \quad (1)$$

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2000 *Mathematics Subject Classification.* Primary: 35B27; Secondary: 34E .

*Key words and phrases.* Homogenization, periodic media, high-contrasts, eigenvalue asymptotics.

The work was supported by Bath Institute for Complex Systems (EPSRC grant GR/S86525/01), by Nuffield grant NAL/32758 and by EPSRC grant EP/E037607/1.

with Dirichlet boundary conditions on the exterior boundary, most generally, both stiffness<sup>1</sup>  $a_\varepsilon$  and mass density  $\rho_\varepsilon$  are  $\varepsilon$ -periodic,  $a_\varepsilon = \rho_\varepsilon = 1$  in the connected matrix and  $a_\varepsilon \sim \varepsilon^\alpha$ ,  $\rho_\varepsilon \sim \varepsilon^\beta$  in the disconnected inclusions. Outside homogenization, the above resembles problems of vibrations with high contrasts in both density and stiffness, e.g. [5, 6]; within homogenization, dynamic or spectral problems with periodic high contrast densities were studied in e.g. [7, 8, 24, 25, 26]. The double-porosity corresponds to  $\alpha = 2$  and  $\beta = 0$ . For  $\beta \neq 0$ , it is not hard to see that it is  $\alpha = \beta + 2$  when the spectral problems at the macro and micro-scales are coupled in a non-trivial way. To explore this, we choose  $\beta = -1$  and  $\alpha = 1$  and show that this leads to some unusually coupled two-scale limit behaviors of the eigenfunctions and the eigenvalues.

Namely, although the limit behavior of the eigenfunctions  $u_\varepsilon$  is still somewhat similar to that of double porosity, i.e. the two-scale limit is a function of only slow variable  $x$  in the matrix and a function of both  $x$  and the fast variable  $y$  in the inclusions, the limit equations themselves are quite different. We show that there exist asymptotic series of eigenvalues  $\lambda^\varepsilon \sim \lambda_0 + \varepsilon\lambda_1$  with  $\lambda_0$  being an arbitrary eigenvalue of a non-standard self-adjoint “microscopic” inclusion problem (see Theorem 3.1). The eigenfunctions corresponding to  $\lambda_0$  are directly related to the two-scale limit  $w_0(x, y)$  of  $u_\varepsilon$  in the inclusions. In fact,  $\lambda_0$  is either a solution of  $\beta(\lambda_0) = |Q_1|\lambda_0$ , where  $\beta(\lambda)$  is a function introduced by Zhikov [29]<sup>2</sup>, or is an eigenvalue of the Dirichlet Laplacian in the inclusion  $Q_0$  with a zero mean eigenfunction. In the matrix  $Q_1$ ,  $u_\varepsilon \sim v^0(x)$ , where  $v^0$  is an eigenfunction of the homogenized operator in  $\Omega$ , whose eigenvalue  $\nu$  determines the second term  $\lambda_1$  in the asymptotics of  $\lambda^\varepsilon$ , see (57). This is first derived via formal asymptotic expansions, but then we prove a non-standard error bound:

$$|\lambda^\varepsilon - \lambda_0 - \varepsilon\lambda_1| \leq C\varepsilon^{5/4},$$

see Theorems 4.6 & 4.7. The proof employs a combination of a high contrast boundary layer analysis with maximum principle arguments and estimates in Hilbert spaces with  $\varepsilon$ -dependent weights. We finally briefly discuss further refinement of the results via the technique of two-scale convergence. Namely, some version of the two-scale compactness result holds, cf. [29], indicating at the presence of gaps in the spectrum for small enough  $\varepsilon$ , see Proposition 1.

The paper is organized as follows. The next section formulates the problem and introduces necessary notation, Section 3 executes formal asymptotic expansion and derives associated homogenized equations. Section 4 proves the error bounds and Section 5 discusses the two-scale convergence approach. Some technical details are assembled in the appendices.

**2. Problem statement and notations.** We consider a model of eigenvibrations for a body occupying a bounded domain  $\Omega$  in  $\mathbb{R}^d$  ( $d = 2, 3, \dots$ ) containing a periodic array of small inclusions, see Figure 1. The size of inclusions is controlled by a small positive parameter  $\varepsilon$ ,  $\varepsilon \rightarrow 0$ . First we introduce necessary notation.

Let  $Q = (0, 1)^d$  be a reference periodicity cell in  $\mathbb{R}^d$ . Let  $\tilde{Q}_0$  be a periodic set of “inclusions”, i.e.  $\tilde{Q}_0 + m = \tilde{Q}_0$ ,  $\forall m \in \mathbb{Z}^d$ , and  $Q_0 = \tilde{Q}_0 \cap Q$  is a reference inclusion with  $C^2$ -smooth boundary  $\Gamma$  lying inside  $Q$ , see Figure 1. Let  $Q_1 = Q \setminus \tilde{Q}_0$ ,  $\tilde{Q}_1 =$

<sup>1</sup> For this scalar model the adopted terminology from linear elasticity is adequate for anti-plane shear; however the model also admits other physical realizations.

<sup>2</sup>See also [9].

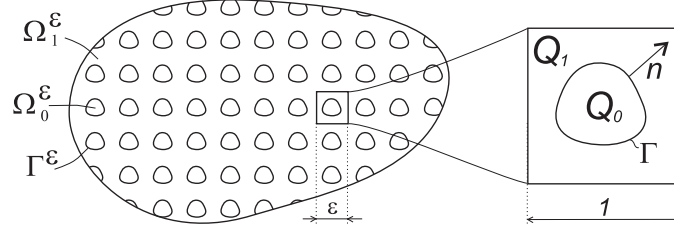


FIGURE 1. The geometry and the periodicity cell

$\mathbb{R}^d \setminus \overline{Q_0}$ ,  $\tilde{\Gamma} = \partial\tilde{Q}_0 = \partial\tilde{Q}_1$ . Introducing  $y = x/\varepsilon$  we refer to  $y$  as to a fast variable, as opposes to the slow variable  $x$ . We denote  $\Omega_0^\varepsilon := \Omega \cap \varepsilon\tilde{Q}_0$ ,  $\Omega_1^\varepsilon := \Omega \cap \varepsilon\tilde{Q}_1 = \Omega \setminus \overline{\Omega_0^\varepsilon}$ ,  $\Gamma^\varepsilon := \varepsilon\tilde{\Gamma} \cap \Omega$ , see Figure 1. The trace on  $\Gamma^\varepsilon$  of a function  $f : \Omega_j^\varepsilon \rightarrow \mathbb{R}^d$  is denoted by  $f|_j$ . Denote by  $n = n(y)$  the outer unit normal to  $Q_0$  on its boundary  $\Gamma$  and by  $n_\varepsilon = n_\varepsilon(x)$  the outer unit normal to  $\Omega_0^\varepsilon$  at  $x \in \Gamma^\varepsilon$ .

Let stiffness  $a_\varepsilon$  and density  $\rho_\varepsilon$  be as follows

$$a_\varepsilon(x) = \begin{cases} 1, & x \in \Omega_1^\varepsilon \\ \varepsilon, & x \in \Omega_0^\varepsilon \end{cases} \quad \text{and} \quad \rho_\varepsilon(x) = \begin{cases} 1, & x \in \Omega_1^\varepsilon \\ \varepsilon^{-1}, & x \in \Omega_0^\varepsilon \end{cases}$$

with a small positive  $\varepsilon$ .

We study the asymptotic behavior of spectral problem

$$\int_\Omega a_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla \phi \, dx - \lambda^\varepsilon \int_\Omega \rho_\varepsilon(x) u_\varepsilon \phi \, dx = 0, \quad \forall \phi \in H_0^1(\Omega); \quad u_\varepsilon \in H_0^1(\Omega), \quad (2)$$

as  $\varepsilon \rightarrow 0$ . If  $\Gamma$  and  $\partial\Omega$  are smooth enough then variational problem (2) can be equivalently represented in a classical form

$$-\operatorname{div}(a_\varepsilon(x) \nabla u_\varepsilon) = \lambda^\varepsilon \rho_\varepsilon(x) u_\varepsilon, \quad x \in \Omega, \quad (3)$$

$$u_\varepsilon|_{\partial\Omega} = 0, \quad (4)$$

implying that at the interfaces  $\Gamma^\varepsilon$  the transmission conditions are satisfied

$$u_\varepsilon|_1 = u_\varepsilon|_0, \quad \frac{\partial u_\varepsilon}{\partial n_\varepsilon}|_1 = \varepsilon \frac{\partial u_\varepsilon}{\partial n_\varepsilon}|_0. \quad (5)$$

For an operator formulation of problem (2) see Section 4.1.

**3. Formal asymptotic expansions.** We seek formal asymptotic expansions for the eigenvalues  $\lambda^\varepsilon$  and eigenfunctions  $u_\varepsilon$  in the form

$$\lambda^\varepsilon \sim \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots, \quad (6)$$

$$u_\varepsilon(x) \sim \begin{cases} v_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 v_2\left(x, \frac{x}{\varepsilon}\right) + \dots, & x \in \Omega_1^\varepsilon, \\ w_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(x, \frac{x}{\varepsilon}\right) + \dots, & x \in \Omega_0^\varepsilon. \end{cases} \quad (7)$$

Here all the functions  $v_j(x, y)$ ,  $w_j(x, y)$ ,  $j \geq 0$ , are required to be periodic in the “fast” variable  $y$ ;  $v_0$  and  $w_0$  are not simultaneously identically zero

$$v_0^2 + w_0^2 \not\equiv 0. \quad (8)$$

In a standard way, the ansatz (6), (7) is then formally substituted into (3)–(5). In particular, from (3), for  $(x, y) \in \Omega \times Q_1$ , we obtain

$$-\Delta_y v_0 = 0, \quad (9)$$

$$-\Delta_y v_1 = 2 \frac{\partial^2 v_0}{\partial x_j \partial y_j}, \quad (10)$$

$$-\Delta_y v_2 = 2 \frac{\partial^2 v_1}{\partial x_j \partial y_j} + \Delta_x v_0 + \lambda_0 v_0, \quad (11)$$

(with  $\Delta_y$  and  $\Delta_x$  denoting the Laplace operators in  $y$  and  $x$ , respectively, and summation implied henceforth with respect to repeated indices). For  $(x, y) \in \Omega \times Q_0$  we have

$$-\Delta_y w_0 = \lambda_0 w_0, \quad (12)$$

$$-\Delta_y w_1 = 2 \frac{\partial^2 w_0}{\partial x_j \partial y_j} + \lambda_1 w_0 + \lambda_0 w_1, \quad (13)$$

$$-\Delta_y w_2 = 2 \frac{\partial^2 w_1}{\partial x_j \partial y_j} + \Delta_x w_0 + \lambda_2 w_0 + \lambda_1 w_1 + \lambda_0 w_2. \quad (14)$$

Further, the first of conditions (5) transforms to

$$w_j(x, y) \Big|_{y \in \Gamma} = v_j(x, y) \Big|_{y \in \Gamma}, \quad x \in \Omega, \quad j = 0, 1, \dots \quad (15)$$

Similarly, the other transmission condition (5) yields

$$\frac{\partial v_0}{\partial n_y} \Big|_{y \in \Gamma} = 0, \quad (16)$$

$$\frac{\partial v_1}{\partial n_y} \Big|_{y \in \Gamma} = - \frac{\partial v_0}{\partial n_x} \Big|_{y \in \Gamma} + \frac{\partial w_0}{\partial n_y} \Big|_{y \in \Gamma}, \quad (17)$$

$$\frac{\partial v_2}{\partial n_y} \Big|_{y \in \Gamma} = - \frac{\partial v_1}{\partial n_x} \Big|_{y \in \Gamma} + \frac{\partial w_1}{\partial n_y} \Big|_{y \in \Gamma} + \frac{\partial w_0}{\partial n_x} \Big|_{y \in \Gamma}, \quad (18)$$

where  $\frac{\partial}{\partial n_y} := n_y \cdot \nabla_y$ ,  $\frac{\partial}{\partial n_x} := n_x \cdot \nabla_x$ , with  $\nabla_y$  and  $\nabla_x$  standing for gradients in  $y$  and  $x$ , respectively. The above has employed the identity

$$\frac{\partial u}{\partial n_\varepsilon} \left( x, \frac{x}{\varepsilon} \right) = \varepsilon^{-1} \frac{\partial u}{\partial n_y}(x, y) + \frac{\partial u}{\partial n_x}(x, y), \quad y = \frac{x}{\varepsilon}. \quad (19)$$

Finally, (4) suggests

$$v_0 \Big|_{x \in \partial \Omega} = w_0 \Big|_{x \in \partial \Omega} = 0. \quad (20)$$

(The boundary layer effect does not generally permit satisfying (4) by  $v_j$  and  $w_j$  for  $j \geq 1$ , as is clarified later.)

Combining (9) and (16), together with the periodicity conditions in  $y$ , implies that  $v_0$  is a constant with respect to  $y$ , i.e.

$$v_0(x, y) \equiv v^0(x).$$

Then, (10) and (17) form the following boundary value problem for  $v_1$

$$-\Delta_y v_1(x, y) = 0 \quad \text{in } \Omega \times Q_1, \quad \frac{\partial v_1}{\partial n_y} \Big|_{y \in \Gamma} = - \frac{\partial v^0}{\partial n_x} \Big|_{y \in \Gamma} + \frac{\partial w_0}{\partial n_y} \Big|_{y \in \Gamma}. \quad (21)$$

The latter is solvable if and only if

$$\int_{\Gamma} \frac{\partial w_0}{\partial n_y} dy = 0. \tag{22}$$

Considering next (12) and (15) gives

$$-\Delta_y w_0 = \lambda_0 w_0 \quad \text{in } \Omega \times Q_0, \quad w_0(x, y) \Big|_{y \in \Gamma} = v^0(x). \tag{23}$$

Since

$$\int_{\Gamma} \frac{\partial w_0}{\partial n_y} dy = \int_{Q_0} \Delta_y w_0 dy = -\lambda_0 \int_{Q_0} w_0 dy,$$

condition (22) implies that

$$\lambda_0 \langle w_0 \rangle = 0, \tag{24}$$

where

$$\langle u \rangle := \int_{Q_0} u(y) dy.$$

We notice that (23)–(24) together with (8) constitute restrictions on possible values of  $\lambda_0$ . Those are described by Theorem 3.1 below. Before, let us consider an auxiliary Dirichlet problem

$$-\Delta_y \phi = \lambda^D \phi \quad \text{in } Q_0, \quad \phi \Big|_{\Gamma} = 0. \tag{25}$$

Let  $\{\lambda_j^D\}_{j=1}^\infty$  be eigenvalues for (25), labeled in the ascending order counting for the multiplicities, and let  $\{\phi_j\}_{j=1}^\infty$  be the corresponding eigenfunctions, orthonormal in  $L_2(Q_0)$ , i.e.

$$\int_{Q_0} \phi_j \phi_k dy = \delta_{jk},$$

where  $\delta_{jk}$  is Kronecker’s delta. Denote by  $\sigma_D$  the spectrum of (25):  $\sigma_D = \bigcup_{j=1}^\infty \lambda_j^D$ .

We additionally introduce the following auxiliary problem:

$$-\Delta_y \eta = \lambda_0 \eta \quad \text{in } Q_0, \quad \eta(y) \Big|_{y \in \Gamma} = 1. \tag{26}$$

Notice that (26) is solvable if and only if either  $\lambda_0 \notin \sigma_D$  or  $\lambda_0 = \lambda_j^D$  with all the associated eigenfunctions  $\phi_j$  having zero mean<sup>3</sup>,  $\langle \phi_j \rangle = 0$ . In the former case  $\eta$  is determined uniquely and (23) implies  $w_0(x, y) = v^0(x)\eta(y)$ . In the latter case  $\eta$  is determined up to an arbitrary eigenfunction  $\phi_j$  associated with  $\lambda_j^D$ , however  $\langle \eta \rangle$  is determined uniquely.

By direct inspection, (23), (24) has a non-trivial solution  $(v^0, w_0)$ , i.e. with (8) holding, if and only if  $\lambda_0$  is an eigenvalue of following problem:

$$-\Delta_y \zeta = \lambda_0 \zeta \quad \text{in } Q_0, \quad \zeta(y) \Big|_{y \in \Gamma} = \text{constant}, \quad \lambda_0 \langle \zeta \rangle = 0. \tag{27}$$

We claim that (27) corresponds to a self-adjoint operator associated with the (symmetric, closed, densely defined, bounded from below) Dirichlet form

$$\alpha(\zeta, h) := \int_{Q_0} \nabla \zeta \cdot \nabla h dy \tag{28}$$

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<sup>3</sup>We remark that the case of eigenfunctions with zero mean is known to be not a “generic” case, i.e. it is unstable via a small perturbation of the shape of  $Q_0$ , see e.g. discussion in [18] and further references therein.

with domain

$$D(\alpha) := \{h \in H^1(Q_0) : h|_{y \in \Gamma} = \text{constant}\}. \quad (29)$$

**Theorem 3.1.** *The problem (27) is equivalent to an eigenvalue problem for a self-adjoint operator in  $L_2(Q_0)$  associated with (28)–(29). The spectrum of (27) is a countable set of real non-negative eigenvalues (of finite multiplicity) with the only accumulation point at  $+\infty$ , with the eigenfunctions complete in  $L_2(Q_0)$  and those corresponding to different  $\lambda_0$  mutually orthogonal.*

*The spectrum consists of all the eigenvalues  $\lambda^D$  of problem (25) with a zero mean eigenfunction and all the solutions of the equation*

$$B(\lambda_0) := \lambda_0 \langle \eta \rangle = \lambda_0 \left( |Q_0| + \lambda_0 \sum_{j=1}^{\infty} \frac{\langle \phi_j \rangle^2}{\lambda_j^D - \lambda_0} \right) = 0 \quad (30)$$

(which are hence all real non-negative). In (30) the summation is with respect to only those  $\lambda_j^D$  for which there exists an eigenfunction with a non-zero mean.

The associated eigenfunctions  $\zeta$  are either proportional to  $\eta$  as in (26) or are eigenfunctions of (25) with zero mean.

*Proof.* In the weak formulation of the eigenvalue problem associated with (28)–(29),

$$\int_{Q_0} \nabla \zeta \cdot \nabla h \, dy = \lambda_0 \int_{Q_0} \zeta h \, dy, \quad \forall h \in D(\alpha), \quad (31)$$

we first set  $h$  to be an arbitrary function from  $C_0^\infty(Q_0)$  which implies  $-\Delta_y \zeta = \lambda_0 \zeta$  in  $Q_0$ , and then set  $h \equiv 1$  yielding  $\lambda_0 \langle \zeta \rangle = 0$ . Conversely, multiplying the first equation of (27) by  $h \in D(\alpha)$  and integrating by parts we obtain (31). Further, since  $D(\alpha)$  is compactly embedded into  $L_2(Q_0)$ , each eigenvalue has a finite multiplicity, the set of all eigenfunctions  $\zeta$  is complete in  $L_2(Q_0)$  and those corresponding to different  $\lambda_0$  are mutually orthogonal.

Obviously, the spectrum of (27) includes those and only those eigenvalues of (25) which have an eigenfunction  $\phi_j$  with zero mean. In this case corresponding eigenfunctions of (27) are given by  $\zeta_j = C\phi_j$ ,  $C \neq 0$ . If  $\lambda_j^D$  does not have a zero-mean eigenfunction, then the solvability of (27) requires  $\zeta|_{y \in \Gamma} = 0$  implying  $\zeta \equiv 0$ .

Considering other possibilities, fix  $\lambda_0$  outside  $\sigma_D$  and let  $\eta$  be the unique solution of (26). Then  $\lambda_0$  is an eigenvalue of (27) if and only if

$$\lambda_0 \langle \eta \rangle = 0, \quad (32)$$

with corresponding eigenfunction given by  $\zeta(y) = C\eta(y)$ ,  $C \neq 0$ .

Via the spectral decomposition, the solution to (26) is found to be, cf. [29]:

$$\eta(y) = 1 + \lambda_0 \sum_{j=1}^{\infty} \frac{\langle \phi_j \rangle}{\lambda_j^D - \lambda_0} \phi_j(y). \quad (33)$$

Substituting (33) further into (32) yields (30).  $\square$

The formula (30) can be transformed to read

$$B(\lambda_0) = \beta(\lambda_0) - |Q_1| \lambda_0 = 0, \quad (34)$$

where function  $\beta(\lambda)$  has been introduced by Zhikov [29] (see also [9]),

$$\beta(\lambda) = \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle \phi_j \rangle^2}{\lambda_j^D - \lambda}, \quad (35)$$

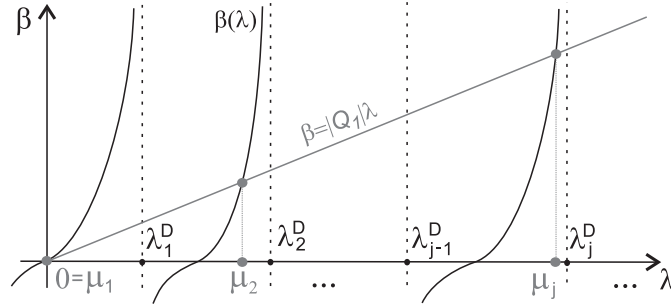


FIGURE 2. The limit eigenvalues  $\lambda_0 = \mu_j$

see Figure 2 ( $\lambda_j^D$  are re-labeled to include only those with a nonzero-mean eigenfunction). This implies that  $\lambda_0$  is either a solution to the nonlinear equation

$$\beta(\lambda) = |Q_1|\lambda, \tag{36}$$

as visualized on Figure 2, or is an eigenvalue of (25) with a zero mean eigenfunction.

**Remark 1.** If  $Q_0$  is a ball of radius  $0 < a < 1/2$ , i.e  $Q_0 = B_a = \{y : |y| < a\} + y_0$ , then we have an explicit representation for  $\beta(\lambda)$ . Indeed, for  $\lambda_0 \notin \sigma_D$  the solution of (26) is radially symmetric and (placing the origin in the ball’s center) reads

$$\eta(y) = |y|^{\frac{2-d}{2}} J_{\frac{d-2}{2}}(\lambda_0^{1/2}|y|) \left( |a|^{\frac{2-d}{2}} J_{\frac{d-2}{2}}(\lambda_0^{1/2}a) \right)^{-1},$$

where  $J_{\frac{d-2}{2}}(z)$  is Bessel’s function. Further, we have

$$\begin{aligned} B(\lambda_0) &= \lambda_0 \langle \eta \rangle = - \int_{\partial B_a} \frac{\partial \eta}{\partial n_y} dy \\ &= -\frac{1}{a} |\Gamma| \left( 1 - d/2 + a\lambda_0^{1/2} J'_{\frac{d-2}{2}}(\lambda_0^{1/2}a) / J_{\frac{d-2}{2}}(\lambda_0^{1/2}a) \right). \end{aligned}$$

Using (35), (33) we obtain

$$\beta(\lambda) = \lambda(1 - |B_a|) - \frac{1}{a} |\Gamma| \left( 1 - d/2 + a\lambda^{1/2} J'_{\frac{d-2}{2}}(\lambda^{1/2}a) / J_{\frac{d-2}{2}}(\lambda^{1/2}a) \right).$$

In particular, for  $d = 3$  we have,

$$\begin{aligned} B(\lambda_0) &= \lambda_0 \langle \eta \rangle = 4\pi a \left( 1 - a\lambda_0^{1/2} \cotan(\lambda_0^{1/2}a) \right), \\ \beta(\lambda) &= \lambda(1 - 4\pi a^3/3) + 4\pi a \left( 1 - a\lambda^{1/2} \cotan(\lambda^{1/2}a) \right). \end{aligned}$$

We next explore in detail the further steps in the method of asymptotic expansions, to determine  $v^0$ , etc. Let us consider a  $K$ -dimensional eigenspace ( $K \geq 1$ ) for a given eigenvalue  $\lambda_0$  of (27), and let  $\zeta_1, \dots, \zeta_K$  be associated linearly independent eigenfunctions. Then, (23) and (24) imply

$$w_0(x, y) = \sum_{k=1}^K c_k(x) \zeta_k(y). \tag{37}$$

Following Theorem 3.1 we distinguish two cases:

(a)  $\lambda_0 \notin \sigma_D$ . In this case (26) and (23) dictate

$$w_0(x, y) = v^0(x)\eta(y), \quad (38)$$

and (8) implies  $v^0 \not\equiv 0$ .

(b)  $\lambda_0 \in \sigma_D$ . The latter means  $\lambda_0 = \lambda_j^D$  for some  $j$ . This includes two further possibilities:

(i) The eigenspace of (25) has an eigenfunction  $\phi_j^*$  with a non-zero mean. Since the solvability conditions for (23) include

$$v^0(x)\langle \phi_j^* \rangle = 0, \quad (39)$$

necessarily  $v^0 \equiv 0$ . Moreover, with  $K_D$  denoting the multiplicity of  $\lambda_j^D$  as of the eigenvalue of the Dirichlet problem (25), necessarily  $K_D \geq 2$ : if  $K_D = 1$  then  $w_0 = C(x)\phi_j^*$  and thus (24) implies  $C(x) \equiv 0$  and  $w_0 \equiv 0$  contradicting to (8). Hence  $w_0$  is given by (37) with  $K = K_D - 1$ , with  $\zeta_k$ ,  $k = 1, \dots, K$  being linearly independent eigenfunctions of (25) with zero mean (such  $K$  eigenfunctions do exist).

(ii) All of the eigenfunctions corresponding  $\lambda_j^D$  have a zero mean. In this case, if  $\langle \eta \rangle \neq 0$  i.e.  $B(\lambda_0) \neq 0$  then  $v^0 \equiv 0$  and  $w_0$  is again given by (37), with  $K = K_D$ . Alternatively, i.e. if  $B(\lambda_0) = 0$  then (37) gives  $w_0$  with  $K = K_D + 1$  and  $\zeta_{K_D+1} = \eta$  is an arbitrary solution of (26), and  $v^0(x) = c_K(x)$  an arbitrary function sufficiently smooth and vanishing at  $\partial\Omega$ .

We henceforth consider Cases (a) and (b) separately.

**3.1. Case (a):**  $\lambda_0 \notin \sigma_D$ . In this case  $\lambda_0$  are solutions of (36). There is a countable set of  $\lambda_0 = \mu_j$ ,  $j = 1, 2, \dots$  as Figure 2 illustrates. Note that this includes  $\lambda_0 = 0$ . Function  $\beta$  blows up at the points  $\lambda_j^D$ , which are eigenvalues of (25) having an eigenfunction with a non-zero mean, monotonically increasing between such points. It also directly follows from (35) that  $\beta(\lambda) > |Q_1|\lambda$  for  $\lambda \in (0, \lambda_1^D)$ , implying  $\lambda_1^D < \mu_2 < \lambda_2^D$ . Let  $\lambda_0$  satisfying (36) be fixed.

We consider problem (21) taking into account (38), i.e.

$$-\Delta_y v_1(x, y) = 0 \quad \text{in } \Omega \times Q_1, \quad \frac{\partial v_1}{\partial n_y} \Big|_{y \in \Gamma} = -\frac{\partial v^0}{\partial n_x} \Big|_{y \in \Gamma} + v^0(x) \frac{\partial \eta}{\partial n_y} \Big|_{y \in \Gamma}, \quad (40)$$

where  $\eta(y)$  solves (26) and is given by (33). Hence  $v_1$  is a solution to a problem depending linearly on  $v^0$  and  $\nabla_x v^0$ , implying

$$v_1(x, y) = v^0(x)\mathcal{N}(y) + \frac{\partial v^0}{\partial x_j} N_j(y) + v_1^*(x), \quad (41)$$

with an arbitrary function  $v_1^*(x)$ . The choice of  $v_1^*$  does not affect the subsequent constructions, so we set for simplicity  $v_1^* \equiv 0$ . In (41) functions  $N_j$  and  $\mathcal{N}$  are solutions to the periodic problems

$$\Delta_y N_j(y) = 0 \quad \text{in } Q_1, \quad \frac{\partial N_j}{\partial n_y} \Big|_{y \in \Gamma} = -n_j(y), \quad (42)$$

and

$$\Delta_y \mathcal{N}(y) = 0 \quad \text{in } Q_1, \quad \frac{\partial \mathcal{N}}{\partial n_y} \Big|_{y \in \Gamma} = \frac{\partial \eta}{\partial n_y} \Big|_{y \in \Gamma}, \quad (43)$$

where  $n_j(y)$  are components of the vector  $n = (n_1(y), \dots, n_d(y))$ . Henceforth the periodicity conditions are implied with respect to  $y$ .



Solvability of (43) requires

$$\int_{\Gamma} \frac{\partial \eta}{\partial n_y} dy = 0,$$

which is equivalent to (32) and is hence already assured. Since the solutions of (42) and (43) are unique up to an arbitrary constant, we fix those by choosing

$$\int_{Q_1} N_j(y) dy = \int_{Q_1} \mathcal{N}(y) dy = 0.$$

We next consider the problem for  $w_1$ , which from (13) and (15) combined with (38) and (41) reads

$$-\Delta_y w_1 - \lambda_0 w_1 = \lambda_1 v^0 \eta + 2 \frac{\partial v^0}{\partial x_j} \frac{\partial \eta}{\partial y_j} \quad \text{in } \Omega \times Q_0, \tag{44}$$

$$w_1 \Big|_{y \in \Gamma} = v^0 \mathcal{N}(y) \Big|_{y \in \Gamma} + \frac{\partial v^0}{\partial x_j} N_j(y) \Big|_{y \in \Gamma}. \tag{45}$$

Since the problem depends linearly on  $v^0$ ,  $\lambda_1 v^0$  and  $\frac{\partial v^0}{\partial x_j}$ , the solution admits representation

$$w_1(x, y) = \frac{\partial v^0}{\partial x_j}(x) \mathcal{M}_j(y) + v^0(x) \mathcal{P}(y) + \lambda_1 v^0(x) \mathcal{R}(y), \tag{46}$$

where functions  $\mathcal{M}_j$ ,  $\mathcal{P}$  and  $\mathcal{R}$  are solutions to the problems

$$-\Delta_y \mathcal{M}_j - \lambda_0 \mathcal{M}_j = 2 \frac{\partial \eta}{\partial y_j}(y) \quad \text{in } Q_0, \quad \mathcal{M}_j \Big|_{\Gamma} = N_j \Big|_{\Gamma}, \tag{47}$$

$$-\Delta_y \mathcal{P} - \lambda_0 \mathcal{P} = 0 \quad \text{in } Q_0, \quad \mathcal{P} \Big|_{\Gamma} = \mathcal{N} \Big|_{\Gamma}, \tag{48}$$

$$-\Delta_y \mathcal{R} - \lambda_0 \mathcal{R} = \eta(y) \quad \text{in } Q_0, \quad \mathcal{R} \Big|_{\Gamma} = 0. \tag{49}$$

Since by the assumption  $\lambda_0 \notin \sigma_D$ , all the problems (47) – (49) are uniquely solvable.

The problem for  $v_2$  is in turn given by (11) and (18), whose solvability condition hence reads

$$\int_{Q_1} \left( \Delta_x v^0 + \lambda_0 v^0 + 2 \frac{\partial^2 v_1}{\partial x_j \partial y_j} \right) dy = \int_{\Gamma} \left( - \frac{\partial v_1}{\partial n_x} + \frac{\partial w_1}{\partial n_y} + \frac{\partial w_0}{\partial n_x} \right) dy, \tag{50}$$

with functions  $v_1$ ,  $w_1$  and  $w_0$  given by (41), (46) and (38), respectively.

Appendix A provides a detailed calculation showing that the above yields the following equations for  $v^0$ :

$$- \operatorname{div} (A^{\text{hom}} \nabla_x v^0) = \nu(\lambda_1) v^0 \quad \text{in } \Omega, \tag{51}$$

$$v^0 \Big|_{\partial \Omega} = 0. \tag{52}$$

Here  $A^{\text{hom}} = \left( A_{jk}^{\text{hom}} \right)_{j,k=1}^d$  is the classical homogenized matrix for periodic perforated domains, see e.g. [19]

$$A_{jk}^{\text{hom}} = |Q_1| \delta_{jk} + \int_{Q_1} \frac{\partial N_k}{\partial y_j} dy; \tag{53}$$

$$\nu(\lambda_1) = \mathcal{C} \lambda_1 + \lambda_0 \left( |Q_1| + \int_{Q_0} \mathcal{P} dy \right), \tag{54}$$

where

$$\mathcal{C} := \int_{Q_0} \eta^2 dy > 0. \tag{55}$$

Note that the problem (51)–(52) involves  $\nu = \nu(\lambda_1)$  as a spectral parameter.

The spectrum of (51)–(52) consists of a countable set of eigenvalues

$$0 < \nu_1 < \nu_2 \leq \dots \leq \nu_m \leq \dots \rightarrow +\infty. \tag{56}$$

Corresponding eigenfunctions  $v_m^0$  form an orthonormal basis in  $L_2(\Omega)$ ,

$$\int_{\Omega} v_l^0 v_m^0 dx = \delta_{lm}.$$

Fixing an eigenvalue  $\nu$  of (51), (52) with corresponding eigenfunction  $v^0$  of unit norm in  $L_2(\Omega)$ , according to (54) we find

$$\lambda_1 = \mathcal{C}^{-1} \left( \nu - \lambda_0 \left( |Q_1| + \int_{Q_0} \mathcal{P} dy \right) \right). \tag{57}$$

The following diagram summarizes the algorithm for constructing the first terms of the asymptotic expansions (for the case  $\lambda_0 \notin \sigma_D$ )

$$\left. \begin{array}{l} N_j \xrightarrow{(53)} A^{\text{hom}} \xrightarrow{(51),(52)} \nu, v^0 \\ \lambda_0 \xrightarrow{(26)} \eta \xrightarrow{(43)} \mathcal{N} \end{array} \right\} \begin{array}{l} \xrightarrow{(57)} \lambda_1 \\ \xrightarrow{(41)} v_1 \\ \xrightarrow{(47)-(49)} \mathcal{M}_j, \mathcal{P}, \mathcal{R} \\ \xrightarrow{(38)} w_0 \end{array} \right\} \xrightarrow{(46)} w_1 \xrightarrow{(11),(18)} v_2.$$

We can additionally construct  $w_2$  from (14) and (15), whose unique solution exists for any choice of  $\lambda_2$ . For purposes of the justification of the first two terms in the asymptotics (the next section) it is sufficient to set  $\lambda_2 = 0$  and fix the corresponding solution  $w_2$ .

This completes constructing a formal asymptotic approximation, which we now summarize. We introduce an approximate eigenvalue

$$\Lambda_\varepsilon = \lambda_0 + \varepsilon \lambda_1, \tag{58}$$

and corresponding approximate eigenfunction

$$W_\varepsilon(x) = \begin{cases} v^0(x) + \varepsilon v_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^2 v_2 \left( x, \frac{x}{\varepsilon} \right), & x \in \Omega_1^\varepsilon, \\ w_0 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon w_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^2 w_2 \left( x, \frac{x}{\varepsilon} \right), & x \in \Omega_0^\varepsilon. \end{cases} \tag{59}$$

The essence of the above formal asymptotic construction is that the action of differential operator  $\mathcal{A}^\varepsilon$  on  $W_\varepsilon$  defined by

$$\mathcal{A}^\varepsilon W_\varepsilon := \operatorname{div} (a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon \tag{60}$$

produces a small right-hand side in both  $\Omega_1^\varepsilon$  and  $\Omega_0^\varepsilon$ , and on the interface  $\Gamma^\varepsilon$  in the following sense.

- Lemma 3.2.**
- (i)  $\max_{\Omega_1^\varepsilon} | \operatorname{div} (a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon | \leq C\varepsilon.$
  - (ii)  $\max_{\Omega_0^\varepsilon} | \operatorname{div} (a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon | \leq C\varepsilon^2.$
  - (iii)  $\max_{\Gamma^\varepsilon} \left| a_\varepsilon \frac{\partial W_\varepsilon}{\partial n} \Big|_0 - a_\varepsilon \frac{\partial W_\varepsilon}{\partial n} \Big|_1 \right| \leq C\varepsilon^2.$

*Proof.* (i) The function  $W_\varepsilon$  explicitly depends on the fast and slow variables in  $\Omega_1^\varepsilon$ . Therefore, we obtain

$$\begin{aligned} & \operatorname{div}(a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon \\ &= \left( \varepsilon^{-2} \Delta_y + \varepsilon^{-1} 2 \frac{\partial^2}{\partial x_j \partial y_j} + \Delta_x + \lambda_0 + \varepsilon \lambda_1 \right) (v^0 + \varepsilon v_1 + \varepsilon^2 v_2) \Big|_{y=\frac{x}{\varepsilon}} \\ &= \left\{ \varepsilon^{-1} \Delta_y v_1(x, y) + \varepsilon^0 \left( \Delta_y v_2 + 2 \frac{\partial^2 v_1}{\partial x_j \partial y_j} + \Delta_x v^0 + \lambda_0 v^0 \right) \right. \\ &\quad + \varepsilon^1 \left( 2 \frac{\partial^2 v_2}{\partial x_j \partial y_j} + \Delta_x v_1 + \lambda_1 v^0 + \lambda_0 v_1 \right) \\ &\quad \left. + \varepsilon^2 (\Delta_x v_2 + \lambda_1 v_1 + \lambda_0 v_2) + \varepsilon^3 \lambda_1 v_2 \right\} \Big|_{y=\frac{x}{\varepsilon}}. \end{aligned} \tag{61}$$

Since  $v_1$  is a solution to (40), the coefficient of  $\varepsilon^{-1}$  vanishes. The same is with the coefficient of  $\varepsilon^0$  since  $v_2$  satisfies (11). Functions  $v^0$ ,  $v_1$  and  $v_2$  are solutions of elliptic problems with smooth enough coefficients to guarantee belonging solutions to  $C^2$ . Thus, maxima for coefficients of  $\varepsilon^1$ ,  $\varepsilon^2$  and  $\varepsilon^3$  in (61) exist.

(ii) Similarly, in  $\Omega_0^\varepsilon$

$$\begin{aligned} & \operatorname{div}(a_\varepsilon \nabla W_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon \\ &= \left( \varepsilon^{-1} \Delta_y + 2 \frac{\partial^2}{\partial x_j \partial y_j} + \varepsilon \Delta_x + \varepsilon^{-1} \lambda_0 + \lambda_1 \right) (w_0 + \varepsilon w_1 + \varepsilon^2 w_2) \Big|_{y=\frac{x}{\varepsilon}} \\ &= \left\{ \varepsilon^{-1} (\Delta_y w_0 + \lambda_0 w_0) + \varepsilon^0 \left( \Delta_y w_1 + 2 \frac{\partial^2 w_0}{\partial x_j \partial y_j} + \lambda_0 w_1 + \lambda_1 w_0 \right) \right. \\ &\quad + \varepsilon^1 \left( \Delta_y w_2 + 2 \frac{\partial^2 w_1}{\partial x_j \partial y_j} + \Delta_x w_0 + \lambda_0 w_2 + \lambda_1 w_1 \right) \\ &\quad \left. + \varepsilon^2 \left( 2 \frac{\partial^2 w_2}{\partial x_j \partial y_j} + \Delta_x w_1 + \lambda_1 w_2 \right) + \varepsilon^3 \Delta_x w_2 \right\} \Big|_{y=\frac{x}{\varepsilon}}. \end{aligned}$$

Since  $w_0(x, y) = v^0(x) \eta(y)$  is chosen according to (26), the coefficient of  $\varepsilon^{-1}$  vanishes. The coefficient of  $\varepsilon^0$  vanishes due to (44). Further,  $w_2$  satisfies (14) with  $\lambda_2 = 0$  and thus the coefficient of  $\varepsilon^1$  is zero as well. Since  $w_1$  and  $w_2$  are solutions of elliptic problems with smooth enough coefficients, the maxima of the coefficients of  $\varepsilon^2$  and  $\varepsilon^3$  exist.

(iii) Using (19), we obtain

$$\begin{aligned} & a_\varepsilon \frac{\partial W_\varepsilon}{\partial n} \Big|_0 - a_\varepsilon \frac{\partial W_\varepsilon}{\partial n} \Big|_1 = \left( \frac{\partial}{\partial n_y} + \varepsilon \frac{\partial}{\partial n_x} \right) (w_0 + \varepsilon w_1 + \varepsilon^2 w_2) \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}} \\ &\quad - \left( \varepsilon^{-1} \frac{\partial}{\partial n_y} + \frac{\partial}{\partial n_x} \right) (v^0 + \varepsilon v_1 + \varepsilon^2 v_2) \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}} \\ &= \varepsilon^0 \left( \frac{\partial w_0}{\partial n_y} - \frac{\partial v_1}{\partial n_y} - \frac{\partial v^0}{\partial n_x} \right) \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}} + \varepsilon^1 \left( \frac{\partial w_1}{\partial n_y} + \frac{\partial w_0}{\partial n_x} - \frac{\partial v_2}{\partial n_y} - \frac{\partial v_1}{\partial n_x} \right) \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}} \\ &\quad + \varepsilon^2 \left( \frac{\partial w_2}{\partial n_y} + \frac{\partial w_1}{\partial n_x} - \frac{\partial v_2}{\partial n_x} \right) \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}} + \varepsilon^3 \frac{\partial w_2}{\partial n_x} \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}}. \end{aligned} \tag{62}$$

The coefficients of  $\varepsilon^0$  and  $\varepsilon^1$  vanish because of (21) and (18) respectively. The rest of the coefficients are smooth enough to guarantee that their maxima for  $x \in \Gamma^\varepsilon$  and  $y \in \Gamma$  exist.  $\square$

3.2. **Case (b):**  $\lambda_0 = \lambda_j^D$ . For simplicity, we consider here only the case of eigenvalues of multiplicity  $K = 1$  with zero mean eigenfunction ( $\phi = \phi_j$ ), assuming additionally  $\lambda_0$  is *not* a solution of (34). This corresponds to a subcase of b(ii) on page 420. All other degenerate cases could be considered similarly.

In this case  $v^0 \equiv 0$  and we can introduce a refined approximation for the eigenfunction

$$W_\varepsilon^*(x) = \begin{cases} \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 v_2\left(x, \frac{x}{\varepsilon}\right), & x \in \Omega_1^\varepsilon, \\ w_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(x, \frac{x}{\varepsilon}\right), & x \in \Omega_0^\varepsilon, \end{cases} \tag{63}$$

where

$$w_0(x, y) = c(x)\phi(y), \quad c(x) \neq 0. \tag{64}$$

**Lemma 3.3.** *For an arbitrary function  $c \in C^3(\Omega)$  there exist smooth  $v_1, w_1, v_2, w_2$  in form (B.1)-(B.4) and a constant  $\lambda_1$  such that  $\Lambda_\varepsilon = \lambda_j^D + \varepsilon\lambda_1$  and  $W_\varepsilon^*$  defined by (63) satisfy*

- (i)  $W_\varepsilon^*(x) \in C(\Omega),$
- (ii)  $\max_{\Omega_1^\varepsilon} |\operatorname{div}(a_\varepsilon \nabla W_\varepsilon^*) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon^*| \leq C_0 \varepsilon,$
- (iii)  $\max_{\Omega_0^\varepsilon} |\operatorname{div}(a_\varepsilon \nabla W_\varepsilon^*) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon^*| \leq C_0 \varepsilon,$
- (iv)  $\max_{\Gamma^\varepsilon} \left| a_\varepsilon \frac{\partial W_\varepsilon^*}{\partial n} \Big|_0 - a_\varepsilon \frac{\partial W_\varepsilon^*}{\partial n} \Big|_1 \right| \leq C_0 \varepsilon^2.$

*Proof.* See Appendix B. □

#### 4. Justification of asymptotics.

4.1. **Operator formulation.** We use a standard notation for Lebesgue and Sobolev spaces:  $L_p^2(\Omega)$  is a  $p$ -weighted  $L^2$ -space of square-integrable functions in  $\Omega$ . Notation  $(\cdot, \cdot)_H$  is used for the inner product in a Hilbert space  $H$ .

Let  $\mathcal{L}^\varepsilon = L_{\rho_\varepsilon}^2(\Omega)$  and  $\mathcal{H}^\varepsilon$  be  $H_0^1(\Omega)$  Sobolev space with inner product

$$(u, v)_{\mathcal{H}^\varepsilon} = \int_\Omega a_\varepsilon(x) \nabla u \cdot \nabla v \, dx + \int_\Omega \rho_\varepsilon(x) uv \, dx.$$

Following a standard procedure, see e.g. [19], we introduce a bounded operator  $\mathcal{B}_\varepsilon : \mathcal{L}^\varepsilon \rightarrow \mathcal{L}^\varepsilon$  such that

$$(\mathcal{B}_\varepsilon f, v)_{\mathcal{H}^\varepsilon} = (f, v)_{\mathcal{L}^\varepsilon}, \quad \forall v \in \mathcal{H}^\varepsilon. \tag{65}$$

In other words  $\mathcal{B}_\varepsilon f = u_\varepsilon$ , where  $u_\varepsilon$  is the unique solution of the problem

$$-\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) + \rho_\varepsilon u_\varepsilon = \rho_\varepsilon f, \quad x \in \Omega, \tag{66}$$

$$u_\varepsilon|_{\partial\Omega} = 0, \tag{67}$$

$$u_\varepsilon \Big|_1 = u_\varepsilon \Big|_0, \quad \frac{\partial u_\varepsilon}{\partial n_\varepsilon} \Big|_1 = \varepsilon \frac{\partial u_\varepsilon}{\partial n_\varepsilon} \Big|_0. \tag{68}$$

Note that the operator  $\mathcal{B}_\varepsilon$  is positive, self-adjoint and compact for any fixed  $\varepsilon > 0$  (since its image is in  $\mathcal{H}^\varepsilon$ ). Eigenvalue problem (2) is equivalent to

$$\mathcal{B}_\varepsilon u_\varepsilon = (\lambda^\varepsilon + 1)^{-1} u_\varepsilon \quad \text{in } \mathcal{L}^\varepsilon. \tag{69}$$

Hence the spectrum of the problem consists of a countable set of eigenvalues

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \dots \leq \lambda_k^\varepsilon \leq \dots \rightarrow +\infty,$$

with the only accumulation point at  $+\infty$ . Moreover, the set of corresponding eigenfunctions is complete in  $\mathcal{L}^\varepsilon$ .

**4.2. Case (a).** In this Section we justify the leading terms of the asymptotic expansions constructed above in case  $\lambda_0 \notin \sigma_D$  and thus  $v^0 \neq 0$ , see Section 3.1. Let  $\lambda_0$  be a solution to equation (36). All the functions  $(\eta, N_j, \mathcal{N}, \mathcal{M}, \mathcal{P}, \mathcal{R}, w_0, w_1, v_1$  and  $w_2, v_2)$  are as defined in Section 3.1. We also fix  $\lambda_1$  according to (57). The approximate eigenvalue  $\Lambda_\varepsilon$  and eigenfunction  $W_\varepsilon$  are given by (58) and (59), respectively.

Notice that although  $W_\varepsilon \in H^1(\Omega)$  since  $W_\varepsilon|_1 = W_\varepsilon|_0$ , it does not satisfy the zero Dirichlet boundary conditions on  $\partial\Omega$ . To remedy for this we introduce the following boundary-layer corrector to our approximation.

**Lemma 4.1.** *There exists a corrector  $V_\varepsilon$  solving the problem*

$$-\operatorname{div}(a_\varepsilon \nabla V_\varepsilon) + \rho_\varepsilon V_\varepsilon = 0 \quad \text{in } \Omega, \tag{70}$$

$$V_\varepsilon|_{\partial\Omega} = -W_\varepsilon|_{\partial\Omega}, \quad V_\varepsilon|_1 = V_\varepsilon|_0, \quad \frac{\partial V_\varepsilon}{\partial n_\varepsilon}|_1 = \varepsilon \frac{\partial V_\varepsilon}{\partial n_\varepsilon}|_0, \tag{71}$$

such that  $U_\varepsilon = W_\varepsilon + V_\varepsilon \in H_0^1(\Omega)$  and  $\max_\Omega |V_\varepsilon| \leq C\varepsilon$ .

*Proof.* Clearly a solution of (70), (71) does exist. On each of the subsets  $\Omega_1^\varepsilon$  and  $\Omega_0^\varepsilon$  the coefficients of (70) are smooth. Then, by the maximum principle, the function  $V_\varepsilon$  can reach its positive maximum or negative minimum only at the boundaries  $\Gamma^\varepsilon$  or  $\partial\Omega$ . Let us prove that this cannot be  $\Gamma^\varepsilon$ . Assume to the contrary the existence of  $x_* \in \Gamma^\varepsilon$  such that  $\max_\Omega |V_\varepsilon| = |V_\varepsilon(x_*)| > 0$ . The strong maximum principle yields that there is no more point inside  $\Omega_1^\varepsilon$  or  $\Omega_0^\varepsilon$  where the maximum is reached. Without loss of generality we assume  $V_\varepsilon(x_*) > V_\varepsilon(x)$  for any  $x \in \Omega \setminus \Gamma^\varepsilon$  and  $V_\varepsilon(x_*) > 0$  (otherwise the point would be a positive maximum for  $-V_\varepsilon$  and we would then consider  $-V_\varepsilon$ ). Then by the virtue of Hopf's Lemma, e.g. [16, p.330], applied in the relevant component of  $\Omega_0^\varepsilon$  we have

$$\frac{\partial V_\varepsilon}{\partial n_\varepsilon} \Big|_{(x_*)} > 0.$$

From transmission conditions (71) we have that the normal derivative on the  $\Omega_1^\varepsilon$  side of domain is also positive. Therefore the value of  $V_\varepsilon$  increases from the point  $x_*$  inside  $\Omega_1^\varepsilon$  in the  $n$ -direction and hence  $x_*$  is not a point of maximum of  $V_\varepsilon$  in  $\Omega_1^\varepsilon$ . The contradiction proves that  $|V_\varepsilon|$  reaches it's maximum at  $\partial\Omega$ . Then, from boundary conditions (71),

$$\begin{aligned} \max_\Omega |V_\varepsilon| &= \max_{\partial\Omega} |V_\varepsilon| = \max_{\partial\Omega} |W_\varepsilon| \\ &\leq \varepsilon \max_{\partial\Omega} \left| v_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon v_2 \left( x, \frac{x}{\varepsilon} \right) \right| + \varepsilon \max_{\partial\Omega} \left| w_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon w_2 \left( x, \frac{x}{\varepsilon} \right) \right| \leq C\varepsilon. \end{aligned}$$

Obviously  $U_\varepsilon = W_\varepsilon + V_\varepsilon$  satisfies zero boundary condition on  $\partial\Omega$  and thus belongs to  $H_0^1(\Omega)$ . □

**Lemma 4.2.** *The constructed corrector  $V_\varepsilon$  satisfies the estimate  $\|V_\varepsilon\|_{\mathcal{L}^\varepsilon} \leq C\varepsilon^{3/4}$ .*

*Proof.* Let  $\chi \in C^\infty(\mathbb{R})$  and  $\chi(t) = 0$ ,  $t \leq 1$  and  $\chi(t) = 1$ ,  $t \geq 2$ . Define a family of cut-off functions:

$$\chi_\varepsilon(x) = \chi\left(\varepsilon^{-1/2} \operatorname{dist}(x, \partial\Omega)\right), \quad x \in \Omega.$$

Then  $\chi_\varepsilon : \Omega \rightarrow \mathbb{R}$  satisfies the properties

- $\chi_\varepsilon(x) = 0$  if  $\operatorname{dist}(x, \partial\Omega) \leq \varepsilon^{1/2}$ ,
- $\chi_\varepsilon(x) = 1$  if  $\operatorname{dist}(x, \partial\Omega) \geq 2\varepsilon^{1/2}$ ,
- $|\nabla\chi_\varepsilon| \leq C\varepsilon^{-1/2}$  and  $|\operatorname{supp} \nabla\chi_\varepsilon| \leq C\varepsilon^{1/2}$ ,

where “supp” denotes function’s support, and  $|\operatorname{supp} \cdot|$  is the measure of the corresponding support. Multiplying (70) by  $\chi_\varepsilon^2 V_\varepsilon$  and integrating by parts, we obtain

$$\int_\Omega a_\varepsilon \nabla V_\varepsilon \cdot \nabla(\chi_\varepsilon^2 V_\varepsilon) dx + \int_\Omega \rho_\varepsilon \chi_\varepsilon^2 V_\varepsilon^2 dx = 0. \quad (72)$$

Then using the identity

$$\nabla V_\varepsilon \cdot \nabla(\chi_\varepsilon^2 V_\varepsilon) = |\nabla(\chi_\varepsilon V_\varepsilon)|^2 - V_\varepsilon^2 |\nabla\chi_\varepsilon|^2,$$

we get from (72)

$$\int_\Omega a_\varepsilon |\nabla(\chi_\varepsilon V_\varepsilon)|^2 dx + \int_\Omega \rho_\varepsilon \chi_\varepsilon^2 V_\varepsilon^2 dx = \int_\Omega a_\varepsilon V_\varepsilon^2 |\nabla\chi_\varepsilon|^2 dx, \quad (73)$$

implying

$$\int_\Omega \rho_\varepsilon \chi_\varepsilon^2 V_\varepsilon^2 dx \leq \int_\Omega a_\varepsilon V_\varepsilon^2 |\nabla\chi_\varepsilon|^2 dx. \quad (74)$$

Lemma 4.1 provides the estimate  $V_\varepsilon^2 \leq C\varepsilon^2$ . Moreover,  $|\operatorname{supp} \nabla\chi_\varepsilon| \leq C\varepsilon^{1/2}$  and  $|\nabla\chi_\varepsilon|^2 \leq C\varepsilon^{-1}$ . Therefore (74) yields

$$\|\chi_\varepsilon V_\varepsilon\|_{\mathcal{L}^\varepsilon}^2 = \int_\Omega \rho_\varepsilon \chi_\varepsilon^2 V_\varepsilon^2 dx \leq C\varepsilon^{3/2}. \quad (75)$$

Similarly we estimate

$$\|(1 - \chi_\varepsilon)V_\varepsilon\|_{\mathcal{L}^\varepsilon}^2 = \int_\Omega \rho_\varepsilon (1 - \chi_\varepsilon)^2 V_\varepsilon^2 dx \leq C\varepsilon^{3/2}, \quad (76)$$

since  $|\operatorname{supp}(1 - \chi_\varepsilon)| \leq C\varepsilon^{1/2}$  and  $|\rho_\varepsilon| \leq C\varepsilon^{-1}$ . Combining (75) and (76), we obtain  $\|V_\varepsilon\|_{\mathcal{L}^\varepsilon} = \|(1 - \chi_\varepsilon)V_\varepsilon + \chi_\varepsilon V_\varepsilon\|_{\mathcal{L}^\varepsilon} \leq C\varepsilon^{3/4}$ .  $\square$

**Lemma 4.3.** *If  $\varphi \in H_0^1(\Omega)$  then*

$$\left(\int_{\Gamma^\varepsilon} |\varphi|^2 dx\right)^{1/2} \leq C\|\varphi\|_{\mathcal{H}^\varepsilon}. \quad (77)$$

*Proof.* Extend function  $\varphi$  by zero to the whole of  $\mathbb{R}^n$ . Then (77) follows upon rescaling  $y = x/\varepsilon$  from the standard trace estimates applied to each connected component of  $\tilde{Q}_0$  (which are shifts of  $Q_0$ ).  $\square$

**Lemma 4.4.** *The corrected approximation  $U_\varepsilon$  satisfies the estimate*

$$\|\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon + 1)^{-1} U_\varepsilon\|_{\mathcal{L}^\varepsilon} \leq \|\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon + 1)^{-1} U_\varepsilon\|_{\mathcal{H}^\varepsilon} \leq C\varepsilon^{3/4}.$$

*Proof.* For an arbitrary  $\varphi \in \mathcal{H}^\varepsilon$  consider

$$\begin{aligned}
|(\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon + 1)^{-1} U_\varepsilon, \varphi)_{\mathcal{H}^\varepsilon}| &= |\Lambda_\varepsilon + 1|^{-1} |(U_\varepsilon - (\Lambda_\varepsilon + 1)\mathcal{B}_\varepsilon U_\varepsilon, \varphi)_{\mathcal{H}^\varepsilon}| \\
&\leq C |(U_\varepsilon, \varphi)_{\mathcal{H}^\varepsilon} - (\Lambda_\varepsilon + 1)(U_\varepsilon, \varphi)_{\mathcal{L}^\varepsilon}| \\
&= C \left| \int_{\Omega} a_\varepsilon \nabla U_\varepsilon \cdot \nabla \varphi \, dx - \Lambda_\varepsilon \int_{\Omega} \rho_\varepsilon U_\varepsilon \varphi \, dx \right| \\
&\leq C \left| \int_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} (\operatorname{div}(a_\varepsilon \nabla U_\varepsilon) + \Lambda_\varepsilon \rho_\varepsilon U_\varepsilon) \varphi \, dx \right| \\
&\quad + C \int_{\Gamma^\varepsilon} \left| a_\varepsilon \frac{\partial U_\varepsilon}{\partial n} \Big|_0 - a_\varepsilon \frac{\partial U_\varepsilon}{\partial n} \Big|_1 \right| |\varphi| \, dx.
\end{aligned} \tag{78}$$

Denote the right-hand side of (78) by  $F_\varepsilon(U_\varepsilon, \varphi)$ . Substituting  $U_\varepsilon = W_\varepsilon + V_\varepsilon$  and taking into account (70) and (71),

$$F_\varepsilon(U_\varepsilon, \varphi) \leq F_\varepsilon(W_\varepsilon, \varphi) + C \left| (\Lambda_\varepsilon + 1) \int_{\Omega} \rho_\varepsilon V_\varepsilon \varphi \, dx \right| \leq F_\varepsilon(W_\varepsilon, \varphi) + C \|V_\varepsilon\|_{\mathcal{L}^\varepsilon} \|\varphi\|_{\mathcal{L}^\varepsilon}.$$

By Lemma 4.2 and obvious inequality  $\|\varphi\|_{\mathcal{L}^\varepsilon} \leq \|\varphi\|_{\mathcal{H}^\varepsilon}$ ,

$$F_\varepsilon(U_\varepsilon, \varphi) \leq F_\varepsilon(W_\varepsilon, \varphi) + C \varepsilon^{3/4} \|\varphi\|_{\mathcal{H}^\varepsilon}. \tag{79}$$

According to Lemma 3.2 (i) and (ii),

$$F_\varepsilon(W_\varepsilon, \varphi) \leq C \varepsilon \|\varphi\|_{L^2(\Omega)} + C \int_{\Gamma^\varepsilon} \left| a_\varepsilon \frac{\partial W_\varepsilon}{\partial n} \Big|_0 - a_\varepsilon \frac{\partial W_\varepsilon}{\partial n} \Big|_1 \right| |\varphi| \, dx.$$

Due to Lemmas 3.2 (iii) and 4.3 the latter yields

$$F_\varepsilon(W_\varepsilon, \varphi) \leq C \varepsilon \|\varphi\|_{\mathcal{L}^\varepsilon} + C \varepsilon^{3/2} \left( \int_{\Gamma^\varepsilon} |\varphi|^2 \, dx \right)^{1/2} \leq C \varepsilon \|\varphi\|_{\mathcal{H}^\varepsilon}. \tag{80}$$

Using (80) and (79) in (78) provides

$$|(\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon + 1)^{-1} U_\varepsilon, \varphi)_{\mathcal{H}^\varepsilon}| \leq C \varepsilon^{3/4} \|\varphi\|_{\mathcal{H}^\varepsilon}$$

for all  $\varphi \in \mathcal{H}^\varepsilon$ . Hence,  $\|\mathcal{B}_\varepsilon U_\varepsilon - (\Lambda_\varepsilon + 1)^{-1} U_\varepsilon\|_{\mathcal{H}^\varepsilon} \leq C \varepsilon^{3/4}$ .  $\square$

**Lemma 4.5.**  $\|U_\varepsilon\|_{\mathcal{L}^\varepsilon} \geq C \varepsilon^{-1/2}$ .

*Proof.* By the triangle inequality we have

$$\|U_\varepsilon\|_{\mathcal{L}^\varepsilon} \geq \|W_\varepsilon\|_{\mathcal{L}^\varepsilon} - \|V_\varepsilon\|_{\mathcal{L}^\varepsilon}. \tag{81}$$

We consider

$$\begin{aligned}
\|W_\varepsilon\|_{\mathcal{L}^\varepsilon}^2 &= \varepsilon^{-1} \int_{\Omega_0^\varepsilon} \left| w_0 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon w_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^2 w_2 \left( x, \frac{x}{\varepsilon} \right) \right|^2 \, dx \\
&\quad + \int_{\Omega_1^\varepsilon} \left| v^0(x) + \varepsilon v_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^2 v_2 \left( x, \frac{x}{\varepsilon} \right) \right|^2 \, dx \\
&= \varepsilon^{-1} \int_{\Omega_0^\varepsilon} \left| w_0 \left( x, \frac{x}{\varepsilon} \right) \right|^2 \, dx + O(1) \\
&= \varepsilon^{-1} \int_{\Omega_0^\varepsilon} \left| v^0(x) \eta \left( \frac{x}{\varepsilon} \right) \right|^2 \, dx + O(1), \quad \varepsilon \rightarrow 0.
\end{aligned} \tag{82}$$

Extending  $\eta : Q_0 \rightarrow \mathbb{R}$  by zero onto entire periodicity cell  $Q$ , by the mean value property we obtain

$$\int_{\Omega_0^\varepsilon} \left| v^0(x) \eta \left( \frac{x}{\varepsilon} \right) \right|^2 dx \rightarrow C_* := \langle \eta^2 \rangle \int_{\Omega} |v^0(x)|^2 dx \quad \text{as } \varepsilon \rightarrow 0, \tag{83}$$

where  $\langle \eta^2 \rangle$  is the mean value of function  $\eta^2$  over  $Q$ , namely

$$\langle \eta^2 \rangle = \int_Q \eta^2(y) dy = \int_{Q_0} \eta^2(y) dy > 0.$$

Since also  $v^0(x) \not\equiv 0$ ,  $C_*$  is positive. Therefore (83) and (82) yield

$$\|W_\varepsilon\|_{\mathcal{L}^\varepsilon} = C_*^{1/2} \varepsilon^{-1/2} + o(\varepsilon^{-1/2}), \quad \varepsilon \rightarrow 0. \tag{84}$$

Due to Lemma 4.2 and (84), it follows from (81) that  $\|U_\varepsilon\|_{\mathcal{L}^\varepsilon} \geq C_*^{1/2} \varepsilon^{-1/2} + o(\varepsilon^{-1/2})$  as  $\varepsilon \rightarrow 0$ . □

**Theorem 4.6.** *Let  $\lambda_0$  be a solution to (36) such that  $\lambda_0 \neq \lambda_j^D$  and  $\lambda_1$  is defined according to (57). Then*

1. *For sufficiently small  $\varepsilon > 0$  there exists an eigenvalue  $\lambda^\varepsilon$  of (2) such that*

$$|\lambda^\varepsilon - \lambda_0 - \varepsilon \lambda_1| \leq C_1 \varepsilon^{5/4}, \tag{85}$$

*with positive constant  $C_1$  independent of  $\varepsilon$ .*

2. *Let  $W_\varepsilon$  be defined by (59) and  $\widetilde{W}_\varepsilon = \|W_\varepsilon\|_{\mathcal{L}^\varepsilon}^{-1} W_\varepsilon$ . Then there exist constants  $d_j(\varepsilon)$  such that*

$$\left\| \widetilde{W}_\varepsilon - \sum_{j \in J_\varepsilon} d_j(\varepsilon) u_j^\varepsilon \right\|_{\mathcal{L}^\varepsilon} \leq C_2 \varepsilon^{5/4}, \tag{86}$$

*where  $J_\varepsilon = \{j : |\lambda_j^\varepsilon - \lambda_0 - \varepsilon \lambda_1| \leq C_1 \varepsilon^{5/4}\}$ , and  $\lambda_j^\varepsilon, u_j^\varepsilon$  are eigenvalues and ( $\mathcal{L}^\varepsilon$ -normalized) eigenfunctions of (2), and the positive constants  $C_1$  and  $C_2$  are independent of  $\varepsilon$ .*

*Proof.* Application of classical lemma on “approximate eigenvalues”, e.g. [28], with  $\widetilde{U}_\varepsilon = \|U_\varepsilon\|_{\mathcal{L}^\varepsilon}^{-1} U_\varepsilon$  as a test function and  $\Lambda_\varepsilon = \lambda_0 + \varepsilon \lambda_1$  as an approximate eigenvalue, ensures, via Lemmas 4.4 & 4.5, the existence of an eigenvalue  $\mu_\varepsilon$  of operator  $\mathcal{B}_\varepsilon$  such that

$$|(\Lambda_\varepsilon + 1)^{-1} - \mu_\varepsilon| \leq C \varepsilon^{5/4}, \tag{87}$$

and delivers the estimate analogous to (86) with  $u_j^\varepsilon$  being eigenfunctions of  $\mathcal{B}_\varepsilon$  and  $\widetilde{W}_\varepsilon$  replaced by  $\widetilde{U}_\varepsilon$ . It suffices to notice that the eigenfunctions of problem (2) and of operator  $\mathcal{B}_\varepsilon$  coincide, their eigenvalues are related via  $\mu_\varepsilon^{-1} = \lambda^\varepsilon + 1$  and that  $\mathcal{L}^\varepsilon$  norm of the difference between  $\widetilde{U}_\varepsilon$  and  $\widetilde{W}_\varepsilon$  can be estimated via the right hand side of (86) (see Lemmas 4.2 and 4.5). □

**Remark 2.** Notice that (86) implies weaker but more transparent interpretations on the approximate eigenfunctions. For example, introducing

$$u(x, y) = \begin{cases} v^0(x), & y \in Q_1, \\ w_0(x, y), & y \in Q_0, \end{cases} \tag{88}$$

we claim that

$$\left\| u \left( x, \frac{x}{\varepsilon} \right) - \sum_{j \in J_\varepsilon} d_j(\varepsilon) u_j^\varepsilon \right\|_{L^2(\Omega)} \leq C \varepsilon^{3/4}, \tag{89}$$

with appropriate  $d_j(\varepsilon)$ . Note that  $\|u(\cdot, \frac{\cdot}{\varepsilon})\|_{L^2(\Omega)} \geq C_0 > 0$ . Then (89) follows from (86) by splitting its left hand side into the parts corresponding to  $\Omega_1^\varepsilon$  and



$\Omega_0^\varepsilon$ , removing the weight, retaining only the main-order terms and then adding the inequalities up.

We also remark that, in principle, the result (86) on the convergence of eigenfunctions could be further sharpened, e.g. using the technique of two-scale convergence, cf. Section 5 below and [13].

**4.3. Case (b).** In this section we assume that  $\lambda_0 = \lambda_j^D$  for some  $j$ , its multiplicity is equal to 1 and the corresponding eigenfunction  $\phi$  has zero mean, i.e.  $\langle \phi \rangle = 0$ , see Section 3.2.

**Theorem 4.7.** *Let  $c \in C^3(\Omega)$ ,  $c = 0$  on  $\partial\Omega$ ,  $\lambda_0$  be not a solution to (36) and  $\lambda_1$  be defined according to (B.17). Then there exist  $\varepsilon_0 > 0$  and constants  $C_1, C_2$  independent of  $\varepsilon$  (but dependent on  $c$ ) such that for any  $0 < \varepsilon \leq \varepsilon_0$ ,*

1. *There exists an eigenvalue  $\lambda^\varepsilon$  of (2) such that*

$$|\lambda^\varepsilon - \lambda_0 - \varepsilon\lambda_1| \leq C_1\varepsilon^{5/4}. \tag{90}$$

2. *Let  $W_\varepsilon^*$  be defined by (63) and  $\widetilde{W}_\varepsilon^* = \|W_\varepsilon^*\|_{\mathcal{L}^\varepsilon}^{-1}W_\varepsilon^*$ . Then there exist constants  $d_j(\varepsilon)$  such that*

$$\left\| \widetilde{W}_\varepsilon^* - \sum_{j \in J_\varepsilon} d_j(\varepsilon)u_j^\varepsilon \right\|_{\mathcal{L}^\varepsilon} \leq C_2\varepsilon^{5/4}, \tag{91}$$

where  $J_\varepsilon = \{j : |\lambda_j^\varepsilon - \lambda_0 - \varepsilon\lambda_1| \leq C_1\varepsilon^{5/4}\}$ , and  $\lambda_j^\varepsilon, u_j^\varepsilon(x)$  are eigenvalues and ( $\mathcal{L}^\varepsilon$ -normalized) eigenfunctions of (2).

*Proof.* Proof of this theorem literally follows the proof of Theorem 4.6 with reference to Lemma 3.3. □

A direct analogue of Remark 2 also holds.

**5. On the eigenfunction convergence.** In this section we give a brief sketch of a possible further refinement of the presented results using the technique of two-scale convergence, see [23, 2, 29].

First, the inclusions intersecting or touching the boundary are “excluded”, e.g. by re-defining  $a_\varepsilon$  and  $\rho_\varepsilon$  there as in the matrix phase ( $a_\varepsilon(x) = \rho_\varepsilon(x) = 1$ ). Denoting now via  $\varepsilon \rightarrow 0$  an appropriate subsequence in  $\varepsilon$ , without relabeling, let  $u_\varepsilon$  and  $\lambda^\varepsilon$  be eigenfunctions and eigenvalues of the original problem, with normalization

$$\int_{\Omega_1^\varepsilon} \nabla u_\varepsilon^2 + \varepsilon^2 \int_{\Omega_0^\varepsilon} \nabla u_\varepsilon^2 = 1. \tag{92}$$

The boundedness of  $u_\varepsilon$  in  $L^2(\Omega)$  is then implied by (92) e.g. via the uniform positivity of the double-porosity operator whose form is given by the left hand side of (92), [29, Thm 8.1]. This implies that, up to a subsequence,  $u_\varepsilon \xrightarrow{2} u(x, y)$  and  $\varepsilon \nabla u_\varepsilon \xrightarrow{2} \nabla_y u(x, y)$ , where  $u \in L^2(\Omega, H_{per}^1)$  and  $\xrightarrow{2}$  denotes weak two-scale convergence. Additionally, since (92) implies  $\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \rightarrow 0$ , [29, Thm 4.1] assures that the two-scale limit is independent of  $y$  in the matrix, i.e. is exactly in the form (88). Further, by [29, Thm 4.2],  $v^0 \in H_0^1(\Omega)$  and

$$\theta_1^\varepsilon \nabla u_\varepsilon \xrightarrow{2} \theta_1(y)(\nabla v^0(x) + p(x, y)), \tag{93}$$

where  $p \in L^2(\Omega, V_{pot})$  with  $\theta_1^\varepsilon$  and  $\theta_1(y)$  denoting the characteristic functions of  $\Omega_1^\varepsilon$  and  $Q_1$ , respectively, and  $V_{pot}$  standing for the space of potential vector fields on  $Q_1$ , i.e. with respect to the Lebesgue measure supported on  $Q_1$ , cf. [29, §3.2].

Let  $\lambda^\varepsilon \rightarrow \lambda_0$  and  $(\lambda^\varepsilon - \lambda_0)/\varepsilon \rightarrow \lambda_1$ . Selecting then in (2) appropriate oscillating test functions  $\phi = \phi_\varepsilon$  one can pass to the limit recovering the weak forms of the equations derived in Section 3. For example, selecting  $\phi_\varepsilon(x) = \varepsilon\psi(x)b(x/\varepsilon)$ ,  $\psi \in C_0^\infty(\Omega)$ ,  $b(y) \in C_{per}^\infty(Q)$  yields

$$\begin{aligned} \int_{\Omega} \int_{Q_1} (\nabla v^0(x) + p(x, y)) \cdot \nabla_y b(y) \psi(x) dy dx + \int_{\Omega} \int_{Q_0} \nabla_y w_0(x, y) \cdot \nabla_y b(y) \psi(x) dy dx \\ = \lambda_0 \int_{\Omega} \int_{Q_0} w_0(x, y) b(y) \psi(x) dy dx. \end{aligned} \quad (94)$$

This can be seen to be a weak form of (23) and (21). Selecting further  $\phi_\varepsilon(x) = \psi(x)$  can be seen, after some careful technical analysis, to recover (51), (52) and (54).

The above implies that as long as  $(v^0)^2 + w_0^2 \not\equiv 0$ ,  $\lambda_0$ ,  $\lambda_1$ ,  $v^0$  and  $w_0$  can only be those constructed in Section 3. This does not however rule out the possibility that  $v^0$  and  $w_0$  are both trivial (equivalently, the two-scale limit  $u(x, y)$  is identically zero). Therefore additional two-scale compactness type arguments are required, cf. [29, Lemma 8.2]. In fact, following literally the argument of Zhikov one observes that the two-scale compactness of the eigenfunctions does hold, i.e.  $u_\varepsilon \xrightarrow{2} u(x, y)$ , where  $\xrightarrow{2}$  denotes strong two-scale convergence, in particular there is a convergence of norms:

$$\|u_\varepsilon - u(x, x/\varepsilon)\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (95)$$

However, this in turn does not rule out the possibility of  $\|u_\varepsilon\| \rightarrow 0$  with the normalization (92), which requires a separate analysis.

We announce here a partial result with this effect, postponing detailed discussions for future.

**Proposition 1.** *Let  $\lambda_\varepsilon \rightarrow \lambda_0$ ,  $(\lambda_\varepsilon - \lambda_0)/\varepsilon \rightarrow \lambda_1$ , and let  $\lambda_0$  be not an eigenvalue of the Dirichlet problem in  $Q_0$ , i.e.  $\lambda_0 \neq \lambda_j^D$ ,  $j \geq 1$ , see (25). Then*

- (i) *In the above setting, necessarily,  $\beta(\lambda_0) \geq |Q_1|\lambda_0$ , i.e. there are gaps developed for small enough  $\varepsilon$  in the spectrum, containing in the limit at least  $\{\lambda : \beta(\lambda) < |Q_1|\lambda\}$ .*
- (ii) *If  $\beta(\lambda_0) = |Q_1|\lambda_0$ , necessarily  $u(x, y) \not\equiv 0$ . Consequently,  $\lambda_1$  can only be one of those described by (56), (57). The eigenfunctions converge strongly, in particular (95) holds. For fixed  $\lambda_0$  and  $\lambda_1$  for any sufficiently small positive constant  $C_*$  for small enough  $\varepsilon$  the multiplicity of the eigenvalues  $\lambda^\varepsilon$  in the interval  $[\Lambda_\varepsilon - C_*\varepsilon, \Lambda_\varepsilon + C_*\varepsilon]$  with  $\Lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1$  does not exceed the multiplicity of  $\nu$  as an eigenvalue of (51), (52).*

We remark that the above statement does not provide a full analogue of Hausdorff convergence of the spectra as in the double porosity case [29, Thm 8.1]. It does ensure however the existence of the gaps (on Figure 2,  $(\lambda_j^D, \mu_{j+1})$ ,  $j \geq 1$ ) and of the spectrum accumulation near the left ends  $\mu_j$ ,  $j \geq 1$ , of the “bands”  $[\mu_j, \lambda_j^D]$ . However it does not clarify whether the “rests” of the bands,  $(\mu_j, \lambda_j^D]$  could be accumulation points. We conjecture that they could. According to (57), (56), for a chosen  $\lambda_0 = \mu_j$  there exist infinitely many  $\lambda_1 = \lambda_1^{(k)}$  and  $\lambda_1^{(k)} \rightarrow +\infty$  as  $k \rightarrow \infty$ . According to Proposition 1, on any band, for any small enough  $\varepsilon$  there exists a finite but infinitely increasing number  $M(\varepsilon)$  of the eigenvalues. The issue is hence, in a sense, whether  $\varepsilon\lambda_1^{(k)}$  may become of order one for large  $k$  ( $k \sim M(\varepsilon)$ ). For  $\lambda_1^{(k)} \sim \varepsilon^{-1}$ , according to (57)  $\nu \sim \varepsilon^{-1}$ , and hence, formally, the solutions  $v^0$  of

the homogenized equation (51) becomes oscillatory on the new scale  $z := x/\varepsilon^{1/2}$ . One can attempt deriving asymptotic expansions similarly to those in Section 3, involving this new scale  $z$ . A preliminary analysis has shown that those have formal asymptotic solutions near every point inside the band. More detailed analysis is beyond the scope of the present work and will be reported elsewhere.

**Appendix A. Derivation of the limit equation for  $v_0$ .** Since

$$-\int_{\Gamma} \frac{\partial v_1}{\partial n_x} dy = -\int_{\Gamma} \frac{\partial v_1}{\partial x_j} n_j dy = \int_{Q_1} \frac{\partial^2 v_1}{\partial x_j \partial y_j} dy,$$

(50) transforms to

$$(\Delta_x v_0 + \lambda_0 v_0)|_{Q_1} + \int_{Q_1} \frac{\partial^2 v_1}{\partial x_j \partial y_j} dy = \int_{\Gamma} \left( \frac{\partial w_1}{\partial n_y} + \frac{\partial w_0}{\partial n_x} \right) dy.$$

Taking into account (41) and (38) this yields

$$\begin{aligned} (\Delta_x v_0 + \lambda_0 v_0)|_{Q_1} + \frac{\partial^2 v_0}{\partial x_j \partial x_k} \int_{Q_1} \frac{\partial N_k}{\partial y_j} dy \\ = \frac{\partial v_0}{\partial x_j} \left( -\int_{Q_1} \frac{\partial \mathcal{N}}{\partial y_j} dy + \int_{\Gamma} \eta n_j dy \right) + \int_{\Gamma} \frac{\partial w_1}{\partial n_y} dy. \end{aligned} \tag{A.1}$$

Since  $\eta(y) = 1$  on  $\Gamma$ ,

$$\int_{\Gamma} \eta n_j dy = \int_{\Gamma} n_j dy = 0. \tag{A.2}$$

We introduce homogenized matrix  $A^{\text{hom}} = (A_{jk}^{\text{hom}})_{j,k=1}^d$  by (53). According to (46) we have

$$\int_{\Gamma} \frac{\partial w_1}{\partial n_y} dy = \frac{\partial v^0}{\partial x_j} \int_{\Gamma} \frac{\partial \mathcal{M}_j}{\partial n_y} dy + v^0 \int_{\Gamma} \frac{\partial \mathcal{P}}{\partial n_y} dy + \lambda_1 v^0 \int_{\Gamma} \frac{\partial \mathcal{R}}{\partial n_y} dy. \tag{A.3}$$

Substituting (A.2) – (A.3) into (A.1) yields

$$-\text{div } A^{\text{hom}} \nabla_x v^0 = \nu(\lambda_1) v^0 + \mathcal{K}_j \frac{\partial v^0}{\partial x_j} \quad \text{in } \Omega, \tag{A.4}$$

with

$$\nu(\lambda_1) = \lambda_0 |Q_1| - \lambda_1 \int_{\Gamma} \frac{\partial \mathcal{R}}{\partial n_y} dy - \int_{\Gamma} \frac{\partial \mathcal{P}}{\partial n_y} dy$$

and

$$\mathcal{K}_j = \int_{Q_1} \frac{\partial \mathcal{N}}{\partial y_j} dy - \int_{\Gamma} \frac{\partial \mathcal{M}_j}{\partial n_y} dy. \tag{A.5}$$

**Lemma A.1.**  $\nu(\lambda_1)$  depends on  $\lambda_1$  with a positive linear coefficient.

*Proof.* We estimate the linear coefficient

$$\mathcal{C} := -\int_{\Gamma} \frac{\partial \mathcal{R}}{\partial n_y} dy.$$

Note that, via (49) and (26),

$$\mathcal{C} = \int_{\Gamma} \left( \mathcal{R} \frac{\partial \eta}{\partial n_y} - \eta \frac{\partial \mathcal{R}}{\partial n_y} \right) dy = \int_{Q_0} (\mathcal{R} \Delta_y \eta - \eta \Delta_y \mathcal{R}) dy = \int_{Q_0} \eta^2 dy > 0.$$

Thus

$$\nu(\lambda_1) = \mathcal{C} \lambda_1 + \lambda_0 |Q_1| - \int_{\Gamma} \frac{\partial \mathcal{P}}{\partial n_y} dy$$

with positive constant  $\mathcal{C}$  (depending on the choice of  $\lambda_0$ ).  $\square$

**Corollary 1.** (i) If  $\lambda_0 = 0$  then  $\eta(y) \equiv 1$  and hence  $\mathcal{C} = |Q_0|$ .

(ii) According to (48) we also have the representation

$$\nu(\lambda_1) = \mathcal{C}\lambda_1 + \lambda_0 \left( |Q_1| + \int_{Q_0} \mathcal{P} dy \right). \quad (\text{A.6})$$

**Lemma A.2.** All  $\mathcal{K}_j$  defined by (A.5) equal zero.

*Proof.* First we prove an auxiliary identity, namely

$$\int_{\Gamma} \left( \frac{\partial \mathcal{M}_j}{\partial n_y} \eta - \mathcal{M}_j \frac{\partial \eta}{\partial n_y} \right) dy = 0. \quad (\text{A.7})$$

Notice for this that the left-hand side is

$$\int_{Q_0} (\Delta \mathcal{M}_j \eta - \mathcal{M}_j \Delta \eta) dy = - \int_{Q_0} 2 \frac{\partial \eta}{\partial y_j} \eta dy = - \int_{Q_0} \frac{\partial \eta^2}{\partial y_j} dy,$$

where equations (47) and (26) have been used. Since  $\eta(y) = 1$  on  $\Gamma$ ,

$$\int_{Q_0} \frac{\partial \eta^2}{\partial y_j} dy = \int_{\Gamma} \eta^2 n_j dy = \int_{\Gamma} n_j dy = 0,$$

which proves (A.7).

Then consider

$$\mathcal{K}_j = - \int_{\Gamma} \mathcal{N} n_j dy - \int_{\Gamma} \frac{\partial \mathcal{M}_j}{\partial n_y} \eta dy,$$

which, according to (42) and (A.7), yields

$$\mathcal{K}_j = \int_{\Gamma} \mathcal{N} \frac{\partial N_j}{\partial n_y} dy - \int_{\Gamma} \mathcal{M}_j \frac{\partial \eta}{\partial n_y} dy. \quad (\text{A.8})$$

Since  $\mathcal{N}$  and  $N_j$  are both harmonic in  $Q_1$ ,

$$\int_{\Gamma} \mathcal{N} \frac{\partial N_j}{\partial n_y} dy = \int_{\Gamma} N_j \frac{\partial \mathcal{N}}{\partial n_y} dy. \quad (\text{A.9})$$

Using (47) and (43) we obtain

$$\int_{\Gamma} \mathcal{M}_j \frac{\partial \eta}{\partial n_y} dy = \int_{\Gamma} N_j \frac{\partial \mathcal{N}}{\partial n_y} dy. \quad (\text{A.10})$$

Substitution of (A.9) and (A.10) into (A.8) proves the lemma.  $\square$

Finally we come to the formulation of homogenized problem for the function  $v_0$ , which comes from (A.4) and boundary condition (4), resulting in (51), (52).

**Appendix B. Proof of Lemma 3.3.** We look for  $v_1, w_1, v_2, w_2$  in the form

$$v_1(x, y) = c(x) \mathcal{V}_1(y), \quad (\text{B.1})$$

$$w_1(x, y) = c(x) \mathcal{W}_1(y) + \frac{\partial c(x)}{\partial x_k} \mathcal{Z}_1^{(k)}(y), \quad (\text{B.2})$$

$$v_2(x, y) = c(x) \mathcal{V}_2(y) + \frac{\partial c(x)}{\partial x_k} \mathcal{P}_k(y), \quad (\text{B.3})$$

$$w_2(x, y) = c(x) \mathcal{W}_2(y) + \frac{\partial c(x)}{\partial x_k} \mathcal{Z}_2^{(k)}(y), \quad (\text{B.4})$$

where  $c$  is an arbitrary smooth function in  $\Omega$  and  $\mathcal{V}_i, \mathcal{W}_i, \mathcal{Z}_i^{(k)}, \mathcal{P}_k$ , with  $i = 1, 2$  and  $k = 1, \dots, d$ , are functions to be found.

Applying differential operator  $\mathcal{A}^\varepsilon$  given by (60) to (63) in  $\Omega_1^\varepsilon$  we obtain

$$\begin{aligned} & \operatorname{div}(a_\varepsilon \nabla W_\varepsilon^*) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon^* \\ &= \left\{ \varepsilon^{-1} c \Delta_y \mathcal{V}_1 + \varepsilon^0 \left( c \Delta_y \mathcal{V}_2(y) + \frac{\partial c}{\partial x_k} \left\{ \Delta_y \mathcal{P}_k(y) + 2 \frac{\partial \mathcal{V}_1}{\partial y_k} \right\} \right) \right. \\ & \quad \left. + \varepsilon^1 \left( 2 \frac{\partial^2 v_2}{\partial x_j \partial y_j} + \Delta_x v_1 + \lambda_0 v_1 \right) + \varepsilon^2 (\Delta_x v_2 + \lambda_1 v_1 + \lambda_0 v_2) + \varepsilon^3 \lambda_1 v_2 \right\} \Big|_{y=\frac{x}{\varepsilon}}. \end{aligned} \quad (\text{B.5})$$

Applying next  $\mathcal{A}^\varepsilon$  to (63) in  $\Omega_0^\varepsilon$ , and using (64) where  $(\Delta_y + \lambda_0)\phi = 0$ , we obtain

$$\begin{aligned} & \operatorname{div}(a_\varepsilon \nabla W_\varepsilon^*) + \Lambda_\varepsilon \rho_\varepsilon W_\varepsilon^* \\ &= \left\{ \varepsilon^0 \left( c [(\Delta_y + \lambda_0) \mathcal{W}_1 + \lambda_1 \phi] + \frac{\partial c}{\partial x_k} [(\Delta_y + \lambda_0) \mathcal{Z}_1^{(k)} + 2 \frac{\partial \phi}{\partial y_k}] \right) \right. \\ & \quad \left. + \varepsilon^1 \left( (\Delta_y + \lambda_0) w_2 + 2 \frac{\partial^2 w_1}{\partial x_j \partial y_j} + \Delta_x w_0 + \lambda_1 w_1 \right) \right. \\ & \quad \left. + \varepsilon^2 \left( 2 \frac{\partial^2 w_2}{\partial x_j \partial y_j} + \Delta_x w_1 + \lambda_1 w_2 \right) + \varepsilon^3 \Delta_x w_2 \right\} \Big|_{y=\frac{x}{\varepsilon}}. \end{aligned} \quad (\text{B.6})$$

Evaluating the jumps of conormal derivatives on  $\Gamma^\varepsilon$ , we obtain

$$\begin{aligned} & a_\varepsilon \frac{\partial W_\varepsilon^*}{\partial n} \Big|_0 - a_\varepsilon \frac{\partial W_\varepsilon^*}{\partial n} \Big|_1 = \varepsilon^0 c \left( \frac{\partial \phi}{\partial n_y} - \frac{\partial \mathcal{V}_1}{\partial n_y} \right) \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}} \\ & + \varepsilon^1 \left( c \left\{ \frac{\partial \mathcal{W}_1}{\partial n_y} - \frac{\partial \mathcal{V}_2}{\partial n_y} \right\} + \frac{\partial c}{\partial x_k} \left\{ \frac{\partial \mathcal{Z}_1^{(k)}}{\partial n_y} + n_k \phi - \frac{\partial \mathcal{P}_k}{\partial n_y} - n_k \mathcal{V}_1 \right\} \right) \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}} \\ & + \varepsilon^2 \left( \frac{\partial w_2}{\partial n_y} + \frac{\partial w_1}{\partial n_x} - \frac{\partial v_2}{\partial n_x} \right) \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}} + \varepsilon^3 \frac{\partial w_2}{\partial n_x} \Big|_{\substack{x \in \Gamma^\varepsilon \\ y \in \Gamma}}. \end{aligned} \quad (\text{B.7})$$

On the other hand function  $W_\varepsilon^*$  is required to be continuous, i.e we have

$$\mathcal{W}_1 = \mathcal{V}_1 \quad \text{on } \Gamma, \quad (\text{B.8})$$

$$\mathcal{Z}_1^{(k)} = 0 \quad \text{on } \Gamma, \quad (\text{B.9})$$

$$\mathcal{W}_2 = \mathcal{V}_2 \quad \text{on } \Gamma, \quad (\text{B.10})$$

$$\mathcal{Z}_2^{(k)} = \mathcal{P}_k \quad \text{on } \Gamma. \quad (\text{B.11})$$

Equating to zero the term of order  $\varepsilon^{-1}$  in (B.5) and the term of order  $\varepsilon^0$  in (B.7), we obtain periodic problem for  $\mathcal{V}_1$ :

$$\Delta_y \mathcal{V}_1 = 0 \quad \text{in } Q_1, \quad \frac{\partial \mathcal{V}_1}{\partial n_y} = \frac{\partial \phi}{\partial n_y} \quad \text{on } \Gamma. \quad (\text{B.12})$$

A solution to this problem exists since  $\langle \phi \rangle = 0$  and we can present it as:

$$\mathcal{V}_1 = \tilde{\mathcal{V}}_1 + \tilde{A}, \quad (\text{B.13})$$

where  $\langle \tilde{\mathcal{V}}_1 \rangle = 0$  and  $\tilde{A}$  is a constant which will be determined later.

Equating to zero the terms of order  $\varepsilon^0$  in (B.6), and using (B.8), (B.9) we obtain problems for  $\mathcal{Z}_1^{(k)}$

$$(\Delta_y + \lambda_0) \mathcal{Z}_1^{(k)} = -2 \frac{\partial \phi}{\partial y_k} \quad \text{in } Q_0, \quad \mathcal{Z}_1^{(k)} = 0 \quad \text{on } \Gamma, \quad (\text{B.14})$$

which admits an explicit solution

$$\mathcal{Z}_1^{(k)}(y) = -y_k \phi(y), \quad (\text{B.15})$$

and for  $\mathcal{W}_1$

$$(\Delta_y + \lambda_0)\mathcal{W}_1 = -\lambda_1 \phi \quad \text{in } Q_0, \quad \mathcal{W}_1 = \mathcal{V}_1 \quad \text{on } \Gamma. \quad (\text{B.16})$$

A solution to the latter exists if and only if

$$\lambda_1 = \int_{\Gamma} \mathcal{V}_1 \frac{\partial \phi}{\partial n_y} dy = - \int_{Q_1} |\nabla \mathcal{V}_1|^2 dy = - \int_{Q_1} |\nabla \tilde{\mathcal{V}}_1|^2 dy, \quad (\text{B.17})$$

and we can present it in the following way:

$$\mathcal{W}_1 = \tilde{\mathcal{W}}_1 + \tilde{A}\eta, \quad (\text{B.18})$$

where  $\tilde{\mathcal{W}}_1$  solves (B.16) with  $\mathcal{V}_1$  replaced by  $\tilde{\mathcal{V}}_1$  (a solution exists for the same reason), and  $\eta$  solves (26) (a solution exists since  $\langle \phi \rangle = 0$ ). Notice that

$$\lambda_0 \langle \eta \rangle = - \int_{\Gamma} \frac{\partial \eta}{\partial n_y} dy \neq 0,$$

otherwise  $\lambda_0$  would be a solution of (36) which contradicts to the assumptions of this section.

Equating to zero the terms of order  $\varepsilon^0$  in (B.5) and of order  $\varepsilon^1$  in (B.7), we obtain problems for  $\mathcal{V}_2$  and  $\mathcal{P}_k$ . For  $\mathcal{V}_2$  we have periodic problem

$$\Delta_y \mathcal{V}_2 = 0 \quad \text{in } Q_1, \quad \frac{\partial \mathcal{V}_2}{\partial n_y} = \frac{\partial \mathcal{W}_1}{\partial n_y} \quad \text{on } \Gamma. \quad (\text{B.19})$$

A solution to this problem exists if and only if

$$0 = \int_{\Gamma} \frac{\partial \mathcal{W}_1}{\partial n_y} dy = \int_{\Gamma} \frac{\partial \tilde{\mathcal{W}}_1}{\partial n_y} dy + \tilde{A} \int_{\Gamma} \frac{\partial \eta}{\partial n_y} dy, \quad (\text{B.20})$$

and consequently

$$\tilde{A} = (\lambda_0 \langle \eta \rangle)^{-1} \int_{\Gamma} \frac{\partial \tilde{\mathcal{W}}_1}{\partial n_y} dy. \quad (\text{B.21})$$

The problem for  $\mathcal{P}_k$  has the form:

$$\Delta_y \mathcal{P}_k(y) = -2 \frac{\partial \mathcal{V}_1}{\partial y_k} \quad \text{in } Q_1, \quad \frac{\partial \mathcal{P}_k}{\partial n_y} = \frac{\partial \mathcal{Z}_1^{(k)}}{\partial n_y} - n_k \mathcal{V}_1 \quad \text{on } \Gamma. \quad (\text{B.22})$$

The solvability condition for this problem is

$$2 \int_{Q_1} \frac{\partial \mathcal{V}_1}{\partial y_k} dy = \int_{\Gamma} \left( \frac{\partial \mathcal{Z}_1^{(k)}}{\partial n_y} - n_k \mathcal{V}_1 \right) dy. \quad (\text{B.23})$$

The left hand side of (B.23) can be transformed as follows,

$$\int_{Q_1} 2 \frac{\partial \mathcal{V}_1}{\partial y_k} dy = -2 \int_{\Gamma} n_k \mathcal{V}_1 dy. \quad (\text{B.24})$$

On the other hand, for the right hand side of (B.23),

$$\begin{aligned} \int_{\Gamma} \left( \frac{\partial \mathcal{Z}_1^{(k)}}{\partial n_y} - n_k \mathcal{V}_1 \right) dy &= \int_{\Gamma} \left( -y_k \frac{\partial \phi}{\partial n_y} - n_k \mathcal{V}_1 \right) dy \\ &= \int_{\Gamma} \left( -y_k \frac{\partial \mathcal{V}_1}{\partial n_y} - n_k \mathcal{V}_1 \right) dy = -2 \int_{\Gamma} n_k \mathcal{V}_1 dy. \end{aligned} \quad (\text{B.25})$$

Here we have used (B.15), (B.12) and the integration by parts. Comparing (B.24) and (B.25) we see that solvability condition (B.23) is satisfied. Finally, we choose  $\mathcal{W}_2$  and  $\mathcal{Z}_2^{(k)}$  as arbitrary smooth functions satisfying (B.10) and (B.11). This ensures, via (B.5)–(B.7), that the assertions of Lemma 3.3 are satisfied.  $\square$

**Acknowledgments.** The work was supported by Bath Institute for Complex Systems (EPSRC grant GR/S86525/01), by Nuffield grant NAL/32758, EPSRC grant EP/E037607/1 and RFBR grant 07-01-00548. The authors acknowledge partial support of Isaac Newton Institute for Mathematical Sciences under the programme “Highly Oscillatory Problems: Computation, Theory and Application” February–July 2007. The authors thank V.V. Zhikov for comments on the manuscript and improving suggestions.

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Received April 2008.

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