NETWORKS AND HETEROGENEOUS MEDIA
©American Institute of Mathematical Sciences
Volume **3**, Number **2**, June **2008**

Website: http://aimSciences.org

pp. **201–219**

ON ALGEBRAIC GRAPH THEORY AND THE DYNAMICS OF INNOVATION NETWORKS

M. D. König, S. Battiston, M. Napoletano and F. Schweitzer

Chair of Systems Design, ETH Zurich,
Kreuzplatz 5, CH-8032 Zurich, Switzerland

Abstract. We investigate some of the properties and extensions of a dynamic innovation network model recently introduced in [37]. In the model, the set of efficient graphs ranges, depending on the cost for maintaining a link, from the complete graph to the (quasi-) star, varying within a well defined class of graphs. However, the interplay between dynamics on the nodes and topology of the network leads to equilibrium networks which are typically not efficient and are characterized, as observed in empirical studies of R&D networks, by sparseness, presence of clusters and heterogeneity of degree. In this paper, we analyze the relation between the growth rate of the knowledge stock of the agents from R&D collaborations and the properties of the adjacency matrix associated with the network of collaborations. By means of computer simulations we further investigate how the equilibrium network is affected by increasing the evaluation time τ over which agents evaluate whether to maintain a link or not. We show that only if τ is long enough, efficient networks can be obtained by the selfish link formation process of agents, otherwise the equilibrium network is inefficient. This work should assist in building a theoretical framework of R&D networks from which policies can be derived that aim at fostering efficient innovation networks.

1. **Introduction.** The field of Network Theory has only recently focused its attention on the study of dynamic models in which the topology of the network endogenously drives the evolution of the network. These models assume that the evolution of the links in the network is driven by the dynamics of a state variable, associated to each node, which depends, through the network, on the state variable of the other nodes [22, 47]. Such an interplay is crucial in many biological systems and especially in socio-economic systems. In biological systems, a Darwinian selection mechanism usually works at a global level: for instance in the context of networks, one can think of a mechanism in which the least fit nodes are replaced (together with their connections) with new nodes that are randomly connected to the remaining nodes [34, 33, 6]. In socio-economic networks, besides the global selection mechanism, there exists a "local" selection mechanism: the nodes in fact represent agents that form or delete links with other agents, based on the utility that those links may provide to them [41, 28, 8].

The foregoing issue has also attracted researchers in computer science [39, 18, 11] as well as social scientists and economists [32, 1, 2, 25, 26] In particular, the study of networks has become increasingly important in the literature on R&D networks

2000 *Mathematics Subject Classification.* 37N40, 05C50.
Key words and phrases. Algebraic Graph Theory, Innovation Networks, Network Formation.

[21, 17, 12]. Here, the evolution of the network is of interest for the investigation of the efficiency and stability of networks of agents exchanging knowledge in R&D collaborations. In such a context, we have recently introduced a new model of network evolution in which the topology and the state variable of nodes co-evolve [37, 38]. The nodes of the network are associated with a dynamical state variable, representing the utility of the agent, that depends on the links (the R&D collaborations) and the utility of the other agents. The network then evolves according to a prescribed link formation rule, which, in turn, depends on the expected increase of utility of the agents.

Independently of the co-evolution of the network and the utility of the agents, we determine exactly the efficient graphs (in which the aggregate utility of the agents is maximized) and we show that there are stable equilibria which are not efficient. This implies that, if the network evolves through the selfish linking behavior of agents, it may not reach an efficient equilibrium. This result is of interest for policy design questions how to establish incentive mechanisms and legal frameworks in order to help the system to reach its efficient equilibrium. Interestingly, the model is also able to reproduce some of the main stylized facts of empirical studies on R&D networks [27, 42, 12] - namely that such networks are sparse, clustered and heterogeneous in degree - and therefore offers a candidate framework to explain the formation of these networks.

In this paper, we consider the same model of [37] and we investigate further some of its properties, in particular, the relation between the utility of the agents and the properties of the adjacency matrix of the network. In addition, here we also introduce a time delay τ in the decision about keeping or removing a link and we investigate by means of computer simulation how the equilibrium reached by the network is affected by increasing τ. The result is interesting in view of more realistic models for the design of policies that may facilitate the formation of efficient innovation networks.

The paper is organized as follows. First (Sec. 2), we introduce the dynamics of knowledge exchange on a static network. Then we review some results from algebraic graph theory and we discuss their implications for our model (Sec. 3). We proceed by showing the existence of inefficient equilibria (Sec. 4-6). We finally report the results of computer simulations of the evolution of the network, in particular, with respect to impact of the evaluation time of the links (Sec. 7). We finally summarize the results and draw some conclusions (Sec. 8).

2. **Knowledge dynamics and utility function of the agents.**

In this Section we describe the dynamics of the state variable of the nodes in a static network. In Section 5 we extend our studies to the endogenous evolution of the network whereby we introduce the rules for the formation of links.

Consider a set of agents, $N = \{1, ..., n\}$, represented as nodes of an undirected graph G, with an associated variable x_i representing the knowledge of agent i. A link ij, represents the transfer of knowledge between agent i and agent j. Knowledge is shared among an individual's direct and indirect acquaintances and the knowledge level of an agent is proportional to the knowledge levels of its neighbors. We assume that knowledge $\mathbf{x} = (x_1, ..., x_n)$ grows, starting from positive values, $x_i(0) > 0 \ \forall i$, according to the following linear ordinary differential equation

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j, \tag{1}$$

where $a_{ij} = \{0, 1\}$ are the elements of the adjacency matrix $\mathbf{A}$ of the graph G. In vector notation we have $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. In the following we will use the terms network and graph as synonyms.

Similar to [7] we assume that the gross return of agent i is proportional to her knowledge growth rate, with proportionality constant set to 1 for sake of simplicity. We also assume that maintaining a link induces a constant cost $c \geq 0$ for both agents connected by the link. Therefore the net return ρ_i of agent i is given by

$$\rho_i(t) = \frac{\dot{x}_i(t)}{x_i(t)} - cd_i, \tag{2}$$

where d_i denotes the degree of agent i. We assume that the utility function of an agent in a given network is her asymptotic net return $u_i = \lim_{t\to\infty} \rho_i(t)$. As we will show in the next Section, $\lim_{t\to\infty} \frac{\dot{x}_i(t)}{x_i(t)} = \lambda_{\mathrm{PF}}(G_i)$ where λ_{PF} is the spectral radius of the block (sub-) matrix in $\mathbf{A}$ corresponding to the connected component G_i to which i belongs to. Therefore, the utility function of agent i in a given network is

$$u_i = \lambda_{\mathrm{PF}}(G_i) - cd_i. \tag{3}$$

As we will see later on, the evolution of the network stems from each agent trying independently to increase her utility by forming or deleting links. Of course when she does so, this affects the utility of the other agents which will react by forming or removing other links. However, before describing the evolution of the network, we want to discuss some implications of our assumptions on the growth of knowledge and the utility function of the agents.

3. Knowledge growth and properties of the adjacency matrix.

Since the knowledge dynamics is linear and the utility function is proportional to the largest real eigenvalue of the graph, there are many mathematical properties immediately available for the static part of the model. In this Section we review the implications of some well known results for matrices and graphs on the dynamics of knowledge growth in the model. We will only focus on undirected graphs and symmetric matrices respectively.

First of all, since the adjacency matrix in (1) is non-negative and in particular it is a Metzler matrix, the vectorial space $\mathbb{R}^n_+$ is invariant for the linear operator $\mathbf{A}$. It follows that for non-negative initial values (as assumed in the model), it is $\dot{\mathbf{x}}(\mathbf{t}) \geq 0$ and $\mathbf{x}(\mathbf{t}) \geq 0$, $\forall t > 0$ [45]. This ensures the following property:

Proposition 1. *The values of knowledge* $\mathbf{x}$ *in (1) are non-negative for all times.*

For convenience of the reader we report below some facts and definitions that we need in the succeeding Sections.

A walk in the graph is an alternating sequence of nodes and links. The k-power of the adjacency matrix is related to walks of length k in the graph. In particular, $\left(\mathbf{A}^k\right)_{ij}$ gives the number of walks of length k from node i to node j [20]. A connected component of a graph is a maximal subgraph in which there exists a walk from every node to every other node. The graph is connected when the only connected component is the graph itself. If the adjacency matrix can be decomposed in blocks, each block corresponds to a connected component.

An $n \times n$ matrix A is said to be a reducible matrix if and only if for some permutation matrix P, the matrix $P^T AP$ is block upper triangular. If a square matrix is

not reducible, it is said to be an irreducible matrix. The adjacency matrix of a connected graph is always irreducible [30] and in particular it cannot be decomposed in multiple blocks. Irreducible matrices can be primitive or cyclic (imprimitive) [45]. This distinction is important because some result about the convergence of the knowledge values holds only for graphs with primitive adjacency matrix.

For a primitive, non-negative matrix $\mathbf{A}$ it is $\mathbf{A}^k > 0$ for some positive integer $k \leq (n-1)n^n$ [30]. This means that, $\mathbf{A}$ is primitive if, for some k, there is a walk of length k from every node to every other node. Notice that this definition is a much more restrictive than the one of irreducible (or connected) graph in which it is required that there exits a walk from every node to every other node, but not necessarily of the same length. A graph is said to be primitive if its associated adjacency matrix is primitive.

The general solution [30, 35, 4] of the system of linear ordinary differential equations in Eq. (1) is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0), \tag{4}$$

where $\mathbf{x}(0)$ is the initial state and $e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}$ is the matrix exponential. The real, symmetric matrix $\mathbf{A}(G)$ is diagonalizable [23] and thus, the matrix exponential can be written as

$$e^{\mathbf{A}t} = \mathbf{S} \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \mathbf{S}^{-1}. \tag{5}$$

$\lambda_{\mathrm{PF}} = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ are the real eigenvalues of $\mathbf{A}$ and S is a non-singular matrix whose columns are the eigenvectors of $\mathbf{A}$. Accordingly, for (4) we get [50]

$$\mathbf{x}(t) = \sum_{i=1}^{n} c_i \mathbf{v}_i e^{\lambda_i t}, \tag{6}$$

where c_i are unknown constants, that are determined by the initial values $\mathbf{x}(0) = \sum_{i=1}^{n} c_i \mathbf{v}_i$, and $\mathbf{v}_1, ..., \mathbf{v}_n$ the eigenvectors corresponding to the eigenvalues.

If the graph G is connected, then the largest real eigenvalue of G dominates all other eigenvalues in the exponent for each component i in Eq. (6). Since the matrix $\mathbf{A} \geq 0$ is non-negative, this eigenvalue coincides with the spectral radius and with the Perron-Frobenius eigenvalue, λ_{PF} (see below). Thus, it is straightforward to see that the ratio $\frac{\dot{x}_i}{x_i}$ is dominated, for $t \to \infty$, by λ_{PF},

$$\lim_{t\to\infty} \frac{\dot{x}_i}{x_i} = \lambda_{\mathrm{PF}}. \tag{7}$$

If the graph is disconnected, the agents in disconnected components G_i have uncoupled equations of the form (1) that can be solved separately. For each agent i, the asymptotic growth rate in Eq. (7) is then given by the largest real eigenvalue of the connected component of i. More precisely, let $G = (V, E)$ be a graph with connected components $G_1, G_2, ..., G_l$. The set of eigenvalues of G, i.e. the spectrum of G, is the union of sets of eigenvalues of the components. Thus, $\lambda_{\mathrm{PF}}(G) = \max_j\{\lambda_{\mathrm{PF}}(G_j)\}$ [43, 10].

In the following, we repeat here the Perron-Frobenius theorem in a formulation convenient to our context [45].

Theorem 3.1 (The Perron-Frobenius Theorem)**.** *Let A be a non-negative matrix. Then (1) the spectral radius is an eigenvalue, (called λ_{PF}) and all other eigenvalues*

are smaller or equal in absolute value; (2) λ_{PF} is associated to one or more non-negative eigenvectors and, (3) λ_{PF} is bounded from below and above as follows: $\min_i \sum_j a_{ij} \leq \lambda_{PF} \leq \max_i \sum_j a_{ij}$.

If, in addition, A is an irreducible matrix, then (4) λ_{PF} has multiplicity 1 and (5) the associated eigenvector is positive.

If, in addition, A is a primitive matrix, then (6) λ_{PF} is strictly greater in absolute value than all other eigenvalues.

Notice that, going from non-negative to irreducible matrices the eigenspace of λ_{PF} reduces from several non-negative eigenvectors to only one positive eigenvector. In the limit of large t, the terms related to the largest real eigenvalues will dominate in Eq. (6). In particular, it can be shown that for large t, the vector $x(t)$ converges (in direction) to a linear combination of Perron eigenvectors (associated to the Perron eigenvalue λ_{PF}) [30], where the specific linear combination may depend on the initial conditions. In particular, if the adjacency matrix is primitive, the Perron eigenvector is unique and there is a unique stable attractor. Interpreting the result in our model, one can say that

Proposition 2. *If the graph of interaction between agents is primitive, there is a unique asymptotic distribution of relative values of knowledge $\mathbf{x}/\sum_{j=1}^{n} x_j$ given by the Perron eigenvector of the adjacency matrix $\mathbf{A}$.*

If the assumption of primitivity of the matrix falls, in particular if the matrix is non-negative but not irreducible, then there are, in general, several Perron eigenvectors and thus several possible equilibria for the relative values of knowledge, depending on the initial condition.

It is useful to look at an alternative but equivalent way to characterize a primitive graph. A graph G is primitive if and only if it is connected and the greatest common divisor of the set of length of all cycles in G is 1 [49]. This means, for instance, that the connected graph consisting of two connected nodes is not primitive as the only cycle has length 2 (since the link is undirected a walk can go forward and backward along the link). Similarly, a chain or a tree is also not primitive, since all cycles have only even length. However, if we add one link in order to form a triangle, the graph becomes primitive. The same is true, if we add links in order to form any cycle of odd length. We can state the following result.

Proposition 3. *If the graph G is connected, the presence of one cycle of odd length is a sufficient condition for the primitivity of G and hence for the uniqueness of the relative knowledge distribution $\mathbf{x}/\sum_{j=1}^{n} x_j$ given by the Perron eigenvector.*

We now discuss the relation between walks in the graph and growth rate of knowledge. In our model, a walk in the graph corresponds to a sequence of agents contributing to their individual knowledge to their neighbors in the walk in order to generate a sequence of recombined knowledge. As mentioned in the beginning, each component of the power k of the adjacency matrix, $\left(\mathbf{A}^k\right)_{ij}$, gives the number of walks of length k from node i to node j. Considering the vector $\mathbf{u} = (1, ..., 1)$, we have that $n_k := \mathbf{u}^T\mathbf{A}^k\mathbf{u}$ is the number of all walks of length k among all nodes in G. Since the adjacency matrix is symmetric we have that $\mathbf{u} = \sum a_i\mathbf{w}_i$ where $\mathbf{w}_i$ is the eigenvector of $\mathbf{A}$ associated with the eigenvalue λ_i. It follows that $n_k = \sum_i |a_i|^2\lambda_i^k$. For large k, we have approximately $n_k \sim \lambda_{\mathrm{PF}}^k$ [9], and we get

$$\frac{n_k - n_{k-1}}{n_{k-1}} \sim \lambda_{\mathrm{PF}} - 1. \tag{8}$$

Thus, the largest real eigenvalue λ_{PF} of the graph measures the growth rate of the number of walks of length k when the length increases by one, as well as the growth factor of the number of knowledge recombinations in the network of collaborations. As we have seen in the first part of this Section, λ_{PF} coincides also with the asymptotic growth rate of knowledge in time. Therefore, the faster the number of walks in the graph (and thus of knowledge recombinations) grows with the length of the walks, the faster also grows in time the knowledge of the agents involved. One should not confuse the two growth rates, one in time and the other with respect to walk length (which does not vary in time, as we are analyzing a static network).

A similar interpretation comes from the Rayleigh-Ritz theorem [30] which states that:

$$\lambda_{\mathrm{PF}} = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T \mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}, \tag{9}$$

where the maximum is obtained for the Perron eigenvector associated with λ_{PF}. Here, $\mathbf{x}$ can be any vector in $\mathbb{R}^n$. $x_i x_j$ can be interpreted as the result of the recombination of the knowledge of agents i and j if they are connected. Accordingly, one can interpret the right-hand side of Eq. (9) as the maximum number of total knowledge recombinations, $\mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i,j} x_i a_{ij} x_j$, normalized to the absolute total knowledge, $\mathbf{x}^T\mathbf{x} = \sum_{i=1}^n x_i^2$. Some other results relate λ_{PF} to the number of links or the degree of the nodes in the graph. For instance, the Perron-Frobenius theorem states that λ_{PF} is bounded from below and above by the minimum and maximum degree respectively ($d_i = \sum_j a_{ij}$ is the degree of node i). This means, that the higher (minimum or maximum) the degree of the nodes in the graph, the higher λ_{PF} and thus the knowledge growth rate. We denote the maximum degree in G by Δ. Then, a better lower bound holds so that $\sqrt{\Delta} \leq \lambda_{\mathrm{PF}}(G) \leq \Delta$. We refer to [15, 14] for other inequalities involving λ_{PF}.

There is also a result about the inequality of the growth rate of knowledge across agents. For a primitive matrix $\mathbf{A}$ one can show [3] that the Perron-Frobenius eigenvector associated with the eigenvalue λ_{PF} is the solution to the following optimization problem

$$\max_{\mathbf{x}>0} \min_{1\leq i\leq n} \frac{\sum_{j=1}^n a_{ij} x_j}{x_i}, \tag{10}$$

where $\sum_{j=1}^n a_{ij} x_j = (\mathbf{A}\mathbf{x})_i = \dot{x}_i$. The Perron eigenvector is the vector that maximizes the minimum growth factor over all agents i and also minimizes the maximum growth factor. By maximizing the minimum growth factor we obtain balanced growth [3]. In terms of our model:

Proposition 4. *If the graph G is primitive, the unique stable distribution of relative knowledge values $\mathbf{x}/\sum_{j=1}^n x_j$ to which the dynamics (1) converges, is also the distribution that minimizes the difference between maximum and minimum growth rates across agents.*

From the results above we can conclude that the utility function of agent i in Eq. (3) increases with the number of walks in the connected component to which agent i belongs to. On the other hand the utility decrease with the degree of the agent. Therefore it is best for an agent to be able to reach the other agents through many walks but to have not too many links. We now compare this utility function with other similar utility functions in the literature on innovation networks that depend on the position of an agent in the network. For instance, the utility function of [32]

is given by

$$u_i = \sum_{i=1}^{n} \delta^{d(i,j)} - cd_i, \tag{11}$$

where $0 \leq \delta \leq 1$ and $d(i,j)$ is the length of the shortest path from node i to node j. Other examples are those introduced by [28, 8] and [18, 11].

The cost term in our utility function (3) is the same as in (11). The difference is in the benefit term: while the latter utility function only considers the shortest path we take into account all walks across. It has been argued that that knowledge gets transferred not only along the shortest path but also along all other paths in the network [48]. Accordingly, all agents to which agent i is indirectly connected to, contribute to the utility of agent i in our model. [1, 36] introduce a utility function of the form

$$u_i = |G_i| - cd_i, \tag{12}$$

where $|G_i|$ is the size of the connected component of agent $i \in G_i$, that is the number of agents who can be reached by agent i in the network G. This utility function takes into account all agents that agent i can reach and it is higher the more agents there are in its connected component. The difference between (12) and our utility function (3) is that, in our model, not only the number of agents that can be reached (size of the connected component $|G_i|$) but also the structure of the component contributes to the utility of the agent.

4. **Efficiency.** In this Section we define the efficiency of the system (the social optimum for all agents) and we show that if cost is not too high ($c < 1/2$), the complete graph is efficient. In the following Section (6) we show that however, during the evolution, the system does not necessarily reach the efficient network and can very well stabilize in inefficient networks. For the investigation of the set of efficient graphs in the whole range of cost, see [37]

Definition 4.1. The performance $\Pi(G)$ of the network G is defined as the sum of the individual utility

$$\begin{aligned} \Pi(G) &= \textstyle\sum_{i=1}^{n} u_i \\ &= \textstyle\sum_{i=1}^{n} (\lambda_{\mathrm{PF}}(G_i) - cd_i) \\ &= \textstyle\sum_{i=1}^{n} \lambda_{\mathrm{PF}}(G_i) - 2mc, \end{aligned} \tag{13}$$

where m denotes the number of edges in G and G_i is the connected component to which agent i belongs.

If G is connected, then there is obviously only one component. The idea of Definition (13) is that, in order to maximize the performance of the system, one has to maximize total knowledge growth while minimizing the total cost. Π is given by the sum of the individual asymptotic net returns, which is just the sum of the asymptotic individual knowledge growth rates $\frac{\dot{x}_i}{x_i}$ minus the total cost for all links.

The network G^* is called *efficient*, if it maximizes Π over the set of all possible graphs with a given number of nodes:

$$G^* = \operatorname*{argmax}_{G} \{\Pi(G) : |V(G)| = n\}. \tag{14}$$

Following [15] , we will denote the star with n nodes (and $n-1$ edges) as $K_{1,n-1}$ and the complete graph with n nodes as K_n.

We can immediately determine the efficient network, in the special case of null costs, $c = 0$. The case $c < 1/2$ requires some more work.

Proposition 5. *If costs are zero, $c = 0$, then the complete graph K_n is the efficient graph. Its performance is given by $\Pi(K_n) = n\lambda_{PF}(K_n) = n(n-1)$.*

Proof. If costs are zero, then total asymptotic net returns are $\Pi = n\lambda_{\rm PF}$. The graph with the highest eigenvalue is the complete graph K_n with $\lambda_{\rm PF}(K_n) = n-1$ [29]. □

Proposition 6. *The complete graph K_n is efficient for $c < \frac{1}{2}$. For costs $c \geq n$ the empty graph is efficient.*

Proof. Since for the complete graph it is $\lambda_{\rm PF} = n-1$ and $m = \frac{n(n-1)}{2}$, its performance is $\Pi(K_n) = n(n-1) - 2\frac{n(n-1)}{2}c = n(n-1)(1-c)$.

On the other hand, the largest real eigenvalue $\lambda_{\rm PF}$ of a graph G with m edges is bounded from above so that $\lambda_{\rm PF} \leq \frac{1}{2}(\sqrt{8m+1} - 1)$ [46]. For the performance of the system we then have

$$\begin{aligned}\Pi =& \sum_{i=1}^{n} \lambda_{\rm PF}(G_i) - 2mc \leq n \max_{1\leq i\leq n} \lambda_{\rm PF}(G_i) - 2mc \\ \leq& \frac{n}{2}(\sqrt{8m+1} - 1) - 2cm := b(n, m, c),\end{aligned} \tag{15}$$

with $n \leq m \leq \binom{n}{2}$. For fixed cost c and number of nodes n, the number of edges maximizing Eq. (15) is given by $m^* = \frac{n^2-c^2}{8c^2}$ if $\frac{n^2-c^2}{8c^2} < \binom{n}{2}$ and $m^* = \frac{n(n-1)}{2}$ if $\frac{n^2-c^2}{8c^2} > \binom{n}{2}$. The graph with the latter number of edges is the complete graph. Inserting m^* into Eq. (15) yields

$$b(n, m^*, c) = \begin{cases} \frac{n}{2}(\sqrt{\frac{n^2-c^2}{c^2} + 1} - 1) - \frac{n^2-c^2}{4c} & c > \frac{n}{2n-1} \\ n(n-1)(1-c) = \Pi(K_n) & c < \frac{n}{2n-1}. \end{cases} \tag{16}$$

The bound for $c \leq \frac{n}{2n-1} \sim \frac{1}{2}$ coincides with the performance of the complete

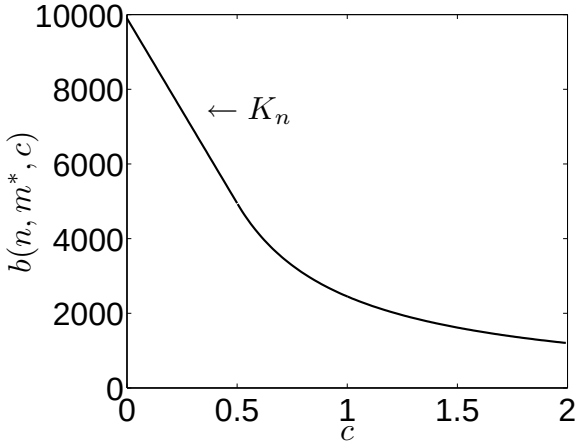


FIGURE 1. Upper bound $b(n, m^*, c)$ of Eq. (16) for $n = 100$ and varying costs c. For $c \leq \frac{n}{2n-1}$ the upper bound is given by the complete graph K_n.

graph, K_n which is therefore the efficient graph. If instead $c = n$ then $m^* = 0$ and the efficient graph is the empty graph. This concludes the proof. Notice that

a similar result can be obtained using an alternative bound for connected graphs, $\lambda_{\mathrm{PF}} \leq \sqrt{2m-n+1}$ due to [29]. □

5. **Network evolution.** In our model we assume that agents have a certain inertia for creating new links and evaluating their existing ones. The rate at which links are formed is much slower than the rate at which knowledge flows (and the knowledge stocks of agents change). In other words, there are two different time scales in our dynamical system: the fast dynamics of the knowledge levels and returns and the slow evolution of the network. The returns immediately reach their quasi-equilibrium state, whereas the network remains unchanged during this short adaptation time. One can say that the variables with the fast dynamics are "slaved" by the variables with the slow dynamics [24] (see [22] for a review)[1]. We assume that the knowledge growth rate has got very close to its asymptotic value when agents create new links.

In the network evolution process, pairs of agents are asynchronously updated. Let T count the number of such updates. At every network update T the following steps are taken: (1) Two agents, not already connected, are uniformly selected at random to form a link and the creation time T (birth date) is recorded for that link. (2) All links that have been previously created and that are as old as τ are evaluated (with τ as an exogenous parameter). For the evaluation of a link, the incident agents compare their current utility at T with the utility before the creation at $T-(\tau+1)$, i.e. before the birth date of the link. The link is maintained only if both agents strictly increase their utility[2], otherwise the link is removed. (3) Finally. the age of all links is increased by one, $T \to T+1$. This process is represented in the following algorithm.

1 Initialization: empty graph
2 **quasi-equilibrium** (fast knowledge growth/decline):
With **A** fixed, the knowledge grows according to Eq. (1) to reach constant growth rates ("balanced" growth).
3 **perturbation**: network update (slow network evolution)
(i) A pair of agents is randomly chosen to create a link.
(ii) The performance of the links attaining an age of τ is evaluated[3]:
if both utilities have increased → keep the link
otherwise → remove the link
4 Stop the evolution if the network is stable[4], otherwise $T \to T+1$ and go to **2**

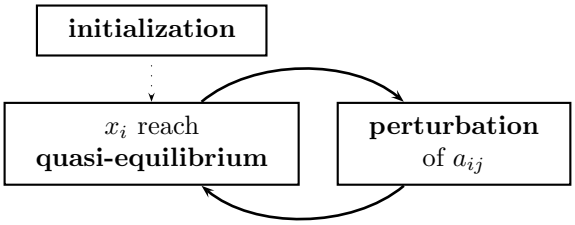
In the version of the model analyzed here the links that pass the evaluation τ network updates after their creation remain in the graph forever. Some of the results presented here still hold if this hypothesis is relaxed and links are evaluated again in the future and possibly deleted, but we do not consider this case in the present paper. Moreover, we assume that the costs for a link are identical for both

[1] This principle has been used e.g by [33, 34] in the context of evolutionary biology. Subsequently [44] have applied their model to an innovation system.

[2] Note that in the mean time the network and also the neighbors of the agents that are evaluating the link may have different knowledge values which in turn affects the utility of the agents. This means that the utility of the agents may have increased due to different reasons than the link that is currently evaluated. In our model agents do not distinguish which of their links is responsible for an increase or decrease in their utility separately but they rather observe the overall effect on their utility by all their links at the same time.

[3] If $\tau = 1$ the link that is created is immediately evaluated afterwards.

[4] The notion of stability is defined in (6.1)



adjacent agents and its creation requires their mutual consent. Finally, the local process of formation of links described above intends to mimic the process by which selfish agents improve their utility through a trial and error method.

6. **Stability.** In this Section we first give a definition of stability as a stationary network resulting from the network formation process described in the previous Section. We then give two examples of stable networks that are not efficient, the star $K_{1,n-1}$ and the clique K_n. Finally we derive an upper bound for the cost of links above which the complete graph is not reachable (and for costs $c < \frac{1}{2}$ it is the efficient network). In the proofs of this Section we assume an evaluation period of $\tau = 1$. This means that links are evaluated immediately after they are created.

The network evolution is in a stable equilibrium if it is bilaterally stable. A network G is bilaterally stable if and only if no pair of agents, i and j, can create a bilateral connection such that the utilities of both agents at the evaluation period τ are higher than the current utilities. More formally,

Definition 6.1. For a fixed value of τ G is bilaterally stable at time t if there does not exist a pair i, j such that both $u_i(t+\tau) > u_i(t)$ and $u_j(t+\tau) > u_j(t)$.

This definition is similar to the notion of pairwise stability introduced earlier by [31]. Let $G + uv$ denote the graph obtained by adding a link uv to the existing graph G. The addition of one edge uv leads to a change in the individual utility u_i (cf. Eq. (3))

$$\begin{aligned} \Delta u_i &= u_i(G+uv) - u_i(G) \\ &= \lambda_{\mathrm{PF}}(G+uv) - c(d_i+1) - (\lambda_{\mathrm{PF}}(G) - cd_i) \\ &= \Delta\lambda_{\mathrm{PF}} - c. \end{aligned} \tag{17}$$

Proposition 7. *For $\tau = 1$ and any value of cost c, there exists a number of nodes n such that the star $K_{1,n-1}$ is bilaterally stable .*

Proof. We consider a graph with consisting in the star $K_{1,n-1}$ as a subgraph and some isolated nodes and we show that if n is large enough, the benefit of any additional link is smaller than a given cost c. As shown in Fig. 2, there are only three types of links that can be added, either between the nodes of the star or by attaching a disconnected node.

(i) By adding a leaf to the central node of the star $K_{1,n-1}$, (link (i) in Fig. 2), the change in eigenvalue is $\Delta\lambda_{\mathrm{PF}} = \sqrt{n} - \sqrt{n-1}$ which is monotonically decreasing with $n > 0$. For any given c if $\sqrt{n} - \sqrt{n-1} < c$ then no new node will be attached to $K_{1,n-1}$.

(ii) If a link is created from a disconnected node to a peripheral node in the star (link (ii) in Fig. 2), then the resulting eigenvalue is smaller than the eigenvalue obtained from the link created to the central node in the star. This comes

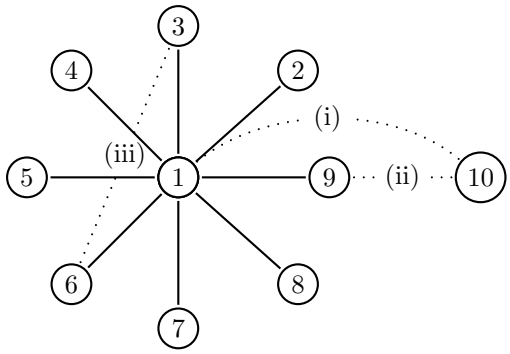


FIGURE 2. A star $K_{1,8}$ and an additional edge, appended either from an isolated node to the nodes in the star or between two nodes in the star.

from the fact that the former adjacency matrix is not stepwise while the latter is, see [5].

(iii) The case of a link between two nodes in the star (link (iii) in Fig. 2) requires a little more work. We use an upper bound due to [40] on the increase of the largest real eigenvalue $\Delta\lambda_{\mathrm{PF}}$ of a connected undirected graph G, if an edge ij is added. The upper bound depends on the component i and j of the eigenvector $\mathbf{x}$ associated to λ_{PF}:

$$\lambda_{\mathrm{PF}}(G+ij) - \lambda_{\mathrm{PF}}(G) < 1 + \delta - \frac{\delta(1+\delta)(2+\delta)}{(x_i+x_j)^2 + \delta(2+\delta+2x_ix_j)}, \tag{18}$$

where δ denotes the minimum degree in the graph G. With this upper bound we can compute the change in individual utility by the addition of an edge, $\Delta u_i = \Delta\lambda_{\mathrm{PF}} - c$.

The characteristic polynomial of the star $K_{1,n-1}$ is given by $(\lambda^2 - (n-1))\lambda^{n-2}$. Thus the largest real eigenvalue is $\lambda_{\mathrm{PF}} = \sqrt{n-1}$. The corresponding normalized eigenvector is given by $\frac{1}{2(n-1)}(1,...,1,\sqrt{n-1},1,...,1)^T$. Applying Eq. (18) to the star $K_{1,n-1}$ gives $\Delta\lambda_{\mathrm{PF}} = (\lambda_{\mathrm{PF}}(K_{1,n-1}+ij) - \lambda_{\mathrm{PF}}(K_{1,n-1})) < 2 - \frac{4n^2}{1+2n^2}$ which leads to $\Delta u_i < \frac{2}{1+2n^2} - c$. For $n \geq \sqrt{\frac{2-c}{2c}}$, Δu_i becomes negative and therefore adding the link is not profitable.

Combining the results above, and noticing that $\sqrt{n} - \sqrt{n-1} > \frac{2}{2n^2+1}$, we can conclude that, for any c, no link of either one of the three types is added to the star for n large enough. □

Notice that the bound of [40] which we used in the first part of the previous Proposition holds only for connected graphs and cannot be used when adding a link that connects a graph to a previously disconnected node. For the next proofs we will use a bound on the increase of the largest real eigenvalue $\Delta\lambda_{\mathrm{PF}}$ when a link is added, which depends only on the number of links and nodes in the graph, regardless of the structure of the links.

Proposition 8. *The change in the largest real eigenvalue, $\Delta\lambda_{PF}$ of a graph G with m edges and n nodes, by adding one edge to the graph is bounded as follows*

$$\Delta\lambda_{PF} \leq \frac{1}{2}(-1 + \sqrt{1+8(m+1)}) - \frac{2m}{n}. \tag{19}$$

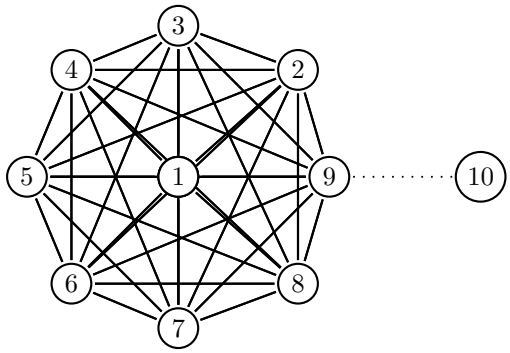


FIGURE 3. A complete graph K_9 and a node appended.

Proof. The average degree of the graph is $\bar{d} = \frac{2m}{n}$. A lower bound on the largest real eigenvalue is given by $\lambda_{\rm PF} \geq \bar{d}$ [15]. An upper bound on the largest real eigenvalue is given by $\lambda_{\rm PF} \leq \frac{1}{2}(-1 + \sqrt{1+8m})$ [46]. Combining the two bounds yields the Proposition. □

Proposition 9. *For $\tau = 1$ and any value of cost c, there exists a number of nodes n such the clique K_n is bilaterally stable.*

Proof. We consider the graph G' obtained by connecting a clique K_n and an isolated node via and edge (see Fig. 3). We consider the increase of the largest real eigenvalue, $\Delta\lambda_{\rm PF} = \lambda_{\rm PF}(G') - \lambda_{\rm PF}(K_n)$ and we apply the bound of prop. 8. Since $\bar{d} = n - 1$ in the clique, we have $\Delta\lambda_{\rm PF} \leq \frac{1}{2}(-1 + \sqrt{1+8(m+1)}) - (n-1)$ which is smaller than c for $n > n^* = \frac{2+c(1+c)}{2c}$, as one can check solving the inequality for $m = \frac{n(n-1)}{2} + 1$. □

There is another bound on the change of $\lambda_{\rm PF}$ for bilateral link deletion or creation: if the undirected connected graphs G and G' differ in only one edge then $|\lambda_{\rm PF}(G) - \lambda_{\rm PF}(G')| \leq 1$ [16]. This bound is weaker than the ones previously introduced, but it is still useful to derive the following Proposition.

Proposition 10. *If costs are higher than one, $c > 1$, and $\tau = 1$, then no agent will create any link. Any graph is bilaterally stable and in particular, the empty graph is bilaterally stable.*

Since Eq. (1) implicitly assumes benefit equal 1 from a collaboration, the case $c > 1$ is somehow an extreme case, because the cost of a link is higher than the benefit and therefore the result above is not surprising.

We now prove that the efficient graph is not necessarily reached by the evolution (see also Fig. 4).

Proposition 11. *For $\tau = 1$ and cost $c < 1/2$ the efficient, complete graph K_n of size $n \geq \frac{2}{c}$ cannot be reached by the network formation process.*

Proof. We apply the bound of Proposition (8) on the change in the largest real eigenvalue, $\Delta\lambda_{\rm PF}$, by adding an edge to the graph G with m edges. Solving the equation $\Delta\lambda_{\rm PF} = c$ for m yields the maximal number m^* of edges that can be added to a graph of n nodes when the cost is c, $m^*(n,c) = \frac{n}{4}(-1 - 2c + n +$

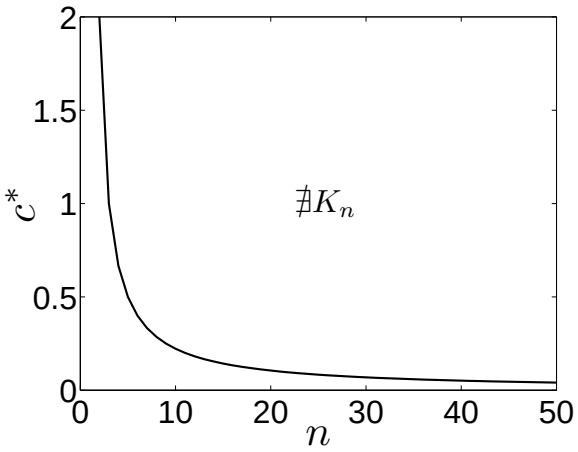


FIGURE 4. Maximal value of cost c for which the complete graph can be obtained as an equilibrium network.

$\sqrt{n^2+9-2n(1+2c)}$). Notice that $m^*(n,c)$ decreases with increasing cost c. Imposing now this expression to be equal to one edge less than the number of edges in a complete graph K_n of n nodes, $\binom{n}{2}-1 = \frac{n(n-1)}{2}-1$, we get $c^* = \frac{2}{n}$. Thus, if costs exceed this value then the increase in eigenvalue corresponding to the creation of the link that would make the graph complete, is smaller than the cost. Notice that c^* decreases with n and tends to 0 for large n, as plotted in Fig. 4, and therefore for any given c there is an n large enough such that the complete graph cannot be reached. □

Similar to previous works of other authors [32, 1, 11] we find stars and cliques to be stable structures for a given value of cost for the links. However, in this model, there is a limit size above which these networks can be stable. Moreover, for a same level of cost, one can obtain both a star or a complete graph (with different n) as stable equilibria of the dynamics. This points to the existence of multiple equilibria, as investigated more thoroughly in [37]

7. **Simulation studies of network evolution.** For multiple realizations (simulations)[5] we study the evolution of the network and the stable equilibrium networks reached by this evolution. In order to characterize the networks, we introduce some simple network measures:

(i) The network density $s(G)$ of a graph G is defined as the number of links m divided by the maximum number of links $\frac{n(n-1)}{2}$, i.e. $s(G) := \frac{2m}{n(n-1)}$. $s(G)$ measures how sparse a network is.

(ii) The relative performance $\pi(G) := \frac{\Pi(G)}{\Pi(K_n)} = \frac{\Pi(G)}{n(n-1)(1-c)}$. $\pi(G)$ measures the relative performance of a network compared to the complete graph. In Section 4 we have shown that for costs $c < \frac{1}{2}$ the complete graph is efficient. Thus, a value of the relative performance smaller than one is a measure of the inefficiency of the network.

[5]When simulating the network evolution the largest real eigenvalue of the network has to be computed many times. Since the largest real eigenvalue of a graph can be computed in polynomial time [29] our model is well suited for numerical investigations.

(iii) The *clustering coefficient* $\mathcal{C}_i$ for agent i is the proportion of links between the agents within its neighborhood N_i divided by the number of links that could possibly exist between them. More formally, let $E(G)$ denote the set of edges of the graph G and $d_i = |N_i|$ the degree of agent i, then

$$\mathcal{C}_i = \frac{2|\{jk : j,k \in N_i \wedge jk \in E\}|}{d_i(d_i-1)}. \tag{20}$$

The total clustering coefficient is the sum of the clustering coefficients for each agent, $\mathcal{C} = \sum_{i=1}^{n} \mathcal{C}_i$. The global clustering coefficient is at most one.

We first study the networks obtained with an evaluation period $\tau = 1$, which means that agents evaluate their links immediately after reaching their balanced growth rates. We then investigate the density and efficiency of the stable equilibrium networks that are reached, if τ is longer than 1. This means that agents are evaluating their bilateral links after several other agents may have created bilateral links.

In Fig. (5), the evolution of the network measures mentioned above (the network density, the relative performance, average degree and the global clustering coefficient) is shown for three particular realizations with $n = 30$[6] agents and different values of cost, $c \in \{0.01, 0.2, 0.5\}$, $c \leq 0.5$. These cost values range from a scenario in which the formation of links is rather cheap over an intermediate level of cost to high costs and they were chosen such that the complete graph is known to be the efficient network. The values we measure are relative quantities with respect to the complete graph. One can see that for $c = 0.2$ and $c = 0.5$, the stable equilibrium network is inefficient, sparse and highly clustered. It is important to notice that those agents with high degree, which bear the cost of many interactions, have smaller utility than those with a smaller degree. This is also indicated by the colors of the nodes in Fig. (6). The agents with small degree are benefiting to a larger extent than the high degree agents. This comes from the properties of the largest real eigenvalue of the adjacency matrix. The eigenvalue of the network, which determines the positive contribution to the individual growth rates, is the same for all the agents in the same component, but the costs are depending on the degree. Accordingly, the nodes with high degree have the same return as the nodes with small degree from the network but they have to incur higher costs.

In Fig. (6)[7] the stable equilibrium networks for 30 agents and three different values of the cost are shown. The links between the nodes in the figure indicate the mutual exchange of knowledge between agents (R&D collaborations). For small costs, $c = 0.1$, the complete and efficient graph is reached. For intermediate costs, $c = 0.2$, a sparse and highly clustered graph with a highly heterogeneous degree distribution is obtained. For high costs, $c = 0.5$, the stable equilibrium network consists of many small clusters. For exceedingly high costs the stable equilibrium graph is identical to the initial, empty graph. One can see that by decreasing the cost the size of the connected components grows. This is consistent with what has been observed in a recent study by [27] on R&D collaborations of firms in the IT industry.

[6] We have chosen a small network with 30 agents to make the network layouts more easily readable and we have used this system sizes for all measures under investigation in order to keep the results consistent. Moreover, the returns from a link are a function of the connected component of an agent and not of the system size. Thus, the size of the connected components formed in the equilibrium depend on the cost and not on the total number of agents. By increasing the system size we therefore do not expect to obtain significantly different results than with the current size.

[7] The graphs were plotted with a network layout algorithm introduced by [19].

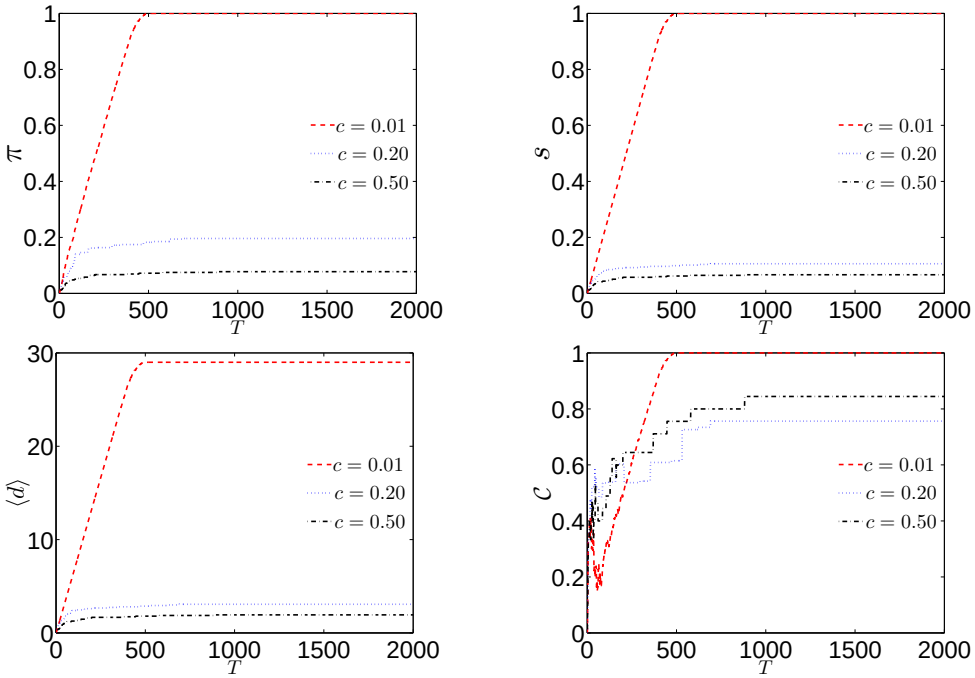


FIGURE 5. Evolution of the network starting from an empty graph until reaching its stable equilibrium configuration, Fig. (6 bottom). The stable equilibrium network for intermediate costs $c = 0.2$ is inefficient, sparse and is highly clustered, while for small costs $c = 0.01$ the efficient complete graph is realized.

In Fig. (7) network density $s(G)$ and relative performance $\pi(G)$ are shown for increasing values of the evaluation period τ (10 realizations for every value of τ) for $n = 30$ agents and intermediate costs $c = 0.2$. If the evaluation period is long enough, the efficient graph, i.e. the complete graph, can be reached. Thus, if agents are evaluating their interactions in the long-term, the performance of the system can be increased up to the efficient state.

8. **Conclusion.** In this paper, we consider the economic model of network evolution introduced in [37] in the context of innovation and R&D collaborations. The model is characterized by two time scales: there is a fast dynamics on the state variable of the nodes, representing their knowledge, and a slow dynamics on the links of the graph. Since the fast dynamics is linear and occurs on a static graph, there is a number of well known results form the theory of matrices that can be applied to the model and we have reviewed the most important of them. For what concerns the evolution of the network we have used some results from the theory of graph spectra to derive some Propositions on the efficiency and stability of the network. In particular, we have provided a simple proof of the existence of equilibria, like the star and the clique, that for $c < 1/2$ are not efficient but are stable. The existence of inefficient equilibria is of interest to economists because it raises the issue of how to design appropriate policies to help the system to reach the efficient equilibria. Our simulations confirm the analytical results and show that the interplay between

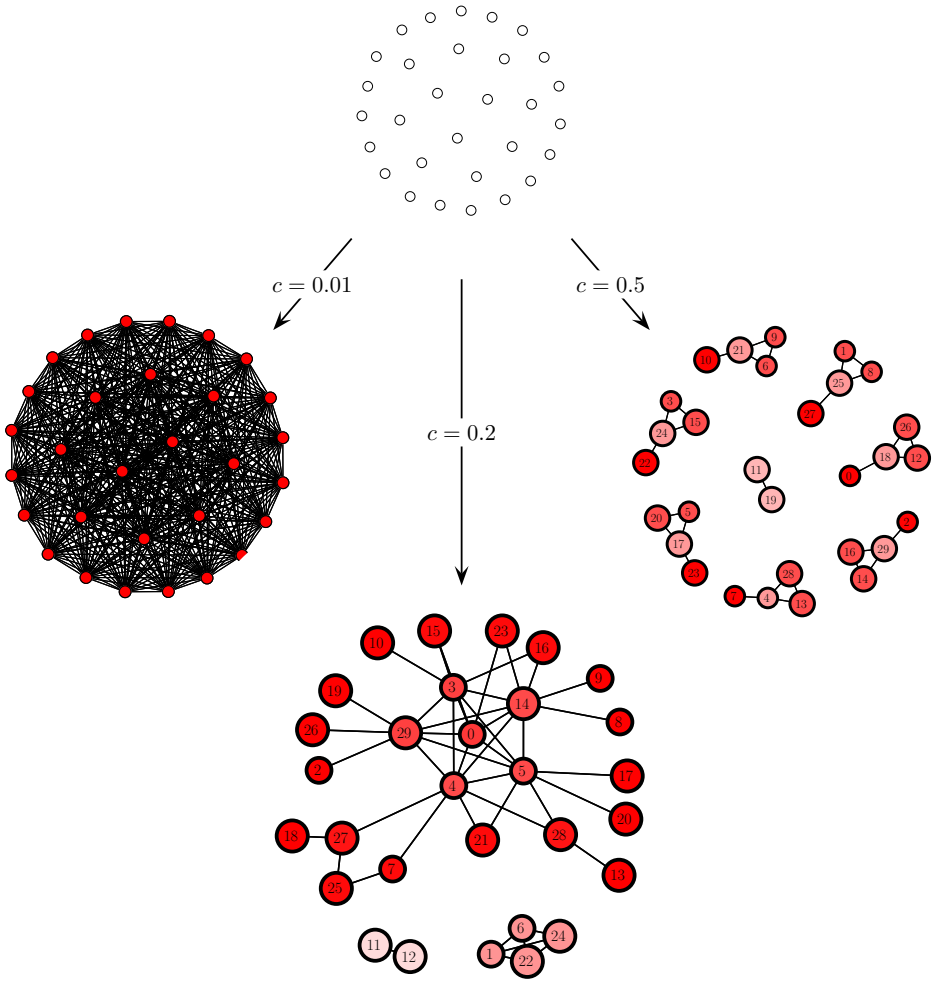


FIGURE 6. Initial (empty) network (top) and stable equilibrium networks for costs $c = 0.01$ (left), $c = 0.2$ (bottom) and $c = 0.5$ (right). For all values of cost the complete graph is most efficient, but only for a very small cost, $c = 0.01$ (left), it is reached in the network evolution. For intermediate costs, (bottom), those agents with a high degree, that are maintaining many links, have smaller utility than those with a small degree. The color saturation of the nodes indicates the utility of the agent compared to the maximum utility.

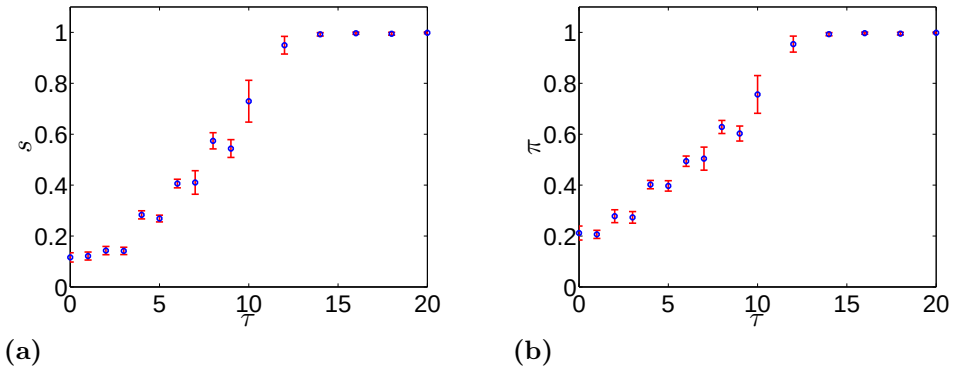


FIGURE 7. Density (**a**) and relative performance (**b**) for the stable equilibrium networks with cost $c = 0.2$, $n = 30$ agents and 10 realizations for every evaluation period $\tau \in [0, 20]$. By increasing the evaluation period τ, the efficient graph, K_n, is reached.

dynamics on the nodes and topology of the network leads in many cases to equilibrium networks which are not efficient and are characterized, as observed in empirical studies of R&D networks, by sparseness, presence of clusters and heterogeneity of degree. In particular, we observe subgraphs of finite size and highly heterogeneous degree distribution among which there are only a few connections. These properties have been observed in empirical studies of innovation networks [13, 27] (for a more systematic comparison with the stylized facts on innovation networks, see [37]).

As an new element, in this paper we also introduce a time τ after which agents evaluate whether to keep or delete a link and we investigate by means of computer simulation how the equilibrium reached by the network is affected by increasing the time τ. If agents evaluate their interactions on a long-term, then they are able to reach an efficient state, which, on the other hand, is not reachable, when collaborations are evaluated in the short-term. In other words, a short-sighted rational behavior in the agents can give rise to inefficient networks, as often happens in reality. Appropriate policy measures could be designed to support economic agents in maintaining interactions even when, in the short run, they may be unprofitable. Our model may serve as a first step towards both a theoretical explanation for the empirical regularities in a R&D networks and a very simple test bed for policy design.

Acknowledgments. We are particularly grateful to Hans Haller for his astute comments and perceptive suggestions. Moreover we are in debited to Koen Frenken, Giorgio Fagiolo, Matthias Feiler, Kerstin Press, Jan Lorentz and Nicolas Carayol for clarifying discussions and helpful comments. Finally we would like to thank the critical audiences at the 5th International EMAEE Conference on Innovation in Manchester, 2007, and the audience at the European Conference on Complex Systems in Dresden, 2007.

REFERENCES

[1] Venkatesh Bala and Sanjeev Goyal, *A noncooperative model of network formation*, Econometrica, **68** (2000), 1181–1229.

[2] Coralio Ballester, Antoni Calvo-Armengol and Yves Zenou, *Who's who in networks. wanted: The key player*, Econometrica, **74** (2006), 1403–1417.
[3] Stephen Boyd, "Linear Dynamical Systems," Lecture Notes, Stanford University, 2006.
[4] Martin Braun, "Differential Equations and Their Applications," An introduction to applied mathematics, Texts in Applied Mathematics, Springer, 4th edition, 1993.
[5] Richard A. Brualdi and Ernie S. Solheid, *On the spectral radius of connected graphs*, Publications de l' Institute Math., **53** (1986), 45–54.
[6] Guido Caldarelli, Andrea Capocci and Diego Garlaschelli, *Self–organized network evolution coupled to extremal dynamics*, Nature Physics, 2007.
[7] Nicolas Carayol and Pascale Roux, "Self-Organizing Innovation Networks: When do Small Worlds Emerge?" Working Papers of GRES - Cahiers du GRES 2003-8, Groupement de Recherches Economiques et Sociales, 2003.
[8] R. Carvalho and G. Iori, *Socioeconomic networks with long-range interactions*, ArXiv e-prints, 706, 2007.
[9] Fan Chung and Linyuan Lu, "Complex Graphs and Networks," American Mathematical Society, 2006.
[10] Fan R. K. Chung, "Spectral Graph Theory," Regional Conference Series in Mathematics, Conference Board of the Mathematical Sciences, 1997.
[11] Jacomo Corbo and David Parkes, *The price of selfish behavior in bilateral network formation*, In "PODC '05: Proceedings of the Twenty-Fourth Annual Acm Symposium on Principles of Distributed Computing," 99–107, New York, NY, USA, 2005. ACM Press.
[12] Robin Cowan and Nicolas Jonard, *Network structure and the diffusion of knowledge*, Journal of Economic Dynamics and Control, **28** (2004), 1557–1575.
[13] Robin Cowan, Nicolas Jonard and Müge Ozman, *Knowledge dynamics in a network industry*, Technological Forecasting and Social Change, **71** (2004), 469–484.
[14] D. Cvetkovic, M. Doob and H. Sachs, "Spectra of Graphs: Theory and Applications," Third edition, Johann Ambrosius Barth, 1995.
[15] D. Cvetkovic and P. Rowlinson, *The largest eigenvalue of a graph: A survey*, Linear and Multinilear Algebra, **28** (1990), 3–33.
[16] Dragos Cvetkovic, Peter Rowlinson and Slobodan Simic, "Eigenspaces of Graphs," volume **66**, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1997.
[17] G. C. M. A. Ehrhardt, M. Marsili and F. Vega-Redondo, *Phenomenological models of socioeconomic network dynamics*, Physical Review E, **74** (2006), 036106.
[18] Alex Fabrikant, Ankur Luthra, Elitza Maneva, Christos H. Papadimitriou and Scott Shenker, *On a network creation game*, In "PODC '03: Proceedings of the Twenty-Second Annual Symposium on Principles of Distributed Computing," 347–351, New York, NY, USA, 2003, ACM Press.
[19] Markus Geipel, *Self-organization applied to dynamic network layout*, International Journal of Modern Physics C, 2007. forthcoming.
[20] Chris D. Godsil and Gordon F. Royle, "Algebraic Graph Theory," Graduate Texts in Mathematics, 207, Springer-Verlag, New York, 2001.
[21] S. Goyal and J. L. Moraga-Gonzalez, *R&D networks*, RAND Journal of Economics, **32** (2001), 686–707.
[22] T. Gross and G. Blasius, *Adaptive coevolutionary networks - a review*, eprint arXiv:0709.1858, (2007).
[23] W. H. Haemers, "Handbook of Linear Algebra," chapter Matrices and Graphs, CRC Press, 2006.
[24] Hermann Haken, "Synergetics - An Introduction; Nonequilibrium Phase Transitions and Self-Organization in Physics, Chemistry and Biology," Springer-Verlag, Berlin-New York, 1977.
[25] Hans Haller, Jurjen Kamphorst and Sudipta Sarangi, *(Non-)existence and scope of nash networks*, Economic Theory, **31** (2007), 597–604.
[26] Hans Haller and Sudipta Sarangi, *Nash networks with heterogeneous links*, Mathematical Social Sciences, **50** (2005), 181–201.
[27] Nobuyuki Hanaki, Ryo Nakajima and Yoshiaki Ogura, "The Dynamics of R&D Collaboration in the IT Industry," Working Paper, 2007.
[28] Petter Holme and Gourab Ghoshal, *Dynamics of networking agents competing for high centrality and low degree*, Physical Review Letters, **96** (2006), (098701).
[29] Y. Hong, *Bounds of eigenvalues of graphs*, Discrete Math., **123** (1993), 65–74.

[30] Roger A. Horn and Charles R. Johnson, "Matrix Analysis," Corrected reprint of the 1985 original, Cambridge University Press, Cambridge, 1990.
[31] Matthew O. Jackson, "A survey of Models of Network Formation: Stability and Efficiency," Working Papers 1161, California Institute of Technology, Division of the Humanities and Social Sciences, 2003.
[32] Matthew O. Jackson and Asher Wolinsky, *A strategic model of social and economic networks*, Journal of Economic Theory, **71** (1996), 44–74.
[33] Sanjay Jain and Sandeep Krishna, *Autocatalytic sets and the growth of complexity in an evolutionary model*, Physical Review Letters, **81** (1998), 5684–5687.
[34] Sanjay Jain and Sandeep Krishna, *A model for the emergence of cooperation, interdependence, and structure in evolving networks*, Proceedings of the National Academy of Sciences, **98** (2001), 543–547.
[35] Hassan K. Khalil, "Nonlinear Systems," Prentice Hall, 2002.
[36] Chongmin Kim and Kam-Chau Wong, *Network formation and stable equilibrium*, Journal of Economic Theory, **133** (2007), 536–549.
[37] Michael D. König, Stefano Battiston, Mauro Napoletano and Frank Schweitzer, *Efficiency and stability of dynamic innovation networks*, forthcoming, 2007.
[38] Michael D. König, Stefano Battiston and Frank Schweitzer, "Innovation Networks - New Approaches in Modeling and Analyzing," chapter Modeling Evolving Innovation Networks, Springer Complexity Series, 2007.
[39] E. Koutsoupias and C. H. Papadimitriou, *Worst-case equilibria*, Lecture Notes in Computer Science, **1563** (1999), 404–413.
[40] Christoph Maas, *Perturbation results for the adjacency spectrum of a graph*, ZAMM, **67** (1987), 428–430.
[41] M. Marsili, F. Vega-Redondo and F Slanina, *The rise and fall of a networked society: A formal model*, Proceedings National Academy of Sciences USA, **101** (2004), 1439–1442.
[42] Walter W. Powell, Douglas R. White, Kenneth W. Koput and Jason Owen-Smith, *Network dynamics and field evolution: The growth of interorganizational collaboration in the life sciences*, American Journal of Sociology, **110** (2005), 1132–1205.
[43] R. F. Robinson and L. R. Foulds, "Digraphs: Theory and Techniques," Gordon and Breach Science Publishers, New York, 1980.
[44] Arora Saurabh and Robin Cowan, *The growth of knowledge and complexity in an evolving network model of technological innovation*, Technical report, UNU-INTECH and MERIT, University of Maastricht, The Netherlands, 2004; Paper presented at "Organisations, Innovation and Complexity: New Perspectives on the Knowledge Economy", 9th-10th September 2004, CRIC, University of Manchester, Manchester,England, UK.
[45] Eugene Seneta, "Non-negative Matrices and Markov Chains," Springer, New York, 2006.
[46] Richard P. Stanley, *A bound on the spectral radius of graphs with e edges*, Linear Algebra and its Applications, **87** (1987), 267–269.
[47] Ian Stewart, *Networking opportunity*, Nature, **427** (2004), 601–604.
[48] Stanley Wasserman and Katherine Faust, "Social Network Analysis: Methods and Applications," Cambridge University Press, 1994.
[49] Junming Xu, "Theory and Application of Graphs," Network Theory and Applications, 10, Kluwer Academic Publishers, Dordrecht, 2003.
[50] Daniel Zwillinger, "Handbook of Differential Equations," Academic Press, 1998.

Received November 2007; revised February 2008.

E-mail address: mkoenig@ethz.ch
E-mail address: sbattiston@ethz.ch
E-mail address: mnapoletano@ethz.ch
E-mail address: fschweitzer@ethz.ch