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HOMOGENIZATION AND CORRECTORS FOR THE WAVE EQUATION IN NON PERIODIC PERFORATED DOMAINS

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Abstract. We consider here the wave equation in a (not necessarily periodic) perforated domain, with a Neumann condition on the boundary of the holes. Assuming H^0-convergence ([3]) on the elliptic part of the operator, we prove two main theorems: a convergence result and a corrector one. To prove the corrector result, we make use of a suitable family of elliptic local correctors given in [4] whose columns are piecewise locally square integrable gradients. As in the case without holes ([2]), some additional assumptions on the data are needed.

1. **Introduction.**

In this paper, we are concerned with some convergence and corrector results for the wave equation in perforated domains, when the elliptic part of the operator converges in the sense of the H^0-convergence. This notion, introduced by M. Briane, A. Damlamian and P. Donato in [3], generalizes the G and H-convergences to perforated domains. The G-convergence was introduced by S. Spagnolo ([13]) for symmetric matrices in order to study second order differential operators with oscillating coefficients. The H-convergence was introduced by F. Murat and L. Tartar ([10]) to treat the general nonsymmetric case.

To describe the problem, let $T > 0$ be given and let $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$ be a perforated domain, obtained by removing from a given bounded domain Ω of $\mathbb{R}^n$ a compact set of holes S_ε.

Consider the following problem:

$$\left\{\begin{array}{rcll} \rho_\varepsilon u_\varepsilon'' - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) & = & f_\varepsilon & \text{on } \Omega_\varepsilon \times (0,T), \\ u_\varepsilon & = & 0 & \text{on } \partial\Omega \times (0,T), \\ A^\varepsilon \nabla u_\varepsilon \,.\, \nu & = & 0 & \text{on } \partial S_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) & = & a_\varepsilon & \text{on } \Omega_\varepsilon, \\ u_\varepsilon'(x,0) & = & b_\varepsilon & \text{on } \Omega_\varepsilon, \end{array}\right. \tag{1}$$

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where ν denotes the unitary outward normal to Ω_ε, $\{\rho_\varepsilon\}$ is a sequence of uniformly bounded and positive functions defined on Ω_ε, $\{f_\varepsilon\}$ is a sequence of functions belonging to $L^2(\Omega_\varepsilon \times (0,T))$ and $\{A^\varepsilon\}$ is a sequence of symmetric bounded matrix fields such that

$$\{(A^\varepsilon, S_\varepsilon)\} \quad H^0\text{-converges to } A^0, \tag{2}$$

for some matrix field A^0 (see Definitions 2.1 and 2.3 in Section 2).

We prove two main results. The first one (Theorem 4.3) concerns the convergence of the solution of problem (1.1) to that of the homogenized problem. The second one (Theorem 4.4), more technical, is a corrector result. Under additional assumptions on the data, we prove that

$$\begin{cases} i) \quad \lim_{\varepsilon \to 0} \|u'_\varepsilon - u'\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} = 0, \\ ii) \quad \lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} = 0, \end{cases} \tag{3}$$

where u is the unique solution of the homogenized problem associated with (1.1) and C^ε is an elliptic local corrector (see Definition 3.1).

As shown in Section 7, the convergence of the energy associated with problem (1.1) (defined in Section 6) to that of the homogenized problem is necessary for proving convergence (1.3). On the other hand, as already observed by S. Brahim-Otsmane, G. A. Francfort and F. Murat in [2], where they have studied the homogenization and the correctors for the wave equation with H-convergent elliptic part, more restrictive hypotheses on the data are needed for the convergence of the energy. This explain the additional assumptions in the corrector result.

We first prove the corrector result for the particular family $(C^\varepsilon)_\varepsilon$ of elliptic local correctors given in [4], whose columns are piecewise locally square integrable gradients. It has the property that on suitable subsets of Ω, it can be approximated by a matrix field C^ε_h of the form

$$C^\varepsilon_h e_i = \nabla W^\varepsilon_{h,i}, \qquad \text{a.e. in } \Omega, \tag{4}$$

where for any $i \in \{1, \ldots, n\}$, the function $W^\varepsilon_{h,i}$ is defined through the solution of a suitable elliptic problem in Ω_ε. These properties play an essential role in the proof of the corrector result (1.3), since it allows us to use compensated compactness arguments (a div-curl type lemma for perforated domains, see Proposition 2.6). We recall in Section 3 the construction of this particular corrector and give some related properties. In Corollary 4.5, we show that (1.3) holds true for a more general family of elliptic local correctors.

Let us mention that in the case of a fixed domain (H-convergence), one always has a global corrector in the whole domain Ω (Definition 3.1). In the presence of holes, this is still true for the global correctors constructed in [3], but some additional regularity assumptions on the geometry of Ω_ε have to be assumed (see Theorem 3.3).

In the last section, we split the solution of (1.1) (denoted by v_ε) as a sum of two functions u_ε and z_ε, where u_ε satisfies a problem for which the corrector result applies and z_ε converges weakly to zero. We show that a strong convergence for z_ε is necessary in order to have the corrector result (1.3) for v_ε. This means (see Proposition 8.2 and Remark 8.1) that the assumptions of the homogenization result are not sufficient.

We refer to [1] for the homogenization and the correctors of the wave equation with periodically oscillating coefficients in a fixed domain (see also [6] chap. 12, for detailed proofs). The corresponding homogenization problem in periodically perforated domains has been studied in [5] by D. Cioranescu and P. Donato, and the corrector result has been proved by A. Nabil in [11].

Plan

2. **Definitions and main properties of H^0-convergence.** Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$ and ε the general term of a sequence of positive number which tends to zero.

For any ε, let S_ε be the set of holes, which is a closed subset included in Ω. We denote by Ω_ε the perforated domain $\Omega \setminus S_\varepsilon$ and by V_ε the space

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon) \mid v_{|\partial\Omega} = 0\},$$

equipped with the H^1-norm.

We will use the following notation:

- χ_E for the characteristic function of the set E of $\mathbb{R}^n$,
- $|E|$ for the Lebesgue measure of the subset E,
- $\widetilde{v_\varepsilon}$ for the zero extension on Ω of any function v_ε defined on Ω_ε,
- ν for the unitary outward normal vector with respect to Ω_ε,
- $\nabla u(x,t)$ for the gradient of a function u with respect to the space variables,
- for any function $f : \Omega \times (0,T) \longrightarrow \mathbb{R}$, f' denotes its time derivative,
- $\mathcal{M}(\alpha, \beta, \Omega)$ denotes for two positive reals $\alpha < \beta$, the set of the $n \times n$ matrix fields A defined on Ω and satisfying

$$\begin{cases} A \text{ measurable on } \Omega, \\ A(x)\lambda\lambda \geq \alpha|\lambda|^2, \ |A(x)\lambda| \leq \beta|\lambda|, \quad \forall\lambda \in \mathbb{R}^n, \ a.e.\ x \in \Omega. \end{cases}$$

Definition 2.1. ([3]) The sequence $\{S_\varepsilon\}_\varepsilon$ is said to be admissible in Ω if any L^∞-weak* limit of $\chi_{\Omega_\varepsilon}$ is positive almost everywhere on Ω and if there exist a positive real c_0 independent of ε and a sequence $\{Q_\varepsilon\}_\varepsilon$ of linear extension operators such that for each ε

$$\begin{cases} Q_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)), \\ (Q_\varepsilon v)_{|\Omega_\varepsilon} = v\,, \quad \forall\, v \in V_\varepsilon, \\ \|\nabla(Q_\varepsilon v)\|_{(L^2(\Omega))^n} \leq c_0 \|\nabla v\|_{(L^2(\Omega_\varepsilon))^n}\,, \ \forall v \in V_\varepsilon. \end{cases} \tag{5}$$

The existence of such operators is proved in [7] for the case of periodic holes. In general, it depends on the geometry of the domain Ω_ε and a necessary and sufficient condition for their existence seems to be an open question. We refer to [3] and [8] for other examples and comments.

Remark 2.1. In the following, we use on V_ε the norm $\|v\|_{V_\varepsilon} = \|\nabla v\|_{(L^2(\Omega_\varepsilon))^n}$. Due to (2.1) and the Poincaré inequality on Ω, this norm is equivalent to the H^1-norm. Indeed, one has

$$\|v\|_{H^1(\Omega_\varepsilon)} \leq \|Q_\varepsilon v\|_{H^1(\Omega)} \leq c_\Omega \|\nabla(Q_\varepsilon v)\|_{(L^2(\Omega))^n} \leq c_\Omega c_0 \|\nabla v\|_{(L^2(\Omega_\varepsilon))^n},$$

where obviously $c_\Omega c_0$ does not depend on ε.

Lemma 2.2. *([3]) Let $\{S_\varepsilon\}$ be an admissible sequence in Ω and $\{Q_\varepsilon\}$ satisfying (2.1).*
Then

$$\left(\varphi_\varepsilon \rightharpoonup \varphi \text{ in } H_0^1(\Omega)\right) \Rightarrow \left(Q_\varepsilon(\varphi_{\varepsilon|_{\Omega_\varepsilon}}) \rightharpoonup \varphi \text{ in } H_0^1(\Omega)\right).$$

In the following, Q_ε^* denotes the adjoint of Q_ε, i.e., the operator in $\mathcal{L}(H^{-1}(\Omega), V_\varepsilon')$ defined by

$$\forall\, g \in H^{-1}(\Omega),\ \forall\, v \in V_\varepsilon, \qquad Q_\varepsilon^* g(v) = \langle g, Q_\varepsilon v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Recall now the definition of the H^0-convergence introduced by M. Briane, A. Damlamian and P. Donato in [3]. It is an extension to perforated domains of H-convergence.

Definition 2.3. ([3]) Let $\{S_\varepsilon\}$ be an admissible sequence in Ω and $\{A^\varepsilon\}$ a sequence in $\mathcal{M}(\alpha, \beta, \Omega)$. The sequence $\{(A^\varepsilon, S_\varepsilon)\}$ is said to H^0-converge to $A^0 \in \mathcal{M}(\alpha', \beta', \Omega)$ (denoted by $(A^\varepsilon, S_\varepsilon) \overset{H^0}{\rightharpoonup} A^0$), if and only if for any function f in $H^{-1}(\Omega)$, the solution v_ε of

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla v_\varepsilon) = Q_\varepsilon^* f & \text{on } \Omega_\varepsilon, \\ (A^\varepsilon \nabla v_\varepsilon).\,\nu = 0 & \text{on } \partial S_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

satisfies the weak convergences

$$\begin{cases} Q_\varepsilon v_\varepsilon \rightharpoonup v & \text{weakly in } H_0^1(\Omega), \\ A^\varepsilon \widetilde{\nabla v_\varepsilon} \rightharpoonup A^0 \nabla v & \text{weakly in } (L^2(\Omega))^n, \end{cases} \tag{7}$$

where v is the unique solution in $H_0^1(\Omega)$ of the following problem:

$$\begin{cases} -\text{div}\,(A^0 \nabla v) = f & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

As shown in Proposition 1.7 of [3], this definition does not depend on the sequence $\{Q_\varepsilon\}$. Let us recall the main properties of H^0-convergence which shall be used in the main result.

Theorem 2.4. ***Compactness*** *([3])*
If $\{S_\varepsilon\}$ is admissible in Ω and if $\{A^\varepsilon\}$ belongs to $\mathcal{M}(\alpha, \beta, \Omega)$, then there exists a subsequence ε' of ε such that $(A^{\varepsilon'}, S_{\varepsilon'}) \overset{H^0}{\rightharpoonup} A^0$, where $A^0 \in \mathcal{M}(\alpha/c_0^2, \beta, \Omega)$ and c_0 is the constant introduced in (2.1).

The next proposition gives an equivalent definition of H^0-convergence.

Proposition 2.5. *([3]) Let $\{A^\varepsilon\}$ a sequence of $\mathcal{M}(\alpha, \beta, \Omega)$ and $\{S_\varepsilon\}$ be admissible in Ω. The following propositions are equivalent:*

a) *The sequence $\{(A^\varepsilon, S_\varepsilon)\}$ H^0-converges to A^0.*

b) *For every sequence of functions* $\{g_\varepsilon\}$ *in* $L^2(\Omega_\varepsilon)$ *such that* $\widetilde{g_\varepsilon} \rightharpoonup g$ *weakly in* $L^2(\Omega)$, *the solution* v_ε *of the problem*

$$\begin{cases} -div\,(A^\varepsilon \nabla v_\varepsilon) &=& g_\varepsilon & on\ \ \Omega_\varepsilon, \\ (A^\varepsilon \nabla v_\varepsilon).\,\nu &=& 0 & on\ \ \partial S_\varepsilon, \\ v_\varepsilon &=& 0 & on\ \ \partial\Omega, \end{cases} \tag{9}$$

satisfies the weak convergences

$$\begin{cases} Q_\varepsilon v_\varepsilon & \rightharpoonup & v & weakly\ in\ \ H_0^1(\Omega), \\ A^\varepsilon \widetilde{\nabla v_\varepsilon} & \rightharpoonup & A^0 \nabla v & weakly\ in\ \ (L^2(\Omega))^n, \end{cases} \tag{10}$$

where v *is the unique solution in* $H_0^1(\Omega)$ *of the following problem:*

$$\begin{cases} -div\,(A^0 \nabla v) &=& g & on\ \ \Omega, \\ v &=& 0 & on\ \ \partial\Omega. \end{cases} \tag{11}$$

The following result gives a "div-curl" lemma for perforated domains.

Proposition 2.6. *([3]) Let* $\{S_\varepsilon\}$ *be admissible in* Ω *and* $\{\xi_\varepsilon\} \subset (L^2(\Omega_\varepsilon))^n$ *a vector field sequence. Suppose that* $\{\widetilde{\xi_\varepsilon}\}$ *is bounded in* $(L^2(\Omega))^n$ *and*

$$\begin{cases} -div\ \xi_\varepsilon = Q_\varepsilon^* f_\varepsilon & on\ \ \Omega_\varepsilon, \\ \xi_\varepsilon\,.\,\nu = 0 & on\ \ \partial S_\varepsilon, \end{cases} \tag{12}$$

where $\{f_\varepsilon\}$ *is compact in* $H^{-1}(\Omega)$.
Then

i) $\{div\ \widetilde{\xi_\varepsilon}\}$ *is compact in* $H^{-1}(\Omega)$ *and if* $\widetilde{\xi_\varepsilon} \rightharpoonup \xi_0$ *in* $L^2(\Omega)$, *then* f_ε *converges to* $f = div\ \xi_0$ *in* $H^{-1}(\Omega)$.

ii) *If* $\{\eta_\varepsilon\} \in (L^2(\Omega))^n$ *is a vector field sequence which converges weakly to some* $\eta_0 \in (L^2(\Omega))^n$ *and satisfies if* $\{curl\ (\eta_\varepsilon)\}$ *is compact in* $H^{-1}(\Omega)$, *then* $\widetilde{\xi_\varepsilon}\,.\,\eta_\varepsilon$ *converges to* $\xi_0\,.\,\eta_0$ *in* $D'(\Omega)$.

3. **The elliptic corrector.** In this section we give the definition of an elliptic corrector. We also recall the construction of a particular local corrector done in [4] and we give some additional properties. This construction plays an essential role in the proof of the corrector result for the wave equation.

First, let us give the definition of an elliptic corrector.

Definition 3.1. ([3]) Let $\{S_\varepsilon\}$ be admissible in Ω and $\{A^\varepsilon\}$ be a sequence of $\mathcal{M}(\alpha, \beta, \Omega)$ such that $\{A^\varepsilon, S_\varepsilon\}$ H^0-converges to A^0. A sequence of matrix fields $\{C^\varepsilon\}$ in $(L^2(\Omega))^{n^2}$ is said to be a global corrector for $\{A^\varepsilon, S_\varepsilon\}$, if for every $f \in H^{-1}(\Omega)$ the following strong convergence holds:

$$\lim_{\varepsilon \to 0} \|\nabla v_\varepsilon - C^\varepsilon \nabla v\|_{(L^1(\Omega_\varepsilon))^n} = 0, \tag{13}$$

where v_ε is the solution of (2.2) associated with f and v the solution of (2.4).

A sequence of matrix fields $\{C^\varepsilon\}$ in $(L^2_{loc}(\Omega))^{n^2}$ is said to be a local corrector for $\{A^\varepsilon, S_\varepsilon\}$ if for every $f \in H^{-1}(\Omega)$ and for every open set $\omega \subset\subset \Omega$, the following strong convergence holds:

$$\lim_{\varepsilon \to 0} \|\nabla v_\varepsilon - C^\varepsilon \nabla v\|_{(L^1(\Omega_\varepsilon \cap \omega))^n} = 0, \tag{14}$$

where v_ε is the solution of (2.2) associated with f and v the solution of (2.4).

Proposition 3.2. *([3]) Let $\{A^\varepsilon, S_\varepsilon\}$ be a sequence which H^0-converges to A^0. Let $\{C^\varepsilon\}$ in $(L^2_{loc}(\Omega))^{n^2}$ be a sequence of matrix fields satisfying for every λ in $\mathbb{R}^n$ and for every open subset $\omega \subset\subset \Omega$*

$$\begin{cases} i) & C^\varepsilon\lambda \rightharpoonup \lambda \quad \text{weakly in } L^2(\omega), \\ ii) & \text{curl } (C^\varepsilon\lambda) \text{ is compact in } H^{-1}(\omega), \\ iii) & \text{div } (\chi_{\Omega_\varepsilon} A^\varepsilon C^\varepsilon\lambda) \text{ is compact in } H^{-1}(\omega). \end{cases} \tag{15}$$

Then

1. *The sequence $\{C^\varepsilon\}$ is a local corrector for $\{A^\varepsilon, S_\varepsilon\}$ and $\chi_{\Omega_\varepsilon} A^\varepsilon C^\varepsilon \rightharpoonup A^0$ weakly in $(L^2_{loc}(\Omega))^{n^2}$.*
2. *If $\{C^\varepsilon\}$ is bounded in $(L^r(\omega))^{n^2}$ for $\omega \subset\subset \Omega$ and some $r \in [2, +\infty]$, and if $\nabla v \in (L^s(\Omega))^n$ for some $s \in [2, +\infty[$, then*
$$\lim_{\varepsilon\to 0} \|\nabla v_\varepsilon - C^\varepsilon \nabla v\|_{(L^q(\Omega_\varepsilon\cap\omega))^n} = 0, \tag{16}$$
for $q = \min\left\{2, \frac{rs}{r+s}\right\}$, where v_ε is the solution of (2.2) associated with $f = -\text{div } (A^0\nabla v)$.
3. *If $\{C^\varepsilon\} \subset L^2(\Omega)$ and (3.3) holds for $\omega = \Omega$, then $\{C^\varepsilon\}$ is a global corrector and (3.4) holds for $\omega = \Omega$.*

Theorem 3.3. *([3]) For every H^0-converging sequence $\{A^\varepsilon, S_\varepsilon\}$, there exists a local corrector $\{C^\varepsilon\}$.*

Moreover, if there exists an open set Ω_1 such that $\overline{\Omega} \subset \Omega_1$ and $\{S_\varepsilon\}$ is admissible in Ω_1, then there exists a global corrector for $\{A^\varepsilon, S_\varepsilon\}$.

Remark 3.1. In [3], the existence of a global corrector is established by constructing a family (C^ε) satisfying (3.3) in Ω as follows:

Let
$$B^\varepsilon = \begin{cases} A^\varepsilon & \text{on } \Omega, \\ I & \text{on } \Omega_1 \setminus \Omega. \end{cases}$$

Since the sequence $\{B^\varepsilon\}$ is clearly in $\mathcal{M}(\alpha, \beta, \Omega_1)$ and $\{S_\varepsilon\}$ is admisssible in Ω_1, by Theorem 2.4 there exist a subsequence of ε (still denoted ε) and a matrix B^0 such that $\{B^\varepsilon, S_\varepsilon\}$ H^0-converges to B^0. Let Q_ε be an extension operator provided by the admissibility of S_ε in Ω_1 and let Q^*_ε be its adjoint. Let φ be a function of $D(\Omega_1)$ equal to 1 on Ω and for every $\lambda \in \mathbb{R}^n$, let w^ε_λ be the solution of the problem

$$\begin{cases} -\text{div } (B^\varepsilon \nabla w^\varepsilon_\lambda) &=& Q^*_\varepsilon(-\text{div } (B^0\nabla(\varphi\lambda\,.\,x))) & \text{on } \Omega_1 \setminus S_\varepsilon, \\ (B^\varepsilon \nabla w^\varepsilon_\lambda).\,\nu &=& 0 & \text{on } \partial S_\varepsilon, \\ w^\varepsilon_\lambda &=& 0 & \text{on } \partial\Omega_1. \end{cases} \tag{17}$$

Since $\{B^\varepsilon\}$ H^0-converges to B^0, we have the following:

$$\begin{cases} Q_\varepsilon w^\varepsilon_\lambda & \rightharpoonup & \omega_\lambda \quad \text{weakly in } H^1_0(\Omega_1), \\ B^\varepsilon \widetilde{\nabla w^\varepsilon_\lambda} & \rightharpoonup & B^0\nabla\omega_\lambda \quad \text{weakly in } (L^2(\Omega_1))^n, \end{cases} \tag{18}$$

where w_λ is the solution of the problem

$$\begin{cases} -\text{div } (B^0\nabla w_\lambda) &=& -\text{div } (B^0\nabla(\varphi\lambda\,.\,x)) & \text{on } \Omega_1, \\ w_\lambda &=& 0 & \text{on } \partial\Omega_1. \end{cases} \tag{19}$$

Because of the uniqueness of the solution of (3.7), we have $w_\lambda = \varphi\lambda\,.\,x$ on Ω_1. Since φ is equal to 1 on Ω, one has

$$\begin{cases} Q_\varepsilon w_\lambda^\varepsilon & \rightharpoonup & \lambda\,.\,x & \text{weakly in } H^1(\Omega), \\ B^\varepsilon \widetilde{\nabla w_\lambda^\varepsilon} & \rightharpoonup & B^0\lambda & \text{weakly in } (L^2(\Omega))^n. \end{cases} \tag{20}$$

Consider the matrix field C^ε defined by

$$C^\varepsilon\lambda = \nabla(Q_\varepsilon w_\lambda^\varepsilon), \quad \forall\,\lambda \in \mathbb{R}^n. \tag{21}$$

As $B^\varepsilon = A^\varepsilon$ on Ω and from (3.5), (3.8), (3.9) and Proposition 2.6 i) applied to $B^\varepsilon w_\lambda^\varepsilon$, the family (C^ε) satisfies the three conditions of (3.3) in Ω.

Proposition 3.4. *If C_1^ε and C_2^ε are two local correctors, then*

$$\lim_{\varepsilon\to 0} \|C_1^\varepsilon - C_2^\varepsilon\|_{(L^1(\Omega_\varepsilon\cap\omega))^{n^2}} = 0, \qquad \forall\,\omega \subset\subset \Omega. \tag{22}$$

Moreover, if C_1^ε and C_2^ε satisfy property (3.4) with $r = 2$, one has for any $q \in [1, 2[$,

$$\lim_{\varepsilon\to 0} \|C_1^\varepsilon - C_2^\varepsilon\|_{(L^q(\Omega_\varepsilon\cap\omega))^{n^2}} = 0, \qquad \forall\,\omega \subset\subset \Omega. \tag{23}$$

Proof. Fix a compact subset ω contained in Ω and let $\varphi \in D(\Omega)$ such that $\varphi = 1$ on ω. Set $u_0(x) = (\lambda\,.\,x)\,\varphi$ where $x \in \Omega$, and λ is a fixed element in $\mathbb{R}^n$.
Let u_ε be the solution of the problem

$$\begin{cases} \operatorname{div}(A^\varepsilon\nabla u_\varepsilon) &=& \operatorname{div}(A^0\nabla u_0) & \text{on } \Omega_\varepsilon, \\ (A^\varepsilon\nabla u_\varepsilon).\,\nu &=& 0 & \text{on } \partial S_\varepsilon, \\ u_\varepsilon &=& 0 & \text{on } \partial\Omega, \end{cases} \tag{24}$$

If $\omega_1 = \operatorname{supp}\varphi$, we have for any $\lambda \in \mathbb{R}^n$

$$\begin{aligned} \|C_1^\varepsilon\lambda - C_2^\varepsilon\lambda\|_{L^1(\Omega_\varepsilon\cap\omega)} &\leq \|C_1^\varepsilon\nabla u_0 - C_2^\varepsilon\nabla u_0\|_{L^1(\Omega_\varepsilon\cap\omega_1)} \\ &\leq \|C_1^\varepsilon\nabla u_0 - \nabla u_\varepsilon\|_{L^1(\Omega_\varepsilon\cap\omega_1)} + \|C_2^\varepsilon\nabla u_0 - \nabla u_\varepsilon\|_{L^1(\Omega_\varepsilon\cap\omega_1)}. \end{aligned}$$

From (3.2), the last term converges to 0 as ε tends to 0. Hence (3.10) holds. Suppose that C_1^ε and C_2^ε satisfy (3.4) with $r = 2$ and let $q \in [1, 2[$ be fixed. Choosing s such that $q = \dfrac{2s}{s+2}$, i.e., $s = \dfrac{2q}{2-q} \in [2, +\infty[$, one has

$$\|C_1^\varepsilon\lambda - C_2^\varepsilon\lambda\|_{L^q(\Omega_\varepsilon\cap\omega)} \leq \|C_1^\varepsilon\nabla u_0 - \nabla u_\varepsilon\|_{L^q(\Omega_\varepsilon\cap\omega_1)} + \|C_2^\varepsilon\nabla u_0 - \nabla u_\varepsilon\|_{L^q(\Omega_\varepsilon\cap\omega_1)}.$$

From (3.4), the right-hand side converges to 0, as ε tends to 0. □

Let us now recall the construction of the local corrector done by G. Cardone, P. Donato and A. Gaudiello in [4].

Let $\{\omega_h\}_{h\in\mathbb{N}}$ be a sequence of increasing subsets of $\mathbb{R}^n$ with smooth boundary and let $\{\varphi_h\}_{h\in\mathbb{N}}$ be a sequence of functions defined on Ω such that

$$\begin{cases} \omega_0 = \emptyset \subset \omega_1 \subset\subset \ldots\, \omega_h \subset\subset \omega_{h+1} \subset\subset \ldots \subset\subset \Omega\,, \\ \bigcup\limits_{h\in\mathbb{N}} \omega_h = \Omega\,, \\ \varphi_h \in D(\Omega) \;\text{ and }\; \varphi_h = 1 \;\text{ on }\; \omega_h, \forall\,h \in \mathbb{N}. \end{cases} \tag{25}$$

For any $h \in \mathbb{N}$, introduce the family $(C_h^\varepsilon)_\varepsilon$ in $(L^2(\Omega))^{n^2}$ defined by

$$C_h^\varepsilon e_i = \nabla(Q_\varepsilon w_{h,i}^\varepsilon) \quad a.e. \text{ on } \Omega,\ \forall\,i \in \{1, \ldots, n\}, \tag{26}$$

where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$ and $w^\varepsilon_{h,i}$ is the unique solution, for any $h \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, of the following problem:

$$\begin{cases} -\text{div } (A^\varepsilon \nabla w^\varepsilon_{h,i}) &= Q^*_\varepsilon(-\text{div } (A^0 \nabla(\varphi_h x_i))) \quad \text{on } \Omega_\varepsilon, \\ (A^\varepsilon \nabla w^\varepsilon_{h,i}).\,\nu &= 0 \quad \text{on } \partial S_\varepsilon, \\ w^\varepsilon_{h,i} &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{27}$$

whose variational formulation is

$$\begin{cases} \text{Find } w^\varepsilon_{h,i} \in V_\varepsilon \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} A^\varepsilon \nabla w^\varepsilon_{h,i} \nabla v\, dx = \int_\Omega A^0 \nabla(\varphi_h x_i) \nabla(Q_\varepsilon v)\, dx, \quad \forall\, v \in V_\varepsilon. \end{cases} \tag{28}$$

Since $\{A^\varepsilon, S_\varepsilon\}$ H^0-converges to A^0 and φ_h is equal to 1 on ω_h, it follows that

$$\begin{cases} \|C^\varepsilon_h\|_{(L^2(\Omega))^{n^2}} \le c_h, \\ \text{where } c_h \text{ is independent of } \varepsilon \text{ but dependent on } h, \end{cases} \tag{29}$$

and

$$\begin{cases} i) & Q_\varepsilon w^\varepsilon_{h,i} \rightharpoonup \varphi_h x_i & \text{in } H^1_0(\Omega), \\ ii) & A^\varepsilon \widetilde{\nabla w^\varepsilon_{h,i}} \rightharpoonup A^0 \nabla(\varphi_h x_i) & \text{in } (L^2(\Omega))^n, \\ iii) & Q_\varepsilon w^\varepsilon_{h,i} \rightharpoonup x_i & \text{in } H^1(\omega_h), \\ iv) & \chi_{\Omega_\varepsilon} A^\varepsilon C^\varepsilon_h \rightharpoonup A^0 & \text{in } (L^2(\omega_h))^{n^2}, \end{cases} \tag{30}$$

Now, define C^ε in $(L^2_{loc}(\Omega))^{n^2}$ as follows:

$$C^\varepsilon = C^\varepsilon_h \quad \text{a.e. in } \omega_h - \overline{\omega_{h-1}}, \quad \forall\, h \in \mathbb{N}^*, \tag{31}$$

where $\{\omega_h\}$ satisfies (3.13) and C^ε_h is given by (3.14).

From (3.13) and (3.17), for any open set $\omega \subset\subset \Omega$, one has

$$\begin{cases} \|C^\varepsilon\|_{(L^2(\omega))^{n^2}} \le c_\omega, \\ \text{where } c_\omega \text{ is independent of } \varepsilon, \text{ but dependent on } \omega. \end{cases} \tag{32}$$

The following result is proved in [4]:

Theorem 3.5. [4] *Let $(A^\varepsilon, S_\varepsilon)_\varepsilon$ be a sequence which H^0-converges to A^0. Let f be in $H^{-1}(\Omega)$, v_ε and v be the solutions of (2.2) and (2.4) respectively, associated with f. Then, if C^ε is given by (3.19), one has for any $q \in [1, 2[$,*

$$\begin{cases} i) \quad \forall\, \xi \in \mathbb{R}^n, \forall \varphi \in D(\Omega), \varphi \ge 0 \text{ in } \Omega, \quad \overline{\lim_{\varepsilon \to 0}} \displaystyle\int_{\Omega_\varepsilon} |C^\varepsilon \xi|^q \varphi\, dx \le c|\xi|^q \int_\Omega \varphi\, dx, \\ ii) \quad \forall\, \phi \in (D(\Omega))^n, \quad \overline{\lim_{\varepsilon \to 0}} \|\nabla v_\varepsilon - C^\varepsilon \phi\|_{(L^q(\Omega_\varepsilon))^n} \le c\, \|\nabla v - \phi\|_{(L^2(\Omega))^n}. \end{cases} \tag{33}$$

This result allows us to prove the existence (not explicitly proved in [4]) of a local corrector result for a sequence $\{A^\varepsilon, S_\varepsilon\}$.

Theorem 3.6. *Under the hypotheses of Theorem 3.5, the sequence $\{C^\varepsilon\}$ given by (3.19) is a local corrector for $\{A^\varepsilon, S_\varepsilon\}$. Moreover, it satisfies property (3.4) with $r = 2$.*

Proof. Let us show that (3.2) holds for the sequence $\{C^\varepsilon\}$ given by (3.19).

Let f be in $H^{-1}(\Omega)$, v_ε and v be the solutions of (2.2) and (2.4) respectively, ω be an open set with $\omega \subset\subset \Omega$, $\eta > 0$ and $\phi \in (D(\Omega))^n$ such that

$$\|\nabla v - \phi\|_{(L^2(\Omega))^n} \le \eta.$$

From Theorem 3.5, it follows that

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}} \|\nabla v_\varepsilon - C^\varepsilon \nabla v\|_{(L^1(\Omega_\varepsilon\cap\omega))^n} \\
&\quad\le \overline{\lim_{\varepsilon\to 0}} \left(\|\nabla v_\varepsilon - C^\varepsilon \phi\|_{(L^1(\Omega_\varepsilon))^n} + \|C^\varepsilon\phi - C^\varepsilon \nabla v\|_{(L^1(\Omega_\varepsilon\cap\omega))^n} \right) \\
&\quad\le \overline{\lim_{\varepsilon\to 0}} \left(c\,\|\nabla v - \phi\|_{(L^2(\Omega))^n} + \|C^\varepsilon\|_{(L^2(\omega))^{n^2}} \|\phi - \nabla v\|_{(L^2(\Omega))^n} \right) \le c\eta + c_\omega \eta,
\end{aligned}$$

where c_ω is the constant introduced in (3.20). This, gives the claimed result, since η is arbitrary.

Now, we prove that $\{C^\varepsilon\}$ satisfies property (3.4) with $r = 2$. To do so, let $\eta > 0$, $\omega \subset\subset \Omega$ and $\phi \in (D(\Omega))^n$ such that

$$\|\nabla v - \phi\|_{\left(L^{\frac{2q}{2-q}}(\Omega)\right)^n} \le \eta,$$

where $q = \frac{2s}{2+s} \in [1, 2[$. From Theorem 3.5 and the Hölder inequality, we have

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}} \|\nabla v_\varepsilon - C^\varepsilon \nabla v\|_{(L^q(\Omega_\varepsilon\cap\omega))^n} \\
&\le \overline{\lim_{\varepsilon\to 0}} \left(\|\nabla v_\varepsilon - C^\varepsilon \phi\|_{(L^q(\Omega_\varepsilon))^n} + \|C^\varepsilon\phi - C^\varepsilon \nabla v\|_{(L^q(\Omega_\varepsilon\cap\omega))^n} \right) \\
&\le \overline{\lim_{\varepsilon\to 0}} \left(c'\,\|\nabla v - \phi\|_{(L^2(\omega))^n} + \|C^\varepsilon\|_{(L^2(\omega))^{n^2}} \|\phi - \nabla v\|_{\left(L^{\frac{2q}{2-q}}(\Omega)\right)^n} \right) \le c'\eta + c_\omega \eta,
\end{aligned}$$

and this concludes the proof, since η is arbitrary. □

In the following, we will use this result:

Proposition 3.7. *For any $\phi \in D(\Omega)$, there exists $l_\phi \in \mathbb{N}$ such that*

$$\lim_{\varepsilon\to 0} \left\| (C^\varepsilon - C^\varepsilon_{l_\phi}) \nabla \phi \right\|_{(L^1(\Omega_\varepsilon))^n} = 0, \tag{34}$$

where $C^\varepsilon_{l_\phi}$ is given by (3.14).

The proof of this proposition is based on the following lemma proved in [4]:

Lemma 3.8. [4] *Under the hypotheses of Theorem 3.5, if $\{\omega_h\}$ is given by (3.13) and C^ε_h by (3.14), one has for any $h, k \in \mathbb{N}$ and for any $\Phi \in (D(\omega_h))^n$*

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} A^\varepsilon((C^\varepsilon_h - C^\varepsilon_{h+k})\Phi)((C^\varepsilon_h - C^\varepsilon_{h+k})\Phi)\, dx = 0. \tag{35}$$

Proof of Proposition 3.7. Since $\phi \in D(\Omega)$, there exist $\omega \subset\subset \Omega$ and an integer l_ϕ such that supp $\phi = \omega$ and $\omega \subset \omega_{l_\phi}$, where ω_{l_ϕ} is an element of the sequence $\{\omega_h\}$ introduced in (3.13). For $\eta > 0$ and for every $h \in \{1, \dots, l_\phi\}$, let $\Phi^h_\eta \in D(\omega_h))^n$ be such that

$$\left\|\nabla \phi - \Phi^h_\eta\right\|_{(L^2(\omega_h))^n} \le \eta.$$

From the Hölder inequality, the ellipticity of A^ε, (3.17) and definition (3.19) of C^ε, one has

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} |(C^\varepsilon - C^\varepsilon_{l_\phi})\nabla\phi|\,dx \le \sum_{h=1}^{l_\phi}\int_{\Omega_\varepsilon\cap\omega_h}|(C^\varepsilon_h - C^\varepsilon_{l_\phi})\nabla\phi|\,dx\\
&\le \sum_{h=1}^{l_\phi}\int_{\Omega_\varepsilon\cap\omega_h}|(C^\varepsilon_h - C^\varepsilon_{l_\phi})(\nabla\phi-\Phi^h_\eta)|\,dx + \sum_{h=1}^{l_\phi}\int_{\Omega_\varepsilon\cap\omega_h}|(C^\varepsilon_h - C^\varepsilon_{l_\phi})\Phi^h_\eta|\,dx\\
&\le \sum_{h=1}^{l_\phi}\eta\left(\int_{\Omega_\varepsilon\cap\omega_h}|(C^\varepsilon_h - C^\varepsilon_{l_\phi})|^2\,dx\right)^{\frac12} + |\Omega|^{\frac12}\sum_{h=1}^{l_\phi}\left(\int_{\Omega_\varepsilon\cap\omega_h}|(C^\varepsilon_h - C^\varepsilon_{l_\phi})\Phi^h_\eta|^2\,dx\right)^{\frac12}\\
&\le \eta\sum_{h=1}^{l_\phi}(c_h + c_{l_\phi}) + |\Omega|^{\frac12}\sum_{h=1}^{l_\phi}\left(\frac{1}{\alpha}\int_{\Omega_\varepsilon}A^\varepsilon((C^\varepsilon_h - C^\varepsilon_{l_\phi})\Phi^h_\eta)((C^\varepsilon_h - C^\varepsilon_{l_\phi})\Phi^h_\eta)\,dx\right)^{\frac12},
\end{aligned}
\tag{36}
$$

where c_h and c_{l_ϕ} are independent of ε.

Since supp $\phi^h_\eta \subset \omega_h$ and $l_\phi \ge h$ for every $h \in \{1,\dots,l_\phi\}$, we can apply Lemma 3.8 to the right-hand side of (3.24) to obtain

$$
0 \le \overline{\lim_{\varepsilon\to0}}\left\|(C^\varepsilon - C^\varepsilon_{l_\phi})\nabla\phi\right\|_{(L^1(\Omega_\varepsilon))^n} \le c\,\eta,
$$

which implies (3.22), since η is arbitrary. □

4. **Statement of the main results.** In this section we state the homogenization and the corrector results.

4.1. **Homogenization of the wave equation.** Consider the problem

$$
\begin{cases}
\rho_\varepsilon u''_\varepsilon - \operatorname{div}(A^\varepsilon\nabla u_\varepsilon) = f_\varepsilon & \text{on } \Omega_\varepsilon\times(0,T),\\
u_\varepsilon = 0 & \text{on } \partial\Omega\times(0,T),\\
A^\varepsilon\nabla u_\varepsilon\,.\,\nu = 0 & \text{on } \partial S_\varepsilon\times(0,T),\\
u_\varepsilon(x,0) = a_\varepsilon & \text{on } \Omega_\varepsilon,\\
u'_\varepsilon(x,0) = b_\varepsilon & \text{on } \Omega_\varepsilon.
\end{cases}
\tag{37}
$$

We make the following hypotheses on the data:

$$
\begin{cases}
\rho_\varepsilon \in L^\infty(\Omega_\varepsilon),\\
\exists\,\lambda_1,\lambda_2\in\mathbb{R} \ \text{ such that } \ 0<\lambda_1\le\rho_\varepsilon\le\lambda_2,
\end{cases}
\tag{38}
$$

$$
\begin{cases}
A^\varepsilon \in \mathcal{M}(\alpha,\beta,\Omega),\ {}^tA^\varepsilon = A^\varepsilon,\\
\{S_\varepsilon\}_\varepsilon \text{ is admissible on } \Omega,
\end{cases}
\tag{39}
$$

and

$$
f_\varepsilon \in L^2((0,T)\times\Omega_\varepsilon),\quad a_\varepsilon\in V_\varepsilon,\quad b_\varepsilon\in L^2(\Omega_\varepsilon).
\tag{40}
$$

Let us introduce the spaces

$$
\begin{aligned}
W_\varepsilon(0,T;V_\varepsilon,L^2(\Omega_\varepsilon)) &= \left\{v\in L^2(0,T;V_\varepsilon)\ \mid\ v'\in L^2(0,T;L^2(\Omega_\varepsilon))\right\},\\
W(0,T;H^1_0(\Omega),L^2(\Omega)) &= \left\{v\in L^\infty(0,T;H^1_0(\Omega))\ \mid\ v'\in L^2(0,T;L^2(\Omega))\right\},
\end{aligned}
$$

which are Banach spaces with respect to their graph norms, defined by

$$
\begin{aligned}
\|v\|_{W_\varepsilon(0,T;V_\varepsilon,L^2(\Omega_\varepsilon))} &= \|v\|_{L^2(0,T;V_\varepsilon)} + \|v'\|_{L^2(\Omega_\varepsilon\times(0,T))},\\
\|v\|_{W(0,T;H^1_0(\Omega),L^2(\Omega))} &= \|v\|_{L^\infty(0,T;H^1_0(\Omega))} + \|v'\|_{L^2(\Omega\times(0,T))}.
\end{aligned}
$$

Recall that $W(0,T;H_0^1(\Omega),L^2(\Omega))$ is dense in $\mathcal{C}^0([0,T];L^2(\Omega))$ and $W_\varepsilon(0,T;V_\varepsilon,L^2(\Omega_\varepsilon))$ is a Hilbert space.

The variational formulation of (4.1) is

$$\begin{cases} \text{Find } u_\varepsilon \in W_\varepsilon(0,T;V_\varepsilon,L^2(\Omega_\varepsilon)) \text{ such that} \\ \langle \rho_\varepsilon u''_\varepsilon, v\rangle_{V'_\varepsilon,V_\varepsilon} + \displaystyle\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v \; dx \;=\; \int_{\Omega_\varepsilon} f_\varepsilon v\, dx \quad \text{in } D'(0,T), \quad \forall\, v\in V_\varepsilon, \\ \qquad u_\varepsilon(x,0) \;=\; a_\varepsilon \qquad \text{on } \Omega_\varepsilon, \\ \qquad u'_\varepsilon(x,0) \;=\; b_\varepsilon \qquad \text{on } \Omega_\varepsilon. \end{cases} \tag{41}$$

It is well known that under hypotheses (4.2)-(4.4), for every fixed ε there exists a unique solution u_ε of (4.1) in $W_\varepsilon(0,T;V_\varepsilon,L^2(\Omega_\varepsilon))$ such that

$$u_\varepsilon \in \mathcal{C}^0([0,T];V_\varepsilon), \quad u'_\varepsilon \in \mathcal{C}^0([0,T];L^2(\Omega_\varepsilon)) \;\text{ and }\; u''_\varepsilon \in L^2(0,T;V'_\varepsilon).$$

Therefore, the initial conditions $u_\varepsilon(0)=a_\varepsilon$ and $u'_\varepsilon(0)=b_\varepsilon$ make sense.

Suppose furthermore that

$$(A^\varepsilon, S_\varepsilon) \xrightharpoonup{H^0} A^0, \tag{42}$$

where $A^0 \in \mathcal{M}(\alpha/c_0^2,\beta,\Omega)$ and c_0 is given by (2.1). Observe that A^ε is symmetric for every ε, so its H^0-limit A^0 is also symmetric.
We make the following assumptions on the data:

$$\widetilde{\rho_\varepsilon} \overset{*}{\rightharpoonup} \rho \quad \text{weakly * in } L^\infty(\Omega), \tag{43}$$

and

$$\begin{cases} i) & \widetilde{f_\varepsilon} \rightharpoonup f \quad \text{weakly in } L^2((0,T)\times\Omega), \\ ii) & Q_\varepsilon a_\varepsilon \rightharpoonup a_0 \quad \text{weakly in } H_0^1(\Omega), \\ iii) & \widetilde{\rho_\varepsilon b_\varepsilon} \rightharpoonup b \quad \text{weakly in } L^2(\Omega). \end{cases} \tag{44}$$

Remark 4.1. As we will see in Theorem 4.3 below, in the homogenized problem appears the limit of $\widetilde{\rho_\varepsilon b_\varepsilon}$ and not that of b_ε. This explains why we make assumption (4.8)iii. This fact was already observed in [2] for a fixed domain.

Proposition 4.1. *Under hypotheses (4.2)-(4.4) and (4.6)-(4.8), there exists a constant c independent of ε such that*

$$\begin{cases} \|u'_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \;\le\; c, \\ \|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega_\varepsilon))} \;\le\; c. \end{cases} \tag{45}$$

The proof of Proposition 4.1 follows the same outline as in the periodic case (see for instance [5]). Indeed, it only makes use of the fact that $A^\varepsilon \in \mathcal{M}(\alpha,\beta,\Omega)$ and that the initial data are bounded.

To describe the asymptotic behavior of problem (4.1), we need to extend not only its solution u_ε on the whole domain Ω but also its time derivative u'_ε. To do that, we need to suppose that the extension operator Q_ε given in Definition 2.1 also acts on functions in $L^2(\Omega_\varepsilon)$.

In the following, we suppose that Q_ε satisfies (2.1) and also

$$\begin{cases} i) & Q_\varepsilon \in \mathcal{L}(L^2(\Omega_\varepsilon),L^2(\Omega)), \\ ii) & (Q_\varepsilon v)_{|\Omega_\varepsilon} = v, \quad \forall\, v\in L^2(\Omega_\varepsilon), \\ iii) & \|Q_\varepsilon v\|_{L^2(\Omega)} \le c_0 \|v\|_{L^2(\Omega_\varepsilon)}, \; \forall\, v \in L^2(\Omega_\varepsilon), \end{cases} \tag{46}$$

where c_0 is the constant introduced in (2.1).

We can now construct a family (P_ε) of extension operators for time-dependent functions.

Theorem 4.2. *Let $\{S_\varepsilon\}$ be admissible in Ω and assume that (4.10) holds. Then, there exists an extension operator*
$P_\varepsilon \in \mathcal{L}\left(L^\infty(0,T;V_\varepsilon), L^\infty(0,T;H_0^1(\Omega))\right) \cap \mathcal{L}\left(L^\infty(0,T;L^2(\Omega_\varepsilon)), L^\infty(0,T;L^2(\Omega))\right)$
such that $\forall\, \phi \in L^\infty(0,T;V_\varepsilon) \cap W^{1,\infty}(0,T;L^2(\Omega_\varepsilon))$

$$\begin{cases} i) & P_\varepsilon \phi = \phi \quad \text{on } \Omega_\varepsilon \times (0,T), \\ ii) & (P_\varepsilon \phi)' = P_\varepsilon \phi' \quad \text{on } \Omega \times (0,T), \\ iii) & \|P_\varepsilon \phi\|_{L^\infty(0,T;L^2(\Omega))} \le c_0 \|\phi\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}, \\ iv) & \|P_\varepsilon \phi'\|_{L^\infty(0,T;L^2(\Omega))} \le c_0 \|\phi'\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}, \\ v) & \|\nabla(P_\varepsilon \phi)\|_{L^\infty(0,T;L^2(\Omega))^n)} \le c_0 \|\nabla \phi\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))^n)}, \end{cases} \tag{47}$$

where c_0 is the constant introduced in (2.1).

Proof. As in the periodic case studied in [5], we construct the extension operator P_ε from Q_ε, treating the variable t as a parameter. Set

$$P_\varepsilon(\phi)(x,t) = (Q_\varepsilon \phi(\cdot,t))(x) \tag{48}$$

for ϕ in $L^\infty(0,T;V_\varepsilon)$ or in $L^\infty(0,T;L^2(\Omega_\varepsilon))$.

With this definition, P_ε belongs to $\mathcal{L}\left(L^\infty(0,T;V_\varepsilon), L^\infty(0,T;H_0^1(\Omega))\right)$ and also to
$\mathcal{L}\left(L^\infty(0,T;L^2(\Omega_\varepsilon), L^\infty(0,T;L^2(\Omega))\right)$. It is clear that if
$\phi \in L^\infty(0,T;V_\varepsilon) \cap W^{1,\infty}(0,T;L^2(\Omega_\varepsilon))$, then one has ii) since Q_ε is independent of t.
To prove iii), let $\phi \in L^\infty((0,T)\times\Omega_\varepsilon))$. One has by (4.10)iii and (4.12)

$$\begin{aligned} \|(P_\varepsilon \phi)\|_{L^\infty(0,T;L^2(\Omega))} &= \sup_{t\in(0,T)} \|(Q_\varepsilon \phi(t))\|_{L^2(\Omega)} \\ &\le \sup_{t\in(0,T)} c_0 \|\phi(t)\|_{L^2(\Omega_\varepsilon)} = c_0 \|\phi\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))}. \end{aligned}$$

Assertions iv) and v) are obtained by using (2.1), (4.10) and (4.12). □

We can now state the convergence result.

Theorem 4.3. *Suppose that (4.2)-(4.4), (4.6)-(4.8) and (4.10) hold. Then, one has the following convergences:*

$$\begin{cases} i) & P_\varepsilon u_\varepsilon \overset{*}{\rightharpoonup} u & \text{weakly * in } L^\infty(0,T;H_0^1(\Omega)), \\ ii) & P_\varepsilon u'_\varepsilon \overset{*}{\rightharpoonup} u' & \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \\ iii) & A^\varepsilon \widetilde{\nabla u_\varepsilon} \overset{*}{\rightharpoonup} A^0 \nabla u & \text{weakly * in } L^\infty(0,T;(L^2(\Omega))^n), \end{cases} \tag{49}$$

where u is the solution of the homogenized problem

$$\begin{cases} \rho\, u'' - div\,(A^0 \nabla u) &= f & \text{on } \Omega \times (0,T), \\ u &= 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= a_0 & \text{on } \Omega, \\ u'(x,0) &= b_0 & \text{on } \Omega, \end{cases} \tag{50}$$

and $b_0 = \dfrac{b}{\rho}$.

The proof is given in Section 5.

4.2. **Corrector result.** In this section we state a corrector result for problem (4.1). Under suitable assumptions, we show that the convergence of u'_ε to u' is strong and the weak convergence of ∇u_ε to ∇u given by Theorem 4.3 can be improved by using the corrector matrix C^ε defined in (3.19). As in the case of a fixed domain (see [2]), we have to make additional hypotheses on the initial data $(a_\varepsilon, b_\varepsilon)$. Here, due to the presence of the holes, we have also to suppose (which is always true for a subsequence) that

$$\chi_{\Omega_\varepsilon} \overset{*}{\rightharpoonup} \chi_0 \quad \text{weakly * in } L^\infty(\Omega), \tag{51}$$

for some $\chi_0 \in L^\infty(\Omega)$ positive a.e. on Ω.

Concerning the data we suppose that $f_\varepsilon \in L^2(\Omega \times (0,T))$, $b_\varepsilon \in L^2(\Omega)$ for every ε and that

$$\begin{cases} i) \quad f_\varepsilon \to \dfrac{f}{\chi_0} \quad \text{strongly in } L^2(\Omega \times (0,T)), \\ ii) \quad b_\varepsilon \to b_0 \quad \text{strongly in } L^2(\Omega), \end{cases} \tag{52}$$

for some $f \in L^2(\Omega \times (0,T))$ and with b_0 defined in (4.14).

We also assume that a_ε is the solution of the following equation:

$$\begin{cases} -\text{div}(A^\varepsilon \nabla a_\varepsilon) = Q^*_\varepsilon(-\text{div}(A^0 \nabla a_0)) & \text{on } \Omega_\varepsilon, \\ A^\varepsilon \nabla a_\varepsilon \,.\, \nu = 0 & \text{on } \partial S_\varepsilon, \\ a_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{53}$$

for some $a_0 \in H^1_0(\Omega)$.

These assumptions are essential in order to prove the convergence of the energy associated with problem (4.1) to that of problem (4.14) (see Section 6). Indeed, we prove that

$$\frac{1}{2}\int_{\Omega_\varepsilon} (\rho_\varepsilon (u'_\varepsilon)^2 + A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon)\, dx \longrightarrow \frac{1}{2}\int_\Omega (\rho (u')^2 + A^0 \nabla u \nabla u)\, dx \quad \text{in } \mathcal{C}^0([0,T]).$$

Remark 4.2. Observe that assumptions (4.15)-(4.17) imply convergences (4.8). Indeed, (4.15) and (4.16)i give immediately (4.8)i. On the other hand, the H^0-convergence of A^ε to A^0 and (4.17) imply

$$Q^\varepsilon a_\varepsilon \rightharpoonup a \quad \text{weakly in } H^1_0(\Omega),$$

where a is the solution of

$$\begin{cases} -\text{div}\,(A^0 \nabla a) = -\text{div}\,(A^0 \nabla a_0) & \text{on } \Omega, \\ a = 0 & \text{on } \partial\Omega. \end{cases}$$

Since this problem has a unique solution, it follows that $a = a_0$.

Finally, from (4.7) and (4.16)ii, one has

$$\widetilde{\rho_\varepsilon} b_\varepsilon \rightharpoonup \rho\, b_0 \quad \text{weakly in } L^2(\Omega),$$

so that (4.8)iii holds for $b = \rho\, b_0$. Consequently, Theorem 4.3 applies and gives convergences (4.13) and the homogenized problem (4.14) with a_0, b_0 and f given by (4.16) and (4.17). □

Now, we can state the corrector result.

Theorem 4.4. *Under hypotheses (4.2)-(4.4), (4.6)-(4.7), (4.10) and (4.15)-(4.17), we have the following corrector result:*

$$\begin{cases} i) \quad \lim_{\varepsilon \to 0} \|u'_\varepsilon - u'\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} = 0, \\ ii) \quad \lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} = 0, \end{cases} \tag{54}$$

where u is the solution of (4.14) and $\{C^\varepsilon\}$ is the corrector given by (3.19).

Corollary 4.5. *Under the assumptions of Theorem 4.4, if $\nabla u \in \mathcal{C}^0([0,T];(L^p(\Omega))^n)$ for some $p > 2$, then (4.18)ii holds for any local corrector satisfying (3.4) with $r = 2$.*

Theorem 4.4 and Corollary 4.5 will be proved in Section 7.

Remark 4.3. Observe that it is quite natural to suppose that (3.4) holds. Indeed, in practice to show that a sequence $\{C^\varepsilon\}$ is a local corrector one makes use of (3.3), which implies (3.4). This condition also holds for the particular corrector constructed in [4] with $r = 2$, as shown in Theorem 3.6.

Remark 4.4. Suppose that the hypotheses of Theorem 3.3 are verified and let $\{C^\varepsilon\}$ be the global corrector constructed in [3] (see Remark 3.1). Then, following the same arguments as in the proof of Theorem 4.4, one can prove that convergences (4.18)ii still hold for this corrector in Ω_ε. Namely, we have

$$\lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{\mathcal{C}^0([0,T];(L^1(\Omega_\varepsilon))^n)} = 0.$$

The proof of Theorem 4.4 is based on this following result, proved in Section 7:

Proposition 4.6. *For all $h \in \mathbb{N}^*$, let $v \in D(\omega_h)$, $\phi \in \mathcal{C}^\infty([0,T])$ and*

$$\begin{aligned} X_{\varepsilon,h}(t) = \frac{1}{2} \int_{\Omega_\varepsilon} \Big(& \rho_\varepsilon (u'_\varepsilon(t) - v\phi'(t))^2 \\ & + A^\varepsilon (\phi(t)\, C^\varepsilon_h \nabla v - \nabla u_\varepsilon(t)) \,.\, (\phi(t)\, C^\varepsilon_h \nabla v - \nabla u_\varepsilon(t)) \Big)\, dx, \end{aligned} \tag{55}$$

where ω_h is an element of the sequence $\{\omega_h\}$ introduced in (3.13) and C^ε_h is the matrix defined in (3.14).

Under the hypotheses of Theorem 4.4, one has

$$X_{\varepsilon,h} \longrightarrow X \quad \text{in } \mathcal{C}^0([0,T]), \tag{56}$$

where for any t in $[0,T]$,

$$X(t) = \frac{1}{2} \int_\Omega \Big(\rho\, (u'(t) - v\phi'(t))^2 + A^0\, (\phi(t)\nabla v - \nabla u(t)) \,.\, (\phi(t)\nabla v - \nabla u(t)) \Big) dx. \tag{57}$$

5. **Proof of Theorem 4.3 (homogenization result).** Let φ be in $D(]0,T[)$. Multiplying (4.1) by φ and integrating by parts the first term over $(0,T)$, we get

$$\int_0^T \rho_\varepsilon u_\varepsilon \varphi''\, dt \; - \; \int_0^T \mathrm{div}\; (A^\varepsilon \nabla u_\varepsilon)\varphi\, dt \; = \; \int_0^T f_\varepsilon \varphi\, dt.$$

For any w_ε in $L^2(\Omega_\varepsilon \times (0,T))$ and for any w in $L^2(\Omega \times (0,T))$, set

$$\widehat{w}_\varepsilon = \int_0^T w_\varepsilon(x,t)\varphi(t)\, dt \quad \text{ and } \quad \widehat{w} = \int_0^T w(x,t)\varphi(t)\, dt,$$

respectively. Therefore, $\widehat{u}_\varepsilon$ satisfies

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla \widehat{u}_\varepsilon) &= \widehat{f}_\varepsilon - \rho_\varepsilon \displaystyle\int_0^T u_\varepsilon \varphi''\,dt \quad \text{on } \Omega_\varepsilon, \\ A^\varepsilon \nabla \widehat{u}_\varepsilon \,.\, \nu &= 0 \quad \text{on } \partial S_\varepsilon, \\ \widehat{u}_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{58}$$

By virtue of estimates (4.9), there exist a function u and a subsequence of ε (still denoted ε) such that

$$\begin{cases} P_\varepsilon u_\varepsilon \overset{*}{\rightharpoonup} u & \text{weakly * in } L^\infty(0,T;H_0^1(\Omega)), \\ P_\varepsilon (u_\varepsilon)' \overset{*}{\rightharpoonup} u' & \text{weakly * in } L^\infty(0,T;L^2(\Omega)). \end{cases}$$

Then

$$\begin{cases} \displaystyle\int_0^T P_\varepsilon u_\varepsilon \varphi\,dt \rightharpoonup \int_0^T u\,\varphi\,dt & \text{weakly in } H_0^1(\Omega), \\ \displaystyle\int_0^T P_\varepsilon u_\varepsilon \varphi''\,dt \rightharpoonup \int_0^T u\,\varphi''\,dt & \text{weakly in } H_0^1(\Omega). \end{cases}$$

Consequently,

$$\begin{cases} \displaystyle\int_0^T P_\varepsilon u_\varepsilon \varphi\,dt \to \int_0^T u\,\varphi\,dt & \text{strongly in } L^2(\Omega), \\ \displaystyle\int_0^T P_\varepsilon u_\varepsilon \varphi''\,dt \to \int_0^T u\,\varphi''\,dt & \text{strongly in } L^2(\Omega). \end{cases}$$

From (4.7) and (4.8)i, we have

$$\begin{cases} \displaystyle\int_0^T (\widetilde{f}_\varepsilon)\,\varphi\,dt \rightharpoonup \int_0^T f\varphi\,dt & \text{weakly in } L^2(\Omega), \\ \widetilde{\rho}_\varepsilon \displaystyle\int_0^T (P_\varepsilon u_\varepsilon)\,\varphi''\,dt \rightharpoonup \rho \int_0^T u\,\varphi''\,dt & \text{weakly in } L^2(\Omega). \end{cases}$$

In view of (4.6) and Proposition 2.5, this implies that $Q_\varepsilon \widehat{u}_\varepsilon$ converges weakly in $H_0^1(\Omega)$ to the solution u^* of the following problem:

$$-\,\text{div}\,(A^0 \nabla u^*) = \widehat{f} - \rho \int_0^T u\varphi''\,dt \quad \text{in } \Omega, \tag{59}$$

and moreover,

$$A^\varepsilon \widetilde{\nabla \widehat{u}_\varepsilon} \rightharpoonup A^0 \nabla u^* \quad \text{weakly in } L^2(\Omega). \tag{60}$$

Let us show that

$$u^* = \widehat{u}. \tag{61}$$

Observe that there exists a subsequence (still denoted ε) such that

$$\chi_{\Omega_\varepsilon} \overset{*}{\rightharpoonup} \chi_0 \quad \text{weakly * in } L^\infty(\Omega).$$

Hence,

$$\begin{cases} \chi_{\Omega_\varepsilon} Q_\varepsilon \widehat{u}_\varepsilon \rightharpoonup \chi_0 u^* & \text{weakly in } L^2(\Omega), \\ \chi_{\Omega_\varepsilon} \displaystyle\int_0^T P_\varepsilon u_\varepsilon \varphi\,dt \rightharpoonup \chi_0 \widehat{u} & \text{weakly in } L^2(\Omega). \end{cases}$$

Since

$$\chi_{\Omega_\varepsilon} Q_\varepsilon \widehat{u}_\varepsilon = \int_0^T \widetilde{u_\varepsilon} \varphi\,dt = \chi_{\Omega_\varepsilon} \int_0^T P_\varepsilon u_\varepsilon \varphi\,dt,$$

then $\chi_0 u^* = \chi_0 \widehat{u}$, which in view of the positiveness of χ_0, gives (5.4).

From (5.3) one obtains (4.13)iii and from (5.2)

$$-\text{div}\,(A^0\nabla\widehat{u}) = \int_0^T f\,\varphi\,dt - \rho\int_0^T u\,\varphi''\,dt \quad \text{in } D'(\Omega),$$

which implies, since φ is arbitrary,

$$\int_\Omega \rho\, u'' v\,dx + \int_\Omega A^0\nabla u\nabla v\,dx = \int_\Omega f v\,dx \quad \text{in } D'(0,T),\ \ \forall v\in H_0^1(\Omega). \tag{62}$$

It remains to check the initial conditions.

Since $W(0,T;H_0^1(\Omega),L^2(\Omega))$ is compactly embedded in $\mathcal{C}^0([0,T];L^2(\Omega))$ (see [12], Corollary 4, p. 85), we have

$$P_\varepsilon u_\varepsilon \longrightarrow u \quad \text{strongly in } C^0([0,T];L^2(\Omega)), \tag{63}$$

which gives

$$P_\varepsilon u_\varepsilon(0) \longrightarrow u(0) \quad \text{strongly in } L^2(\Omega).$$

From (4.12) and (4.8)ii, we get

$$P_\varepsilon u_\varepsilon(0) = Q_\varepsilon a_\varepsilon \rightharpoonup a_0 \quad \text{weakly in } L^2(\Omega),$$

so that

$$u(0) = a_0 \ \text{ on } \Omega.$$

On the other hand, to identify $u'(0)$, choose $v\in H_0^1(\Omega)$ and $\varphi\in\mathcal{C}^\infty([0,T])$ such that $\varphi'(0)=\varphi(T)=\varphi'(T)=0$ and $\varphi(0)=1$.

An integration by parts over (0,T) in (4.5) gives

$$\int_0^T\!\!\int_\Omega A^\varepsilon\widetilde{\nabla u_\varepsilon}\nabla v\varphi\,dxdt - \int_0^T\!\!\int_\Omega \widetilde{f_\varepsilon} v\,\varphi\,dxdt \;=\; -\int_0^T\!\!\int_{\Omega_\varepsilon}\rho_\varepsilon u_\varepsilon'' v\varphi\,dxdt$$

$$= \int_0^T\!\!\int_{\Omega_\varepsilon}\rho_\varepsilon u_\varepsilon' v\varphi'\,dxdt + \int_{\Omega_\varepsilon}\rho_\varepsilon b_\varepsilon v\varphi(0)\,dx$$

$$= -\int_0^T\!\!\int_\Omega \widetilde{\rho_\varepsilon} P_\varepsilon u_\varepsilon v\varphi''\,dxdt + \int_\Omega \widetilde{\rho_\varepsilon b_\varepsilon} v\,dx.$$

Passing to the limit as ε tends to 0 and using (4.7), (4.8) and (4.13), we obtain

$$\int_0^T\!\!\int_\Omega A^0\nabla u\nabla v\varphi\,dxdt - \int_0^T\!\!\int_\Omega f v\varphi\,dxdt \;=\; -\int_0^T\!\!\int_\Omega \rho\,u\,v\,\varphi''\,dxdt + \int_\Omega b\,v\,dx$$

$$= -\int_0^T\!\!\int_\Omega \rho\,u''\,v\,\varphi\,dxdt - \int_\Omega \rho\,u'(0)v\,dx + \int_\Omega b\,v\,dx. \tag{64}$$

Observe now that in (5.5) one can choose $\varphi\in\mathcal{C}^\infty([0,T])$, which together with (5.7) and assumption (4.2) gives

$$u'(0) = \frac{b}{\rho}.$$

The uniqueness of the solution of the limit wave equation (4.14) implies that in (4.13) the whole sequence $P_\varepsilon u_\varepsilon$ converges to u and this concludes the proof. □

6. **Convergence of the energy.** In this section, we study the asymptotic behavior of the energy associated with problem (4.1). Take u'_ε as test function in (4.5) (this is formal, but a standard density argument makes the computation rigorous, see [6] Chap. 12, Prop. 12.7). We have after an integration by parts in t over $(0,T)$,

$$\int_{\Omega_\varepsilon} (\frac{1}{2}\rho_\varepsilon\, |u'_\varepsilon|^2 + \frac{1}{2}A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon)\; dx =$$

$$\frac{1}{2}\int_{\Omega_\varepsilon} \rho_\varepsilon\, |b_\varepsilon|^2 \;\; dx \;\; + \frac{1}{2}\int_{\Omega_\varepsilon} A^\varepsilon \nabla a_\varepsilon \nabla a_\varepsilon \;\, dx \;\; + \int_0^T\!\!\int_{\Omega_\varepsilon} f_\varepsilon u'_\varepsilon \;\, dxdt.$$

Set

$$e_\varepsilon(t) = \frac{1}{2}\int_{\Omega_\varepsilon} \Big[\rho_\varepsilon\, (u'_\varepsilon(t))^2 + A^\varepsilon \nabla u_\varepsilon(t) \nabla u_\varepsilon(t)\Big]\, dx, \quad \forall t \in [0,T] \tag{65}$$

and

$$E_\varepsilon = \frac{1}{2}\int_{\Omega_\varepsilon} \big[\rho_\varepsilon (b_\varepsilon)^2 + A^\varepsilon \nabla a_\varepsilon \nabla a_\varepsilon\big]\, dx. \tag{66}$$

The quantity e_ε is called the energy associated with problem (4.1).

Observe that

$$e_\varepsilon(t) - E_\varepsilon = \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon u'_\varepsilon \, dxds, \quad \forall\, t \in [0,T]. \tag{67}$$

The corresponding e^0 and E^0, associated with the solution u of the homogenized wave equation (4.14), are defined by

$$e^0(t) = \frac{1}{2}\int_{\Omega} \big[\rho\, (u'(t))^2 + A^0 \nabla u(t) \nabla u(t)\big]\, dx, \quad \forall\, t \in [0,T], \tag{68}$$

$$E^0 = \frac{1}{2}\int_{\Omega} \big[\rho (b_0)^2 + A^0 \nabla a_0 \nabla a_0\big]\, dx \tag{69}$$

and one has (using a density argument similar to that needed for e_ε)

$$e^0(t) = E^0 + \int_0^t \int_{\Omega} f u' \, dxds, \quad \forall\, t \in [0,T]. \tag{70}$$

Proposition 6.1. *Let e_ε be defined in (6.1). Under the hypotheses of Theorem 4.3, there exists a continuous function $e(t)$ and a subsequence of e_ε (still defined ε) such that*

$$e_\varepsilon \longrightarrow e \qquad \textit{strongly in } \mathcal{C}^0([0,T]).$$

The proof of this proposition can be done using the same arguments as that given in [11] for the periodic case. In general we do not have $e(t) = e^0(t)$. But under the additional hypotheses (4.15)-(4.17), this equality holds true, as shown in the next result.

Theorem 6.2. *Under the hypotheses of Theorem 4.4, we have the following strong convergence of the energy:*

$$e_\varepsilon \longrightarrow e^0 \qquad \textit{strongly in } \mathcal{C}^0([0,T]),$$

where e_ε and e^0 are defined by (6.1) and (6.4), respectively.

Proof. Let e be given by Proposition 6.1. We have to show that

$$e(t) = e^0(t), \quad \forall\, t \in [0,T].$$

From (6.1), (6.2) and (6.3), it follows that

$$e_\varepsilon(t) = \frac{1}{2}\int_\Omega \widetilde{\rho}_\varepsilon (b_\varepsilon)^2\, dx + \frac{1}{2}\int_{\Omega_\varepsilon} A^\varepsilon \nabla a_\varepsilon \nabla a_\varepsilon\, dx + \int_0^t\!\!\int_\Omega f_\varepsilon \chi_{\Omega_\varepsilon} P_\varepsilon u'_\varepsilon\, dxds. \tag{71}$$

Taking a_ε as test function in (53) and using Remark 4.2, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A^\varepsilon \nabla a_\varepsilon \nabla a_\varepsilon dx \;=\; \lim_{\varepsilon\to 0}\int_\Omega A^0 \nabla a_0 \nabla (Q_\varepsilon a_\varepsilon)\, dx \;=\; \int_\Omega A^0 \nabla a_0 \nabla a_0 dx. \tag{72}$$

From (4.15) and using the strong convergence (5.6), one has

$$\chi_{\Omega_\varepsilon} P_\varepsilon u_\varepsilon \rightharpoonup \chi_0 u \quad \text{ weakly in } L^2(\Omega\times(0,T)). \tag{73}$$

On the other hand, for any $\psi \in D(\Omega)$ and $\zeta \in D(0,T)$

$$\int_0^T\!\!\int_\Omega \chi_{\Omega_\varepsilon}(P_\varepsilon u_\varepsilon)'\psi\,\zeta\, dxdt = -\int_0^T\!\!\int_\Omega \chi_{\Omega_\varepsilon} P_\varepsilon u_\varepsilon \psi\,\zeta'\, dxdt.$$

This, together with (6.9), gives

$$\lim_{\varepsilon\to 0} -\int_0^T\!\!\int_\Omega \chi_{\Omega_\varepsilon} P_\varepsilon u_\varepsilon \psi\,\zeta'\, dxdt = -\int_0^T\!\!\int_\Omega \chi_0 u\psi\,\zeta'\, dxdt = \int_0^T\!\!\int_\Omega \chi_0 u'\psi\,\zeta\, dxdt.$$

Hence, by density

$$\chi_{\Omega_\varepsilon} P_\varepsilon u'_\varepsilon \rightharpoonup \chi_0 u' \quad \text{ weakly in } L^2(\Omega\times(0,T)),$$

and from (4.16)i, it follows that

$$\lim_{\varepsilon\to 0}\int_0^t\!\!\int_\Omega f_\varepsilon \chi_{\Omega_\varepsilon} P_\varepsilon u'_\varepsilon\, dxds = \int_0^t\!\!\int_\Omega f u'\, dxds. \tag{74}$$

From (4.7) and (4.16)ii, we have

$$\lim_{\varepsilon\to 0}\int_\Omega \widetilde{\rho}_\varepsilon (b_\varepsilon)^2\, dx = \int_\Omega \rho (b_0)^2\, dx. \tag{75}$$

Combining (6.8), (6.10), (6.11) and passing to the limit in (6.7), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} e_\varepsilon(t) &= \frac{1}{2}\int_\Omega \rho(b_0)^2 dx + \int_\Omega A^0\nabla a_0 \nabla a_0 dx + \int_0^t\!\!\int_\Omega f u'\, dxds \\ &= e^0(t),\end{aligned}$$

which is the claimed result. □

Remark 6.1. Observe that once the corrector result of Theorem 4.4 is proved, we can show that each term which composes the expression (6.1) of $e_\varepsilon(t)$ converges to the corresponding one in (6.4).

7. **Proof of Theorem 4.4 (corrector result).** Has already mentioned, the proof of Theorem 4.4 makes use of Proposition 4.6. We begin by its proof.

Proof of Proposition 4.6. Let $v \in D(\omega_h)$ and $\phi \in \mathcal{C}^\infty([0,T])$. If $X_{\varepsilon,h}$ is given by (4.19), we can write

$$X_{\varepsilon,h} = X^1_{\varepsilon,h} + X^2_{\varepsilon,h} + X^3_{\varepsilon,h},$$

where for every $t \in [0,T]$,

$$\begin{cases} X^1_{\varepsilon,h}(t) = \dfrac{1}{2}\displaystyle\int_{\Omega_\varepsilon}\Big[\rho_\varepsilon\,(u'_\varepsilon(t))^2 + A^\varepsilon \nabla u_\varepsilon(t)\nabla u_\varepsilon(t)\Big]\,dx = e_\varepsilon(t), \\ X^2_{\varepsilon,h}(t) = -\displaystyle\int_{\Omega_\varepsilon}\left[\rho_\varepsilon u'_\varepsilon(t)\,\phi'(t)\,v + \phi(t) A^\varepsilon C^\varepsilon_h \nabla v \nabla u_\varepsilon(t)\right]dx, \\ X^3_{\varepsilon,h}(t) = \dfrac{1}{2}\displaystyle\int_{\Omega_\varepsilon}\left[\rho_\varepsilon(\phi'(t))^2\,v^2 + A^\varepsilon\phi(t)C^\varepsilon_h\nabla v\,\phi(t)C^\varepsilon_h\nabla v\right]dx. \end{cases} \tag{76}$$

We study the limit of each term separately.

The limit of $X^1_{\varepsilon,h}$: In Theorem 6.2 this limit was computed in $\mathcal{C}^0([0,T])$, it is the energy of the problem limit given by (6.4).

The limit of $X^3_{\varepsilon,h}$: Let us prove the convergence

$$X^3_{\varepsilon,h} \longrightarrow \frac{1}{2}\int_\Omega\left[\rho(\phi')^2v^2 + A^0\phi\nabla v\,\phi\nabla v\right]\,dx \quad \text{in } \mathcal{C}^0([0,T]). \tag{77}$$

From (3.17), (4.2) and (4.3), one has

$$2\left\|X^3_{\varepsilon,h}\right\|_{L^\infty(0,T)} \le \|\phi'\|^2_{L^\infty(0,T)}\,\|\widetilde{\rho_\varepsilon}\|_{L^\infty(\Omega)}\,\|v\|^2_{L^2(\Omega)} + \\ \|\phi\|^2_{L^\infty(0,T)}\,\|A^\varepsilon\|_{(L^\infty(\Omega))^{n^2}}\,\|C^\varepsilon_h\nabla v\|^2_{L^2(\Omega)^n} \le c,$$

where c is independent of ε. Similarly, in view of the regularity of ϕ, one can prove that the time derivative of $X^3_{\varepsilon,h}$ is bounded. Thus, from the Ascoli-Arzela theorem, there exists a subsequence (still denoted ε) such that $X^3_{\varepsilon,h}$ converges strongly in $\mathcal{C}^0([0,T])$ to a function X^3_h. To identify this limit, it is sufficient to identify the pointwise limit of $X^3_{\varepsilon,h}$. Observe first that from (4.7), one has for every $t \in [0,T]$,

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\rho_\varepsilon\,(\phi'(t))^2\,v^2\,dx = \lim_{\varepsilon\to 0}\int_\Omega\widetilde{\rho_\varepsilon}\,(\phi'(t))^2\,v^2\,dx = \int_\Omega\rho\,(\phi'(t))^2\,v^2\,dx. \tag{78}$$

From definition (3.14) of C^ε_h, we can write

$$C^\varepsilon_h\nabla v = \sum_{i=1}^n \nabla(Q_\varepsilon w^\varepsilon_{h,i})\frac{\partial v}{\partial x_i}, \qquad \text{a.e. on } \Omega. \tag{79}$$

Using (7.4) yields

$$\begin{aligned}\int_{\Omega_\varepsilon}A^\varepsilon C^\varepsilon_h\nabla v\,C^\varepsilon_h\nabla v\,dx &= \sum_{i,j=1}^n\int_{\Omega_\varepsilon}A^\varepsilon\nabla w^\varepsilon_{h,i}\frac{\partial v}{\partial x_i}\nabla w^\varepsilon_{h,j}\frac{\partial v}{\partial x_j}\,dx \\ &= \sum_{i,j=1}^n\int_\Omega\left(\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}A^\varepsilon\widetilde{\nabla w^\varepsilon_{h,i}}\nabla(Q_\varepsilon w^\varepsilon_{h,j})\right)dx. \end{aligned} \tag{80}$$

Now, we apply Proposition 2.6 to the last term.

Due to (3.15) and (3.18)ii, we can apply Proposition 2.6 i) to $A^\varepsilon\nabla w^\varepsilon_{h,i}$ to get

$$\text{div}\,(A^\varepsilon\widetilde{\nabla w^\varepsilon_{h,i}}) \longrightarrow \text{div}\,(A^0\nabla(\varphi_h x_i)) \quad \text{strongly in } H^{-1}(\Omega), \tag{81}$$

for every $i \in \{1,\dots,n\}$.

On the other hand, $\text{curl}\,(\nabla(Q_\varepsilon\omega^\varepsilon_{h,j})) = 0$ for every $j \in \{1,\dots,n\}$.

Now, we can pass to the limit in (7.5) using Proposition 2.6 ii). Since $v \in D(\omega_h)$ and $\varphi_h = 1$ on ω_h, one has

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A^\varepsilon\phi(t)C^\varepsilon_h\nabla v\,\phi(t)C^\varepsilon_h\nabla v\,dx\\
&= (\phi(t))^2\sum_{i,j=1}^n\int_\Omega\left(\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}A^0\nabla(\varphi_h x_i)\nabla(\varphi_h x_j)\right)dx\\
&= (\phi(t))^2\sum_{i,j=1}^n\int_\Omega\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\,A^0 e_i e_j\,dx = \int_\Omega A^0\phi(t)\nabla v\,\phi(t)\nabla v\,dx,
\end{aligned}$$

for every $t \in [0,T]$. This, together with (7.3), gives (7.2).

The limit of $X^2_{\varepsilon,h}$: To show that the convergence of $X^2_{\varepsilon,h}$ takes place in $\mathcal{C}^0([0,T])$, we cannot use the same argument as for $X^3_{\varepsilon,h}$. Indeed, since we have no uniform estimate on u''_ε, we have no estimate on $(X^2_{\varepsilon,h})'$. We study then separately the limit of the two terms which compose $X^2_{\varepsilon,h}$.

Let us prove first that

$$\int_{\Omega_\varepsilon}\widetilde{\rho_\varepsilon}v\,u'_\varepsilon\phi'\,dx \longrightarrow \int_\Omega \rho\,v\,u'\phi'\,dx \quad \text{in } \mathcal{C}^0([0,T]). \tag{82}$$

To do so, we first identify its limit in $L^\infty(0,T)$ and then we show that the convergence takes place in $C^0([0,T])$. Let ψ be in $D(0,T)$, one has

$$\begin{aligned}
\left\langle\int_{\Omega_\varepsilon}\rho_\varepsilon v u'_\varepsilon\phi'\,dx\,,\,\psi\right\rangle_{L^\infty(0,T),L^1(0,T)} &= \int_0^T\int_{\Omega_\varepsilon}\rho_\varepsilon v u'_\varepsilon\phi'\,\psi\,dx\,dt\\
&= -\int_0^T\int_\Omega\widetilde{\rho_\varepsilon}vP_\varepsilon u_\varepsilon\,(\phi'\psi' + \phi''\,\psi)\,dx\,dt.
\end{aligned}$$

Then, using (5.6) and (4.7), we can pass to the limit to get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\left\langle\int_{\Omega_\varepsilon}\rho_\varepsilon v u'_\varepsilon\phi'\,dx\,,\,\psi\right\rangle_{L^\infty(0,T),L^1(0,T)} &= -\int_0^T\int_\Omega\rho\,v\,u\,(\phi'\psi' + \phi''\psi)\,dx\,dt\\
&= \int_0^T\int_\Omega\rho\,v u'\phi'\psi\,dx\,dt = \left\langle\int_\Omega\rho\,v u'\phi'\,dx\,,\,\psi\right\rangle_{L^\infty(0,T),L^1(0,T)},
\end{aligned}$$

which, by density, gives

$$\int_{\Omega_\varepsilon}\rho_\varepsilon v u'_\varepsilon\phi'\,dx \overset{*}{\rightharpoonup} \int_\Omega\rho v u'\phi'\,dx \quad \text{weakly * in } L^\infty(0,T). \tag{83}$$

On the other hand, to show that this convergence is strong in $\mathcal{C}^0([0,T])$, we use the compact embedding of $H^1(0,T)$ into $\mathcal{C}^0([0,T])$. From (4.5), we can write

$$\frac{\partial}{\partial t}\int_{\Omega_\varepsilon}\rho_\varepsilon u'_\varepsilon\phi' v\,dx = \int_{\Omega_\varepsilon}\rho_\varepsilon u'_\varepsilon\,\phi'' v\,dx + \int_{\Omega_\varepsilon}f_\varepsilon\phi' v\,dx - \int_{\Omega_\varepsilon}A^\varepsilon\nabla u_\varepsilon\phi'\nabla v\,dx.$$

From (4.2)-(4.4), (4.7), (4.13) and (4.16)i, one has

$$\left|\frac{\partial}{\partial t}\int_{\Omega_\varepsilon}\rho_\varepsilon u_\varepsilon'(t)\phi'(t)v\,dx\right| \leq \ \|\phi''\|_{L^\infty(0,T)}\|\rho_\varepsilon\|_{L^\infty(\Omega)}\left\|P_\varepsilon u_\varepsilon'(t)\right\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$

$$+\ \|\phi'\|_{L^\infty(0,T)}\Big(\|f_\varepsilon(t)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\ +\ \|A^\varepsilon\|_{(L^\infty(\Omega))^{n^2}}\|\nabla(P_\varepsilon u_\varepsilon(t))\|_{(L^2(\Omega))^n}\|\nabla v\|_{(L^2(\Omega))^n}\Big).$$

Thus,

$$\int_0^T\left|\frac{\partial}{\partial t}\int_{\Omega_\varepsilon}\rho_\varepsilon u_\varepsilon'\phi' v\,dx\right|^2 \leq \ c,$$

where c is independent of ε. Convergence (7.7) follows from this inequality and (7.8).

Let us now prove the convergence of the second term of $X^2_{\varepsilon,h}$, namely

$$\int_{\Omega_\varepsilon}\phi\,A^\varepsilon C^\varepsilon_h\nabla v\nabla u_\varepsilon\,dx \longrightarrow \int_\Omega\phi\,A^0\nabla v\nabla u\,dx \quad \text{in } \mathcal{C}^0([0,T]). \tag{84}$$

To do so, we adapt to our case some arguments used in [2]. From (7.4), the variational formulation of (3.15), definition (4.12) of P_ε , we have

$$\begin{aligned}
\int_{\Omega_\varepsilon}A^\varepsilon\phi(t)\,C^\varepsilon_h\nabla v\nabla u_\varepsilon(t)\,dx \ &= \sum_{i=1}^n\int_{\Omega_\varepsilon}A^\varepsilon\phi(t)\,\nabla w^\varepsilon_{h,i}\frac{\partial v}{\partial x_i}\nabla u_\varepsilon(t)\,dx\\
=\sum_{i=1}^n\int_{\Omega_\varepsilon}&\left(A^\varepsilon\phi(t)\,\nabla w^\varepsilon_{h,i}\nabla\left(\frac{\partial v}{\partial x_i}\,u_\varepsilon(t)\right)-A^\varepsilon\phi(t)\,\nabla w^\varepsilon_{h,i}\nabla\left(\frac{\partial v}{\partial x_i}\right)u_\varepsilon(t)\right)dx\\
=\sum_{i=1}^n&\left\langle -\mathrm{div}\ (A^0\phi(t)\nabla(\varphi_h x_i))\,,\,Q_\varepsilon\left(\left.\frac{\partial v}{\partial x_i}\right|_{\Omega_\varepsilon}u_\varepsilon(t)\right)\right\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\\
&\qquad -\sum_{i=1}^n\int_\Omega A^\varepsilon\phi(t)\,\widetilde{\nabla w^\varepsilon_{h,i}}\nabla\left(\frac{\partial v}{\partial x_i}\right)P_\varepsilon u_\varepsilon(t)\,dx\\
=\sum_{i=1}^n&\langle G_i\,,k^\varepsilon_i\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}(t)\ -\ \langle H_\varepsilon,\,P_\varepsilon u_\varepsilon\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}(t),
\end{aligned} \tag{85}$$

where

$$\begin{cases}
G_i=-\phi(t)\,\mathrm{div}\ (A^0\nabla(\varphi_h x_i))\ , \qquad k^\varepsilon_i=P_\varepsilon\left(\left.\frac{\partial v}{\partial x_i}\right|_{\Omega_\varepsilon}u_\varepsilon\right), \quad \forall\, i\in\{1,\dots,n\}\\
\qquad\qquad \text{and} \quad H_\varepsilon=\displaystyle\sum_{i=1}^n A^\varepsilon\phi\,\widetilde{\nabla w^\varepsilon_{h,i}}\nabla\left(\frac{\partial v}{\partial x_i}\right).
\end{cases} \tag{86}$$

For simplicity, the duality pairing $\langle\cdot,\cdot\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}$ will be denoted $\langle\cdot,\cdot\rangle$.

In view of (3.18)ii, $\{H_\varepsilon\}$ is a bounded sequence in $\mathcal{C}^\infty([0,T]);L^2(\Omega))$ and furthermore,

$$H_\varepsilon \overset{*}{\rightharpoonup} H_0=\sum_{i=1}^n A^0\nabla(\varphi_h x_i)\,\phi\,\nabla\left(\frac{\partial v}{\partial x_i}\right) \qquad \text{weakly * in } L^\infty(0,T;L^2(\Omega)). \tag{87}$$

Let us show that

$$\langle H_\varepsilon-H_0,P_\varepsilon u_\varepsilon\rangle\longrightarrow 0 \quad \text{strongly in } C^0([0,T]). \tag{88}$$

First, notice that from (5.6) and (7.12), we have for any t fixed in [0,T],

$$\lim_{\varepsilon\to 0}\langle H_\varepsilon - H_0, P_\varepsilon u_\varepsilon\rangle(t) = 0. \tag{89}$$

Moreover, due to (4.13) and (7.12), there exists a constant c independent of ε such that

$$\begin{cases} \|\langle H_\varepsilon - H_0, P_\varepsilon u_\varepsilon\rangle\|_{C^0([0,T])} \le c, \\ \|(\langle H_\varepsilon - H_0, P_\varepsilon u_\varepsilon\rangle)'\|_{C^0([0,T])} \le c. \end{cases} \tag{90}$$

Then, from (7.14), (7.15) and the Ascoli-Arzela theorem, we obtain (7.13).

Let us also prove that for any $i \in \{1, \dots, n\}$,

$$k_i^\varepsilon \longrightarrow k_i = \frac{\partial v}{\partial x_i}\, u \quad \text{ strongly in } \mathcal{C}^0([0,T]; L^2(\Omega)). \tag{91}$$

This is not obvious because in general, we do not have $P_\varepsilon\left(\left.\dfrac{\partial v}{\partial x_i}\right|_{\Omega_\varepsilon} u_\varepsilon\right) = \dfrac{\partial v}{\partial x_i}\, P_\varepsilon u_\varepsilon$.

In view of (4.9) and (4.11), k_i^ε and $(k_i^\varepsilon)'$ are bounded in $L^\infty(0,T; H_0^1(\Omega))$ and in $L^\infty(0,T; L^2(\Omega))$, respectively. This implies that there exist a subsequence of ε (still denoted ε) and a function k_i^0 such that

$$\begin{cases} k_i^\varepsilon \overset{*}{\rightharpoonup} k_i^0 & \text{weakly * in } L^\infty(0,T; H_0^1(\Omega)), \\ (k_i^\varepsilon)' \overset{*}{\rightharpoonup} (k_i^0)' & \text{weakly * in } L^\infty(0,T; L^2(\Omega)). \end{cases} \tag{92}$$

Using definition (4.12) of P_ε, we have for any t fixed in $[0,T]$,

$$P_\varepsilon\left(\left.\frac{\partial v}{\partial x_i}\right|_{\Omega_\varepsilon} u_\varepsilon\right)(t) = Q_\varepsilon\left(\left.\left(\frac{\partial v}{\partial x_i}\, Q_\varepsilon u_\varepsilon(t)\right)\right|_{\Omega_\varepsilon}\right). \tag{93}$$

Due to convergences (4.13), we can apply Lemma 2.2 to (7.18) to get $k_i^0 = k_i$. This, together with estimates (7.17), gives (7.16).

Using (5.6), (7.10)-(7.13) and (7.16), for every t fixed in [0,T], one has

$$\lim_{\varepsilon\to 0}\left(\sum_{i=1}^n \langle G_i, k_i^\varepsilon\rangle(t) - \langle H_\varepsilon, P_\varepsilon u_\varepsilon\rangle(t)\right) = \sum_{i=1}^n \langle G_i, k_i\rangle(t) - \langle H_0, u\rangle(t). \tag{94}$$

Since supp $v \subset \omega_h$ and $\varphi_h = 1$ on ω_h,

$$\int_\Omega A^0\phi(t)\nabla(\varphi_h x_i)\nabla\left(\frac{\partial v}{\partial x_i}\, u(t)\right) dx = \int_\Omega A^0 e_i\phi(t)\nabla\left(\frac{\partial v}{\partial x_i}\, u(t)\right) dx, \tag{95}$$

and

$$\int_\Omega A^0\phi(t)\nabla(\varphi_h x_i)\nabla\left(\frac{\partial v}{\partial x_i}\right) u(t)\, dx = \int_\Omega A^0 e_i\phi(t)\nabla\left(\frac{\partial v}{\partial x_i}\right) u(t)\, dx. \tag{96}$$

This, in view of (7.10), (7.11) and (7.19), gives for every t fixed in $[0,T]$,

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\phi(t)A^\varepsilon C_h^\varepsilon\nabla v\nabla u_\varepsilon(t)\, dx = \int_\Omega \phi(t)A^0\nabla v\nabla u(t)\, dx. \tag{97}$$

Finally, let us check that this convergence takes place in $\mathcal{C}^0([0,T])$.

Let $\eta > 0$, since $\mathcal{C}^0([0,T];L^2(\Omega))$ is dense in $\mathcal{C}^0([0,T],H^{-1}(\Omega))$, we can find $(H,L_i)_{1\le i\le n}$ in $(\mathcal{C}^0([0,T];L^2(\Omega))^{n+1}$ such that

$$\begin{cases} \|H_0 - H\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))} \le \eta, \\ \|G_i - L_i\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))} \le \eta. \end{cases} \tag{98}$$

We can write

$$\begin{cases} \langle H_\varepsilon, P_\varepsilon u_\varepsilon\rangle - \langle H_0, u\rangle = \langle H_\varepsilon - H_0, P_\varepsilon u_\varepsilon\rangle + \langle H_0 - H, P_\varepsilon u_\varepsilon\rangle \\ \qquad\qquad + \langle H, P_\varepsilon u_\varepsilon - u\rangle + \langle H - H_0, u\rangle, \\ \langle G_i, k_i^\varepsilon\rangle - \langle G_i, k_i\rangle = \langle G_i - L_i, k_i^\varepsilon\rangle + \langle L_i, k_i^\varepsilon - k_i\rangle + \langle L_i - G_i, k_i\rangle . \end{cases} \tag{99}$$

Then, there exists a constant $C > 0$ independent of η such that

$$\begin{aligned} &\|\langle H_\varepsilon, P_\varepsilon u_\varepsilon\rangle - \langle H_0, u\rangle\|_{\mathcal{C}^0([0,T])} + \sum_{i=1}^n \|\langle G_i, k_i^\varepsilon\rangle - \langle G_i, k_i\rangle\|_{\mathcal{C}^0([0,T])} \\ &\quad\le \|\langle H_\varepsilon - H_0, P_\varepsilon u_\varepsilon\rangle\|_{\mathcal{C}^0([0,T])} + \|H_0 - H\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))} \|P_\varepsilon u_\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} \\ &\qquad + \|H\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \|P_\varepsilon u_\varepsilon - u\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \\ &\qquad + \|H - H_0\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))} \|u\|_{L^\infty(0,T;H_0^1(\Omega))} \\ &\qquad + \sum_{i=1}^n \Big(\|G_i - L_i\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))} \|k_i^\varepsilon\|_{L^\infty(0,T;H_0^1(\Omega))} \\ &\qquad + \|L_i\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \|k_i^\varepsilon - k_i\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \\ &\qquad + \|L_i - G_i\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))} \|k_i\|_{L^\infty(0,T;H_0^1(\Omega))}\Big) \\ &\quad\le C \Big(\|\langle H_\varepsilon - H_0, P_\varepsilon u_\varepsilon\rangle\|_{\mathcal{C}^0([0,T])} + \|P_\varepsilon u_\varepsilon - u\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \\ &\qquad\qquad + \|k_i^\varepsilon - k_i\|_{\mathcal{C}^0([0,T];L^2(\Omega))} + \eta\Big). \end{aligned}$$

From (4.13), (7.13), (7.16) and since η is arbitrary, we deduce that the left-hand side tends to 0 as $\varepsilon \to 0$, so that

$$\sum_{i=1}^n \langle G_i, k_i^\varepsilon\rangle - \langle H_\varepsilon, P_\varepsilon u_\varepsilon\rangle \longrightarrow \sum_{i=1}^n \langle G_i, k_i\rangle - \langle H_0, u\rangle \quad \text{strongly in } \mathcal{C}^0([0,T]).$$

This, together with (7.22) gives (7.9).

By using Theorem 6.2, (7.2), (7.7) and (7.9), we conclude that X_h^ε converges strongly to X in $\mathcal{C}^0([0,T])$. □

Proof of Theorem 4.4. Let $\{C^\varepsilon\}$ be the sequence introduced in Section 3 by (3.19). To prove the result we use from Proposition 3.7, Proposition 4.6 and the following classical density result:
$\forall\, u \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)),\ \forall\, \eta > 0, \quad \exists\, (\phi, v) \in \mathcal{C}^\infty([0,T]) \times D(\Omega),$

$$\begin{cases} i) \qquad \|u' - \phi' v\|^2_{C^0([0,T];L^2(\Omega))} \le \eta, \\ ii) \quad \|\nabla u - \phi \nabla v\|^2_{C^0([0,T];(L^2(\Omega))^n)} \le \eta. \end{cases} \tag{100}$$

Let $\eta > 0$ and (ϕ, v) be defined by (7.25). Take h in $\mathbb{N}$ such that supp $v \subset \omega_h$. From (4.2), (4.3) and definition (4.19) of $X_{\varepsilon,h}$, we have for every $t \in [0,T]$,

$$\frac{\lambda_1}{2} \|u_\varepsilon'(t) - \phi'(t) v\|^2_{L^2(\Omega_\varepsilon)} + \frac{\alpha}{2} \|\phi(t) C_h^\varepsilon \nabla v - \nabla u_\varepsilon(t)\|^2_{(L^2(\Omega_\varepsilon))^n} \le X_{\varepsilon,h}(t).$$

Then, due to Proposition 4.5, there exists a constant c such that

$$\begin{aligned}&\overline{\lim_{\varepsilon\to 0}}\left\{\|u'_\varepsilon-\phi'\,v\|^2_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))}+\|\phi\,C^\varepsilon_h\nabla v-\nabla u_\varepsilon\|^2_{\mathcal{C}^0([0,T];(L^2(\Omega_\varepsilon))^n)}\right\}\\&\qquad\le c\,\|X\|_{\mathcal{C}^0([0,T])}\,.\end{aligned}\tag{101}$$

In view of (4.2), (4.3) and definition (4.21) of X, we have

$$\|X\|_{\mathcal{C}^0([0,T])}\le\frac{\lambda_2}{2}\,\|u'-\phi'v\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))}+\frac{\beta}{2}\,\|\phi\nabla v-\nabla u\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))}\,.\tag{102}$$

Let ω be an open set with $\omega\subset\subset\Omega$. From (3.20), (7.25)-(7.27) and the Hölder inequality, there exist positive constants c_1,c_2 and c_3 such that

$$\begin{aligned}&\frac{\lambda_1}{2}\,\|u'_\varepsilon-u'\|^2_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))}+\frac{\alpha}{2}\,\|C^\varepsilon\nabla u-\nabla u_\varepsilon\|^2_{C^0([0,T];(L^1(\Omega_\varepsilon\cap\omega))^n)}\\
&\le c_1\Big(\|u'_\varepsilon-\phi'v\|^2_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))}+\|u'-\phi'v\|^2_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))}\\
&\quad+\|C^\varepsilon\nabla u-\phi\,C^\varepsilon\nabla v\|^2_{C^0([0,T];(L^1(\Omega_\varepsilon\cap\omega))^n)}+\|\phi\,C^\varepsilon\nabla v-\phi\,C^\varepsilon_h\nabla v\|^2_{C^0([0,T];L^1(\Omega_\varepsilon\cap\omega))^n)}\\
&\qquad\qquad+\|\phi\,C^\varepsilon_h\nabla v-\nabla u_\varepsilon\|^2_{C^0([0,T];L^1(\Omega_\varepsilon\cap\omega))^n)}\Big)\\
&\le c_2\Big(\|X\|^2_{C^0([0,T])}+\eta+\|C^\varepsilon\|^2_{(L^2(\omega))^{n^2}}\,\|\nabla u-\phi\nabla v\|^2_{C^0([0,T];(L^2(\Omega_\varepsilon))^n)}\\
&\qquad\qquad+\|(C^\varepsilon_h-C^\varepsilon)\nabla v\|^2_{(L^1(\Omega_\varepsilon))^n}\,\|\phi\|^2_{L^\infty(0,T)}\Big)\\
&\le c_3\Big(\|u'-\phi'v\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))}+\|\nabla u-\phi\nabla v\|^2_{\mathcal{C}^0([0,T];(L^2(\Omega))^n)}+\eta\\
&\qquad+\|(C^\varepsilon_h-C^\varepsilon)\nabla v\|^2_{(L^1(\Omega_\varepsilon))^n}\Big)\;\le\;c_3\Big(3\eta+\|(C^\varepsilon_h-C^\varepsilon)\nabla v\|^2_{(L^1(\Omega_\varepsilon))^n}\Big)\,.\end{aligned}$$

By virtue of Proposition 3.7, there exists a positive constant c_4 such that

$$0\le\overline{\lim_{\varepsilon\to 0}}\Big(\|u'_\varepsilon-u'\|^2_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))}+\|C^\varepsilon\nabla u-\nabla u_\varepsilon\|^2_{C^0([0,T];(L^1(\Omega_\varepsilon\cap\omega))^n)}\Big)\le c_4\eta.$$

Since η is arbitrary, this implies (4.18)i and (4.18)ii for the sequence $\{C^\varepsilon\}$ defined by (3.19). □

Proof of Corollary 4.5. Suppose that $\nabla u\in\mathcal{C}^0([0,T];(L^p(\Omega))^n)$ where $p>2$. Let $(D^\varepsilon)_\varepsilon$ be an elliptic local corrector satisfying (3.4) with $r=2$. From the Hölder inequality, for any open set ω with $\omega\subset\subset\Omega$, one has

$$\begin{aligned}&\|\nabla u_\varepsilon-D^\varepsilon\nabla u\|_{\mathcal{C}^0([0,T];(L^1(\Omega_\varepsilon\cap\omega))^n)}\\
&\le\|\nabla u_\varepsilon-C^\varepsilon\nabla u\|_{\mathcal{C}^0([0,T];(L^1(\Omega_\varepsilon\cap\omega))^n)}+\|(C^\varepsilon-D^\varepsilon)\nabla u\|_{\mathcal{C}^0([0,T];(L^1(\Omega_\varepsilon\cap\omega))^n)}\\
&\le\|\nabla u_\varepsilon-C^\varepsilon\nabla u\|_{\mathcal{C}^0([0,T];(L^1(\Omega_\varepsilon\cap\omega))^n)}\\
&\qquad\qquad+\|(C^\varepsilon-D^\varepsilon)\|_{\left(L^{\frac{p}{p-1}}(\Omega_\varepsilon\cap\omega)\right)^{n^2}}\,\|\nabla u\|_{\mathcal{C}^0([0,T];(L^p(\Omega))^n)}\,.\end{aligned}$$

Applying Proposition 3.4 for $q=\frac{p}{p-1}\in[1,2[$ and Theorem 3.6, yields

$$\lim_{\varepsilon\to 0}\|\nabla u_\varepsilon-D^\varepsilon\nabla u\|_{\mathcal{C}^0([0,T];(L^1(\Omega_\varepsilon\cap\omega))^n)}=0,$$

which concludes the proof. □

8. **The general case.** In this section, as in the case of a fixed domain studied by S. Brahim-Otsmane, G. A. Francfort and F. Murat in [2] and following the ideas they developed, we show that the hypotheses of Theorem 4.3 are not sufficient to have the corrector result stated in Theorem 4.4.
Consider the problem

$$\begin{cases} \rho_\varepsilon v_\varepsilon'' - \operatorname{div}(A^\varepsilon \nabla v_\varepsilon) &= f_\varepsilon \quad \text{on } \Omega_\varepsilon \times (0,T), \\ v_\varepsilon &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ A^\varepsilon \nabla v_\varepsilon \,.\, \nu &= 0 \quad \text{on } \partial S_\varepsilon \times (0,T), \\ v_\varepsilon(x,0) &= a_\varepsilon \quad \text{on } \Omega_\varepsilon, \\ v_\varepsilon'(x,0) &= b_\varepsilon \quad \text{on } \Omega_\varepsilon, \end{cases} \tag{103}$$

under the hypotheses of Theorem 4.3, namely (4.2)-(4.4), (4.6)-(4.8) and (4.10). We also assume that

$$\chi_{\Omega_\varepsilon} \overset{*}{\rightharpoonup} \chi_0 \quad \text{weakly * in } L^\infty(\Omega). \tag{104}$$

Theorem 4.3 implies

$$\begin{cases} i) \quad P_\varepsilon v_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{weakly * in } L^\infty(0,T;H_0^1(\Omega)), \\ ii) \quad P_\varepsilon v_\varepsilon' \overset{*}{\rightharpoonup} u' \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \end{cases} \tag{105}$$

where u is the solution of the homogenized problem (4.14).

Let us show that in general we do not have the convergences

$$\begin{cases} i) \quad \lim_{\varepsilon\to 0} \|v_\varepsilon' - u'\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} = 0, \\ ii) \quad \lim_{\varepsilon\to 0} \|\nabla v_\varepsilon - C^\varepsilon \nabla u\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} = 0. \end{cases} \tag{106}$$

To do so, decompose v_ε in the form

$$v_\varepsilon = u_\varepsilon + z_\varepsilon,$$

where u_ε solves a problem whose data satisfy the assumptions of the corrector result and $P_\varepsilon u_\varepsilon$ has the same weak* limit u than $P_\varepsilon v_\varepsilon$.
In this decomposition, u_ε is the solution of the problem

$$\begin{cases} \rho_\varepsilon u_\varepsilon'' - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) &= \dfrac{f}{\chi_0} \quad \text{on } \Omega_\varepsilon \times (0,T), \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ A^\varepsilon \nabla u_\varepsilon \,.\, \nu &= 0 \quad \text{on } \partial S_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) &= d_\varepsilon \quad \text{on } \Omega_\varepsilon, \\ u_\varepsilon'(x,0) &= b_0 \quad \text{on } \Omega_\varepsilon, \end{cases} \tag{107}$$

where b_0 is defined in (4.14) and d_ε satisfies

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla d_\varepsilon) &= Q_\varepsilon^*(-\operatorname{div}(A^0 \nabla a_0)) \quad \text{on } \Omega_\varepsilon, \\ A^\varepsilon \nabla d_\varepsilon \,.\, \nu &= 0 \quad \text{on } \partial S_\varepsilon, \\ d_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{108}$$

On the other hand, z_ε is the solution of

$$\begin{cases} \rho_\varepsilon z_\varepsilon'' - \operatorname{div}(A^\varepsilon \nabla z_\varepsilon) &= f_\varepsilon - \dfrac{f}{\chi_0} \quad \text{on } \Omega_\varepsilon \times (0,T), \\ z_\varepsilon &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ A^\varepsilon \nabla z_\varepsilon \,.\, \nu &= 0 \quad \text{on } \partial S_\varepsilon \times (0,T), \\ z_\varepsilon(x,0) &= a_\varepsilon - d_\varepsilon \quad \text{on } \Omega_\varepsilon, \\ z_\varepsilon'(x,0) &= b_\varepsilon - b_0 \quad \text{on } \Omega_\varepsilon. \end{cases} \tag{109}$$

Notice that due to the admissibility of S_ε, the term $\frac{f}{\chi_0}$ is well defined.

We have the following result:

Theorem 8.1. *Under the assumptions of Theorem 4.3 and (8.2), we have*

$$\begin{cases} i) & P_\varepsilon u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{weakly * in } L^\infty(0,T;H^1_0(\Omega)), \\ ii) & P_\varepsilon u'_\varepsilon \overset{*}{\rightharpoonup} u' \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \end{cases} \tag{110}$$

$$\begin{cases} i) & \lim_{\varepsilon\to 0} \|u'_\varepsilon - u'\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} = 0, \\ ii) & \lim_{\varepsilon\to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} = 0, \end{cases} \tag{111}$$

$$\begin{cases} i) & P_\varepsilon z_\varepsilon \overset{*}{\rightharpoonup} 0 \quad \text{weakly * in } L^\infty(0,T;H^1_0(\Omega)), \\ ii) & P_\varepsilon z'_\varepsilon \overset{*}{\rightharpoonup} 0 \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \end{cases} \tag{112}$$

where u is the solution of the homogenized problem (4.14) associated with (8.1) and (8.5) and C^ε is the corrector matrix given by (3.19).

Proof. First, from definition (8.6) of d_ε and the H^0-convergence of A^ε to A^0, one has

$$Q_\varepsilon d_\varepsilon \rightharpoonup a_0 \quad \text{weakly in } H^1_0(\Omega). \tag{113}$$

On the other hand,

$$\chi_{\Omega_\varepsilon} \frac{f}{\chi_0} \rightharpoonup f \quad \text{weakly in } L^2(\Omega \times (0,T)).$$

This, together with (8.11), allows us to apply Theorem 4.3 to (8.5). Thus, $P_\varepsilon u_\varepsilon$ converges to the solution u^0 of the corresponding homogenized problem. Since the data are same as in (4.14), the uniqueness of the solution of problem (4.14) implies that $u^0 = u$, which gives (8.8).

Moreover, since the data of problem (8.5) satisfy also the hypotheses of Theorem 4.4, we have (8.9). Finally, convergence (8.10) follows from (8.3) and (8.8). □

Proposition 8.2. *Under the hypotheses of Theorem 8.1, the corrector result (8.4) holds if and only if*

$$\begin{cases} i) & \lim_{\varepsilon\to 0} \|z'_\varepsilon\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} = 0, \\ ii) & \lim_{\varepsilon\to 0} \|\nabla z_\varepsilon\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} = 0. \end{cases} \tag{114}$$

Proof. Suppose that convergences (8.12) are satisfied. Then

$$\begin{aligned} &\|v'_\varepsilon - u'\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} + \|\nabla v_\varepsilon - C^\varepsilon \nabla u\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} \\ &\le \|u'_\varepsilon - u'\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} + \|z'_\varepsilon\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} + \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} \\ &\qquad + \|\nabla z_\varepsilon\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} . \end{aligned}$$

In view of (8.9) and (8.12), we get (8.4).

Conversely, we have

$$\begin{aligned} &\|z'_\varepsilon\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} + \|\nabla z_\varepsilon\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} \\ &\le \|v'_\varepsilon - u'\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} + \|u'_\varepsilon - u'\|_{\mathcal{C}^0([0,T];L^2(\Omega_\varepsilon))} + \|\nabla v_\varepsilon - C^\varepsilon u\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} \\ &\qquad + \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{\mathcal{C}^0([0,T];(L^1_{loc}(\Omega_\varepsilon))^n)} , \end{aligned}$$

which tends to 0, by virtue of (8.4) and (8.9). □

Remark 8.1. We are now able to show that the assumptions of Theorem 8.1 do not necessarily provide convergences (8.4). Indeed, if (8.4) holds, from Proposition 8.2 one deduces that

$$\begin{cases} \lim_{\varepsilon\to 0} \|z'_\varepsilon(0)\|_{L^2(\Omega_\varepsilon)} = \lim_{\varepsilon\to 0} \|b_\varepsilon - b_0\|_{L^2(\Omega_\varepsilon)} = 0, \\ \lim_{\varepsilon\to 0} \|\nabla z_\varepsilon(0)\|_{L^1_{loc}(\Omega_\varepsilon)} = \lim_{\varepsilon\to 0} \|\nabla a_\varepsilon - \nabla d_\varepsilon\|_{L^1_{loc}(\Omega_\varepsilon)} = 0, \end{cases}$$

that is,

$$\begin{cases} i) \quad \lim_{\varepsilon\to 0} \|\rho\, b_\varepsilon - b\|_{L^2(\Omega_\varepsilon)} = 0, \\ ii) \quad \lim_{\varepsilon\to 0} \|\nabla(Q_\varepsilon a_\varepsilon) - \nabla(Q_\varepsilon d_\varepsilon)\|_{L^1_{loc}(\Omega)} = 0. \end{cases} \tag{115}$$

But, from (4.8)iii we only know that $\widetilde{\rho_\varepsilon b_\varepsilon} \rightharpoonup b$ weakly in $L^2(\Omega)$, which do not imply (8.13)i. Furthermore, in view of (8.11) and (4.8)ii, the function $Q_\varepsilon a_\varepsilon - Q_\varepsilon d_\varepsilon$ converges weakly to 0 in $H^1_0(\Omega)$ and in general, this convergence is not strong. Consequently, we cannot expect a corrector result for v_ε under the assumptions of Theorem 8.1.

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REFERENCES

[1] A. Bensoussan, J. L. Lions and G. Papanicolaou, "Asymptotic Analysis for Periodic Structures," Amsterdam, North Holland, 1978.

[2] S. Brahim-Otsmane, G. A. Francfort and F. Murat, *Correctors for the homogenization of the wave and heat equations*, J. Maths Pures et Appl., **71** (1992), 197–231.

[3] M. Briane, A. Damlamian and P. Donato, *H-convergence for perforated domains* in "Nonlinear Partial Differential Equations and Their Applications," Collège de France seminar vol. XIII, D. Cioranescu & J.-L. Lions eds., Pitman Research Notes in Mathematics Series, Longman, New York, **391** (1998), 62–100.

[4] G. Cardone, P. Donato and A. Gaudiello, *A compactness result for elliptic equations with subquadratic growth in perforated domains*, Nonlinear analysis, Theory, Methods & Applications, **32** (1998), 335–361.

[5] D. Cioranescu and P. Donato, *Exact internal controllability in perforated domains*, J. Math. Pures et Appl., **68** (1989), 185–213.

[6] D. Cioranescu and P. Donato, "An Introduction to Homogenization," Oxford Lecture Series in Mathematics and its Applications, vol.17, Oxford University Press, 1999.

[7] D. Cioranescu and J. Saint Jean Paulin, *Homogenization in open sets with holes*, Journal of Mathematical Analysis and Applications, **71** (1979), 590–607.

[8] A. Damlamian and P. Donato, *Which sequences of holes are admissible for periodic homogenization with Neumann boundary condition?* in "Control, Optimization and Calculus of Variations (8)," Special Volume: A Tribute to Jacques-Louis Lions, Tome 2, (2002), 555–585 (electronic).

[9] F. Murat, "H-Convergence," Séminaire d'analyse fonctionnelle et numérique, Université d'Alger, 1978.

[10] F. Murat and L. Tartar, *H-Convergence*, Topics in "Mathematical Modelling of Composite Materials," ed. A. Cherkaev and R. Kohn, Birkhäuser, Boston, (1997), 21–43.

[11] A. Nabil, *A corrector result for the wave equations in perforated domains*, Homogenization and applications to material sciences, GAKUTO Internat. Ser. Math. Sci. Appl., Gakkotosho, Tokyo, **9** (1995), 309–321.

[12] J. Simon, *Compact sets in* $L^p(0,T,B)$, Ann. Mat. Pura Appl., **146** (1987), 65–96.

[13] S. Spagnolo, *Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore*, Ann. Sc. Norm. Sup. Pisa, **21** (1967), 657–699.

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