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# ASYMPTOTICAL COMPLIANCE OPTIMIZATION FOR CONNECTED NETWORKS

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Abstract. We consider the problem of the optimal location of a Dirichlet region in a two-dimensional domain $\Omega$ subject to a force $f$ in order to minimize the compliance of the configuration. The class of admissible Dirichlet regions among which we look for the optimum consists of all one-dimensional connected sets (networks) of a given length $L$. Then we let $L$ tend to infinity and look for the $\Gamma$-limit of suitably rescaled functionals, in order to identify the asymptotical distribution of the optimal networks. The asymptotically optimal shapes are discussed as well and links with average distance problems are provided.

1. **Introduction.** We consider the problem of finding the best location of the Dirichlet region $\Sigma$ for a two-dimensional membrane $\Omega$ subjected to a given vertical force $f$. The vertical displacement of the membrane satisfies the elliptic equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{on } \Sigma \cup \partial\Omega, \end{cases}$$

and the rigidity of the membrane is measured through the compliance functional

$$\mathcal{C}(\Sigma) = \int_\Omega f u_\Sigma \, dx$$

where $u_\Sigma$ denotes the unique solution of the equation above. The maximal rigidity is obtained by minimizing the compliance functional $\mathcal{C}(\Sigma)$ in the class of admissible regions $\Sigma$.

The admissible class for the control variables $\Sigma$ we consider is the class of all closed connected subsets of $\Omega$ whose one-dimensional Hausdorff measure $\mathcal{H}^1(\Sigma)$ does not exceed a given number $L$. The existence of an optimal configuration $\Sigma_L$ for the optimization problem described above is well known; for instance it can be seen as a consequence of the Sverák compactness result (see [8]).

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We are interested in the asymptotic behaviour of $\Sigma_L$ as $L \to +\infty$; more precisely our goal is to obtain the limit distribution of $\Sigma_L$ as a limit probability measure that minimize the $\Gamma$-limit functional of the suitably rescaled compliances.

Similar results have been obtained for *location problems* (see [1]) and for *irrigation problems* (see [7]) where the functional which has to be minimized has a much simpler form and reduces to

$$\int_\Omega \operatorname{dist}(x,\Sigma) f(x)\,dx$$

(this kind of minimization problems is called *average distance problems*). The asymptotic analysis for the compliance optimization has been studied in the paper [3], where the Dirichlet region was searched among the arrays of a finite number of balls having prescribed total measure. Here we follow the same scheme. For capacitary reasons, in the case of a linear equation with the Laplacian, we are restricted to the case of dimension two. We present anyway the result in the more general case of the $p-$compliance, i.e. where the state equation is given by the $p$-Laplacian. This would a priori allow us to deal with the more general case of $\mathbb{R}^d$ with $p > d-1$, but some extra difficulties arise with $d > 2$ and we will consequently stick to the planar case.

2. **The compliance under length constraints.** In the following $p > 1$ is fixed and $q = p/(p-1)$ denotes the conjugate exponent of $p$. For any open set $\Omega \subset \mathbb{R}^2$ and $L \in ]0,\infty[$ we define

$$\mathcal{A}_L(\Omega) = \big\{\Sigma \subset \overline{\Omega}\ :\ \Sigma \text{ compact and connected, } 0 < \mathcal{H}^1(\Sigma) \le L\big\}.$$

Given $\Omega \subset \mathbb{R}^2$ and $f \in L^q(\Omega)$, for any compact set $\Sigma \subset \overline{\Omega}$ with positive $p$-capacity (any set in $\mathcal{A}_L$ satisfies this assumption) we define the function $u_{f,\Sigma,\Omega}$ (the dependence on $p$ will be neglected up to Section 5) as the solution of the problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega\setminus\Sigma \\ u = 0 & \text{on } \Sigma\cup\partial\Omega, \end{cases}$$

in the weak sense, which means $u \in W_0^{1,p}(\Omega\setminus\Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx = \int_\Omega f\phi\,dx \qquad \forall\phi\in W_0^{1,p}(\Omega\setminus\Sigma). \tag{1}$$

Notice that $f \ge 0$ implies $u_{f,\Sigma,\Omega} \ge 0$, by the maximum principle. For $f \ge 0$, we define the $p$-compliance functional over subsets of a given domain $\Omega$ as

$$\begin{aligned} F_p(\Sigma,f,\Omega) &= \left(1-\frac{1}{p}\right)\int_\Omega f u_{f,\Sigma,\Omega}\,dx = \left(1-\frac{1}{p}\right)\int_\Omega |\nabla u_{f,\Sigma,\Omega}|^p\,dx \\ &= \max\Big\{\int_\Omega f\phi\,dx - \frac{1}{p}\int_\Omega |\nabla\phi|^p\,dx\ :\ \phi\in W_0^{1,p}(\Omega\setminus\Sigma)\Big\}. \end{aligned} \tag{2}$$

The connectedness assumption on the admissible $\Sigma$ and the bound on their total length give the necessary compactness to obtain the following existence result (see [8] for the case $p = 2$ and [2] for the general case).

**Theorem 1.** *For any $L > 0$, if $\Omega$ is any bounded open subset of $\mathbb{R}^2$ and $f \ge 0$ belongs to $L^q(\Omega)$, the problem*

$$\min\big\{F_p(\Sigma,f,\Omega)\ :\ \Sigma\in\mathcal{A}_L(\Omega)\big\} \tag{3}$$

*admits a solution.*

The goal of this paper is is to study the asymptotic behavior of optimal networks $\Sigma_L$ of problem (3) as $L \to +\infty$. In particular, we are interested in the asymptotic distribution of $\Sigma_L$ in $\Omega$, i.e. in its density of lenght per unit area. To do this it is convenient to associate to every $\Sigma \in \mathcal{A}_L(\Omega)$ a probability measure on $\overline{\Omega}$, given by

$$\mu_\Sigma = \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)}$$

and to define a functional $F_L : \mathcal{P}(\overline{\Omega}) \to [0, +\infty]$ by

$$F_L(\mu) = \begin{cases} L^q F_p(\Sigma, f, \Omega) & \text{if } \mu = \mu_\Sigma,\ \Sigma \in \mathcal{A}_L(\Omega); \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

The scaling factor $L^q$ is needed in order to avoid the functionals to degenerate to the trivial limit functional which vanishes everywhere. Anyway, such a coefficient does not affect the choice of the minimizers, and comes out from the estimate

$$\min \left\{ F_p(\Sigma, f, \Omega) \ : \ \Sigma \in \mathcal{A}_L(\Omega) \right\} \approx L^q \tag{5}$$

as $L \to +\infty$. Here in this paper, we will start by simply guessing the estimate (5), and we will prove it later as a consequence of our $\Gamma$-convergence result (see below the consequences of Theorem 2).

We will give a $\Gamma$-convergence result for the sequence $(F_L)_L$, when endowing the space $\mathcal{P}(\overline{\Omega})$ with the weak* topology of probability measures. To introduce the limit functional $F$ we need to define the quantity:

$$\theta := \inf \left\{ \liminf_{L \to +\infty} L^q F_p(\Sigma_L, 1, Y) \ : \ \Sigma_L \in \mathcal{A}_L(Y) \right\}, \tag{6}$$

where $Y = (0,1)^2$ is the unit square in $\mathbb{R}^2$. By (5) $\theta$ is a positive and finite real number. When the dependence of $\theta$ on $p$ will be crucial, we will use the notation $\theta(p)$.

We may now define the candidate limit functional $F$ by setting, for $\mu \in \mathcal{P}(\overline{\Omega})$

$$F(\mu) = \theta \int_\Omega \frac{f^q}{\mu_a^q}\, dx, \tag{7}$$

where $\mu_a$ denotes the density of the absolutely continuous part of $\mu$ with respect to the Lebesgue measure. It is evident from (7) that the value of the constant $\theta$ does not affect the minimization problem for $F$. The result we will prove is the following.

**Theorem 2.** *Given any bounded open set $\Omega \subset \mathbb{R}^d$ and a non-negative function $f \in L^q(\Omega)$, the family of functionals $(F_L)_L$ in (4) $\Gamma$-converges to $F$ as $L \to +\infty$ with respect to the weak* topology on $\mathcal{P}(\overline{\Omega})$.*

The consequences of such a $\Gamma$-convergence result, by means of the general theory (see [4]), are the following:

- if $\Sigma_L$ is a solution of the minimization problem (3) it holds, up to subsequences, $\mu_{\Sigma_L} \rightharpoonup \mu$ as $L \to +\infty$, where $\mu$ is a minimizer of $F$;
- since $F$ has a unique minimizer in $\mathcal{P}(\overline{\Omega})$, we have actually full convergence of the whole family $\mu_{\Sigma_L}$ to the unique minimizer $\mu$, which is given by $\mu = cf^{q/(q+1)}\, dx$ (and $c$ is calculated so that $\mu$ is a probability measure, i.e. $c = 1/\int_\Omega f^{q/(q+1)}\, dx$);
- since the minimal value of $F$ may be calculated once we know the optimal $\mu$ and it equals $\theta c^{-(q+1)}$, the sequence of the values $\inf \big\{ F(\Sigma, f, \Omega) \ :$

$\Sigma \in \mathcal{A}_L(\Omega)\big\}$ is asymptotically equivalent to $L^{-q} \inf \big\{F(\mu) \;:\; \mu \in \mathcal{P}(\overline{\Omega})\big\} = L^{-q}\theta c^{-(q+1)}$.

3. **The Γ-convergence result.** We will prove Theorem 2 in several steps, the most important two corresponding to the Γ-lim inf and Γ-lim sup inequalities.

3.1. **The Γ-liminf inequality.** In the following proposition we prove that the Γ-liminf functional is minorized by the candidate limit $F$ introduced in (7).

**Proposition 1.** *Under the same hypotheses of Theorem 2, denoting by $F^-$ the functional $\Gamma\text{-}\liminf_L F_L$, it holds $F^-(\mu) \geq F(\mu)$ for any $\mu \in \mathcal{P}(\overline{\Omega})$. This means that, for any sequence $(\Sigma_L)_L$ such that $\mu_{\Sigma_L}$ weakly* converges to $\mu$ and $\Sigma_L \in \mathcal{A}_L(\Omega)$, we have*

$$\liminf_{L\to+\infty} L^q \int_\Omega f u_{f,\Sigma_L,\Omega}\, dx \geq F(\mu).$$

*Proof.* First of all, let us fix $\varepsilon > 0$ and, in analogy to what performed in [7], define the set $G_{\varepsilon,L}$ as follows: if $aY$ is a square large enough to contain $\Omega$, the set $G_{\varepsilon,L}$ is a regular grid composed by $n$ horizontal lines and $n$ vertical lines with $n = \lfloor \frac{\varepsilon L}{4a} \rfloor$, so that the total lenght of the grid in the square $aY$ is approximately $\varepsilon L$ and the step size of the grid is approximately proportional to $(\varepsilon L)^{-1}$; then we intersect the grid with $\Omega$.

Now we define $\Sigma'_L = \Sigma_L \cup G_{\varepsilon,L}$ and we set $u'_L = u_{f,\Sigma'_L,\Omega}$. Since $u_L \geq u'_L$, it is sufficient to estimate from below the integrals $L^q \int_\Omega f u'_L dx$. The utility of the new sequence $(u'_L)_L$ lies in the fact that $(L^q u'_L)_L$ is bounded in $L^p(\Omega)$. In fact we have $0 \leq u'_L \leq u_{f,G_{\varepsilon,L},\Omega}$ and, by Lemma 1, we have

$$\|u_{f,G_{\varepsilon,L},\Omega}\|_{L^p(\Omega)} \leq C(\varepsilon, f) L^{-q}.$$

This implies that $(L^q u'_L)_L$ is bounded in $L^p(\Omega)$ and so, up to a subsequence, we have $L^q u'_L \rightharpoonup w$. Hence

$$\lim_{L\to+\infty} L^q \int_\Omega f u'_L\, dx = \int_\Omega f w\, dx,$$

so that it is sufficient to estimate $w$ from below. We will prove that, for almost any $x_0$, we have

$$w(x_0) \geq \frac{f(x_0)^{1/(p-1)}}{(\mu_a(x_0) + \varepsilon)^q}. \tag{8}$$

To do this, we first estimate the average of $w$ on a cube $Q$ centered at point $x_0 \in \Omega$. We will assume that $x_0$ is a Lebesgue point for $f$ and that it satisfies the condition $|Q|^{-1}\mu(Q) \to \mu_a(x_0)$ as $|Q|$ shrinks around $x_0$. These assumptions are verified for almost any point $x_0 \in \Omega$. We also assume that $f(x_0) > 0$, since otherwise (8) would be trivial. We have

$$\int_Q w\, dx = \lim_{L\to+\infty} L^q \int_Q u'_L\, dx.$$

We use

$$u'_L \geq u_{f,\Sigma'_L,Q} \geq u_{f(x_0),\Sigma'_L,Q} - |u_{f,\Sigma'_L,Q} - u_{f(x_0),\Sigma'_L,Q}| \text{ in } Q,$$

where the first inequality comes from the fact that we add Dirichlet boundary conditions on $Q$. Lemmas 2 and 3 show that

$$\int_Q |u_{f,\Sigma'_L,Q} - u_{f(x_0),\Sigma'_L,Q}| dx \leq L^{-q}|Q| r(Q),$$

where $r(Q)$ tends to 0 when the cube $Q$ shrinks to $x_0$, whenever $x_0$ is a Lebesgue point for $f$.

Hence here we will only estimate the other term. We define the number $l(L,Q) = \mathcal{H}^1(\Sigma'_L \cap Q)$ and notice that

$$u_{f(x_0),\Sigma'_L,Q} = f(x_0)^{1/(p-1)} u_{1,\Sigma'_L,Q}.$$

Let us denote, for simplicity, the functions $u_{1,\Sigma'_L,Q}$ by $v_L$. By a change of variables, if $\lambda$ is the side of the square $Q$ and we define $v_{L,\lambda}(x) = \lambda^{-q} v_L(\lambda x)$ (thinking for a while that both cubes are centered at the origin), it holds $v_{L,\lambda} = u_{1,\lambda^{-1}\Sigma'_L,Y}$. We notice that

$$\lambda^{-1}\Sigma'_L \in \mathcal{A}_{l(L,Q)/\lambda}(Y);$$

moreover, we have $l(L,Q) \to +\infty$, since

$$l(L,Q) \geq \mathcal{H}^1(G_{\varepsilon,L} \cap Q) \approx \varepsilon L |Q|. \tag{9}$$

We may also estimate the ratio between $l(L,Q)$ and $L$, by using (9) and the fact that $L^{-1}\mathcal{H}^1(\Sigma_L \cap Q) = \mu_L(Q)$. From $\mu_L \rightharpoonup \mu$ we have $\limsup_L \mu_L(Q) \leq \mu(\overline{Q})$, so that

$$\limsup_L \frac{l(L,Q)}{L} \leq \mu(\overline{Q}) + \varepsilon|Q|. \tag{10}$$

Now, by using the definition of $\theta$ and the change of variables $y = \lambda x$, it is not difficult to check that

$$\liminf_{L\to+\infty} l(L,Q)^q \int_Q v_L\,dx = \liminf_{L\to+\infty} l(L,Q)^q \lambda^{2+q} \int_Y v_{L,\lambda}\,dx \geq \lambda^{2+2q}\theta.$$

Hence we get

$$\begin{aligned}
\liminf_{L\to+\infty} L^q \int_Q v_L\,dx &\geq \liminf_{L\to+\infty} \left(\frac{L}{l(L,Q)}\right)^q \liminf_{L\to+\infty} l(L,Q)^q \int_Q v_L\,dx \\
&\geq \lambda^{2+2q}\theta \left(\frac{1}{\mu(\overline{Q}) + \varepsilon|Q|}\right)^q \\
&= \theta|Q| \left(\frac{|Q|}{\mu(\overline{Q}) + \varepsilon|Q|}\right)^q,
\end{aligned}$$

where we have used that $|Q| = \lambda^2$. This implies

$$|Q|^{-1} \int_Q w\,dx \geq -r(Q) + f(x_0)^{1/(p-1)} \left(\frac{|Q|}{\mu(\overline{Q}) + \varepsilon|Q|}\right)^q.$$

Now we let $Q$ shrink towards $x_0$, thus obtaining, if $x_0$ satisfies our previous assumptions

$$w(x_0) \geq f(x_0)^{1/(p-1)}\theta \left(\frac{1}{\mu_a(x_0) + \varepsilon}\right)^q.$$

We get thus

$$\liminf_{L\to+\infty} L^q \int_\Omega f u_L\,dx \geq \int_\Omega f w\,dx \geq \int_\Omega \frac{f^{1+1/(p-1)}}{(\mu_a + \varepsilon)^q}\theta\,dx,$$

and our original aim is achieved when we let $\varepsilon \to 0$:

$$\liminf_{L\to+\infty} L^q \int_\Omega f u_L\,dx \geq \theta \int_\Omega \frac{f^q}{\mu_a^q}. \quad \square$$

**Lemma 1.** *The following facts hold.*

1. *There exists a constant $C$ such that, for all functions $v \in W_0^{1,p}(Y)$ we have $\int_Y |v|^p\,dx \le C \int_Y |\nabla v|^p\,dx$.*
2. *If we replace $Y$ by a square $Q$ whose side is $\lambda$ the same is true with the constant $\lambda^p C$ instead of $C$.*
3. *As a consequence, for any $\varepsilon > 0$, any $0 < L < \infty$, any domain $\Omega$ and any function $v \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,L}) \subset W_0^{1,p}(\Omega)$ (where $G_{\varepsilon,L}$ is the grid introduced in the proof of Proposition 1) we have $\|v\|_{L^p(\Omega)} \le C(\varepsilon)L^{-1}\|v\|_{W_0^{1,p}(\Omega)}$ for a suitable constant $C(\varepsilon)$.*
4. *As a further consequence, if we also take $f \in L^q(\Omega)$ with $f \ge 0$, the function $u_{f,G_{\varepsilon,L},\Omega}$ satisfies $\|u_{f,G_{\varepsilon,L},\Omega}\|_{L^p(\Omega)} \le L^{-q}C(\varepsilon)\|f\|_{L^q(\Omega)}^{1/(p-1)}$.*

*Proof.* The first assertion is the well-known Poincaré inequality. The second one is obtained by just a scaling of the first. To prove the third, let us extend the function $v$ to a large cube $aY \supset \Omega$ by setting the value 0 outside $\Omega$ (we recall that we have Dirichlet boundary conditions on $\Omega$ so that such an extension belongs to $W_0^{1,p}(aY)$). Then we consider the squares $Q_j$ which come from the subdivision of $aY$ into the cubes given by the grid $G_{\varepsilon,L}$. Their side is of the order of $L^{-1}$. Notice that the extended function vanishes on the boundary of each square $Q_j$. By applying the second statement of this lemma, we get

$$\int_{Q_j} |v|^p\,dx \le C(\varepsilon)L^{-p}\int_{Q_j} |\nabla v|^p\,dx,$$

and, by summing over $j$, we get

$$\int_{aY} |v|^p\,dx \le C(\varepsilon)L^{-p}\int_{aY} |\nabla v|^p\,dx.$$

Since $v$ vanishes outside $\Omega$ we may restrict the integrals to $\Omega$ and raise to the power $1/p$, thus getting the desired inequality. Here and in the sequel, the norm $\|v\|_{W_0^{1,p}(Q)}$ will simply denote the $L^p$ norm of the gradient $\|\nabla v\|_{L^p(Q)}$.

We have now to prove the fourth assertion. By using the weak version of the PDE which defines $u_{f,G_{\varepsilon,L},\Omega}$ we get

$$\int_\Omega |\nabla u_{f,G_{\varepsilon,L},\Omega}|^p\,dx = \int_\Omega u_{f,G_{\varepsilon,L},\Omega} f\,dx \le \|u_{f,G_{\varepsilon,L},\Omega}\|_{L^p(\Omega)}\|f\|_{L^q(\Omega)}.$$

we then get, by recalling $u_{f,G_{\varepsilon,L}} \in W_0^{1,p}(\Omega \setminus G_{\varepsilon,L})$,

$$\begin{aligned}\|u_{f,G_{\varepsilon,L},\Omega}\|^p_{W_0^{1,p}(\Omega)} &\le \|u_{f,G_{\varepsilon,L},\Omega}\|_{L^p(\Omega)}\|f\|_{L^q(\Omega)}\\ &\le C(\varepsilon)L^{-1}\|u_{f,G_{\varepsilon,L},\Omega}\|_{W_0^{1,p}(\Omega)}\|f\|_{L^q(\Omega)},\end{aligned}$$

which gives the thesis. □

**Lemma 2.** *Assume $p \ge 2$. If $f$, $g \in L^q(\Omega)$ and $u_f$ and $u_g$ denote the respective solutions of the $p$-Laplacian Equation with Dirichlet boundary conditions on $\Sigma'_L$, then*

$$L^q\|u_f - u_g\|_{L^1(\Omega)} \le C\|f-g\|_{L^q(\Omega)}^{1/(p-1)}|\Omega|^{1/q},$$

*where the constant $C$ only depends on $p$. In particular, if $\Omega = Q$ (a cube centered at $x_0$), $g = f(x_0)$ and $x_0$ is a Lebesgue point for $f$, we have*

$$L^q\|u_f - u_g\|_{L^1(Q)} \le C|Q|\left(\frac{\int_Q |f(x)-f(x_0)|^q\,dx}{|Q|}\right)^{1/p} = |Q|r(Q).$$

*Proof.* The starting point is the inequality

$$\|u_f - u_g\|^p_{W^{1,p}_0(\Omega)} \le C\|u_f - u_g\|_{L^p(\Omega)}\|f - g\|_{L^q(\Omega)},$$

that follows from the monotonicity inequality

$$|z - w|^p \le C(|z|^{p-2}z - |w|^{p-2}w) \cdot (z - w),$$

which is valid for $p \ge 2$ and for any pair of vectors $(z, w)$ (see for instance [5]).

Thanks to Lemma 1, we also know the inequality $\|v\|_{L^p(\Omega)} \le CL^{-1}\|v\|_{W^{1,p}(\Omega)}$, which is valid for any function $v$ vanishing on $\Sigma'_L$. Since the function $u_f - u_g$ vanishes on $\Sigma'_L$ we get

$$\|u_f - u_g\|^p_{W^{1,p}_0(\Omega)} \le CL^{-1}\|u_f - u_g\|_{W^{1,p}_0(\Omega)}\|f - g\|_{L^q(\Omega)},$$

which implies $\|u_f - u_g\|_{W^{1,p}_0(\Omega)} \le CL^{-1/(p-1)}\|f - g\|^{1/(p-1)}_{L^q(\Omega)}$, and then

$$\begin{aligned}\|u_f - u_g\|_{L^1(\Omega)} &\le |Q|^{1/q}\|u_f - u_g\|_{L^p(\Omega)}\\ &\le C|Q|^{1/q}L^{-1}\|u_f - u_g\|_{W^{1,p}_0(\Omega)}\\ &\le C|Q|^{1/q}L^{-q}\|f - g\|^{1/(p-1)}_{L^q(\Omega)}.\end{aligned}$$

This shows the first part of thesis; the second one is just a simple consequence. □

**Lemma 3.** *Assume $p \le 2$. If $f, g \in L^q(\Omega)$ and $u_f$ and $u_g$ denote the respective solutions of the $p$-Laplacian equation with Dirichlet boundary conditions on $\Sigma'_L$, then*

$$L^q\|u_f - u_g\|_{L^1(\Omega)} \le C\|f - g\|_{L^q(\Omega)}|\Omega|^{1/q}\left(\|f\|^q_{L^q(\Omega)} + \|g\|^q_{L^q(\Omega)}\right)^{(2-p)/p},$$

*where the constant $C$ only depends on $p$. In particular, if $\Omega = Q$ (a cube centered at $x_0$), $g = f(x_0)$ and $x_0$ is a Lebesgue point for $f$ with $f(x_0) \ne 0$, we have*

$$L^q\|u_f - u_g\|_{L^1(Q)} \le C|f(x_0)|^{(2-p)/(p-1)}|Q|\left(\frac{\int_Q |f(x) - f(x_0)|^q\,dx}{|Q|}\right)^{1/q} = |Q|r(Q).$$

*Proof.* Here the starting point is the inequality

$$\begin{aligned}\|u_f - u_g\|^{2p}_{W^{1,p}_0(\Omega)} \quad &\le \quad C\|u_f - u_g\|^p_{L^p(\Omega)}\|f - g\|^p_{L^q(\Omega)}\\ &\qquad \cdot\left(\|u_f\|^p_{W^{1,p}_0(\Omega)} + \|u_g\|^p_{W^{1,p}_0(\Omega)}\right)^{2-p}\end{aligned} \tag{11}$$

that follows from the monotonicity inequality

$$|z - w|^p(|z| + |w|)^{p-2} \le C(|z|^{p-2}z - |w|^{p-2}w) \cdot (z - w),$$

which is valid for $p \le 2$ and for any pair of vectors $(z, w)$ (see for instance [5]). This implies, by choosing $z = \nabla u_f$ and $w = \nabla u_g$, integrating, and using the weak formulation of the $p$-Laplacian equation:

$$\int_Q |\nabla u_f - \nabla u_g|^p(|\nabla u_f| + |\nabla u_g|)^{p-2}\,dx \le \int_Q (u_f - u_g)(f - g)\,dx.$$

The inequality (11) comes out as a consequence of a suitable Hölder inequality.

We start by estimating the term $\|u_f\|^p_{W^{1,p}_0(\Omega)}$. Since we have $\int_Q |\nabla u_f|^p\,dx = \int_Q f u_f\,dx$ we get

$$\|u_f\|^p_{W^{1,p}_0(\Omega)} \le \|u_f\|_{L^p(\Omega)}\|f\|_{L^q(\Omega)} \le CL^{-1}\|u_f\|_{W^{1,p}_0(\Omega)}\|f\|_{L^q(\Omega)}$$

(we use the fact that $u_f$ vanishes on $\Sigma'_L$) and we deduce

$$\|u_f\|_{W_0^{1,p}(\Omega)} \le CL^{-1/(p-1)}\|f\|_{L^q(\Omega)}^{1/(p-1)}.$$

A similar estimate holds for the term $\|u_g\|^p_{W_0^{1,p}(\Omega)}$. We come back to (11), insert the inequalities we have just proved and estimate $\|u_f - u_g\|_{L^p(\Omega)}$ by $\|u_f - u_g\|_{W_0^{1,p}(\Omega)}$, obtaining

$$\begin{aligned}\|u_f - u_g\|^p_{W_0^{1,p}(\Omega)} &\le CL^{-p/2}\|u_f-u_g\|^{p/2}_{W_0^{1,p}(\Omega)}\|f-g\|^{p/2}_{L^q(\Omega)}L^{-q(1-p/2)}\\ &\quad\cdot\left(\|f\|^q_{L^q(\Omega)}+\|g\|^q_{L^q(\Omega)}\right)^{1-p/2}.\end{aligned}$$

This implies, by simplifying and raising to the power $2/p$:

$$\|u_f - u_g\|_{W_0^{1,p}(\Omega)} \le CL^{-1}\|f-g\|_{L^q(\Omega)}L^{(p-2)/(p-1)}\left(\|f\|^q_{L^q(\Omega)}+\|g\|^q_{L^q(\Omega)}\right)^{(2-p)/p}.$$

The estimate on the $L^1(\Omega)$ norm is obtained as usual by passing first to the $L^p(\Omega)$ norm (up to a factor $|Q|^{1/q}$) and then to the $W_0^{1,p}(\Omega)$ norm (up to a factor $L^{-1}$):

$$\|u_f - u_g\|_{L^1(\Omega)} \le \frac{C|Q|^{1/q}}{L^2}\|f-g\|_{L^q(\Omega)}L^{(p-2)/(p-1)}\left(\|f\|^q_{L^q(\Omega)}+\|g\|^q_{L^q(\Omega)}\right)^{(2-p)/p},$$

and the first part of the thesis easily follows.

For the second one it is sufficient to notice that, if $x_0$ is a Lebesgue point for $f$ and $g = f(x_0)$, one gets (supposing $f(x_0) \ne 0$)

$$\frac{\|f\|_{L^q(Q)}}{\|g\|_{L^q(Q)}} = 1 + r(Q).$$

This allows to write $|Q|^{1/q}f(x_0)$ instead of $\|f\|_{L^q(Q)}$, making an error which is negligible (and of the form $|Q|r(Q)$). In this way we get the inequality in the second statement and the proof is concluded. □

3.2. **The Γ-limsup inequality.** To get also the Γ-lim sup inequality, we need this crucial lemma.

**Lemma 4.** *Given $\Sigma_0 \in \mathcal{A}_{L_0}(Y)$, a domain $\Omega \subset \mathbb{R}^2$ and $f \in L^q(\Omega)$, we consider the sequence of sets*

$$\Sigma^k = \bigcup_{y\in\ k^{-1}Z^2}\left(y + k^{-1}\Sigma_0\right)\cap\overline{\Omega}.$$

*We have $\Sigma^k \in \mathcal{A}_{l(k,\Omega)}(\Omega)$, where $l(k,\Omega) \approx |\Omega|kL_0$. Then we consider the sequence $(u_k)_k$, given by*

$$u_k = k^q u_{f,\Sigma^k,\Omega}.$$

*If we assume $\partial Y \subset \Sigma_0$, then we have $u_k \rightharpoonup c(\Sigma_0)f^{1/(p-1)}$ as $k \to \infty$, where the weak convergence is in the $L^p(\Omega)$ sense and $c(\Sigma_0)$ is a constant given by $\int_Y u_{1,\Sigma_0,Y}\,dx$.*

*Proof.* First, we notice that the sequence $(u_k)_k$ is bounded in $L^p(\Omega)$, thanks to Lemma 1. Let us now consider an arbitrary weakly convergent subsequence (not relabeled) and its limit $w_{f,\Sigma_0,\Omega}$. It is easy to see that the pointwise value of this limit function depends only on the local behaviour of $f$. In fact, the key assumption $\partial Y \subset \Sigma_0$ produces small cubes around each point $x \in \Omega$ which do not affect each other. So, if $f = \sum_i f_i 1_{A_i}$ is piecewise constant (the pieces $A_i$ being disjoint open sets, for instance), it happens that for large $k$ the value of $u_k$ at $x \in A_i$ depends only on $f_i$. Thanks to the rescaling properties of $\Delta_p$ it turns out that, for a piecewise constant function $f$, we have $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)}w_{1,\Sigma_0,\Omega}$. It is indeed clear that in the case $f = 1$, since we are simply homogenizing the function $u_{1,\Sigma_0,Y}$, the limit of the whole sequence $(u_k)_k$ exists, does not depend on the global geometry of $\Omega$, but it is a constant and it is the same constant as if there was $Y$ instead of $\Omega$. Then the constant is easy to be computed and coincides with the constant $c(\Sigma_0)$ appearing in the statement. It remains now just to show that the equality $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)}c(\Sigma_0)$ is true for any function $f \in L^q(\Omega)$. The convergence of the whole sequence will then follow easily by uniqueness of the limit of subsequences. To get the result for a generic $f$, take a sequence $(f_n)_n$ of piecewise constant functions approaching it in $L^q(\Omega)$. Up to subsequences, we have $k^q u_{f,\Sigma^k,\Omega} \rightharpoonup w_{f,\Sigma_0,\Omega}$ and $k^q u_{f_n,\Sigma^k,\Omega} \rightharpoonup f_n^{1/(p-1)}c(\Sigma_0)$ as $k \to \infty$. Moreover, thanks to Lemma 2 or Lemma 3 (depending on $p$), we also have

$$\|k^q u_{f,\Sigma^k,\Omega} - k^q u_{f_n,\Sigma^k,\Omega}\|_{L^1(\Omega)} \leq R(\|f - f_n\|_{L^q(\Omega)}),$$

where $\lim_{t\to 0} R(t) = 0$ (actually we have either $R(t) \approx t^{1/(p-1)}$ or $R(t) \approx t$, depending on $p$). If we take into account that the $L^1(\Omega)$-norm is l.s.c. with respect to the weak $L^p(\Omega)$ convergence (actually we may express it through $\|v\|_{L^1(\Omega)} = \sup\{\int v\phi\, dx : \|\phi\|_{L^\infty(\Omega)} \leq 1\}$), we get, passing to the limit as $k \to \infty$,

$$\|w_{f,\Sigma_0,\Omega} - f_n^{1/(p-1)}c(\Sigma_0)\|_{L^1(\Omega)} \leq R(\|f - f_n\|_{L^q(\Omega)}).$$

We pass now to the limit as $n \to \infty$, and we may assume that we have pointwise a.e. convergence of $f_n$ to $f$ (up to subsequences again), so that, again by semicontinuity (Fatou's Lemma), we get $w_{f,\Sigma_0,\Omega} = f^{1/(p-1)}c(\Sigma_0)$, which proves the claim. ◻

Now we want to build efficient sets $\Sigma_0$ satisfying the key assumption of our previous Lemma, that is $\partial Y \subset \Sigma_0$ (we call *boundary-covering sets* those sets for which such an inclusion holds).

**Remark 1.** This is the point where we strongly use the two-dimensional setting we have chosen. In higher dimension, it is not possible to cover all the boundary by means of a finite length. A possible strategy to overcome this difficulty could be "almost-covering" the boundary of $[0,1]^d$ by means of a grid of finite length and then estimating the difference between the solution with Dirichlet boundary conditions on this grid and on the faces of the cubes.

**Lemma 5.** *For any $\varepsilon > 0$ there exists $L_0 > 0$ such that for any $L > L_0$ we find a set $\Sigma \in \mathcal{A}_L(Y)$ which is boundary-covering, with*

$$L^q \int_Y u_{1,\Sigma,Y}\, dx < (1+\varepsilon)\theta.$$

*Proof.* By definition of $\theta$, we may find $\Sigma_1 \in \mathcal{A}_{L_1}(Y)$ such that

$$L_1^q \int_Y u_{1,\Sigma_1,Y}\, dx < (1+\delta)\theta$$

and, moreover, the number $L_1$ may be chosen as large as we want. Now we enlarge the set $\Sigma_1$ to get a new set $\Sigma_2$ which is boundary-covering: we add to $\Sigma_1$ the boundary of $Y$ and some segments to connect it to the original set. The new length $L_2 = \mathcal{H}^1(\Sigma_2)$ does not exceed $L_1 + 5$. It is possible to choose $L_1$ so that

$$\left(\frac{L_1 + 5}{L_1}\right)^q \leq 1 + \delta.$$

This implies

$$L_2^q \int_Y u_{1,\Sigma_2,Y}\,dx \le (1+\delta)^2\theta.$$

Now, if we are given a large length $L$, we just need to homogenize the set $\Sigma_2$. By homogenization of order $k$ of a set $S \subset Y$ into a domain $A$ we mean the set $A \cap k^{-1}(\mathbb{Z}^2 + S)$. Here we take the homogenization of order $k = \lfloor L/L_2 \rfloor$ of $\Sigma_2$ into $Y$, which is a set

$$\Sigma \in \mathcal{A}_{kL_2}(Y) \subset \mathcal{A}(n)(Y).$$

For this set $\Sigma$ it holds $(kL_2)^q \int_Y u_{1,\Sigma,Y}\,dx = L_2^q \int_Y u_{1,\Sigma_2,Y}\,dx$, thanks to the rescaling properties we have used so far. Hence

$$L^q \int_Y u_{1,\Sigma,Y}\,dx \le \left(\frac{k+1}{k}\right)^q (1+\delta)^2\theta.$$

If $L > L_2\delta^{-1}$, then $k > \delta^{-1}$ and $1 + 1/k < 1+\delta$, so that we get

$$L^q \int_Y u_{1,\Sigma,Y}\,dx \le (1+\delta)^{2+q}\theta.$$

It is now sufficient to choose $\delta$ sufficiently small so that $(1+\delta)^{2+q} < 1+\varepsilon$ and then set $L_0 = L_2\delta^{-1}$. □

We are now ready to prove the $\Gamma$-lim sup main part. We will start from a very particular class of measures. Let us call piecewise constant those probability measures $\mu \in \mathcal{P}(\overline{\Omega})$ which are of the form

$$\mu = \rho \cdot dx, \text{ with } \rho \in L^1(\Omega), \int_\Omega \rho\,dx = 1, \ \rho > 0,$$

for a piecewise constant function $\rho = \sum_{i=0}^m \rho_i I_{\Omega_i}$, the pieces $\Omega_i$ being disjoint Lipschitz open subsets with the possible exception of $\Omega_0 = \Omega \setminus \bigcup_{i=1}^m \Omega_i$.

**Proposition 2.** *Under the same hypotheses of Theorem 2, we have*

$$F^+(\mu) \le F(\mu), \text{ where } F^+ = \Gamma\text{-}\limsup_{L\to+\infty} F_L,$$

*for any a piecewise constant measure $\mu \in \mathcal{P}(\overline{\Omega})$. This means that, for any such a measure $\mu$ and any $\varepsilon > 0$, there exists a family of sets $(\Sigma_L)_L$ such that $\mu_{\Sigma_L}$ weakly-* converges to $\mu$, $\Sigma_L \in \mathcal{A}_L(\Omega)$ and moreover*

$$\limsup_{L\to+\infty} L^q \int_\Omega f u_{f,\Sigma_L,\Omega}\,dx \le (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^q}\,dx.$$

*Proof.* First of all, apply Lemma 5 and take a boundary-covering set $\Sigma_0 \in \mathcal{A}_{L_0}(Y)$ such that

$$L_0^q \int_Y u_{1,\Sigma_0,Y}\,dx < (1+\varepsilon)\theta.$$

Now define the sets $\Sigma_L^i$ by homogenizing into $\Omega_i$ the set $\Sigma_0$ of order $k(L,i)$ i.e.

$$\Sigma_L^i = \Omega_i \cap k(L,i)^{-1}(\mathbb{Z}^2 + \Sigma_0).$$

Then we choose $\Sigma_L = \bigcup_i \Sigma_L^i \cup \bigcup_i \partial\Omega_i$ and set $L_1 = \mathcal{H}^1\left(\bigcup_i \partial\Omega_i\right)$. The family $\Sigma_L$ is admissible (i.e. $\Sigma_L \in \mathcal{A}_L(\Omega)$ and $\mu_{\Sigma_L} \rightharpoonup \mu$) if we have, as $L \to +\infty$,

$$\sum_{i=0}^{m} |\Omega_i| k(L,i) L_0 + L_1 \le L \text{ and is asymptotic to } L;$$

$$\frac{k(L,i)L_0}{L} \to \rho_i \text{ for } i = 0, \dots, m.$$

These conditions are satisfied if we set

$$k(L,i) = \left\lfloor \frac{\rho_i(L - L_1)}{L_0} \right\rfloor.$$

We want now to estimate the values $F_L(\Sigma_L)$. We have covered the internal boundaries of the sets $\Omega_i$ in order to get a local behaviour in which different zones $\Omega_i$ are independent on each other. The quantity we want to estimate is

$$L^q \int_\Omega f u_{f,\Sigma_L,\Omega}\, dx = \sum_{i=0}^{m} \left(\frac{L}{k(L,i)}\right)^q \int_{\Omega_i} f k(L,i)^q u_{f,\Sigma_L^i,\Omega_i}\, dx.$$

The disintegration of the integral here performed gives the possibility to apply on each $\Omega_i$ Lemma 4, which provides the weak convergence in $L^p$

$$k(L,i)^q u_{f,\Sigma_L^i,\Omega_i} \rightharpoonup c(\Sigma_0) f^{1/(p-1)}.$$

The factors $(L/k(L,i))^q$ out of the integrals converge to $(L_0/\rho_i)^q$ as $L \to +\infty$. By our choice of $\Sigma_0$ we have $L_0^q c(\Sigma_0) < (1+\varepsilon)\theta$, so that we get

$$\limsup_{L\to+\infty} L^q \int_{\Omega_i} f u_{f,\Sigma_L^i,\Omega_i}\, dx \le (1+\varepsilon)\theta \rho_i^{-q} \int_{\Omega_i} f^q\, dx,$$

and, summing up,

$$\limsup_{L} L^q \int_\Omega f u_{f,\Sigma_L,\Omega}\, dx \le (1+\varepsilon)\theta \int_\Omega \frac{f^q}{\rho^q}\, dx. \quad \square$$

Extending our result to non piecewise constant measures is a simple consequence of a general result in Γ-convergence theory stating that it is sufficient to verify the lim sup inequality on a class which is dense in energy (see [4]). Hence, we only need to prove the following

**Proposition 3.** *For any $\mu \in \mathcal{P}(\overline{\Omega})$ there exists a sequence $(\mu_n)_n$ of piecewise constant measures such that $\mu_n \rightharpoonup \mu$ and $\lim_n F(\mu_n) = F(\mu)$.*

*Proof.* First we prove the thesis in the case $\mu = \rho dx$ with $\rho \ge c > 0$. Take a sequence $\rho_n$ of piecewise constant functions strongly converging in $L^1(\Omega)$ to $\rho$, satisfying $\rho_n \ge c$, and set $\mu_n = \rho_n\, dx$. Since $\rho_n^{-q}$ is bounded and we may also suppose convergence a.e. of $\rho_n$ to $\rho$, the fact that $F(\mu_n) \to F(\mu)$ follows immediately.

It is now sufficient to prove that any measure $\mu$ may be approximated weakly* by absolutely continuous measures $\mu_n$ with densities bounded from below and with $F(\mu_n) \to F(\mu)$. Notice that in general, due to the lower semicontinuity of the functional $F$, it is sufficient to check the reverse inequality

$$F(\mu) \ge \limsup_n F(\mu_n).$$

In particular, the inequality above is trivial whenever $F(\mu) = +\infty$. Assume now $F(\mu) < +\infty$ and take $\mu = \rho dx + \mu^s$, where $\mu^s$ is the singular part of $\mu$ w.r.t. the Lebesgue measure and $\rho$ the density of the absolutely continuous part. Take

$\mu_n = \big((1-1/n)\rho + a_n + \phi_n\big)\,dx$, where $a_n = n^{-1}\int_\Omega \rho\,dx$ and $\phi_n\,dx \rightharpoonup \mu^s$ with $\int_\Omega \phi_n\,dx = \int_{\overline{\Omega}} d\mu^s$. Notice that, since $F(\mu) < +\infty$, the density $\rho$ cannot vanish, so that $a_n > 0$ and $\rho_n = (1-1/n)\rho + a_n + \phi_n$ is bounded from below by the positive constant $a_n$. With this choice of $\mu_n$ we have $\mu_n \to \mu$ and

$$F(\mu_n) = \int_\Omega \frac{f^q}{\big((1-\frac{1}{n})\rho + a_n + \phi_n\big)^q}\,dx \le \int_\Omega \frac{f^q}{\big((1-\frac{1}{n})\rho\big)^q}\,dx = (1-\frac{1}{n})^{-q}F(\mu)$$

and the inequality on the lim sup is proved. □

4. **The value of $\theta$ and optimal sequences.** In this section we want to estimate the value of the constant $\theta$ and prove in particular that it is neither zero nor $+\infty$, so that the limit problem is not trivial. From now on, we will explicitly stress the dependence of $\theta$ on $p$ and we will write $\theta(p)$. Once again, every time we write $q$ we mean $q = q(p) = p/(p-1)$.

To prove that $\theta(p)$ is finite for any $p \in ]1,\infty[$ it is sufficient to take its definition and consider a particular sequence of sets $\Sigma$. Fix a set $\Sigma_L \in \mathcal{A}_L(Y)$ which is boundary-covering. Then, for each $n$ take its homogenization of order $k$ in the square $Y$. This gives a sequence $(\Sigma_n)_n$ with $\mathcal{H}^1(\Sigma_n) \le nL$. We use the homogeneity of $\Delta_p$ (which is what we have done so far) to get from (6)

$$\theta(p) \le \liminf_n (nL)^q F_p(\Sigma_n, 1, Y) = L^q F_p(\Sigma, 1, Y). \tag{12}$$

This is sufficient to get $\theta(p) < +\infty$.

To prove that $\theta(p) > 0$ we make a comparison to a similar problem treated in [7], where the functional considered is (in the simplest case, of a two-dimensional square with $f = 1$):

$$\Sigma \mapsto D_r(\Sigma) = \int_Y d(x,\Sigma)^r\,dx.$$

It is not difficult to prove an estimate concerning our compliance functional $F_p$ and this average distance functional $D_r$, for $r = q$.

**Lemma 6.** *For any set $\Sigma \subset \Omega$ the inequality*

$$F_p(\Sigma, 1, Y) \ge q^{-(q+1)} D_q(\Sigma \cup \partial Y)$$

*holds true. In particular, one has $\theta(p) \ge q^{-(q+1)}\theta_q > 0$, where for each $r$ the number $\theta_r$ is the constant defined in [7] and which is proved to be equal to $1/(2^r(r+1))$. Therefore*

$$\theta(p) \ge \frac{(2q)^{-q}}{q(1+q)}\,. \tag{13}$$

*Proof.* We have, for every real number $A$ and for every $r > 1$

$$\begin{aligned} F_p(\Sigma, 1, Y) &= \max\left\{\int_Y (v - \frac{1}{p}|\nabla v|^p)\,dx \;:\; v \in W_0^{1,p}(Y \setminus \Sigma)\right\} \\ &\ge \int_Y (Ad^r - \frac{1}{p}|\nabla (Ad^r)|^p)\,dx, \end{aligned}$$

where the function $d$ is given by $d(x) = d(x, \Sigma \cup \partial Y)$ and enjoys the property $|\nabla d| = 1$ (and consequently $|\nabla d^r| = rd^{r-1}$). Take $r = (r-1)p$, i.e. $r = q$; hence we get

$$F_p(\Sigma, 1, Y) \ge (A - A^p\left(\frac{q^p}{p}\right)^p)\int_Y d^q\,dx.$$

Optimizing on $A$ (the optimal choice is $A = q^{-q}$, as we can find by derivation) gives the thesis.

To get the second part of the statement, it is sufficient to notice that adding $\partial Y$ is asymptotically irrelevant and use the same asymptotic results of [7]. $\square$

To get the exact value of $\theta(p)$ one should find explicitly asymptotically optimal sequences. This is a very typical question in this kind of problems, and it has been approached, for instance, in the case of average distance functionals, in [6] when placing points (location problems) and in [7] when placing connected one-dimensional sets (irrigation problems). However all the results are very limited and typically two-dimensional.

In the compliance case for $p = 2$ we are unable to give a precise answer. Two main candidates seem interesting: the case of uniform grids and the case of parallel lines (actually, parallel lines are not connected, but it is sufficient to add a segment on one side of the square and they become a connected comb). This second case is proved to be optimal in the Mosconi-Tilli case. By comparing these two, we find that in our case as well a comb configuration performs better than a square-grid one.

It is in fact sufficient to evaluate the compliance in the following two configuration: a square with Dirichlet boundary conditions, and a square with Dirichlet conditions on two opposite sides and Neumann conditions on the remaining ones. Obviously, the first one gives a better result, but uses twice the length. It is consequently necessary to check whether its perfomance is better than the other up to a factor 4 (i.e. the length ratio to the power of $q$). Notice that, when homogenizing, there is a superposition of the lenghts which are contained in the boundary of the square.

Since we want to prove that the square with Dirichlet boundary conditions on the full boundary loses against the case of mixed conditions, we only give an estimation of the compliance in this case. Let us call $u$ the solution of $-\Delta u = 1$ on $Y$ with zero boundary conditions on $\partial Y$ and let us consider the function $\bar{u}(x, y) = xy(1-x)(1-y)$ as well. We have $u \geq \bar{u}$, because the two functions share the same values on $\partial Y$, while $-\Delta \bar{u}(x, y) = 2x(1 - x) + 2y(1 - y) \leq 1$. Hence, if $\Sigma = \partial Y$ we have

$$\begin{aligned} L^2 F_2(\Sigma, 1, Y) &= \frac{16}{2} \int_Y u \, dx \, dy \geq 8 \int_Y \bar{u} \, dx \, dy \\ &= 8 \Big( \int_0^1 x(1 - x) \, dx \Big)^2 = \frac{2}{9} \, . \end{aligned}$$

On the other hand, if $\Sigma$ is made of only two opposite sides, we may compute explicitly the solution of $-\Delta u = 1$, obtaining

$$u(x, y) = \frac{x(1 - x)}{2} \, ,$$

so that

$$L^2 F_2(\Sigma, 1, Y) = \frac{4}{2} \int_Y u \, dx \, dy = \int_0^1 x(1 - x) \, dx = \frac{1}{6} \, .$$

Therefore, we conclude that for $p = 2$ we may estimate $\theta(2)$ by using this parallel lines configuration. When we homogenize it, we have to add one side of the square to get it connected, thus obtaining a comb structure. Anyway the solution with Dirichlet boundary conditions on $\Sigma \cup \partial Y$ is less or equal than the solution with Dirichlet on the parallel lines and Neumann on the two remaining sides of the square, which is the homogenization of the one we computed. Taking into account that, in computing the total lenght of the comb configuration we have an asymptotically irrelevant lenght one on the side we add for granting connectedness, and a complete superposition of the boundary lenghts (so that the total lenght is asymptotically half of the lenght $nL$ which we usaullay have when homogenizing), we get a factor $1/4 = (1/2)^2$ that gives $\theta(2) \leq 1/24$. Notice that this factor 2 in the lenght is common to the two configurations we examined and can be easily seen if one replaces the fours-sides configuration by the two medians of the square and the two-sides by one median only (with Neumann boundary conditions on the sides of the square). These two new configurations homogenize exactly as the previous ones, but have half the length.

**Conjecture.** In analogy to the two-dimensional result by Mosconi and Tilli, we conjecture that the comb configuration is asymptotically optimal and, consequently, that $\theta = 1/24$.

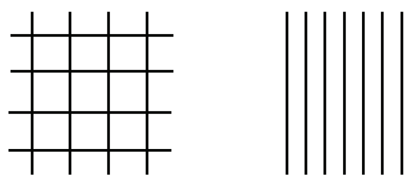

FIGURE 1. A grid is less performant than a comb structure, that we conjecture to be the optimal one.

5. **Average distance as a limit as $p \to \infty$.** In this section our general presentation of the problem for any $p > 1$ is exploited to let $p \to \infty$: this allows us to compare it to some average distance problem. In some sense, the limit of these problems as $p \to \infty$ corresponds to the minimization of the functional $D_1$ introduced in the previous section.

The aim of this section is to complete the previous results to show a commutative $\Gamma$-convergence diagram: if we fix $p$ and let the length constraint $L$ tend to $+\infty$ we get a limit functional depending on $p$, given by (7). We want to show that, both at the finite level of fixed $L$ and at the asymptotic level of the limit functional, we have $\Gamma$-convergence as $p \to \infty$ to the corresponding functional arising in the average distance theory.

The following Lemma is well-known.

**Lemma 7.** *Let $\Omega$ be a fixed domain, $p_0 < \infty$ a fixed exponent with conjugate $q_0 = p_0/(p_0 - 1)$ and $f \in L^{q_0}(\Omega)$ a nonnegative function. Then the sequence of functionals $K_p : W_0^{1,p_0}(\Omega) \to [0, +\infty]$ given by*

$$K_p(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, dx - \int_\Omega fv \, dx$$

*$\Gamma$-converges as $p \to \infty$, with respect to the weak convergence in $W_0^{1,p_0}(\Omega)$, to the functional $K_\infty$ given by*

$$K_\infty(v) = \begin{cases} -\displaystyle\int_\Omega fv \, dx & \text{if } \|\nabla v\|_{W^{1,\infty}} \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

*In particular we have*

$$\lim_{p\to\infty} \min\{K_p(v) \ : \ v \in W_0^{1,p_0}(\Omega)\} = -\int_\Omega f(x)d(x,\partial\Omega)dx.$$

**Remark 2.** All the results of this section are provided under the assumption $f \in L^{q_0}(\Omega)$, even if they could make sense for $f \in L^1(\Omega)$ (since for $p > 2$ all functions in $W^{1,p}(\Omega)$ actually belong to $L^\infty(\Omega)$ because they are Hölder continuous). The reason lies in the fact that in some estimates we used to prove our results in previous sections we explicitly exploited the duality between $W^{1,p}$ and $L^q$ functions.

**Theorem 3.** *Fix $L > 0$, an exponent $p_0 > 2$ with conjugate $q_0$, and a nonnegative function $f \in L^{q_0}(\Omega)$. Consider the functionals*

$$C_p(\Sigma) := F_p(\Sigma, f, \Omega) \qquad \text{for all } \Sigma \in \mathcal{A}_L(\Omega),$$

*where $\mathcal{A}_L(\Omega)$ is endowed with the Hausdorff convergence. As $p \to \infty$ we have $\Gamma$-convergence of $(C_p)_p$ to the average distance functional $D$ given by*

$$D(\Sigma) = \int_\Omega d(x, \Sigma \cup \partial\Omega) f(x)\, dx.$$

*Proof.* To prove the $\Gamma$-limsup inequality we will prove pointwise convergence. This is to be done by fixing $\Sigma$, regarding the compliance as a maximum, and considering $\Gamma$-convergence on these problems, which would give as a byproduct the convergence of the optimal values. This $\Gamma$-convergence follows from Lemma 7, changing the signs in the functionals and applying it to the domain $\Omega \setminus \Sigma$.

For the $\Gamma$-liminf inequality take $\Sigma_p \to \Sigma$ and the corresponding potentials $u_p$. It is easy to see that this sequence is bounded in $W^{1,p_0}(\Omega)$ and, since $p_0 > 2$, thanks to the compact embedding in $C^0$ of the Sobolev space $W^{1,p_0}(\Omega)$, we can also suppose $u_p \to u$ uniformly. If we prove, for almost any $x_0 \in \Omega$ such that $f(x_0) > 0$, the inequality $u(x_0) \geq d(x_0, \Sigma \cup \partial\Omega)$, the goal is achieved. To prove this inequality take $x_0 \notin \Sigma \cup \partial\Omega$ and a radius $r < d(x_0, \Sigma \cup \partial\Omega)$. Since $\Sigma_p$ converges in the Hausdorff topology to $\Sigma$ it will eventually hold $r < d(x_0, \Sigma_p \cup \partial\Omega)$ as well. Hence, if we take the solutions $v_p$ of the $p$-Laplacian equation

$$\begin{cases} -\Delta_p v_p = f & \text{in } B(x_0, r), \\ v_p = 0 & \text{on } \partial B(x_0, r), \end{cases}$$

we have the inequality $v_p \leq u_p$. Hence it is sufficient to estimate the uniform limits of $v_p$. Since $v_p$ is bounded in $W^{1,p_0}(B(x_0, r))$ we may suppose weak (and hence uniform) convergence to a function. By the $\Gamma$-convergence result of Lemma 7, we know that such a limit must optimize the limit problem, i.e. it must realize the maximum of $\int_\Omega vf\, dx$ among all the 1-Lipschitz function $v$ vanishing on $\partial B(x_0, r)$. This maximum is realized by the function $x \mapsto d(x, \partial B(x_0, r))$, which is the highest among these functions, but it could be realized by other functions as well. Those maximizing functions $v$ should satisfy $v(x) = d(x, \partial B(x_0, r))$ a.e. on $\{f > 0\}$. Yet, if $f(x_0) > 0$ and $x_0$ is a Lebesgue point for $f$, using the continuity of $v$ and of the distance function (which are both Lipschitz continuous) we obtain $v(x_0) = d(x_0, \partial B(x_0, r)) = r$. Actually, by using again the $1-$Lipschitz behaviour of $v$, this proves the equality $v(x) = d(x, \partial B(x_0, r))$ for any $x \in B(x_0, r)$. This easily proves that the uniform limit $u$ of the functions $u_p$ must satisfy $u(x_0) \geq r$ and, letting $r$ tend to $d(x_0, \Sigma \cup \partial\Omega)$, we get the desired inequality and the $\Gamma$-liminf inequality we were looking for. ∎

**Remark 3.** Theorem 3 could have been stated replacing the Hausdorff convergence by the weak convergence of the measures $\mu_{\Sigma_p}$. In fact it is easy to prove that $\mu_{\Sigma_p} \rightharpoonup \mu_\Sigma$ implies $\Sigma_p \to \Sigma$ in the Hausdorff topology. This means that the Γ-liminf inequality we proved remains true, and the Γ-limsup as well stays valid because we only used pointwise convergence, i.e. we were not concerned with any topology on the sets $\Sigma$.

**Theorem 4.** *Fix a nonnegative function $f \in L^{q_0}(\Omega)$ and consider the sequence of functionals $C_{p,\infty}$ on $\mathcal{P}(\Omega)$ (endowed with the weak topology) given, for $p > 1$, by*

$$C_{p,\infty}(\mu) := \int_\Omega \left(\frac{f}{\mu_a}\right)^q dx, \qquad \text{where } q = \frac{p}{p-1}.$$

*Consider as well the functional $C_{\infty,\infty}$ given by*

$$C_{\infty,\infty}(\mu) := \int_\Omega \frac{f}{\mu_a}\, dx.$$

*As $p \to \infty$ we have Γ-convergence of $(C_{p,\infty})_p$ to $C_{\infty,\infty}$.*

*Proof.* This result follows straightforward, because we are considering the $L^q$ norms of the same functions $f/\mu_a$. This means that the inequality

$$\|v\|^q_{L^q} \ge \|v\|_{L^1} |\Omega|^{-1/(p-1)}$$

is sufficient to deal with the Γ-lim inf inequality: if we have $\mu_p \rightharpoonup \mu$ we have

$$\liminf_p \|f/(\mu_p)_a\|^q_{L^q} \ge \liminf_p \|f/(\mu_p)_a\|_{L^1} |\Omega|^{-1/(p-1)} \ge \|f/\mu_a\|_{L^1},$$

where the last inequality comes from the semicontinuity of the limit functional and from the fact that $|\Omega|^{-1/(p-1)} \to |\Omega|^0 = 1$.

On the other hand, the Γ-lim sup inequality will follow in this case too from pointwise convergence, which in turn follows from the convergence of the $L^q$ norm to the $L^1$ norm. □

To complete the framework of the convergence as $p \to \infty$, we just need to control the constants $\theta(p)$.

**Remark 4.** As a consequence of what we have proven, it holds $\theta(p) \to 1/4$ as $p \to \infty$. This can be seen from the estimates (12) and (13). For an upper bound, one has to use Lemma 7 as well: in fact we have

$$\limsup_{p\to\infty} \theta(p) \le \limsup_{p\to\infty} L^q F_p(\Sigma, 1, Y) = L \int_Y d(x, \Sigma \cup \partial Y) dy,$$

and, by choosing a set $\Sigma$ composed by $m-1$ equally spaced vertical bars and the whole perimeter of $Y$, we get in the right hand side a value less than or equal to $\frac{m+3}{4m}$. By letting $m \to \infty$ we get $\limsup_{p\to\infty} \theta(p) \le 1/4$. For the lower bound, just use (13) to obtain

$$\liminf_{p\to\infty} \theta(p) \ge \liminf_{p\to\infty} \frac{(2q)^{-q}}{1+q} = \frac{1}{4}.$$

The commutative Γ-convergence result we highlighted in this section is resumed in the following diagram.

$$\begin{array}{ccc} \mathcal{A}_L(\Omega) \ni \Sigma \mapsto L^q \left(1 - \frac{1}{p}\right) \int_\Omega f u^{(p)}_{f,\Sigma,\Omega} dx & \xrightarrow[L\to\infty]{} & \mathcal{P}(\Omega) \ni \mu \mapsto \theta(p) \int_\Omega \frac{f^q}{\mu_a^q} dx \\ {\scriptstyle p\to\infty} \downarrow & & \downarrow {\scriptstyle p\to\infty} \\ \mathcal{A}_L(\Omega) \ni \Sigma \mapsto L \int_\Omega f d(x, \Sigma \cup \partial\Omega) dx & \xrightarrow[L\to\infty]{} & \mathcal{P}(\Omega) \ni \mu \mapsto \frac{1}{4} \int_\Omega \frac{f}{\mu_a} dx \end{array}$$

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