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NULL CONTROLLABILITY OF DEGENERATE PARABOLIC OPERATORS WITH DRIFT

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Abstract. We give null controllability results for some degenerate parabolic equations in non divergence form with a drift term in one space dimension. In particular, the coefficient of the second order term may degenerate at the extreme points of the space domain. For this purpose, we obtain an observability inequality for the adjoint problem using suitable Carleman estimates.

1. **Introduction.** Recently, interest in the null controllability of degenerate parabolic equations has increased. Indeed, as pointed out by several authors, many problems that are relevant for applications are described by *degenerate* parabolic equations, with degeneracy occurring at the boundary of the space domain. For instance, degenerate equations can be obtained as suitable linearizations of the Prandtl equations, see [25]. In a different context, degenerate operators have been extensively studied since Feller's investigations in [17], [18], where the main motivation was the relevance of the previous equations in transition probabilities.

The case of parabolic equations in *divergence form* is well-understood (see, e.g., [1], [4], [6] - [10], [24], [25]): for all $T > 0$ and $u_0 \in L^2(0,1)$ there is a control $f \in L^2((0,T)\times(0,1))$ such that the solution of

$$\begin{cases} u_t - (a(x)u_x)_x + c(t,x)u = f(t,x)\chi_\omega(x), & (t,x) \in (0,T)\times(0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{1}$$

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satisfies $u(T,x) = 0$ for all $x \in [0,1]$. Here, $a \in C^0[0,1]$ satisfies $a(0) = a(1) = 0$, $a > 0$ in $(0,1)$, $c \in L^\infty((0,T) \times (0,1))$ and χ_ω is the characteristic function of a non-empty interval $\omega = (\alpha, \beta) \subset\subset [0,1]$. For the uniformly parabolic case we refer, e. g., to [2], [12], [13], [19], [20], [22], [29] and [31]. Several results have also been obtained for semilinear versions of (1), see, for example, [1], [4], [6], [27], [28].

However, many problems arising in applications (see, e.g., [21], [23] and [30]) are described by degenerate parabolic equations that are *not in divergence form*. In such a context, a null controllability result was obtained in [5] for the following problem:

$$\begin{cases} u_t - a(x)u_{xx} + c(t,x)u = f(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{2}$$

The main goal of this paper is to provide a full analysis of the null controllability problem for

$$\begin{cases} u_t - a(x)u_{xx} - b(x)u_x + c(t,x)u = f(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{3}$$

where $c \in L^\infty((0,T)\times(0,1))$, a and $b \in C^0[0,1]$. Here, a is allowed to degenerate at $x = 0$ and $x = 1$, as long as suitable assumptions are satisfied near these points (see Hypothesis 3.1). A model example of such a degenerate coefficient a is the function

$$a(x) = x^{K_1}(1-x)^{K_2}\ , \quad K_1, K_2 \in (0,2), \tag{4}$$

whereas, for b, we can take

$$b(x) = x^{k_1}(1-x)^{k_2}, \tag{5}$$

where k_i are such that $k_i \geq 0$ and $k_i > (K_i - 1)$ for $i = 1,2$. We observe that the restriction $K_1, K_2 \in (0,2)$ is natural if we want to obtain global null controllability: if K_1 or $K_2 \geq 2$, then the model fails to be null controllable (see [5] for details).

We underline the fact that we cannot consider bu_x as a small perturbation of au_{xx} (see [16]). Therefore, the problem cannot be solved by a straightforward adaptation of the recalled results of [5]. In order to deal with the well-posedness of (3) we refer to [14], [15], [16], [26] and Section 2 of this paper.

The paper is organized as follows:
- in Section 2, we prove the well-posedness of the linear problem (3) when $c \equiv 0$;
- in Section 3, we state Carleman estimates for the adjoint problem of (3) when $c \equiv 0$;
- in Section 4, we prove the observability inequality for the adjoint problem (3) and, as a consequence, we give a null controllability result for (3) when $c \equiv 0$;
- in Section 5, we extend the previous results to (3) when $c \neq 0$.

2. **Well-posedness.** Let $T > 0$, $Q := (0,T) \times (0,1)$, $\omega := (\alpha, \beta) \subset\subset (0,1)$ be a non-empty given interval, we consider the degenerate parabolic problem

$$\begin{cases} u_t - a(x)u_{xx} - b(x)u_x = f(t,x)\chi_\omega(x), & (t,x) \in Q, \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{6}$$

Here $a, b \in C^0[0,1]$ are such that $a(0) = a(1) = 0$, $a > 0$ on $(0,1)$ and $b/a \in L^1(0,1)$.

In order to study the well-posedness of (6), let us recall the well-known weight function

$$\eta(x) := exp\left\{\int_{\frac{1}{2}}^{x} \frac{b(y)}{a(y)}dy\right\}, \quad x \in [0,1],$$

introduced by Feller in a related contex [17] and used by several authors, see, e.g. [16], [26]. Define

$$\sigma(x) := a(x)\eta^{-1}(x),$$

and observe that if u is sufficiently smooth, e.g. $u \in W^{2,1}_{\rm loc}(0,1)$, then

$$Au := au_{xx} + bu_x = \sigma(\eta u_x)_x,$$

for almost every $x \in (0,1)$. For this purpose, let us consider the following Hilbert spaces

$$L^2_{\frac{1}{\sigma}}(0,1) := \left\{u \in L^2(0,1) \,\big|\, \|u\|_{\frac{1}{\sigma}} < \infty\right\}, \qquad \|u\|^2_{\frac{1}{\sigma}} := \textstyle\int_0^1 u^2 \frac{1}{\sigma} dx,$$

$$H^1_{\frac{1}{\sigma}}(0,1) := L^2_{\frac{1}{\sigma}}(0,1) \cap H^1_0(0,1), \qquad \|u\|^2_{1,\frac{1}{\sigma}} := \|u\|^2_{\frac{1}{\sigma}} + \textstyle\int_0^1 u_x^2 dx,$$

$$H^2_{\frac{1}{\sigma}}(0,1) := \left\{u \in H^1_{\frac{1}{\sigma}}(0,1) \,\big|\, Au \in L^2_{\frac{1}{\sigma}}(0,1)\right\}, \quad \|u\|^2_{2,\frac{1}{\sigma}} := \|u\|^2_{1,\frac{1}{\sigma}} + \|Au\|^2_{\frac{1}{\sigma}}.$$

Observe that since $b/a \in L^1(0,1)$, $\eta \in C^0[0,1] \bigcap C^1(0,1)$ is a strictly positive function. Thus, in the sense of Banach spaces, one has that

$$\left\{L^2_{\frac{1}{\sigma}}(0,1),\ H^1_{\frac{1}{\sigma}}(0,1),\ H^2_{\frac{1}{\sigma}}(0,1)\right\} \simeq \left\{L^2_{\frac{1}{a}}(0,1),\ H^1_{\frac{1}{a}}(0,1),\ H^2_{\frac{1}{a}}(0,1)\right\},$$

where the last triplet is the triplet related to well-posedness as in [5].

Lemma 1. *For all* $(u,v) \in H^2_{\frac{1}{\sigma}}(0,1) \times H^1_{\frac{1}{\sigma}}(0,1)$ *one has*

$$< Au, v >_{\frac{1}{\sigma}} = -\int_0^1 \eta u_x v_x dx. \tag{7}$$

Proof. First, we claim that the space $H^1_c(0,1) := \{v \in H^1(0,1) \mid \mathrm{supp}\{v\} \subset (0,1)\}$ is dense in $H^1_{\frac{1}{\sigma}}(0,1)$. Indeed, if we consider the sequence $(v_n)_{n\geq 4}$, where $v_n := \xi_n v$ for a fixed function $v \in H^1_{\frac{1}{\sigma}}(0,1)$ and

$$\xi_n(x) := \begin{cases} 0, & x \in\ [0,1/n] \bigcup [1-1/n,1], \\ 1, & x \in\ [2/n, 1-2/n], \\ nx-1, & x \in\ (1/n, 2/n), \\ n(1-x)-1, & x \in\ (1-2/n, 1-1/n), \end{cases}$$

then one has that $v_n \to v$ in $H^1_{\frac{1}{\sigma}}(0,1)$ (see [5]). Now, set

$$\Phi(v) := \int_0^1 \left((au_{xx} + bu_x)v\frac{1}{\sigma} + \eta u_x v_x\right)dx,$$

with $u \in H^2_{\frac{1}{\sigma}}(0,1)$. Then, Φ is a bounded linear functional on $H^1_{\frac{1}{\sigma}}(0,1)$. Moreover, $\Phi = 0$ on $H^1_c(0,1)$. Indeed, let $v \in H^1_c(0,1)$, one has that

$$\int_0^1 (au_{xx} + bu_x)v\frac{1}{\sigma}dx = \int_0^1 \sigma(\eta u_x)_x v\frac{1}{\sigma}dx = -\int_0^1 \eta u_x v_x dx.$$

Thus, $\Phi = 0$ on $H^1_{\frac{1}{\sigma}}(0,1)$, that is, (7) holds. ∎

The following theorems refine Theorems 1.1 and 1.2 by Barbu-Favini-Romanelli [3] for the case of $n = 1$.

Theorem 1. *The operator* $(A, D(A))$ *given by*

$$Au = au_{xx} + bu_x, \qquad D(A) = H^2_{\frac{1}{\sigma}}(0,1)$$

is m-dissipative and self adjoint in $L^2_{\frac{1}{\sigma}}(0,1)$.

Proof. By (7) we have that A is dissipative and selfadjoint in $L^2_{\frac{1}{\sigma}}(0,1)$. Let $q(u,v)$ the quadratic form in $H^1_{\frac{1}{\sigma}}(0,1) \times H^1_{\frac{1}{\sigma}}(0,1)$ defined as $q(u,v) := \int_0^1 \eta u_x v_x dx$. Then, using the Lax-Milgram Theorem, as in [5] one proves that A is maximal. □

As usual, one can prove the following well-posedness theorem.

Theorem 2. *For all* $f \in L^2(Q)$ *and* $u_0 \in L^2_{\frac{1}{\sigma}}(0,1)$, *there exists a unique weak solution* $u \in \mathcal{U} := C^0\left([0,T]; L^2_{\frac{1}{\sigma}}(0,1)\right) \cap L^2\left(0,T; H^1_{\frac{1}{\sigma}}(0,1)\right)$ *of* (6). *Moreover, one has*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2_{\frac{1}{\sigma}}(0,1)} + \int_0^T \|u\|^2_{H^1_{\frac{1}{\sigma}}(0,1)} dt \le C\left(\|u_0\|^2_{L^2_{\frac{1}{\sigma}}(0,1)} + \int_0^T \|f\|^2_{L^2_{\frac{1}{\sigma}}(\omega)} dt\right),$$

for a positive constant C.

Lastly, we remember the following results that will be helpful in the rest of the paper and, with additional assumptions on the degenerate function $a(x)$, we can prove a characterization for the space $H^1_{\frac{1}{\sigma}}(0,1)$ (for the proof we refer to [5]).

Hypothesis 2.1. *The function* $a \in C^0[0,1]$ *is such that* $a(0) = a(1) = 0$, $a > 0$ *on* $(0,1)$ *and there exist* $K_1, K_2 \in (0,2)$ *such that*

1) *the function* $x \longmapsto \dfrac{a(x)}{x^{K_1}}$ *is nonincreasing near zero;*
2) *the function* $x \longmapsto \dfrac{a(x)}{(1-x)^{K_2}}$ *is nondecreasing near one.*

Lemma 2. *Assume that Hypothesis 2.1 is satisfied. Then,*

1) $\lim_{x\to 0^+} x^2/a(x) = \lim_{x\to 1^-}(1-x)^2/a(x) = 0$;
2) *if* $w \in H^2_{\frac{1}{\sigma}}(J_1)$ *and* $b/a \in L^1(0,1)$, *then*

   $\lim_{x\to 0^+} x w_x^2(x) = \lim_{x\to 1^-}(x-1)w_x^2(x) = 0$;
3) *the following Hardy-Poincaré inequality holds*

$$\int_0^1 v^2 \frac{1}{a} dx \le C \int_0^1 v_x^2 dx \quad \forall\, v \in H^1_0(0,1).$$

where C *is a positive constant. Moreover, if* $b/a \in L^1(0,1)$ *then the Banach spaces* $H^1_{\frac{1}{\sigma}}(0,1)$ *and* $H^1_0(0,1)$ *coincide.*

3. **Carleman Estimates for Degenerate Parabolic Problems.** In this section we prove crucial estimates of Carleman's type, that will be useful to prove the observability inequality for the adjoint problem of (6).

3.1. **Statement of the main results.** Given $T>0$, $J_1 := (0, j_1)$ and $J_2 := (j_2, 1)$ proper subintervals of $(0,1)$ and $h \in L^2\big(0,T; L^2_{\frac{1}{\sigma}}(0,1)\big)$, we consider, for $i=1,2$, the parabolic problems

$$\begin{cases} v_t + a(x)v_{xx} + b(x)v_x = h(t,x), & (t,x) \in Q_i := (0,T)\times J_i, \\ v(t,\partial J_i) = 0, & t \in (0,T)\,. \end{cases} \tag{8}$$

Here, the coefficients a, b satisfy the following assumption.

Hypothesis 3.1. *The function $a \in C^0[0,1]\bigcap C^3(0,1)$ is such that $a(0)=a(1)=0$, $a>0$ on $(0,1)$; the function $b \in C^0[0,1]\bigcap C^2(0,1)$ is such that $b/a \in L^1(0,1)$. There exists $\varepsilon \in (0,1)$ such that*

1.a) *the function* $\dfrac{x(b-a_x)}{a} \in L^\infty(0,\varepsilon)$*;*

1.b) *there exists $K_1 \in (0,2)$ such that* $\dfrac{xa_x(x)}{a(x)} \le K_1 \quad \forall\, x \in (0,\varepsilon)$*;*

1.c) *there exists a function $C_1 = C_1(\varepsilon') > 0$, defined in $(0,\varepsilon)$, such that*

$$C_1(\varepsilon') \to 0 \quad as \quad \varepsilon' \to 0^+,$$

and

$$\left|\left(\frac{x(b(x)-a_x(x))}{a(x)}\right)_{xx} - \frac{b(x)}{a(x)}\left(\frac{x(b(x)-a_x(x))}{a(x)}\right)_x\right| \le C_1(\varepsilon')\frac{1}{x^2}, \quad \forall\, x \in (0,\varepsilon');$$

2.a) *the function* $\dfrac{(x-1)(b-a_x)}{a} \in L^\infty(1-\varepsilon,1)$*;*

2.b) *there exists $K_2 \in (0,2)$ such that* $\dfrac{(x-1)a_x(x)}{a(x)} \le K_2 \quad \forall x \in (1-\varepsilon,1)$*;*

2.c) *there exists a function $C_2 = C_2(\varepsilon') > 0$, defined in $(0,\varepsilon)$, such that*

$$C_2(\varepsilon') \to 0 \quad as \quad \varepsilon' \to 0^+, \quad \forall x \in (1-\varepsilon',1)$$

and

$$\left|\left(\frac{(x-1)(b(x)-a_x(x))}{a(x)}\right)_{xx} - \frac{b(x)}{a(x)}\left(\frac{(x-1)(b(x)-a_x(x))}{a(x)}\right)_x\right| \le C_2(\varepsilon')\frac{1}{(x-1)^2}\,.$$

We observe that Hypotheses 3.1.1.b and 3.1.2.b are equivalent to Hypothesis 2.1 (see [5]).

Now, as in [5], let us introduce the weight functions

$$\begin{cases} \varphi_i(t,x) := \theta(t)(p_i(x) - 2\|p_i\|_{L^\infty(J_i)}), \quad i=1,2, \\ p_1(x) := \displaystyle\int_0^x \frac{y}{a(y)}\, e^{Ry^2}dy, \quad p_2(x) := \int_{j_2}^x \frac{y-1}{a(y)}\, e^{R(y-1)^2}dy, \quad R>0, \\ \theta(t) := \dfrac{1}{[t(T-t)]^4}. \end{cases} \tag{9}$$

Observe that $\varphi_i(t,x) < 0 \quad \forall\,(t,x) \in Q_i$, $\varphi_i(t,x) \to -\infty$ as $t \to 0^+, T^-$ and, by the assumptions on $a(x)$, one has that $p_i \in C^4(0,1) \bigcap W^{1,1}(J_{_i})$ (for $i = 1,2$).

Our main results are the following.

Theorem 3. *Assume that Hypothesis 3.1.1 is satisfied for some $\varepsilon \in (0,1)$ such that $\varepsilon < j_{_1}$. Then, there exist two positive constants C and s_0 such that every solution v of (8) in*

$$\mathcal{V}_1 := L^2\big(0,T; H^2_{\frac{1}{\sigma}}(J_{_1})\big) \cap H^1\big(0,T; H^1_{\frac{1}{\sigma}}(J_{_1})\big)$$

satisfies, for all $s \geq s_0$,

$$\int_{Q_{_1}} \eta\left(s\theta v_x^2 + s^3\theta^3\Big(\frac{x}{a}\Big)^2 v^2\right) e^{2s\varphi_1}\,dxdt$$
$$\leq C\int_{Q_{_1}} h^2\frac{e^{2s\varphi_1}}{\sigma}\;dxdt + 2s\,C\int_0^T \theta(t)\Big[\eta x v_x^2 e^{2s\varphi_1}\Big](t,j_{_1})dt.$$

Theorem 4. *Assume that Hypothesis 3.1.2 is satisfied for some $\varepsilon \in (0,1)$ such that $1-\varepsilon > j_{_2}$. Then, there exist two positive constants C and s_0 such that every solution v of (8) in*

$$\mathcal{V}_2 := L^2\big(0,T; H^2_{\frac{1}{\sigma}}(J_{_2})\big) \cap H^1\big(0,T; H^1_{\frac{1}{\sigma}}(J_{_2})\big)$$

satisfies, for all $s \geq s_0$,

$$\int_{Q_{_2}} \eta\left(s\theta v_x^2 + s^3\theta^3\Big(\frac{x-1}{a}\Big)^2 v^2\right) e^{2s\varphi_2}\,dxdt$$
$$\leq C\int_{Q_{_2}} h^2\frac{e^{2s\varphi_2}}{\sigma}\;dxdt + \;2sC\int_0^T \theta(t)\Big[\eta(1-x) v_x^2 e^{2s\varphi_2}\Big](t,j_{_2})dt.$$

We will prove only Theorem 3 since the proof of Theorem 4 is analogous.

3.2. **Proof of Theorem 3.** In order to prove Theorem 3 the following result is necessary:

Proposition 1. *Assume that Hypothesis 3.1.1 is satisfied. Then there exists $l \in \mathbb{R}$ such that*

$$\lim_{x\to 0^+} x\left(\frac{x(b(x)-a_x(x))}{a(x)}\right)_x = l.$$

Proof. Set $\rho(x) := \frac{x(b(x)-a_x(x))}{a(x)}$, we have that ρ satisfies Hypothesis 3.1.1.c if and only if

$$\rho_{xx}(x) - \frac{b(x)}{a(x)}\rho_x(x) = \frac{C_1}{x^2}\gamma(x) sgn\left\{\rho_{xx}(x) - \frac{b(x)}{a(x)}\rho_x(x)\right\}, \qquad x \in (0,\varepsilon), \quad (10)$$

where γ is a suitable continuous function such that $\gamma(x) \in [0,1]$.

Now, set $\overline{\gamma} := \gamma\ sgn\left\{\rho_{xx} - \frac{b}{a}\rho_x\right\}$, for some fixed $x \in (0,\varepsilon)$ and $h > 0$ such that $x + h \in (0,\varepsilon)$, by classical representation formula of the solutions of (10) one has that

$$\rho_x(x+h) = \exp\left\{\int_x^{x+h}\frac{b(y)}{a(y)}dy\right\}\left(\rho_x(x) + C_1\int_x^{x+h}\exp\left\{-\int_x^s \frac{b(y)}{a(y)}dy\right\}\frac{\overline{\gamma}(s)}{s^2}ds\right). \quad (11)$$

Finally, by (11) and assumptions, there exists a positive constant C such that

$$\begin{aligned} x\,|\rho_x(x)| \le & x\,|\rho_x(x+h)| \exp\left\{-\int_x^{x+h} \frac{b(y)}{a(y)}dy\right\} \\ & + xC_1 \int_x^{x+h} \exp\left\{-\int_x^s \frac{b(y)}{a(y)}dy\right\} \frac{ds}{s^2} \\ \le & C\left(x\,|\rho_x(x+h)| + x\int_x^{x+h} \frac{dy}{y^2}\right). \end{aligned} \tag{12}$$

Passing to the limit, the conclusion follows. □

Now, we define, for $s > 0$, the function

$$w(t,x) := e^{s\varphi_1(t,x)}v(t,x)$$

where v is the solution of (8) in $\mathcal{V}_1$; observe that, since $v \in \mathcal{V}_1$, $w \in \mathcal{V}_1$. Setting, for simplicity, $\varphi := \varphi_1$ and $p := p_1$, one has that w satisfies

$$\begin{cases} (e^{-s\varphi}w)_t + a(x)(e^{-s\varphi}w)_{xx} + b(x)(e^{-s\varphi}w)_x = h(t,x), & (t,x) \in Q_1, \\ w(0,x) = w(T,x) = 0, & x \in J_1, \\ w(t,0) = w(t,j_i) = 0, & t \in (0,T). \end{cases} \tag{13}$$

Defining $Lv := v_t + av_{xx} + bv_x$ and $L_s w := e^{s\varphi}L(e^{-s\varphi}w)$, the equation of (13) can be recast as follows

$$L_s w = L_s^+ w + L_s^- w = e^{s\varphi}h,$$

where

$$\begin{cases} L_s^+ w := Aw - s\varphi_t w + s^2 a\varphi_x^2 w, \\ L_s^- w := w_t - sA\varphi w - 2sa\varphi_x w_x. \end{cases}$$

Moreover, set $< u, v >_{L^2_{\frac{1}{\sigma}}(Q_1)} := \displaystyle\int_{Q_1} uv\frac{1}{\sigma}dxdt$, one has

$$\|L_s^+ w\|^2_{L^2_{\frac{1}{\sigma}}(Q_1)} + \|L_s^- w\|^2_{L^2_{\frac{1}{\sigma}}(Q_1)} + 2 < L_s^+ w, L_s^- w >_{L^2_{\frac{1}{\sigma}}(Q_1)} = \|he^{s\varphi}\|^2_{L^2_{\frac{1}{\sigma}}(Q_1)}. \tag{14}$$

Lemma 3. *The following identity holds*

$$\left.\begin{aligned} < L_s^+ w, L_s^- w >_{L^2_{\frac{1}{\sigma}}(Q_1)} = & s\int_{Q_1} \eta(a\varphi_{xx} + (a\varphi_x)_x)w_x^2 dxdt \\ & + s^3 \int_{Q_1} \eta\varphi_x^2(a\varphi_{xx} + (a\varphi_x)_x)w^2 dxdt \\ & - 2s^2 \int_{Q_1} \eta\varphi_x\varphi_{xt}w^2 dxdt + \frac{s}{2}\int_{Q_1} \eta\frac{\varphi_{tt}}{a}w^2 dxdt \\ & - \frac{s}{2}\int_{Q_1} \eta\left((A\varphi)_{xx} - \frac{b}{a}(A\varphi)_x\right) w^2 dxdt \end{aligned}\right\} \{D.T.\} \tag{15}$$

$$\{B.T.\}\begin{cases} -\dfrac{1}{2}\displaystyle\int_0^{j_1}\eta\Big[w_x^2\Big]_0^T dx+\int_0^T\Big[\eta w_x w_t\Big]_0^{j_1}dt+\frac{s}{2}\int_0^T\Big[\eta(A\varphi)_x w^2\Big]_0^{j_1}dt\\ -s\displaystyle\int_0^T\Big[\eta a\varphi_x w_x^2\Big]_0^{j_1}dt-s\int_0^T\Big[\eta A\varphi w w_x\Big]_0^{j_1}dt\\ +\dfrac{1}{2}\displaystyle\int_0^{j_1}\eta\Big[(s^2\varphi_x^2-s\frac{\varphi_t}{a})w^2\Big]_0^T dx-\int_0^T\Big[\eta(s^3a\varphi_x^3-s^2\varphi_x\varphi_t)w^2\Big]_0^{j_1}dt.\end{cases}$$

Proof. It results, integrating by parts,

$$\begin{aligned}
&< Aw, L_s^- w >_{L^2_{\frac{1}{\sigma}}(Q_1)}\\
=&\int_{Q_1}(\eta w_x)_x w_t dxdt - s\int_{Q_1}(\eta w_x)_x A\varphi w dxdt - 2s\int_{Q_1}(\eta w_x)_x a\varphi_x w_x dxdt\\
=&-\int_{Q_1}\eta w_{xt}w_x dxdt+\int_0^T\big[\eta w_x w_t\big]_0^{j_1}dt+s\int_{Q_1}\eta(A\varphi w)_x w_x dxdt\\
&-s\int_0^T\big[\eta A\varphi w w_x\big]_0^{j_1}dt+s\int_{Q_1}\eta((a\varphi_x)_x-b\varphi_x)w_x^2dxdt-s\int_0^T\big[\eta a\varphi_x w_x^2\big]_0^{j_1}dt\\
=&-\frac{1}{2}\int_0^{j_1}\eta\big[w_x^2\big]_0^T dx+\int_0^T\big[\eta w_x w_t\big]_0^{j_1}dt\\
&+s\int_{Q_1}\eta A\varphi w_x^2 dxdt-\frac{1}{2}s\int_{Q_1}(\eta(A\varphi)_x)_x w^2 dxdt+\frac{1}{2}s\int_0^T\big[\eta(A\varphi)_x w^2\big]_0^{j_1}dt\\
&-s\int_0^T\big[\eta A\varphi w w_x\big]_0^{j_1}dt+s\int_{Q_1}\eta((a\varphi_x)_x-b\varphi_x)w_x^2dxdt-s\int_0^T\big[\eta a\varphi_x w_x^2\big]_0^{j_1}dt.
\end{aligned} \tag{16}$$

Therefore, integrating again by parts,

$$\begin{aligned}
&< -s\varphi_t w + s^2 a\varphi_x^2 w, L_s^- w >_{L^2_{\frac{1}{\sigma}}(Q_1)}\\
=&\int_{Q_1}\eta\big(s^2\varphi_x^2-s\frac{\varphi_t}{a}\big)ww_t dxdt\\
&-\int_{Q_1}\eta A\varphi\big(s^3\varphi_x^2-s^2\frac{\varphi_t}{a}\big)w^2dxdt-2\int_{Q_1}\eta a\varphi_x\big(s^3\varphi_x^2-s^2\frac{\varphi_t}{a}\big)ww_x dxdt\\
=&\frac{1}{2}\int_{Q_1}\eta\big(-s^2\varphi_x^2+s\frac{\varphi_t}{a}\big)_t w^2dxdt+\frac{1}{2}\int_0^{j_1}\eta\big[\big(s^2\varphi_x^2-s\frac{\varphi_t}{a}\big)w^2\big]_0^T dx\\
&-s^3\int_{Q_1}\eta\varphi_x^2A\varphi w^2dxdt+s^2\int_{Q_1}\eta A\varphi\frac{\varphi_t}{a}w^2dxdt\\
&+\int_{Q_1}\big(\eta(s^3a\varphi_x^3-s^2\varphi_x\varphi_t)\big)_x w^2dxdt-\int_0^T\big[\eta\big(s^3a\varphi_x^3-s^2\varphi_x\varphi_t\big)w^2\big]_0^{j_1}dt\\
=&\frac{s}{2}\int_{Q_1}\eta\frac{\varphi_{tt}}{a}w^2dxdt+s^2\int_{Q_1}\big(\eta\frac{\varphi_t}{a}A\varphi-\eta\varphi_x\varphi_{xt}-(\eta\varphi_x\varphi_t)_x\big)w^2dxdt\\
&+s^3\int_{Q_1}\big((\eta a\varphi_x^3)_x-\eta\varphi_x^2A\varphi\big)w^2dxdt+\frac{1}{2}\int_0^{j_1}\eta\big[\big(s^2\varphi_x^2-s\frac{\varphi_t}{a}\big)w^2\big]_0^T dx
\end{aligned} \tag{17}$$

$$
\begin{aligned}
&-\int_0^T \left[\eta\big(s^3 a\varphi_x^3 - s^2\varphi_x\varphi_t\big)w^2\right]_0^{j_1} dt\\
=&\frac{s}{2}\int_{Q_1} \eta\frac{\varphi_{tt}}{a}w^2 dxdt - 2s^2\int_{Q_1} \eta\varphi_x\varphi_{xt}w^2 dxdt\\
&+ s^3\int_{Q_1} \eta\varphi_x^2\big((a\varphi_x)_x + a\varphi_{xx}\big)w^2 dxdt + \frac{1}{2}\int_0^{j_1} \eta\big[\big(s^2\varphi_x^2 - s\frac{\varphi_t}{a}\big)w^2\big]_0^T dx\\
&-\int_0^T \left[\eta\big(s^3 a\varphi_x^3 - s^2\varphi_x\varphi_t\big)w^2\right]_0^{j_1} dt.
\end{aligned}
$$

Adding (16)-(17), (15) follows immediately. □

The next lemma holds.

Lemma 4. *The boundary terms in* (15) *become*

$$
\{B.T.\} = -se^{Rj_1^2}\int_0^T \eta(j_1)\theta(t)j_1 w_x^2(t,j_1)dt. \tag{18}
$$

Proof. Using the definition of φ and the fact that $w(t,j_1)=0$, the boundary terms of $< L_s^+ w, L_s^- w >_{L^2_{\frac{1}{\sigma}}(Q_1)}$ become

$$
\begin{aligned}
\{B.T.\} \;=\; &-\frac{1}{2}\int_0^{j_1} \eta\Big[w_x^2\Big]_0^T dx + \int_0^T \Big[\eta w_x w_t\Big]_0^{j_1} dt\\
&+\frac{1}{2}\int_0^{j_1} \eta\Big[\Big(s^2\theta^2\Big(\frac{x}{a}\Big)^2 e^{2Rx^2} - \frac{s}{a}\,\dot\theta\big(p(x) - 2\|p\,\|_{L^\infty(J_1)}\big)\Big)w^2\Big]_0^T dx\\
&-s\int_0^T \theta(t)\Big[\eta e^{Rx^2} x w_x^2\Big]_0^{j_1} dt - \frac{s}{2}\int_0^T \theta(t)\Big[\eta\Big(\frac{x(b-a_x)}{a}\Big)_x w^2\Big](t,0)dt\\
&-s\int_0^T \theta(t)\Big[\eta\Big(1 + \frac{x(b-a_x)}{a}\Big)ww_x\Big](t,0)dt\\
&+s^3\int_0^T \theta^3(t)\Big[\eta\frac{x^3}{a^2}w^2\Big](t,0)dt\\
&+2s^2\|p\|_{L^\infty(J_1)}\int_0^T \theta(t)\dot\theta(t)\Big[\eta\frac{x}{a}w^2\Big](t,0)dt.
\end{aligned}
$$

Since $w\in\mathcal{V}_1$, where $\mathcal{V}_1$ is as in (10), $w\in C^0\big([0,T];H^1_{\frac{1}{\sigma}}(J_1)\big)$. Thus $w_x(x,0)$, $w_x(x,T)$ and $\int_0^{j_1}\eta\big[w_x^2\big]_0^T dx$ are well defined and, using the boundary conditions of w, it results that

$$
\int_0^{j_1} \eta\Big[w_x^2\Big]_0^T dx = 0.
$$

Moreover, since $w\in H^1\big(0,T;H^1_{\frac{1}{\sigma}}(J_1)\big)$, $w_t(t,0)$ and $w_t(t,j_1)$ are well defined. Now, by Lemma 2, we have that $\lim_{x\to 0}\sqrt{x}w_x(t,x)=0$. Since $w_{tx}(t,x)\in L^2(J_1)$, then, by Hölder's inequality,

$$
|w_t(t,x)| \le \int_0^x |w_{tx}(t,y)|dy \le \sqrt{x}\left(\int_0^x |w_{tx}(t,y)|^2 dy\right)^{1/2}.
$$

Thus, if $w \in \mathcal{V}_1$ then $\int_0^T [w_x w_t]_0^{j_1}\, dt$ is well defined and it is 0. Now, we consider the term

$$\frac{1}{2}\int_0^{j_1} \eta\Big[\Big(s^2\theta^2\Big(\frac{x}{a}\Big)^2 e^{2Rx^2} - \frac{s}{a}\,\dot\theta\big(p(x) - 2\|p\|_{L^\infty(J_1)}\big)\Big)w^2\Big]_0^T dx.$$

Since $w \in \mathcal{V}_1$, then $w \in C^0([0,T]; L^2_{\frac{1}{\sigma}}(J_1))$. Thus $w(0,x)$ and $w(T,x)$ are well defined and $w(0,x) = w(T,x) = 0$. This implies that

$$\frac{1}{2}\int_0^{j_1} \eta\Big[\Big(s^2\theta^2\Big(\frac{x}{a}\Big)^2 e^{2Rx^2} - \frac{s}{a}\,\dot\theta\big(p(x) - 2\|p\,\|_{L^\infty(J_1)}\big)\Big)w^2\Big]_0^T dx = 0.$$

By Lemma 2

$$-s\int_0^T \theta(t)\Big[\eta e^{Rx^2} x w_x^2\Big]_0^{j_1} dt = -s\int_0^T \theta(t)\Big[\eta e^{Rx^2} x w_x^2\Big](t, j_1)dt.$$

Thus, the boundary terms become

$$\begin{aligned}\{B.T.\} = &- s e^{Rj_1^2}\int_0^T \eta(j_1)\theta(t) j_1 w_x^2(t,j_1)dt - \frac{s}{2}\int_0^T \theta(t)\left[\eta\left(\frac{x(b-a_x)}{a}\right)_x w^2\right](t,0)dt\\ &- s\int_0^T \theta(t)\left[\eta\left(1 + \frac{x(b-a_x)}{a}\right) w w_x\right](t,0)dt\\ &+ s^3\int_0^T \theta^3(t)\left[\eta\frac{x^3}{a^2}w^2\right](t,0)dt\\ &+ 2s^2\|p\,\|_{L^\infty(J_1)}\int_0^T \theta(t)\dot\theta(t)\left[\eta\frac{x}{a}w^2\right](t,0)dt.\end{aligned}$$

By Proposition 1,

$$\left|\theta(t)\left[\left(\frac{x(b-a_x)}{a}\right)_x w^2\right](t,\epsilon)\right| \le \theta(t)\left|\epsilon\left(\frac{x(b-a_x)}{a}\right)_x(\epsilon)\right|\int_0^\epsilon w_x^2(t,y)dy \to 0$$

as $\epsilon \to 0^+$. Thus

$$\begin{aligned}&\frac{s}{2}\int_0^T \theta(t)\left[\eta\left(\frac{x(b-a_x)}{a}\right)_x w^2\right](t,0)dt\\ &= \lim_{\epsilon\to 0}\frac{s}{2}\int_0^T \theta(t)\left[\eta\left(\frac{x(b-a_x)}{a}\right)_x w^2\right](t,\epsilon)dt = 0.\end{aligned}$$

Moreover, by assumption, it results

$$\begin{aligned}&\left|\theta(t)\left[\left(1+\frac{x(b-a_x)}{a}\right) w w_x\right](t,\epsilon)\right|\\ &\le \theta(t)\left(1 + \left\|\frac{x(b-a_x)}{a}\right\|_{L^\infty(J_1)}\right)|w_x(\epsilon,t)|\left(\epsilon\int_0^\epsilon |w_x(t,x)|^2 dx\right)^{\frac{1}{2}} \to 0\end{aligned}$$

as $\epsilon \to 0^+$, thus

$$\begin{aligned}&s\int_0^T \theta(t)\left[\eta\left(1 + \frac{x(b-a_x)}{a}\right) w w_x\right](t,0)dt\\ &= \lim_{\epsilon\to 0} s\int_0^T \theta(t)\left[\eta\left(1 + \frac{x(b-a_x)}{a}\right) w w_x\right](t,\epsilon)dt = 0.\end{aligned}$$

Now, by Lemma 2,

$$\left|\theta(t)\dot\theta(t)\left[\frac{x}{a}\ w^2\right](t,\epsilon)\right| \le\ \theta(t)|\dot\theta(t)|\frac{\epsilon^2}{a(\epsilon)}\int_0^{\epsilon} w_x^2(t,y)dy \to 0,$$

as $\epsilon \to 0^+$, thus

$$\begin{aligned} &2s^2\|p\ \|_{L^\infty(J_1)}\int_0^T \theta(t)\dot\theta(t)\left[\eta\frac{x}{a}\ w^2\right](t,0)dt \\ &=\lim_{\epsilon\to 0} 2s^2\|p\ \|_{L^\infty(J_1)}\int_0^T \theta(t)\dot\theta(t)\left[\eta\frac{x}{a}\ w^2\right](t,\epsilon)dt = 0. \end{aligned}$$

Finally,

$$s^3\int_0^T \theta^3(t)\left[\eta\frac{x^3}{a^2}\ w^2\right](t,0)dt = \lim_{\epsilon\to 0} s^3\int_0^T \theta^3(t)\left[\eta\frac{x^3}{a^2}\ w^2\right](t,\epsilon)dt = 0.$$

In fact, by Hölder's inequality, it results $w^2(t,x) \le x\int_0^x w_x^2(t,y)dy$. Thus

$$\int_0^T \theta^3(t)\left[\eta\frac{x^3}{a^2}\ w^2\right](t,\epsilon)dt \le \int_0^T \theta^3(t)\left[\eta\frac{x^4}{a^2}\int_0^x w_x^2dy\right](t,\epsilon)dt$$

and again, by Lemma 2,

$$\int_0^T \theta^3(t)\left[\eta\frac{x^4}{a^2}\int_0^x w_x^2dy\right](t,\epsilon)dt \to 0, \quad \text{as} \quad \epsilon\to 0^+.$$

□

The crucial step is to prove now the following estimate.

Lemma 5. *The distributed terms of* (15) *satisfy the following estimate*

$$\begin{aligned} &s(2-K_1)\int_{Q_1}\eta\theta w_x^2dxdt + s^3(2-K_1)\int_{Q_1}\eta\theta^3\left(\frac{x}{a}\right)^2 w^2dxdt \\ &-C\left(s^2\int_{Q_1}\eta\theta^3\left(\frac{x}{a}\right)^2 w^2dxdt + sC_1(\varepsilon')\int_{Q_1}\eta\theta w_x^2dxdt + s\Lambda(\varepsilon')\int_{Q_1}\eta\frac{\theta^{\frac{3}{2}}}{a}w^2dxdt\right) \\ &\le\quad s\int_{Q_1}\eta(a\varphi_{xx}+(a\varphi_x)_x)w_x^2dxdt + s^3\int_{Q_1}\eta\varphi_x^2(a\varphi_{xx}+(a\varphi_x)_x)w^2dxdt \\ &-2s^2\int_{Q_1}\eta\varphi_x\varphi_{xt}w^2dxdt + \frac{s}{2}\int_{Q_1}\eta\frac{\varphi_{tt}}{a}w^2dxdt - \frac{s}{2}\int_{Q_1}\eta\left((A\varphi)_{xx}-\frac{b}{a}(A\varphi)_x\right)w^2dxdt, \end{aligned}$$

where C is a positive constant and $\Lambda(\varepsilon')$ is a suitable positive function defined in $(0,\varepsilon)$ (see (22)*)* .

Proof. Using the definition of φ, the distributed terms of $< L_s^+ w, L_s^- w >_{L^2_{\frac{1}{\sigma}}(Q_1)}$ take the form

$$
\begin{aligned}
\{D.T.\} = \;& s\int_{Q_1} \eta\theta\Big(2 - \frac{xa_x}{a} + 4Rx^2\Big)e^{Rx^2} w_x^2 dxdt \\
&+ s^3 \int_{Q_1} \eta\theta^3\Big(\frac{x}{a}\Big)^2\Big(2 - \frac{xa_x}{a} + 4Rx^2\Big)e^{3Rx^2} w^2 dxdt \\
&- 2s^2 \int_{Q_1} \eta\theta\dot\theta\Big(\frac{x}{a}\Big)^2 e^{2Rx^2} w^2 dxdt + \frac{s}{2}\int_{Q_1} \eta\frac{\ddot\theta}{a}\Big(p - 2\|p\|_{L^\infty(J_1)}\Big)w^2 dxdt \\
&- \frac{s}{2}\int_{Q_1} \eta\theta\left(\Big(\frac{x(b-a_x)}{a}\Big)_{xx} - \frac{b}{a}\Big(\frac{x(b-a_x)}{a}\Big)_x\right) e^{Rx^2} w^2 dxdt \\
&- s\int_{Q_1} \eta\frac{\theta}{a}Rxb\left(\Big(\frac{x(b-a_x)}{a}\Big) + 3 + 2x^2R\right) e^{Rx^2} w^2 dxdt \\
&- s\int_{Q_1} \eta\theta R\left(2x\Big(\frac{x(b-a_x)}{a}\Big)_x + (1+2Rx^2)\Big(\frac{x(b-a_x)}{a}\Big)\right) e^{Rx^2} w^2 dxdt \\
&- s\int_{Q_1} \eta\theta R(3 + 12Rx^2 + 4R^2x^4)e^{Rx^2} w^2 dxdt.
\end{aligned} \tag{19}
$$

Because of Hypothesis 3.1.1.b

$$2 - \frac{xa_x}{a} \geq 2 - K_1 > 0 \quad \forall\, x \in (0,\varepsilon);$$

thus there exists $R > 0$ such that

$$2 - \frac{xa_x}{a} + 4Rx^2 \geq 2 - K_1 \quad \forall\, x \in J_1.$$

Moreover, for all $x \in J_1$ one has that

$$
\begin{aligned}
& R\left|xb\left(\Big(\frac{x(b-a_x)}{a}\Big) + 3 + 2x^2R\right)\right| \\
\leq & R\,\|b\|_{L^\infty(J_1)}\left(\left\|\Big(\frac{x(b-a_x)}{a}\Big)\right\|_{L^\infty(J_1)} + 3 + 2R\right) =: C_{R,1}.
\end{aligned}
$$

Using Hypothesis 3.1.1.a and Proposition 1, for all $x \in J_1$ one has

$$
\begin{aligned}
& R\left|\left(2x\Big(\frac{x(b-a_x)}{a}\Big)_x + (1+2Rx^2)\Big(\frac{x(b-a_x)}{a}\Big) + (3 + 12Rx^2 + 4R^2x^4)\right)\right| \leq \\
& R\left(2\left\|x\Big(\frac{x(b-a_x)}{a}\Big)_x\right\|_{L^\infty(J_1)} + (1+2R)\left\|\Big(\frac{x(b-a_x)}{a}\Big)\right\|_{L^\infty(J_1)} + (3 + 12R + 4R^2)\right) \\
& =: C_{R,2}.
\end{aligned}
$$

Then, set $\rho(x) := \frac{x(b(x)-a_x(x))}{a(x)}$ as in Proposition 1, and applying Hypothesis 3.1.1.c, one has

$$
\begin{aligned}
&\{D.T.\}\\
\geq & s(2-K_1)\int_{Q_1}\eta\theta w_x^2dxdt + s^3(2-K_1)\int_{Q_1}\eta\theta^3\left(\frac{x}{a}\right)^2 w^2dxdt\\
&-2s^2e^{2R}\int_{Q_1}\eta\theta|\dot{\theta}|\left(\frac{x}{a}\right)^2 w^2dxdt - s\|p\|_{L^\infty(J_1)}\int_{Q_1}\eta\frac{|\ddot{\theta}|}{a}w^2dxdt\\
&-\frac{s}{2}e^RC_1(\varepsilon')\int_{Q_1}\eta\frac{\theta}{x^2}w^2dxdt - \frac{s}{2}e^R\max_{[\varepsilon',j_1]}|\rho_{xx}-(b/a)\rho_x|\int_{Q_1}\eta\theta w^2dxdt\\
&-se^RC_{R,1}\int_{Q_1}\eta\frac{\theta}{a}w^2dxdt - se^RC_{R,2}\int_{Q_1}\eta\theta w^2dxdt,
\end{aligned}
\tag{20}
$$

where $\varepsilon' \in (0,\varepsilon)$. Observing that there exists $C_T > 0$ such that $\theta|\dot{\theta}| \leq C_T\theta^3$, $|\ddot{\theta}| \leq C_T\theta^{\frac{3}{2}}$ and $\theta \leq C_T\theta^{\frac{3}{2}}$, one can deduce the next estimate:

$$
\begin{aligned}
&\{D.T.\}\\
\geq & s(2-K_1)\int_{Q_1}\eta\theta w_x^2dxdt + s^3(2-K_1)\int_{Q_1}\eta\theta^3\left(\frac{x}{a}\right)^2 w^2dxdt\\
&-2s^2e^{2R}C_T\int_{Q_1}\eta\theta^3\left(\frac{x}{a}\right)^2 w^2dxdt - se^RC_1(\varepsilon')\int_{Q_1}\eta\frac{\theta}{x^2}w^2dxdt\\
&-se^RC_T\left(\max_{[\varepsilon',j_1]}|\rho_{xx}-(b/a)\rho_x| + C_{R,2}\right)\|a\|_{L^\infty(0,1)}\int_{Q_1}\eta\frac{\theta^{\frac{3}{2}}}{a}\, w^2dxdt\\
&-se^RC_T\left(C_{R,1}+\|p\,\|_{L^\infty(J_1)}\right)\int_{Q_1}\eta\frac{\theta^{\frac{3}{2}}}{a}\, w^2dxdt.
\end{aligned}
\tag{21}
$$

By Hardy's inequality (see, e.g., [11]) and because η is continuous and strictly positive, it is possible to estimate the last term of (21) in the following way

$$\int_{Q_1}\eta\frac{\theta}{x^2}w^2dxdt \leq C_H\frac{\sup_{J_1}\{\eta\}}{\inf_{J_1}\{\eta\}}\int_{Q_1}\eta\theta w_x^2dxdt.$$

Here, C_H is a positive constant. Finally, set

$$\Lambda(\varepsilon') := \left(\max_{[\varepsilon',j_1]}|\rho_{xx}-(b/a)\rho_x| + C_{R,2}\right)\|a\|_{L^\infty(0,1)} + C_{R,1} + \|p\,\|_{L^\infty(J_1)} \tag{22}$$

we obtain the conclusion. □

Proposition 2. *There exist two positive constants C and s_0 such that, for all $s \geq s_0$, all solutions w of (13) in $\mathcal{V}_1$ satisfy*

$$
\begin{aligned}
&\int_{Q_1}\eta\left(s\theta w_x^2 + s^3\theta^3\left(\frac{x}{a}\right)^2\right)w^2dxdt\\
\leq & C\left(\int_{Q_1}h^2\frac{e^{2s\varphi}}{\sigma}\,dxdt + 2s\int_0^T\eta(j_1)\theta(t)j_1w_x^2(t,j_1)dt\right).
\end{aligned}
$$

Proof. By (14) and by Lemmas 4 and 5 it is sufficient to estimate only the term $-sC\Lambda(\varepsilon')\int_{Q_1}\eta\frac{\theta^{\frac{3}{2}}}{a}w^2dxdt$. For $\lambda>0$ it results

$$\int_{Q_1}\eta\frac{\theta^{\frac{3}{2}}}{a}w^2dxdt=\int_{Q_1}\left(\frac{1}{\lambda}\eta\theta^2\left(\frac{x}{a}\right)^2w^2\right)^{\frac{1}{2}}\left(\lambda\eta\frac{\theta}{x^2}w^2\right)^{\frac{1}{2}}dxdt$$
$$\leq\frac{1}{\lambda}\int_{Q_1}\eta\theta^2\left(\frac{x}{a}\right)^2w^2dxdt+\lambda\int_{Q_1}\eta\frac{\theta}{x^2}w^2dxdt.$$

By Hardy's inequality one has

$$\int_{Q_1}\eta\frac{\theta^{\frac{3}{2}}}{a}\,w^2dxdt\leq\frac{1}{\lambda}\int_{Q_1}\eta\theta^2\left(\frac{x}{a}\right)^2w^2dxdt+\lambda C_H\frac{\sup_{J_1}\{\eta\}}{\inf_{J_1}\{\eta\}}\int_{Q_1}\eta\theta w_x^2dxdt,$$

for some positive constant C_H. Thus, for s_0 large enough and λ small enough,

$$C_\lambda\left(s\int_{Q_1}\eta\theta w_x^2dxdt+s^3\int_{Q_1}\eta\theta^3\left(\frac{x}{a}\right)^2w^2dxdt\right)-2se^{Rj_1^2}\int_0^T\eta(j_1)\theta(t)j_1w_x^2(t,j_1)dt$$
$$\leq\int_{Q_1}h^2\frac{e^{2s\varphi}}{\sigma}\,dxdt,$$

for some positive constant C_λ and for all $s\geq s_0$. □

Recalling the definition of w, we have $v=e^{-s\varphi}w$ and $v_x=(w_x-s\varphi_xw)e^{-s\varphi}$. Thus, Theorem 3 follows immediately by Proposition 2.

4. **Observability and controllability of linear equations.** In this section we will prove, as a consequence of the Carleman estimates established in Section 3, an observability inequality for the adjoint problem

$$\begin{cases}v_t+a(x)v_{xx}+b(x)v_x=0, & (t,x)\in Q,\\ v(t,0)=v(t,1)=0, & t\in(0,T),\\ v(T,x)=v_T(x)\in L^2_{\frac{1}{\sigma}}(0,1)\end{cases}\tag{23}$$

of (6). In particular, the following result holds.

Proposition 3. *Assume that Hypothesis 3.1 is satisfied. Then there exists a positive constant C_T such that every solution $v\in\mathcal{U}$ of (23) satisfies*

$$\int_0^1v^2(0,x)\frac{1}{\sigma}dx\leq C_T\int_0^T\int_\omega v^2\frac{1}{\sigma}dxdt.\tag{24}$$

Here $\mathcal{U}:=C^0\big([0,T];L^2_{\frac{1}{\sigma}}(0,1)\big)\cap L^2\big(0,T;H^1_{\frac{1}{\sigma}}(0,1)\big)$.

Before proving this proposition we will give some results that will be very helpful to this aim. As a first step we introduce the following class of functions

$$\mathcal{W}:=\Big\{v\text{ solution of (23)}\;\Big|\;v_T\in D(A^2)\Big\}$$

where

$$D(A^2)=\Big\{u\in H^1_{\frac{1}{\sigma}}(0,1)\;\Big|\;Au\in H^2_{\frac{1}{\sigma}}(0,1)\Big\}.$$

Obviously,

$$\mathcal{W}\subset C^1\big([0,T]\,;\,H^2_{\frac{1}{\sigma}}(0,1)\big)\subset\mathcal{V}:=L^2\big(0,T;H^2_{\frac{1}{\sigma}}(0,1)\big)\cap H^1\big(0,T;H^1_{\frac{1}{\sigma}}(0,1)\big)\subset\mathcal{U}.$$

Proposition 4 (Caccioppoli's inequality)**.** *Let ω' and ω two open subintervals of $(0,1)$ such that $\omega' \subset\subset \omega \subset\subset (0,1)$. Let $s>0$ and $\psi(t,x) := \theta(t)\Psi(x)$, where θ is defined in (9) and $\Psi \in C^1(0,1)$ is a strictly negative function. Then, there exists a positive constant C such that*

$$\int_0^T \int_{\omega'} v_x^2 e^{2s\psi} dxdt \;\le\; C \int_0^T \int_{\omega} v^2 dxdt, \tag{25}$$

for every solution v of the adjoint problem (23).

Proof. Let us consider a smooth function $\xi : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1, & \text{for all } x \in [0,1], \\ \xi(x) = 1, & x \in \omega', \\ \xi(x) = 0, & x \in (0,1) \setminus \omega. \end{cases}$$

Then,

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt}\left(\int_0^1 (\xi e^{s\psi})^2 v^2 dx\right) dt = \int_Q 2s\psi_t(\xi e^{s\psi})^2 v^2 + 2(\xi e^{s\psi})^2 v(-\sigma(\eta v_x)_x)\, dxdt \\ &= 2s\int_Q \psi_t(\xi e^{s\psi})^2 v^2 dxdt + 2\int_Q \left(\xi^2 e^{2s\psi}\sigma\right)_x \eta v v_x dxdt + 2\int_Q (\xi^2 e^{2s\psi} a) v_x^2 dxdt. \end{aligned}$$

Hence,

$$\begin{aligned} &2\int_Q (\xi^2 e^{2s\psi} a) v_x^2 dxdt \\ &= -\,2s\int_Q \psi_t \left(\xi e^{s\psi}\right)^2 v^2 dxdt - 2\int_Q \left(\xi^2 e^{2s\psi}\sigma\right)_x \frac{\xi e^{s\psi}\sqrt{\sigma}}{\xi e^{s\psi}\sqrt{\sigma}}\, \eta v v_x\, dxdt \\ &\le -\,2s\int_Q \psi_t(\xi e^{s\psi})^2 v^2 dxdt + 4\int_Q \left(\xi e^{s\psi}\sqrt{\sigma}\,\right)_x^2 \eta v^2 dxdt + \int_Q (\xi^2 e^{2s\psi} a) v_x^2 dxdt. \end{aligned}$$

Thus,

$$\inf_{\omega'}\{a\} \int_0^T \int_{\omega'} e^{2s\psi} v_x^2 dxdt \le \sup_{\omega\times(0,T)} \left\{ \left| 4\eta \left(\xi e^{s\psi}\sqrt{\sigma}\right)_x^2 - 2s\psi_t(\xi e^{s\psi})^2 \right| \right\} \int_0^T \int_\omega v^2 dxdt.$$

□

As a consequence of Proposition 24 one has:

Lemma 6. *Assume that Hypothesis 3.1 is satisfied. Let T_0, T_1 be such that $0 < T_0 < T_1 < T$. Then there exists a positive constant $C = C(T_0, T_1)$ such that every solution $v \in \mathcal{W}$ of (23) satisfies*

$$\int_{T_0}^{T_1} \int_0^1 v^2 \frac{1}{\sigma} dxdt \le C \int_0^T \int_\omega v^2 \frac{1}{\sigma} dxdt.$$

Proof. Let us consider a smooth function $\xi : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1, & \text{for all } x \in [0,1], \\ \xi(x) = 1, & x \in [0, (2\alpha+\beta)/3], \\ \xi(x) = 0, & x \in [(\alpha+2\beta)/3, 1]. \end{cases}$$

We define $w(t,x) := \xi(x)v(t,x)$ where $v \in \mathcal{W}$. Then w satisfies

$$\begin{cases} w_t + aw_{xx} + bw_x = (a\xi_{xx} + b\xi_x)v + 2a\xi_x v_x =: h, & (t,x) \in (0,T) \times (0,\beta), \\ w(t,0) = w(t,\beta) = 0, & t \in (0,T). \end{cases}$$

Setting $\omega' := ((2\sigma+\beta)/3, (\sigma+2\beta)/3)$ and using Proposition 4, it results

$$\begin{aligned} \int_Q h^2 \frac{e^{2s\psi}}{\sigma}\, dxdt &= \int_0^T \int_{\omega'} ((a\xi_{xx} + b\xi_x)v + 2a\xi_x v_x)^2 \frac{e^{2s\psi}}{\sigma}\, dxdt \\ &\le C \int_0^T \int_{\omega'} ((a\xi_{xx} + b\xi_x)^2 v^2 + 4a^2\xi_x^2 v_x^2)\, e^{2s\psi} dxdt \\ &\le C \int_0^T \int_{\omega'} v^2 dxdt + C \int_0^T \int_{\omega'} v_x^2 e^{2s\psi} dxdt \\ &\le C \int_0^T \int_{\omega} v^2 dxdt \le C \int_0^T \int_{\omega} v^2 \frac{1}{\sigma} dxdt, \end{aligned} \tag{26}$$

for some positive constant C. Applying the previous inequality with $\psi = \varphi_1$ and Theorem 3 with $J_1 = (0,\beta)$, one has

$$\begin{aligned} \int_0^T \int_\omega v^2 \frac{1}{\sigma} dxdt \ge C \int_Q h^2 \frac{e^{2s_0\varphi_1}}{\sigma}\, dxdt \quad &\ge s_0 C \int_0^T \int_0^\beta \eta\theta w_x^2\, e^{2s_0\varphi_1} dxdt \\ &\ge C \int_{T_0}^{T_1} \int_0^\beta \eta w_x^2 dxdt. \end{aligned}$$

By Lemma 2 it follows

$$\int_0^T \int_\omega v^2 \frac{1}{\sigma} dxdt \ge C \int_{T_0}^{T_1} \int_0^\beta w^2 \frac{1}{\sigma} dxdt \ge C \int_{T_0}^{T_1} \int_0^{(2\alpha+\beta)/3} v^2 \frac{1}{\sigma} dxdt. \tag{27}$$

Consider now $z(t,x) := \big(1 - \xi(x)\big)v(t,x)$. Then z satisfies

$$\begin{cases} z_t + az_{xx} + bz_x = -h, & (t,x) \in (0,T) \times (\alpha,1), \\ z(t,\alpha) = z(t,1) = 0, & t \in (0,T), \end{cases}$$

and, as before, using (26) with $\psi = \varphi_2$, Theorem 4 with $J_2 = (\alpha,1)$ and Lemma 2, it results

$$\int_0^T \int_\omega v^2 \frac{1}{\sigma} dxdt \ge C \int_{T_0}^{T_1} \int_{(\alpha+2\beta)/3}^1 v^2 \frac{1}{\sigma} dxdt. \tag{28}$$

By (27) and (28), one has

$$\int_0^T \int_\omega v^2 \frac{1}{\sigma} dxdt \ge C \int_{T_0}^{T_1} \int_{\omega^c} v^2 \frac{1}{\sigma} dxdt,$$

where $\omega^c := (0,1) \setminus \omega$. Thus

$$\begin{aligned} 2\int_0^T \int_\omega v^2 \frac{1}{\sigma} dxdt \quad &\ge \int_0^T \int_\omega v^2 \frac{1}{\sigma} dxdt + \int_{T_0}^{T_1} \int_\omega v^2 \frac{1}{\sigma} dxdt \\ &\ge C \int_{T_0}^{T_1} \int_{\omega^c} v^2 \frac{1}{\sigma} dxdt + \int_{T_0}^{T_1} \int_\omega v^2 \frac{1}{\sigma} dxdt \\ &\ge (1 \wedge C) \int_{T_0}^{T_1} \int_0^1 v^2 \frac{1}{\sigma} dxdt, \end{aligned}$$

for some positive constant C. □

Lemma 7. *Assume that Hypothesis 3.1 is satisfied. Let T_0, T_1 be such that $0 < T_0 < T_1 < T$. Then every solution $v \in \mathcal{W}$ of (23) satisfies,*

$$\int_0^1 v^2(0,x)\frac{1}{\sigma}dx \le \frac{1}{T_1 - T_0}\int_{T_0}^{T_1}\int_0^1 v^2\frac{1}{\sigma}dxdt.$$

Proof. Multiplying (23) by $\dfrac{v}{\sigma}$ and integrating over $(0,1)$, one has

$$\begin{aligned}0 &= \int_0^1 v_t(t,x)v(t,x)\frac{1}{\sigma}dx + \int_0^1 (\eta v_x(t,x))_x v(t,x)dx\\ &= \frac{1}{2}\frac{d}{dt}\int_0^1 v^2(t,x)\frac{1}{\sigma}dx - \int_0^1 \eta v_x^2(t,x)dx.\end{aligned}$$

Then

$$\frac{d}{dt}\int_0^1 v^2(t,x)\frac{1}{\sigma}dx = 2\int_0^1 \eta v_x^2(t,x)dx \ge 0 \quad \forall\, t \in [0,T],$$

that is the function $t \mapsto \displaystyle\int_0^1 v^2(t,x)\frac{1}{\sigma}dx$ is nondecreasing for all $t \in [0,T]$. □

Proof of Proposition 3. As a direct consequence of the Lemmas 6 and 7 we have that the observability inequality (24) hold for all $v \in \mathcal{W}$. Now, let $v_T \in L^2_{\frac{1}{\sigma}}(0,1)$ and v the solution of (23) associated to v_T. Since $D(A^2)$ is densely contained in $L^2_{\frac{1}{\sigma}}(0,1)$, there exists a sequence $(v_T^n)_n \subset D(A^2)$ which converges to v_T in $L^2_{\frac{1}{\sigma}}(0,1)$. Consider now the solution v_n associated to v_T^n. Obviously, $(v_n)_n$ converges to v in $L^\infty(0,T;L^2_{\frac{1}{\sigma}}(0,1)) \cap L^2(0,T;H^1_{\frac{1}{\sigma}}(0,1))$ (see, e.g.,[2]) and

$$\int_0^1 v_n^2(0,x)\frac{1}{\sigma}dx \le C_T\int_0^T\int_\omega v_n^2\frac{1}{\sigma}dxdt.$$

Clearly,

$$\lim_{n\to+\infty}\int_0^T\int_\omega v_n^2\frac{1}{\sigma}dxdt = \int_0^T\int_\omega v^2\frac{1}{\sigma}dxdt$$

and

$$\lim_{n\to+\infty}\int_0^1 v_n^2(0,x)\frac{1}{\sigma}dx = \int_0^1 v^2(0,x)\frac{1}{\sigma}dx.$$

□

Assuming that Hypothesis 3.1 is satisfied, using the observability property proved in Proposition 3 and a standard technique, one can prove a null controllability result for the linear degenerate problem (6):

Theorem 5. *Assume that Hypotheses 3.1 is satisfied. Then, given $T > 0$ and $u_0 \in L^2_{\frac{1}{\sigma}}(0,1)$, there exists $f \in L^2(Q)$ such that the solution u of (6) satisfies*

$$u(T,x) = 0 \quad \textit{for every} \quad x \in [0,1].$$

Moreover,

$$\int_Q \chi_\omega f^2\frac{1}{\sigma}dxdt \le C\int_0^1 u_0^2\frac{1}{\sigma}dx,$$

for some positive constant C.

5. **A linear extension.** In this section, we will extend the results established in the previous sections to the following degenerate parabolic problem

$$\begin{cases} u_t - a(x)u_{xx} - b(x)u_x + c(t,x)u = f(t,x)\chi_\omega(x), & (t,x) \in Q, \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{29}$$

where, as before, $Q := (0,T) \times (0,1)$, $T > 0$ is fixed, $\omega := (\alpha, \beta) \subset\subset (0,1)$ is a non-empty assigned interval, $(f, u_0) \in L^2(Q) \times L^2_{\frac{1}{\sigma}}(0,1)$, $a \in C^0[0,1]$ is such that $a(0) = a(1) = 0$, $a > 0$ on $(0,1)$ and $b \in C^0[0,1]$ is such that $b/a \in L^1(0,1)$. Furthermore we assume that the potential $c = c(t,x)$ is essentially bounded on Q.

Using a perturbation argument, one can prove that Theorem 2 still hold for (29), that is (29) is well-posed in the sense of semigroup theory:

Theorem 6. *For all $f \in L^2(Q)$ and $u_0 \in L^2_{\frac{1}{\sigma}}(0,1)$, there exists a unique weak solution $u \in \mathcal{U} := C^0\left([0,T]; L^2_{\frac{1}{\sigma}}(0,1)\right) \cap L^2\left(0,T; H^1_{\frac{1}{\sigma}}(0,1)\right)$ of (29). Moreover, one has*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2_{\frac{1}{\sigma}}(0,1)} + \int_0^T \|u\|^2_{H^1_{\frac{1}{\sigma}}(0,1)} dt \le C\left(\|u_0\|^2_{L^2_{\frac{1}{\sigma}}(0,1)} + \int_0^T \|f\|^2_{L^2_{\frac{1}{\sigma}}(\omega)} dt\right),$$

for a positive constant C.

Now, we have to prove that the observability property and the null controllability result obtained in Section 4 still hold for the adjoint problem of (29). To this purpose first we have to establish for

$$\begin{cases} v_t + a(x)v_{xx} + b(x)v_x - c(x,t)v = h(t,x), & (t,x) \in Q_i := (0,T) \times J_i, \\ v(t, \partial J_i) = 0, & t \in (0,T) \end{cases} \tag{30}$$

Carleman estimates similar to the ones proved in Theorems 3 and 4. Here, as in Section 3, $T > 0$ is fixed, $J_1 := (0, j_1)$ and $J_2 := (j_2, 1)$ are proper subintervals of $(0,1)$ and $h \in L^2\big(0,T; L^2_{\frac{1}{\sigma}}(0,1)\big)$. Then, one has the following:

Proposition 5. *Assume that the potential $c \in L^\infty(Q)$ and that Hypothesis 3.1 holds for some $\varepsilon \in (0,1)$ such that $\varepsilon < j_1$ and $1 - \varepsilon > j_2$. Then, there exist two positive constants C and s_0, such that, for all $s \ge s_0$, the following Carleman estimates hold*

$$\begin{aligned} &\int_{Q_1} \eta\left(s\theta v_x^2 + s^3\theta^3\left(\frac{x}{a}\right)^2 v^2\right) e^{2s\varphi_1} dxdt \\ \le &C\int_{Q_1} h^2 \frac{e^{2s\varphi_1}}{\sigma}\, dxdt + s\,C\int_0^T \theta(t)\left[\eta x v_x^2 e^{2s\varphi_1}\right](t, j_1)dt, \end{aligned}$$

for all solution $v \in \mathcal{V}_1$ of (30) and

$$\begin{aligned} &\int_{Q_2} \eta\left(s\theta v_x^2 + s^3\theta^3\left(\frac{x-1}{a}\right)^2 v^2\right) e^{2s\varphi_2} dxdt \\ \le &C\int_{Q_2} h^2 \frac{e^{2s\varphi_2}}{\sigma}\, dxdt \; + \; sC\int_0^T \theta(t)\left[\eta(1-x) v_x^2 e^{2s\varphi_2}\right](t, j_2)dt, \end{aligned}$$

for all solution $v \in \mathcal{V}_2$ of (30).

Proof. We will prove only the first estimate since the proof of the second one is analogous. Rewrite the equation of (30) as $v_t + av_{xx} + bv_x = \bar{h}$, where $\bar{h} := h + cv$. Then, as a consequence of Theorem 3, there exist two positive constants C and $s_0 > 0$, such that, for all $s \geq s_0$,

$$\begin{aligned}&\int_{Q_1} \eta \left(s\theta v_x^2 + s^3\theta^3 \left(\frac{x}{a}\right)^2 v^2 \right) e^{2s\varphi_1} dxdt \\ \leq &C \int_{Q_1} |\bar{h}|^2 \frac{e^{2s\varphi_1}}{\sigma} \, dxdt + s\, C \int_0^T \theta(t) \left[\eta x v_x^2 e^{2s\varphi_1} \right](t, j_1) dt.\end{aligned} \tag{31}$$

By the definition of $\bar{h}$ the term $\int_{Q_1} |\bar{h}|^2 \frac{e^{2s\varphi_1}}{\sigma} \, dxdt$ can be estimated in the following way

$$\int_{Q_1} |\bar{h}|^2 \frac{e^{2s\varphi_1}}{\sigma} \, dxdt \leq 2 \int_{Q_1} \left(|h|^2 + |c|^2 v^2 \right) \frac{e^{2s\varphi_1}}{\sigma} \, dxdt. \tag{32}$$

But, as a consequence of Lemma 2,

$$\begin{aligned}\int_{Q_1} |c|^2 v^2 \frac{e^{2s\varphi_1}}{\sigma} \, dxdt &\leq \|c\|_\infty^2 \int_{Q_1} (e^{s\varphi_1} v)^2 \frac{1}{\sigma} dxdt \leq C \int_{Q_1} \eta (e^{s\varphi_1} v)_x^2 dxdt \\ &\leq C \int_{Q_1} \eta \; e^{2s\varphi_1} v_x^2 dxdt + Cs^2 \int_{Q_1} \eta\theta^2 e^{2s\varphi_1} \left(\frac{x}{a}\right)^2 v^2 dxdt.\end{aligned}$$

Using this last inequality in (32), it follows

$$\begin{aligned}\int_{Q_1} |\bar{h}|^2 \frac{e^{2s\varphi_1}}{\sigma} \, dxdt \leq &2 \int_{Q_1} |h|^2 \frac{e^{2s\varphi_1}}{\sigma} \, dxdt + C \int_{Q_1} \eta e^{2s\varphi_1} v_x^2 dxdt \\ &+ Cs^2 \int_{Q_1} \eta\theta^2 e^{2s\varphi_1} \left(\frac{x}{a}\right)^2 v^2 dxdt.\end{aligned} \tag{33}$$

Substituting in (31), one can conclude

$$\begin{aligned}&\int_{Q_1} \eta \left(s\theta v_x^2 + s^3\theta^3 \left(\frac{x}{a}\right)^2 v^2 \right) e^{2s\varphi_1} dxdt \leq C \Big(\int_{Q_1} v |h|^2 \frac{e^{2s\varphi_1}}{\sigma} \, dxdt + \int_{Q_1} \eta e^{2s\varphi_1} v_x^2 dxdt \\ &+ s^2 \int_{Q_1} \eta\theta^2 e^{2s\varphi_1} \left(\frac{x}{a}\right)^2 v^2 dxdt + s \int_0^T \theta(t) \left[\eta x v_x^2 e^{2s\varphi_1} \right](t, j_1) dt \Big).\end{aligned}$$

Hence, for all $s \geq s_0$, where s_0 is assumed sufficiently large, the first estimate of Proposition 5 is proved. □

As a consequence of the previous Carleman estimates, one can deduce an observability inequality for the adjoint problem

$$\begin{cases} v_t + a(x)v_{xx} + b(x)v_x - c(t,x)v = 0, & (t,x) \in Q, \\ v(t,0) = v(t,1) = 0, & t \in (0,T), \\ v(T) = v_T \in L^2_{\frac{1}{\sigma}}(0,1) \end{cases} \tag{34}$$

of (29). Without loss of generality we can assume that $c \geq 0$. (Otherwise one can reduce the problem to this case introducing $\tilde{u} := e^{-\lambda t} u$ for a suitable $\lambda > 0$.) Moreover, we observe that in a way analogous to the proof of Proposition 4, it is possible to prove that the Caccioppoli's inequality (25) is satisfies for all solution of (34).

Proposition 6. *Assume that the potential $c \in L^\infty(Q)$ and that Hypothesis 3.1 is satisfied. Then, there exists a positive constant C_T such that every solution $v \in \mathcal{U}$ of (34) satisfies*

$$\int_0^1 v^2(0,x)\frac{1}{\sigma}dx \le C_T\int_0^T\int_\omega v^2\frac{1}{\sigma}dxdt. \tag{35}$$

Proof. As in the proof of Lemma 7 and using the fact that $c \ge 0$, it results that every $v \in \mathcal{W}' := \{v \text{ solution of (34)} : v_T \in D(A^2)\}$ satisfies

$$\int_0^1 v^2(0,x)\frac{1}{\sigma}dx \le \frac{1}{T_1 - T_0}\int_{T_0}^{T_1}\int_0^1 v^2\frac{1}{\sigma}dxdt,$$

for all $0 < T_0 < T_1 < T$. Moreover, proceeding as in Lemma 6 and applying Proposition 5, one has

$$\int_{T_0}^{T_1}\int_0^1 v^2\frac{1}{\sigma}dxdt \le C\int_0^T\int_\omega v^2\frac{1}{\sigma}dxdt.$$

for some positive constant C and for all $v \in \mathcal{W}'$.

Now, proceeding as in the proof of Proposition 3, one obtains the conclusion. □

Finally, using Proposition 6 and a standard technique, one can extend the null controllability result established in Theorem 5:

Theorem 7. *Assume that the potential $c \in L^\infty(Q)$ and that Hypothesis 3.1 is satisfied. Then, given $T > 0$ and $u_0 \in L^2_{\frac{1}{\sigma}}(0,1)$, there exists $f \in L^2(Q)$ such that the solution u in $\mathcal{U}$ of (29) satisfies*

$$u(T,x) = 0 \quad \text{for every} \quad x \in [0,1].$$

Moreover,

$$\int_Q \chi_\omega f^2\frac{1}{\sigma}dxdt \le C\int_0^1 u_0^2\frac{1}{\sigma}dx,$$

for some positive constant C.

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