NETWORKS AND HETEROGENEOUS MEDIA
©American Institute of Mathematical Sciences
Volume **2**, Number **4**, December **2007**

Website: http://aimSciences.org

pp. **597–626**

STABLE SYNCHRONIZATION OF RIGID BODY NETWORKS

Sujit Nair

Control and Dynamical Systems
107-81, California Institute of Technology
Pasadena, CA 91125, USA

Naomi Ehrich Leonard

Mechanical and Aerospace Engineering
Princeton University
Princeton, NJ 08544, USA

Abstract. We address stable synchronization of a network of rotating and translating rigid bodies in three-dimensional space. Motivated by applications that require coordinated spinning spacecraft or diving underwater vehicles, we prove control laws that stably couple and coordinate the dynamics of multiple rigid bodies. We design decentralized, energy shaping control laws for each individual rigid body that depend on the relative orientation and relative position of its neighbors. Energy methods are used to prove stability of the coordinated multi-body dynamical system. To prove exponential stability, we break symmetry and consider a controlled dissipation term that requires each individual to measure its own velocity. The control laws are illustrated in simulation for a network of spinning rigid bodies.

1. **Introduction.** In this paper, we derive a decentralized control methodology to coordinate and stabilize a network of rigid bodies moving in three-dimensional space. Coordination here refers to synchronization of the orientations and positions of the rigid body network. A motivating application is the use of a coordinated cluster of satellites carrying telescopes for astronomical interferometry. The goal is to synchronize the motion of the satellites so that using the telescopes together enhances resolution. Our results provide provably stable control laws that align the orientations and synchronize the angular velocities of a network of n spinning rigid bodies.

We are likewise motivated by the application of a fleet of sensor-equipped underwater vehicles that move together in an organized pattern to identify and track features in the ocean. An important goal is to synchronize the motion of the vehicles so that resolution of the sensing array is optimized to minimize estimation error in the sampled environment. In the case that the vehicles are used as an acoustic array, synchronization of vehicle orientation also becomes critical. Our results provide

2000 *Mathematics Subject Classification.* Primary: 70Q05, 70E55, 70E15; Secondary: 37J15, 93D15.

Key words and phrases. Stable synchronization, coordinated control, rigid body dynamics, Lagrangian reduction.

Research supported in part by Office of Naval Research grants N0014-02-1-0826 and N00014–04–1–0534.

provably stable control laws that align the orientations and positions and synchronize the angular and translational velocities of a network of n rigid bodies moving in three dimensions. The underwater vehicle dynamics used to design the control laws are idealized; they assume potential flow and ignore the influence of ocean currents. Application of the results to ocean-going vehicles requires additional attention to the dynamic effect of the real ocean environment. One possibility, for instance, is to use the idealized approach here as a motion planner, complemented with a robust low-level controller designed to carry out the plan in the presence of currents. A relevant and successful prior field demonstration of control of vehicles in the ocean is described in [9].

In response to the growing interest in applications of robotic networks, there has been much research activity on coordinated control of groups. The majority of research has focused on networks of individuals modeled as particles, see, for example, [21, 14, 31, 8, 16, 33] and references therein, or as nonholonomic systems as in [7]. These simplified models are well justified as the focus is the role of interconnection on collective motion independent of the dynamics of the individual agents. Graph theoretical tools are used to study limited and possibly time-varying communication topologies. In [1] the authors consider a group of mobile robots modelled by point masses and a parameter dependent control law. Depending upon the parameter, the group can undergo a rigid or an elastic transformation. The parameter is chosen to shape the kinetic energy metric of the system and is close in spirit to the geometric approach of this paper.

Simple particle models fall short, however, when the coordination problem requires attitude synchronization. A number of researchers have investigated attitude coordination of multiple satellites with rigid body dynamics, e.g., [25, 19, 12, 2, 13, 36, 18] and references therein. These works typically make use of an externally provided trajectory or a leader-follower approach. Additionally, many of the works make use of the non-unique quaternion representation for attitude. In some of the earlier works the authors compute synchronization error by comparing quaternions and angular velocities with respect to different reference frames. This problem is rectified in [36] where comparisons are made with respect to a common reference frame.

In a number of network control problems, it may be undesirable to decouple synchronization from stabilization of individual dynamics. The coordination of multiple satellites is one example where particle models are insufficient since the required coordination is defined in terms of rigid body states. Other problems arise when the individual systems are underactuated and/or have unstable dynamics. For networks of autonomous systems such as unmanned helicopters or underwater vehicles, stability of individual dynamics can be important and challenging, and it may not always be possible (or desirable) to decouple the stabilization problem of individual dynamics from the coordination problem.

In previous work we have proven control laws to address stable synchronization of a class of underactuated mechanical systems with otherwise unstable dynamics [28]. In this case the integral treatment of coordination with individual stabilization is critical; an energy shaping approach leads to a non-trivial definition of coupling variable (i.e., not the naive choice) in order to both stabilize the otherwise unstable dynamics and stably coordinate the network of systems. This point is well illustrated in the case of synchronization of a network of moving carts, each balancing an inverted pendulum. The input directly controls the motion of the cart but not the angle of the pendulum. A decoupled approach might suggest to use a control law so that each cart stabilizes its upright pendulum and then in parallel a coupling control term that is a function of relative cart positions to synchronize cart motion. However, stabilization and synchronization of the motion are proven with a coupling term that depends on the relative measurement of a function of both cart position and pendulum angle.

In the present paper we address the synchronization and stabilization problems for rigid body dynamics. Unlike earlier efforts cited above, we work directly on the configuration manifold avoiding non-unique representations of orientation such as quaternions. Building on previous efforts, e.g., [20, 34, 10, 29], our approach makes use of symmetry, reduction, energy shaping and energy methods. We consider first the case in which each individual in the network has configuration space $SO(3)$ as is the case for a free rigid spacecraft. For a network of n such rigid bodies, the total configuration space is $SO(3)^n = SO(3) \times \ldots \times SO(3)$ (n times). In the second case, each individual has configuration space $SE(3)$ as is the case for a rigid underwater vehicle. For a network of n such rigid bodies, the total configuration space is $SE(3)^n = SE(3) \times \ldots \times SE(3)$ (n times). For both cases, we assume that the uncontrolled dynamics for each individual rigid body are Lagrangian where the Lagrangian is quadratic in velocity. A potential is introduced as a function of relative orientations to couple the rigid bodies. In the $SE(3)$ case, a second potential term is defined as a function of relative positions. These potentials break some but not all of the symmetry in the multi-body system. To compute the control laws that derive from these potentials, we identify the remaining symmetry and determine the corresponding reduced equations for the coupled system dynamics.

In [10], the authors also consider a network of n rigid bodies with configuration space $SO(3)^n$ or $SE(3)^n$ and coupled with a control law that derives from a symmetry-breaking potential dependent on relative orientation and position. Determination of the corresponding symmetry, reduced space and Hamiltonian (Lie-Poisson) structure is the main focus of the paper. Due to the inability to find all Casimirs (invariants of the the Lie-Poisson dynamics, independent of the Hamiltonian), stability of relative equilibria are proven using the Energy-Casimir method only in a limited number of cases. In the present paper, we compute the reduction using the Lagrangian framework and we are able to prove stability more generally using the Energy-Momentum method and Routh reduction [22].

In order to prove exponential stability we use an external reference direction of motion which further breaks symmetry. We include an additional control term that requires that each individual measure its own velocity. This is a limitation of the approach. In [32] the authors prove asymptotic stability of synchronization in the $SO(3)^n$ case where the control term does not require absolute velocities but instead uses relative angular velocities.

Synchronization of spinning rigid bodies is most challenging in the case that the bodies spin about their unstable (middle) axis. Likewise, synchronization of translating (underwater) vehicles moving along their unstable (e.g., middle or long) axis requires stabilizing control. We address first the cases in which the desired motion of each individual rigid body is already stable (e.g., spin around the short axis) and we introduce potential shaping control only to provide the desired synchronization. Finally, we show how to use control that shapes the kinetic energy to stabilize motion that is unstable for each individual rigid body and how to combine this with the potential shaping control term to provide stable multi-body synchronization.

Throughout this paper, we assume a fixed, connected, undirected (bi-directional) graph for the communication between the rigid bodies. We present our method for a particular representative from this class of communication graphs; however, the results are easily extended to the whole class. Consider a graph with n nodes, each corresponding to a rigid body and an edge between node i and node j if there is a communication link between body i and body j. We consider the case in which the communication is represented by a graph that is a chain, i.e., there is a (bi-directional) edge between nodes j and $j-1$ and between nodes j and $j+1$ for $j = 2, \ldots, n-1$. In [32] the authors have made progress in extending the $SO(3)^n$ results to more general communication topologies using a consensus-based approach.

In this paper, we do not require the mass and inertia matrices of different rigid bodies to be equal. This is in contrast with [28] for example, where the authors had to assume equal mass matrices for asymptotic stability analysis purposes. We do not consider the problem of collision avoidance or fuel cost optimization here but these are of future interest.

The organization of the paper is as follows. In §2, we define the configuration manifolds and Lagrangians and set the notation to be used in the rest of the paper. In §3, we define the potentials used to couple the rigid body network. The corresponding coupling control laws are derived, following [5], by calculating the reduced equations of motion for the network. In §4, we prove Lyapunov stability for the synchronized relative equilibria in the case that the motion for each individual system is stable. We introduce the symmetry-breaking potential that depends on an external reference direction of motion since we require this for proving exponential stability. Using it already in this section reduces the symmetry to an Abelian group, thereby allowing us to apply the Routh stability criteria. After adding a control term to emulate dissipation, exponential stability of the relative equilibria is proved in §5. Simulations illustrate the results. In §6 we derive a control that shapes kinetic energy and can be used to stabilize otherwise unstable motions of the individual rigid bodies. We then show how the kinetic shaping control is used with the potential shaping control to provide stable synchronization even in the case that the individuals have unstable dynamics. We make final remarks and discuss future directions in §7.

2. **Rigid body models.**

In this section we define the configuration spaces, state spaces, Lagrangians, uncontrolled equations of motion and synchronized motions for the rigid bodies studied in the paper. We also set the notation used throughout the rest of the paper.

2.1. $SO(3)^n$ **network.**

For a free rigid body in space, the configuration space is the set of all possible orientations of the body. This set is the Lie group $SO(3)$ which consists of all the rotation matrices R given by

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid \det(R) = 1, R^T R = \mathbb{I}_{3\times 3}\}$$

where $\mathbb{I}_{3\times 3}$ is the 3×3 identity matrix. The state space for the body is $TSO(3)$, where a particular element $(R, \boldsymbol{\omega}) \in TSO(3)$ denotes the orientation of the rigid body in inertial space and the angular velocity of the rigid body in inertial space. The angular velocity of the body in the body frame is denoted by $\boldsymbol{\Omega} \in \mathbb{R}^3$. In the language of Lie groups, $\boldsymbol{\omega}$ is the right translate of the element $\dot{R} \in T_R SO(3)$ to the tangent space at the identity $T_I SO(3)$ denoted by $\mathfrak{so}(3)$ and $\boldsymbol{\Omega}$ is the left translate of $\dot{R}$ to $\mathfrak{so}(3)$, i.e.,

$$\dot{R} = \hat{\boldsymbol{\omega}} R, \quad \dot{R} = R\hat{\boldsymbol{\Omega}}.$$

Here $\hat{}$ is a map from $\mathbb{R}^3$ to $\mathfrak{so}(3)$: given a vector $\boldsymbol{a} \in \mathbb{R}^3$, $\hat{\boldsymbol{a}} \in \mathfrak{so}(3)$ denotes a matrix such that $\hat{\boldsymbol{a}}\boldsymbol{x} = \boldsymbol{a} \times \boldsymbol{x}$ for any vector $\boldsymbol{x} \in \mathbb{R}^3$.

We are interested in a network of n such rigid bodies. Let $R_i \in SO(3)$ describe the orientation of the i^{th} rigid body, $i = 1, \ldots, n$. The n rigid body system has phase space $T(SO(3)^n)$ with coordinates

$$(R_1, \ldots, R_n, \dot{R}_1, \ldots, \dot{R}_n).$$

The angular velocity of the i^{th} body in the inertial frame is denoted by $\boldsymbol{\omega}_i \in \mathbb{R}^3$ and in the i^{th} body frame by $\boldsymbol{\Omega}_i \in \mathbb{R}^3$. Let $I_i \in \mathbb{R}^{3\times 3}$ be the moment of inertia matrix for the i^{th} body and let the (k, l) entry of this matrix be $I_{i,kl}$. We assume the body fixed frame is chosen so that I_i is a diagonal matrix and we assume that it has diagonal entries where $I_{i,11} > I_{i,22} > I_{i,33}$, i.e., $I_{i,11}$ corresponds to the short axis and $I_{i,33}$ corresponds to the long axis. Note that we do not assume that the rigid bodies have the same moment of inertia matrices.

The Lagrangian for each rigid body (before coupling) is defined by its kinetic energy $(1/2)\boldsymbol{\Omega}_i^T I_i \boldsymbol{\Omega}_i$. The equations of motion for its dynamics are

$$I_i \dot{\boldsymbol{\Omega}}_i = (I_i \boldsymbol{\Omega}_i) \times \boldsymbol{\Omega}_i + \boldsymbol{u}_{\tau i}, \tag{2.1}$$

for $i = 1, \ldots, n$, where $\boldsymbol{u}_{\tau i} \in \mathbb{R}^3$ is the vector of external control torques for the i^{th} rigid body.

Our goal is to design $\boldsymbol{u}_{\tau i}$, $i = 1, \ldots, n$ to couple n spinning rigid bodies using potentials designed to align (synchronize) their orientations in inertial space and drive the axis of rotation of each to a common prescribed direction. Without loss of generality we let the prescribed common direction be $\boldsymbol{e}_1 = (1, 0, 0)^T$ in inertial space and the prescribed angular rate equal to 1. This is an arbitrary choice; any other desired direction and angular rate can be stabilized using the same methods derived in this paper. Until §6 we assume that each rigid body is to spin about its short axis. The desired synchronized network motion is given by the relative equilibrium:

$$\begin{aligned} R_1 &= \ldots = R_n = R_e, \\ R_e \boldsymbol{e}_1 &= \boldsymbol{e}_1 \\ \boldsymbol{\Omega}_i &= \boldsymbol{\omega}_i = \boldsymbol{e}_1. \end{aligned} \tag{2.2}$$

In §6 we address the case in which each rigid body is to spin about its unstable (middle) axis. As mentioned in the introduction, we do not consider collision avoidance in this paper. These issues are important even in the case of $SO(3)$ when physical displacements in inertial space are not a concern. For the $SO(3)$ case, collision avoidance can mean avoiding certain orientations in space. For example, if one has sensitive imaging sensors for interferometry purposes, orientations which point the sensor to a bright source like the sun are to be avoided.

We note that we can have a system where each body has the same rotation matrix, but different "physical orientations". This is because, for each body, its rotation matrix is defined with respect to a reference orientation in space given by the identity matrix. For different choices of reference orientation, the bodies will, after synchronization, have the same values of rotation matrices but different physical orientation. This is related to the freedom in choosing the matrix K in [10]. In [10], the bodies have the same reference configuration but different rotation matrices. The synchronized state for the network corresponds to $R_{i+1}^T K R_i = \mathbb{I}_{3\times 3}$, i.e., the bodies have different physical orientation if $K \neq \mathbb{I}$.

2.2. $SE(3)^n$ **network.** We now consider the case in which the rigid bodies can rotate and translate in three dimensions. The configuration space for a single rigid body is the Lie group $SE(3)$, the space of rigid body motions. An element in $SE(3)$ is denoted by $(R, \boldsymbol{b})$ with $R \in SO(3)$, $\boldsymbol{b} \in \mathbb{R}^3$ and group multiplication

$$(R, \boldsymbol{b}) \cdot (R_1, \boldsymbol{b}_1) = (RR_1, R\boldsymbol{b}_1 + \boldsymbol{b}).$$

Here, R represents the orientation of the body and $\boldsymbol{b}$ the vector to the body from the origin of an inertial frame. For this action, the inverse of $(R, \boldsymbol{b})$ is $(R^T, -R^T\boldsymbol{b})$. It is useful to represent elements in $SE(3)$ in matrix form so that the group action is represented by matrix multiplication as follows:

$$\begin{bmatrix} R & \boldsymbol{b} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R' & \boldsymbol{b}' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR' & R\boldsymbol{b}' + \boldsymbol{b} \\ 0 & 1 \end{bmatrix}.$$

The angular and linear velocities of the body in the inertial space are obtained by computing the right translate to the tangent space at the identity $T_I SE(3)$, denoted by $\mathfrak{se}(3)$, of an element belonging to the tangent space of $SE(3)$ at a particular point $(R, \boldsymbol{b})$ as follows:

$$\begin{bmatrix} \dot{R} & \dot{\boldsymbol{b}} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} R^T & -R^T\boldsymbol{b} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \dot{\boldsymbol{b}} - \hat{\boldsymbol{\omega}}\boldsymbol{b} \\ 0 & 0 \end{bmatrix}.$$

Here, $\boldsymbol{\omega} \in \mathbb{R}^3$ is the angular velocity of the body in inertial space and $\dot{\boldsymbol{b}}$ is its linear velocity in inertial space. The velocity components in the body frame are similarly obtained using the left translate. They are denoted by $\boldsymbol{\Omega} \in \mathbb{R}^3$ and $\boldsymbol{v} \in \mathbb{R}^3$ and calculated as follows:

$$\begin{bmatrix} R^T & -R^T\boldsymbol{b} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{R} & \dot{\boldsymbol{b}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{\boldsymbol{b}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\Omega}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix}.$$

We are interested in a network of n such rigid bodies. Let $(R_i, \boldsymbol{b}_i) \in SE(3)$ describe the orientation and position of the i^{th} rigid body, $i = 1, \ldots, n$. The n rigid body system has phase space $T(SE(3)^n)$ with coordinates

$$(R_1, \ldots, R_n, \dot{R}_1, \ldots, \dot{R}_n, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n, \dot{\boldsymbol{b}}_1, \ldots, \dot{\boldsymbol{b}}_n).$$

The angular velocity of the i^{th} body in the inertial frame is denoted by $\boldsymbol{\omega}_i \in \mathbb{R}^3$ and in the i^{th} body frame is denoted by $\boldsymbol{\Omega}_i \in \mathbb{R}^3$. The linear velocity of the i^{th} body in the inertial frame is $\dot{\boldsymbol{b}}_i \in \mathbb{R}^3$ and in the i^{th} body frame is denoted by $\boldsymbol{v}_i \in \mathbb{R}^3$.

We let the rigid bodies be immersed in a fluid defined by potential flow. Then to each rigid body there is a moment of inertia matrix $I_i \in \mathbb{R}^{3\times 3}$ and a mass matrix $M_i \in \mathbb{R}^{3\times 3}$ that includes rigid body and fluid terms [11]. Let the (k, l) entry of these matrices be $I_{i,kl}$ and $M_{i,kl}$, respectively. We assume the mass is distributed uniformly and the body frame chosen so that both I_i and M_i are diagonal for all $i = 1, \ldots, n$. We further assume that $I_{i,11} > I_{i,22} > I_{i,33}$ and $M_{i,11} > M_{i,22} > M_{i,33}$. Note that we do not assume that the rigid bodies have the same moment of inertia matrices or mass matrices.

The Lagrangian for each rigid body (before coupling) is defined by its kinetic energy $(1/2)\boldsymbol{\Omega}_i^T I_i \boldsymbol{\Omega}_i + (1/2)\boldsymbol{v}_i^T M_i \boldsymbol{v}$. The equations of motion for its dynamics are

$$\begin{aligned} I_i\dot{\boldsymbol{\Omega}}_i &= (I_i\boldsymbol{\Omega}_i) \times \boldsymbol{\Omega}_i + (M_i\boldsymbol{v}_i) \times \boldsymbol{v}_i + \boldsymbol{u}_{\tau i} \\ M_i\dot{\boldsymbol{v}}_i &= (M_i\boldsymbol{v}_i) \times \boldsymbol{\Omega}_i + \boldsymbol{u}_{fi} \end{aligned} \tag{2.3}$$

for $i = 1, \dots, n$, where $\boldsymbol{u}_{\tau i} \in \mathbb{R}^3$ is the vector of external control torques and $\boldsymbol{u}_{fi} \in \mathbb{R}^3$ the vector of the external control forces for the i^{th} rigid body.

Our goal is to design $\boldsymbol{u}_{\tau i}$ and $\boldsymbol{u}_{fi}$, $i = 1, \dots, n$, to couple n moving rigid bodies using potentials designed to align (synchronize) their orientations and relative positions in inertial space and drive the axis of translation of each to a common prescribed direction. Without loss of generality we let the prescribed common direction be $\boldsymbol{e}_1 = (1, 0, 0)^T$ in inertial space and the prescribed angular rate and linear speed equal to 1. This is an arbitrary choice; any other desired direction and angular rate and linear speed can be stabilized using the same methods derived in this paper. Until §6 we assume that each rigid body is to translate and rotate along its short axis. The desired synchronized network motion is given by the relative equilibrium:

$$\begin{aligned} R_1 &= \dots = R_n = R_e, \\ \boldsymbol{b}_1 &= \boldsymbol{b}_2 - \boldsymbol{d}_{12} \dots = \boldsymbol{b}_n - \boldsymbol{d}_{1n} = \boldsymbol{b}_e, \\ R_e \boldsymbol{e}_1 &= \boldsymbol{e}_1 \\ \boldsymbol{b}_e &\parallel \boldsymbol{e}_1 \\ \boldsymbol{\Omega}_i &= \boldsymbol{\omega}_i = \boldsymbol{e}_1, \\ \boldsymbol{v}_i &= \boldsymbol{e}_1. \end{aligned} \tag{2.4}$$

In (2.4), $\boldsymbol{d}_{ij} \in \mathbb{R}^3$ are fixed vectors determining the constant desired interpositioning between the bodies i and j. Since the vectors $\boldsymbol{d}_{ij}$ are constant, we can choose new coordinates $\boldsymbol{b}_1, \tilde{\boldsymbol{b}}_2 = \boldsymbol{b}_2 - \boldsymbol{d}_{12}, \dots, \tilde{\boldsymbol{b}}_n = \boldsymbol{b}_n - \boldsymbol{d}_{1n}$ without changing the form of the Lagrangian. Further, the dissipation controller that we design in (5.14) depends only on the time derivative of $\boldsymbol{b}_i$ which is equivalent to the time derivative of $\tilde{\boldsymbol{b}}_i$. Hence, without loss of generality, we can set the vectors $\boldsymbol{d}_{ij}$ in (2.4) to be zero. However, the terms $\boldsymbol{d}_{ij}$ will be important in incorporating collision avoidance schemes into the present framework. In §6 we address the case in which each rigid body is to translate and rotate along its middle (unstable) axis.

3. **Reduction for rigid body networks.** We define the potential that couples the rigid bodies in the network as a function of relative orientations and relative positions that are available given the communication graph. As mentioned in the introduction, we use a chain as the communication graph, i.e., the connected, undirected graph on n nodes depicted as follows:

$$1 \longleftrightarrow 2 \longleftrightarrow \cdots\cdots \longleftrightarrow n-1 \longleftrightarrow n$$

The coupling potentials $\tilde{V}_1$ and $\tilde{V}_2$ designed below in (3.1) and (3.21) for the $SO(3)^n$ and $SE(3)^n$ networks, respectively, are consistent with the above communication graph.

We define the controlled system dynamics to be those given by the Lagrangian that is the sum of the original kinetic energy and the coupling potential. We identify the symmetry in the system and then derive the reduced equations of motion for the network using the method of Lagrangian reduction [5]. This procedure yields the coupling control inputs; these correspond to the terms in the equations of motion associated with the coupling potentials.

The central idea in Lagrangian reduction, following [5], is to start with the variational formulation of the network mechanics and then split the variations into a horizontal and a vertical part. The vertical directions are the ones which preserve the symmetry structure and the horizontal directions are orthogonal (with respect to the kinetic energy metric) to the vertical directions. These variations gives rise to horizontal and vertical equations of motion, respectively. In [10], the reduction for the rigid body network is carried out using semi-direct product reduction theory.

Stability of the synchronized rigid body networks using the control laws derived in this section is proved in §4 and §5.

3.1. **Reduction for $SO(3)^n$ network.** For the rigid body network on $SO(3)^n$, we define the coupling potential to be

$$\tilde{V}_1 = \sigma_1 \sum_{i=1}^{n-1} \mathrm{tr}(R_{i+1}^T R_i) \tag{3.1}$$

where $\sigma_1 \in \mathbb{R}$. Note that $\tilde{V}_1$ depends only on relative orientations that can be measured given the fixed communication topology assumed above. When $\sigma_1 < 0$, the global minimum of $\tilde{V}_1$ is $R_1 = R_2 = \cdots = R_n$ as desired. The potential (3.1) resembles the potential used in [4] and [30]. In these works, the authors use the potential for asymptotic tracking of prescribed attitude. However, the authors of [4] and [30] choose to cancel the natural dynamics of the system; whereas we choose to preserve the Lagrangian structure of the system. The potential $\tilde{V}_1$ is also used in [24] to study the topological structure of $SO(3)$.

We define the controlled system by the Lagrangian dynamics with Lagrangian L equal to the sum of the kinetic energies of each individual system minus the coupling potential:

$$L = \frac{1}{2}\sum_{i=1}^{n}\left(\boldsymbol{\Omega}_i^T I_i \boldsymbol{\Omega}_i\right) - \sigma_1 \sum_{i=1}^{n-1} \mathrm{tr}(R_{i+1}^T R_i). \tag{3.2}$$

L has $SO(3)$ as its symmetry group with the symmetry action given by

$$R\cdot(R_1,\ldots,R_n,R_1\widehat{\boldsymbol{\Omega}}_1,\ldots,R_n\widehat{\boldsymbol{\Omega}}_n) = (RR_1,\ldots,RR_n,RR_1\widehat{\boldsymbol{\Omega}}_1,\ldots,RR_n\widehat{\boldsymbol{\Omega}}_n).$$

For $i = 1,\ldots,n-1$, let

$$X_i = R_{i+1}^T R_i,$$

i.e., X_i is the relative orientation of individuals i and $i+1$. Figure 3.1 illustrates the relative orientations in a system of three rigid bodies. Since

$$\dot{X}_i = X_i\widehat{\boldsymbol{\Omega}}_i - \widehat{\boldsymbol{\Omega}}_{i+1}X_i \tag{3.3}$$

we have that

$$\begin{aligned} \hat{\boldsymbol{w}}_i := \dot{X}_i X_i^{-1} &= X_i\widehat{\boldsymbol{\Omega}}_i X_i^{-1} - \widehat{\boldsymbol{\Omega}}_{i+1} \\ &= \widehat{X_i\boldsymbol{\Omega}_i} - \widehat{\boldsymbol{\Omega}}_{i+1}. \end{aligned} \tag{3.4}$$

Then $\boldsymbol{w}_i = X_i\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_{i+1}$ is the difference between the angular velocities of the i^{th} and $(i+1)^{\text{th}}$ bodies represented in the $(i+1)^{\text{th}}$ body frame.

We identify the reduced space as

$$\begin{aligned} &\left[R_1,R_2,\ldots,R_n,R_1\widehat{\boldsymbol{\Omega}}_1,R_2\widehat{\boldsymbol{\Omega}}_2,\ldots,R_n\widehat{\boldsymbol{\Omega}}_n\right]_{SO(3)} \\ = \;& (X_1,X_2,\ldots,X_{n-1},\dot{X}_1X_1^{-1},\dot{X}_2X_2^{-1},\ldots,\dot{X}_{n-1}X_{n-1}^{-1},\boldsymbol{\Omega}_1) \\ = \;& (X_1,X_2,\ldots,X_{n-1},\boldsymbol{w}_1,\boldsymbol{w}_2,\ldots,\boldsymbol{w}_{n-1},\boldsymbol{\Omega}_1). \end{aligned} \tag{3.5}$$

Here, we have used the notion of a principal connection on a principle bundle. In our case, the "value" of the principle connection is the angular velocity of the first rigid body in its body frame and belongs to the Lie algebra $\mathfrak{so}(3)$. Note that there is no unique choice for the principal connection. Each such choice gives rise to its own

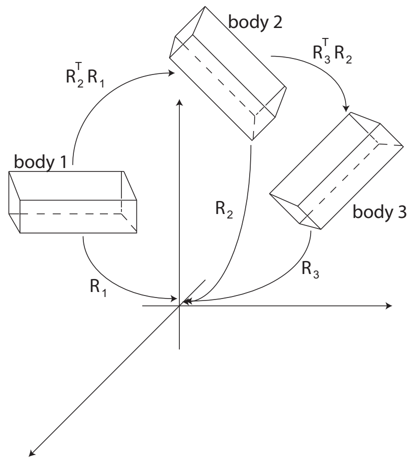

FIGURE 3.1. Illustration of relative orientation $X_1 = R_2^T R_1$ of body 1 with respect to body 2 and relative orientation $X_2 = R_3^T R_2$ of body 2 with respect to body 3.

set of horizontal and vertical equations, which after change of variables is equivalent to the corresponding equations we derive in this section.

The full (unreduced) space is $T(SO(3)^n)$ and the symmetry group is $SO(3)$. The Lagrangian reduced space is the sum of two bundles. They are the tangent bundle of $SO(3)^n/SO(3)$ and another bundle with base space $SO(3)^n/SO(3)$ and fibre given by $\mathfrak{so}(3)$. Let

$$(X, \boldsymbol{w}, \boldsymbol{\Omega}_1) := (X_1, \ldots, X_{n-1}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}, \boldsymbol{\Omega}_1).$$

We denote by l the Lagrangian (3.2) on the reduced space such that

$$l(X, \boldsymbol{w}, \boldsymbol{\Omega}_1) = \frac{1}{2} \sum_{i=1}^{n} \left(\boldsymbol{\Omega}_i^T I_i \boldsymbol{\Omega}_i \right) - V(X) \tag{3.6}$$

where $\boldsymbol{\Omega}_j$, $j = 2, \ldots, n$ can be expressed in terms of $(X, \boldsymbol{w}, \boldsymbol{\Omega}_1)$ using the recursion

$$\boldsymbol{\Omega}_j = X_{j-1} \boldsymbol{\Omega}_{j-1} - \boldsymbol{w}_{j-1}$$

and

$$V(X) = \sigma_1 \sum_{i=1}^{n-1} \mathrm{tr} X_i.$$

The equations of motion are derived using the variational formulation of mechanics. An arbitrary variation in the full configuration space can be split into a vertical part and a horizontal part once a connection is chosen. Here the variations are split using the identification given by (3.5). The vertical variations correspond to variations in the symmetry group direction, i.e., to variations in $\boldsymbol{\Omega}_1$. These variations preserve the shape of the system. The horizontal variations correspond to variations in the reduced space, which in our case is the $(X, \boldsymbol{w})$ space. The horizontal variations do not preserve the shape but instead preserve the total momentum of the system [5].

The variation $\delta \boldsymbol{\Omega}_1$ is computed as follows. Let t_0, t_1 be the initial and final time, respectively, of the paths in the variations. Let $\widehat{\boldsymbol{\eta}} = R_1^T \delta R_1$, where δR_1 is the variation of a path R_1 in $SO(3)$ with fixed end points, i.e., $\delta R_1(t_0) = \delta R_1(t_1) = 0$. Since $\widehat{\boldsymbol{\Omega}}_1 = R_1^T \dot{R}_1$, we have

$$\begin{aligned}
\delta\widehat{\boldsymbol{\Omega}}_1 &= R_1^T \delta\dot{R}_1 - R_1^T \delta R_1 R_1^T \dot{R}_1 \\
&= R_1^T \delta\dot{R}_1 - R_1^T \delta R_1 R_1^T \dot{R}_1 + R_1^T \dot{R}_1 R_1^T \delta R_1 - R_1^T \dot{R}_1 R_1^T \delta R_1 \\
&= \widehat{\dot{\boldsymbol{\eta}}} + \left[\widehat{\boldsymbol{\Omega}}_1, \widehat{\boldsymbol{\eta}}\right].
\end{aligned}$$

Therefore, $\delta\boldsymbol{\Omega}_1 = \dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}_1 \times \boldsymbol{\eta}$. To calculate δX_i, define $\widehat{\boldsymbol{\lambda}}_i = -R_{i+1}^T \delta R_i + X_i R_i^T \delta R_i X_i^{-1}$. Now,

$$\begin{aligned}
\delta X_i &= \delta\left(R_{i+1}^T R_i\right) \\
&= -R_{i+1}^T \delta R_{i+1} R_{i+1}^T R_i + R_{i+1}^T \delta R_i \\
&= -R_{i+1}^T \delta R_{i+1} X_i + X_i R_i^T \delta R_i \\
&= \widehat{\boldsymbol{\lambda}_i} X_i.
\end{aligned}$$

We use the expression for δX_i to calculate $\delta \boldsymbol{w}_i$ as follows:

$$\begin{aligned}
\delta\widehat{\boldsymbol{w}}_i &= \delta\left(\dot{X}_i X_i^{-1}\right) \\
&= \delta\left(\dot{X}_i\right) X_i^{-1} - \dot{X}_i X_i^{-1} \delta X_i X_i^{-1} \\
&= \widehat{\dot{\boldsymbol{\lambda}}}_i + \delta X_i X_i^{-1} \widehat{\boldsymbol{w}}_i - \widehat{\boldsymbol{w}}_i \delta X_i X_i^{-1} \\
&= \widehat{\dot{\boldsymbol{\lambda}}}_i + \widehat{\boldsymbol{\lambda}}_i \widehat{\boldsymbol{w}}_i - \widehat{\boldsymbol{w}}_i \widehat{\boldsymbol{\lambda}}_i.
\end{aligned}$$

The splitting of variations into a vertical variation δ_V and horizontal variation δ_H is then given as

$$\delta_V(X, \boldsymbol{w}, \boldsymbol{\Omega}_1) = ((X, \boldsymbol{w}, \boldsymbol{\Omega}_1), (0, 0, \dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}_1 \times \boldsymbol{\eta})) \tag{3.7}$$
$$\delta_H(X, \boldsymbol{w}, \boldsymbol{\Omega}_1) = \left((X, \boldsymbol{w}, \boldsymbol{\Omega}_1), (\widehat{\boldsymbol{\lambda}} X, \dot{\boldsymbol{\lambda}} - \boldsymbol{w} \times \boldsymbol{\lambda}, 0)\right) \tag{3.8}$$

where $\boldsymbol{\eta}, \boldsymbol{\lambda} \in \mathbb{R}^3$, $\boldsymbol{\eta}(t_i) = 0, \boldsymbol{\lambda}_j(t_i) = 0$ for $i = 0, 1$ and $j = 1, \ldots, n-1$ and

$$\begin{aligned}
\widehat{\boldsymbol{\lambda}} X &:= \left(\widehat{\boldsymbol{\lambda}}_1 X_1, \ldots, \widehat{\boldsymbol{\lambda}}_{n-1} X_{n-1}\right) \\
\dot{\boldsymbol{\lambda}} - \boldsymbol{w} \times \boldsymbol{\lambda} &:= \left(\dot{\boldsymbol{\lambda}}_1 - \boldsymbol{w}_1 \times \boldsymbol{\lambda}_1, \ldots, \dot{\boldsymbol{\lambda}}_{n-1} - \boldsymbol{w}_{n-1} \times \boldsymbol{\lambda}_{n-1}\right).
\end{aligned}$$

This splitting gives rise to a vertical equation and a horizontal equation of motion, respectively, by taking the vertical and horizontal variations of the Lagrangian l. The vertical equation is the momentum conservation equation and is derived as follows using the fact

$$\delta_V \boldsymbol{\Omega}_i = X_i \delta_V \boldsymbol{\Omega}_{i-1} = X_i X_{i-1} \delta_V \boldsymbol{\Omega}_{i-2} = \ldots = R_i^T R_1 \delta_V \boldsymbol{\Omega}_1.$$

We compute the vertical variation of the Lagrangian l as

$$\begin{aligned}\delta_V \int_{t_0}^{t_1} l(X, \boldsymbol{w}, \boldsymbol{\Omega}_1) dt &= \delta_V \int_{t_0}^{t_1} \left(\frac{1}{2} \sum_{i=1}^{n} \boldsymbol{\Omega}_i^T I_i \boldsymbol{\Omega}_i - V(X) \right) dt \\ &= \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} \boldsymbol{\Omega}_i^T I_i \delta_V \boldsymbol{\Omega}_i \right) dt \\ &= \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} \boldsymbol{\Omega}_i^T I_i R_i^T R_1 \right) \delta_V \boldsymbol{\Omega}_1 dt \\ &= \int_{t_0}^{t_1} \boldsymbol{a}^T (\dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}_1 \times \boldsymbol{\eta}) dt \\ &= \int_{t_0}^{t_1} (-\dot{\boldsymbol{a}}^T + (\boldsymbol{a} \times \boldsymbol{\Omega}_1)^T) \boldsymbol{\eta}(t) dt\end{aligned}$$

where we have used in the last step integration by parts and $\delta_V \boldsymbol{\Omega}_1 = \dot{\boldsymbol{\eta}} + \boldsymbol{\Omega}_1 \times \boldsymbol{\eta}$, where $\boldsymbol{\eta}(t)$ is arbitrary with $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}(t_1) = 0$. Here,

$$\boldsymbol{a} = \left(\sum_{i=1}^{n} \boldsymbol{\Omega}_i^T I_i R_i^T R_1 \right)^T = I_1 \boldsymbol{\Omega}_1 + R_1^T R_2 I_2 \boldsymbol{\Omega}_2 + \ldots + R_1^T R_n I_n \boldsymbol{\Omega}_n, \tag{3.9}$$

the total angular momentum as seen in body 1 frame. Setting $\delta_V \int_{t_0}^{t_1} l = 0$ we get

$$\dot{\boldsymbol{a}} = \boldsymbol{a} \times \boldsymbol{\Omega}_1.$$

This is the (vertical) equation for conservation of total angular momentum in inertial space as seen from the body 1 frame.

Before calculating the horizontal equation of motion corresponding to the horizontal variation of l, we first prove the following useful lemma. Let $\boldsymbol{e}_1 = (1, 0, 0)^T$, $\boldsymbol{e}_2 = (0, 1, 0)^T$, $\boldsymbol{e}_3 = (0, 0, 1)^T$.

Lemma 1. *Let $\boldsymbol{b} \in \mathbb{R}^3$ and $R \in SO(3)$ where $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ are the column vectors of R. Then,* $\mathrm{tr}(R\hat{\boldsymbol{b}}) = \boldsymbol{b} \cdot \boldsymbol{v}$, *where $\boldsymbol{v} = \boldsymbol{c}_1 \times \boldsymbol{e}_1 + \boldsymbol{c}_2 \times \boldsymbol{e}_2 + \boldsymbol{c}_3 \times \boldsymbol{e}_3$ is the eigenvector of R corresponding to eigenvalue 1 when $R \neq \mathbb{I}_{3\times 3}$.*

Proof: We have

$$\begin{aligned}\mathrm{tr}(R\hat{\boldsymbol{b}}) &= \mathrm{tr}(\hat{\boldsymbol{b}}R) \\ &= \mathrm{tr}(\hat{\boldsymbol{b}}[\boldsymbol{c}_1\ \boldsymbol{c}_2\ \boldsymbol{c}_3]) \\ &= \mathrm{tr}([\boldsymbol{b} \times \boldsymbol{c}_1\ \ \boldsymbol{b} \times \boldsymbol{c}_2\ \ \boldsymbol{b} \times \boldsymbol{c}_3]) \\ &= \boldsymbol{e}_1 \cdot (\boldsymbol{b} \times \boldsymbol{c}_1) + \boldsymbol{e}_2 \cdot (\boldsymbol{b} \times \boldsymbol{c}_2) + \boldsymbol{e}_3 \cdot (\boldsymbol{b} \times \boldsymbol{c}_3) \\ &= \boldsymbol{b} \cdot (\boldsymbol{c}_1 \times \boldsymbol{e}_1 + \boldsymbol{c}_2 \times \boldsymbol{e}_2 + \boldsymbol{c}_3 \times \boldsymbol{e}_3).\end{aligned}$$

Now let $\boldsymbol{v} = \boldsymbol{c}_1 \times \boldsymbol{e}_1 + \boldsymbol{c}_2 \times \boldsymbol{e}_2 + \boldsymbol{c}_3 \times \boldsymbol{e}_3$. Then,

$$\begin{aligned}\boldsymbol{b} \cdot (R\boldsymbol{v}) &= (R^T \boldsymbol{b}) \cdot \boldsymbol{v} \\ &= \mathrm{tr}(R\widehat{R^T\boldsymbol{b}}) \\ &= \mathrm{tr}(RR^T\hat{\boldsymbol{b}}R) \\ &= \mathrm{tr}(\hat{\boldsymbol{b}}R) \ = \ \mathrm{tr}(R\hat{\boldsymbol{b}}) \ = \ \boldsymbol{b} \cdot \boldsymbol{v}.\end{aligned}$$

Since $\boldsymbol{b}$ is arbitrary, $R\boldsymbol{v} = \boldsymbol{v}$, i.e., $\boldsymbol{v}$ is the eigenvector of R corresponding to eigenvalue 1. ☐

Next we calculate the horizontal variation of $V(X)$. Using Lemma 1, (3.8) and the fact that trace is a linear operator, we get

$$\begin{aligned} \delta_H V(X) &= \delta_H \sigma_1 \mathrm{tr}(\sum_{i=1}^{n-1} X_i) \\ &= \sigma_1 \sum_{i=1}^{n-1} \mathrm{tr}(\delta_H X_i) \\ &= \sigma_1 \sum_{i=1}^{n-1} \mathrm{tr}(\hat{\boldsymbol{\lambda}}_i X_i) \\ &= \sigma_1 \sum_{i=1}^{n-1} (\boldsymbol{u}_{\tau i}^{ps})^T \boldsymbol{\lambda}_i \end{aligned} \tag{3.10}$$

where for $i = 1, \ldots, n-1$

$$\boldsymbol{u}_{\tau i}^{ps} = (\boldsymbol{\Delta}_i \times \boldsymbol{e}_1 + \boldsymbol{\Sigma}_i \times \boldsymbol{e}_2 + \boldsymbol{\Gamma}_i \times \boldsymbol{e}_3) \tag{3.11}$$

and $\boldsymbol{\Delta}_i, \boldsymbol{\Sigma}_i, \boldsymbol{\Gamma}_i$ are the column vectors of $X_i = R_{i+1}^T R_i$. The superscript "ps" refers to "potential shaping". From Lemma 1, we get that $\boldsymbol{u}_{\tau i}^{ps}$ is the eigenvector of X_i corresponding to eigenvalue 1.

Next, we calculate $\delta_H \boldsymbol{\Omega}_i$ for $i > 1$ using (3.8). Since $\boldsymbol{\Omega}_{i+1} = X_i \boldsymbol{\Omega}_i - \boldsymbol{w}_i$, we get

$$\begin{aligned} \delta_H \boldsymbol{\Omega}_{i+1} &= -\delta_H \boldsymbol{w}_i + (\delta_H X_i)\boldsymbol{\Omega}_i + X_i \delta_H \boldsymbol{\Omega}_i \\ &= -(\dot{\boldsymbol{\lambda}}_i - \boldsymbol{w}_i \times \boldsymbol{\lambda}_i) + \widehat{\boldsymbol{\lambda}}_i X_i \boldsymbol{\Omega}_i + X_i \delta_H \boldsymbol{\Omega}_i. \end{aligned}$$

Using this recursively with $\delta_H \boldsymbol{\Omega}_1 = 0$ from (3.8) and the identity $Y(\boldsymbol{z}_1 \times \boldsymbol{z}_2) = (Y\boldsymbol{z}_1) \times (Y\boldsymbol{z}_2)$ for $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^3$ and rotation matrix $Y \in SO(3)$, we get

$$\begin{aligned} \delta_H \boldsymbol{\Omega}_{i+1} &= -\dot{\boldsymbol{\lambda}}_i + \boldsymbol{w}_i \times \boldsymbol{\lambda}_i + \boldsymbol{\lambda}_i \times (X_i \boldsymbol{\Omega}_i) - x_i \dot{\boldsymbol{\lambda}}_{i-1} + X_i \boldsymbol{w}_{i-1} \times X_i \boldsymbol{\lambda}_{i-1} \\ &\quad + X_i \boldsymbol{\lambda}_{i-1} \times (X_i X_{i-1} \boldsymbol{\Omega}_{i-1}) + X_i X_{i-1} \delta_H \boldsymbol{\Omega}_{i-1} \\ &= X_{i+1}^{-1} \sum_{j=1}^{i} X_{i+1} X_i \cdots X_{j+2} X_{j+1} \left(-\dot{\boldsymbol{\lambda}}_j + \boldsymbol{w}_j \times \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_j \times (X_j \boldsymbol{\Omega}_j) \right) \\ &= \sum_{j=1}^{i} R_{i+1}^T R_{j+1} \left(-\dot{\boldsymbol{\lambda}}_j + \boldsymbol{w}_j \times \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_j \times (X_j \boldsymbol{\Omega}_j) \right). \end{aligned} \tag{3.12}$$

Further,

$$
\begin{aligned}
\int_{t_0}^{t_1}\sum_{i=2}^{n}\boldsymbol{\Omega}_i^T I_i \delta_H \boldsymbol{\Omega}_i &= \int_{t_0}^{t_1}\sum_{i=2}^{n}\boldsymbol{\Omega}_i^T I_i \sum_{j=1}^{i-1} R_i^T R_{j+1}\left(-\dot{\boldsymbol{\lambda}}_j + \boldsymbol{w}_j \times \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_j \times (X_j \boldsymbol{\Omega}_j)\right) dt \\
&= \int_{t_0}^{t_1}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right)^T\left(-\dot{\boldsymbol{\lambda}}_j + \boldsymbol{w}_j \times \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_j \times (X_j \boldsymbol{\Omega}_j)\right) dt \\
&= \int_{t_0}^{t_1}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\Big(\frac{d}{dt}\left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right)^T \boldsymbol{\lambda}_j \\
&\quad + \left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right)^T \left(\boldsymbol{w}_j \times \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_j \times (X_j \boldsymbol{\Omega}_j)\right)\Big) dt \\
&= \int_{t_0}^{t_1}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\boldsymbol{\lambda}_j^T\Big(\frac{d}{dt}\left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right) \\
&\quad + \left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right) \times (\boldsymbol{w}_j - X_j \boldsymbol{\Omega}_j)\Big) dt \\
&= \int_{t_0}^{t_1}\sum_{j=1}^{n-1}\boldsymbol{\lambda}_j^T \sum_{i=j+1}^{n}\Big(\frac{d}{dt}\left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right) \\
&\quad + \left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right) \times (\boldsymbol{w}_j - X_j \boldsymbol{\Omega}_j)\Big) dt
\end{aligned} \tag{3.13}
$$

where we have used integration by parts and the fact that $\boldsymbol{\lambda}(t)$ vanishes at t_0, t_1.

Using (3.6), (3.10), (3.12) and (3.13), we compute the horizontal variation of Lagrangian l as

$$
\begin{aligned}
\delta_H \int_{t_0}^{t_1} l(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{\Omega}_1) dt &= \delta_H \int_{t_0}^{t_1} \frac{1}{2}\left(\sum_{i=1}^{n}\boldsymbol{\Omega}_i^T I_i \boldsymbol{\Omega}_i - 2V(X)\right) dt \\
&= \int_{t_0}^{t_1} \Big(\sum_{i=2}^{n}\boldsymbol{\Omega}_i^T I_i \delta_H \boldsymbol{\Omega}_i - \delta_H V(X)\Big) dt \\
&= \int_{t_0}^{t_1}\sum_{j=1}^{n-1}\boldsymbol{\lambda}_j^T \sum_{i=j+1}^{n}\Big(\frac{d}{dt}\left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right) \\
&\quad + \left(R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right) \times (\boldsymbol{w}_j - X_j \boldsymbol{\Omega}_j) - \sigma_1 \boldsymbol{u}_{\tau j}^{ps}\Big) dt.
\end{aligned}
$$

We set $\delta_H \int_{t_0}^{t_1} l = 0$ and note that this holds for arbitrary $\boldsymbol{\lambda}_j(t)$. Again, using the fact that $\boldsymbol{\Omega}_{i+1} = X_i \boldsymbol{\Omega}_i - \boldsymbol{w}_i$, we get the horizontal equations to be for $j = 1, \ldots, n-1$

$$
\frac{d}{dt}\left(\sum_{i=j+1}^{n} R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right) = \left(\sum_{i=j+1}^{n} R_{j+1}^T R_i I_i \boldsymbol{\Omega}_i\right) \times (\boldsymbol{\Omega}_{j+1}) + \sigma_1 \boldsymbol{u}_{\tau j}^{ps}. \tag{3.14}
$$

Since j goes from 1 to $n-1$, we have $n-1$ horizontal vector equations. When $j = n-1$, (3.14) becomes

$$
I_n \dot{\boldsymbol{\Omega}}_n = (I_n \boldsymbol{\Omega}_n) \times (\boldsymbol{\Omega}_n) + \sigma_1 \boldsymbol{u}_{\tau, n-1}^{ps}.
$$

Let $\boldsymbol{u}_{\tau n}^{ps} = \boldsymbol{u}_{\tau 0}^{ps} = 0$. We now show that if

$$
I_j \dot{\boldsymbol{\Omega}}_j = (I_j \boldsymbol{\Omega}_j) \times (\boldsymbol{\Omega}_j) + \sigma_1 (\boldsymbol{u}_{\tau, j-1}^{ps} - \boldsymbol{u}_{\tau j}^{ps}) \tag{3.15}
$$

for $j = k+1, \ldots, n$ where $k \geq 1$, then (3.15) also holds for $j = k$. For $j = 1, \ldots, n$ let

$$\boldsymbol{a}_{j-1} = \left(\sum_{i=j}^{n} R_j^T R_i I_i \boldsymbol{\Omega}_i \right) = I_j \boldsymbol{\Omega}_j + R_j^T \left(\sum_{i=j+1}^{n} R_i I_i \boldsymbol{\Omega}_i \right). \tag{3.16}$$

Note that $\boldsymbol{a}_0 = \boldsymbol{a}$ defined in (3.9). We have

$$\dot{\boldsymbol{a}}_{j-1} = I_j \dot{\boldsymbol{\Omega}}_j - \widehat{\boldsymbol{\Omega}}_j R_j^T \left(\sum_{i=j+1}^{n} R_i I_i \boldsymbol{\Omega}_i \right) + R_j^T \sum_{i=j+1}^{n} \left(R_i \widehat{\boldsymbol{\Omega}}_i (I_i \boldsymbol{\Omega}_i) + R_i I_i \dot{\boldsymbol{\Omega}}_i \right). \tag{3.17}$$

Using (3.14), (3.16) and (3.17), we get

$$I_j \dot{\boldsymbol{\Omega}}_j + R_j^T \sum_{i=j+1}^{n} R_i \left(\widehat{\boldsymbol{\Omega}}_i (I_i \boldsymbol{\Omega}_i) + I_i \dot{\boldsymbol{\Omega}}_i \right) = (I_j \boldsymbol{\Omega}_j) \times \boldsymbol{\Omega}_j + \sigma_1 \boldsymbol{u}^{ps}_{\tau, j-1}. \tag{3.18}$$

Now we use the assumption that for $j = k+1, \ldots, n$ for any integer k, $n > k \geq 1$, (3.15) is satisfied. Then, for $j = k, \ldots, n$ (3.18) implies

$$I_j \dot{\boldsymbol{\Omega}}_j + R_j^T \sum_{i=j+1}^{n} R_i \left(\sigma_1 (\boldsymbol{u}^{ps}_{\tau, i-1} - \boldsymbol{u}^{ps}_{\tau i}) \right) = (I_j \boldsymbol{\Omega}_j) \times \boldsymbol{\Omega}_j + \sigma_1 \boldsymbol{u}^{ps}_{\tau, j-1}. \tag{3.19}$$

Now use the fact that $\boldsymbol{u}^{ps}_{\tau i}$ is the eigenvector of $X_i = R_{i+1}^T R_i$ corresponding to eigenvalue 1, i.e., $R_{i+1} \boldsymbol{u}^{ps}_{\tau i} = R_i \boldsymbol{u}^{ps}_{\tau i}$. This implies that all the terms in the summation in (3.19) cancel except for $\sigma_1 R_j^T R_{j+1} \boldsymbol{u}^{ps}_{\tau j} = \sigma_1 \boldsymbol{u}^{ps}_{\tau j}$. So (3.19) becomes

$$I_j \dot{\boldsymbol{\Omega}}_j = (I_j \boldsymbol{\Omega}_j) \times \boldsymbol{\Omega}_j + \sigma_1 (\boldsymbol{u}^{ps}_{\tau, j-1} - \boldsymbol{u}^{ps}_{\tau, j}). \tag{3.20}$$

Equation (3.20) is satisfied for $j = 1, \ldots, n$.

Equation (3.20) together with (3.3) for the dynamics of relative orientations X_i, $i = 1, \ldots, n-1$ give the complete, reduced, closed-loop equations of motion in $T(SO(3)^n)/SO(3)$. Since $\boldsymbol{\Delta}_i, \boldsymbol{\Sigma}_i, \boldsymbol{\Gamma}_i$ are the column vectors of X_i, (3.3) can be equivalently written for $i = 1, \ldots, n-1$ as

$$\frac{d}{dt} \left[\begin{array}{ccc} \boldsymbol{\Delta}_i & \boldsymbol{\Sigma}_i & \boldsymbol{\Gamma}_i \end{array} \right] = \left[\begin{array}{ccc} \boldsymbol{\Delta}_i & \boldsymbol{\Sigma}_i & \boldsymbol{\Gamma}_i \end{array} \right] \widehat{\boldsymbol{\Omega}}_i - \widehat{\boldsymbol{\Omega}}_{i+1} \left[\begin{array}{ccc} \boldsymbol{\Delta}_i & \boldsymbol{\Sigma}_i & \boldsymbol{\Gamma}_i \end{array} \right].$$

The closed-loop dynamics are the Lagrangian dynamics corresponding to the Lagrangian (3.2). By comparing these dynamics with equation (2.1), the control inputs can be read off as the last term on the right side of (3.20) .

3.2. **Reduction for $SE(3)^n$ network.** For the rigid body network on $SE(3)^n$, we define the coupling potential to be

$$\tilde{V}_2 = \sum_{i=1}^{n-1} \left(\sigma_1 \mathrm{tr}(R_{i+1}^T R_i) + \frac{\sigma_2}{2} \| \boldsymbol{b}_i - \boldsymbol{b}_{i+1} \|^2 \right), \tag{3.21}$$

where $\sigma_1, \sigma_2 \in \mathbb{R}$. Note that $\tilde{V}_2$ depends only on relative orientations and relative positions that can be measured given the communication graph. When $\sigma_1 < 0$ and $\sigma_2 > 0$, the global minimum of $\tilde{V}_2$ is $R_1 = R_2 = \cdots = R_n$ and $\boldsymbol{b}_1 = \boldsymbol{b}_2 = \cdots = \boldsymbol{b}_n$ as desired.

We define the controlled system by the Lagrangian dynamics with Lagrangian L' equal to the sum of the kinetic energies of each individual system minus the coupling potential:

$$L' = \sum_{i=1}^{n} \frac{1}{2}\big(\mathbf{\Omega}_i^T I_i \mathbf{\Omega}_i + \boldsymbol{v}_i^T M_i \boldsymbol{v}_i\big) - \sigma_1 \mathrm{tr}(\sum_{i=1}^{n-1} R_{i+1}^T R_i) - \frac{\sigma_2}{2}\sum_{i=1}^{n-1} \|\boldsymbol{b}_i - \boldsymbol{b}_{i+1}\|^2. \quad (3.22)$$

L' has $SE(3)$ as its symmetry group with the symmetry action given by

$$\begin{aligned} & (\bar{R}, \bar{\boldsymbol{b}}) \cdot (R_1, \ldots, R_n, \dot{R}_1, \ldots, \dot{R}_n, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n, \dot{\boldsymbol{b}}_1, \ldots, \dot{\boldsymbol{b}}_n) \\ = \ & (\bar{R}R_1, \ldots, \bar{R}R_n, \bar{R}\dot{R}_1, \ldots, \bar{R}\dot{R}_n, \bar{R}\boldsymbol{b}_1 + \bar{\boldsymbol{b}}, \ldots, \bar{R}\boldsymbol{b}_n + \bar{\boldsymbol{b}}, \bar{R}\dot{\boldsymbol{b}}_1, \ldots, \bar{R}\dot{\boldsymbol{b}}_n). \end{aligned}$$

For $i = 1, \ldots, n-1$, $X_i = R_{i+1}^T R_i$ and $\boldsymbol{w}_i = \dot{X}_i X_i^{-1} = X_i \mathbf{\Omega}_i - \mathbf{\Omega}_{i+1}$ as defined in §3.1 and let $\boldsymbol{y}_i = \boldsymbol{b}_{i+1} - \boldsymbol{b}_i$. We identify the reduced space as

$$\begin{aligned} & \Big[R_1, R_2, \ldots, R_n, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n, R_1\widehat{\mathbf{\Omega}}_1, R_2\widehat{\mathbf{\Omega}}_2, \ldots, R_n\hat{\mathbf{\Omega}}_n, \dot{\boldsymbol{b}}_1, \ldots, \dot{\boldsymbol{b}}_n\Big]_{SE(3)} \\ = \ & (X_1, \ldots, X_{n-1}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{n-1}, \dot{X}_1 X_1^{-1}, \ldots, \dot{X}_{n-1}X_{n-1}^{-1}, \dot{\boldsymbol{y}}_1, \ldots, \dot{\boldsymbol{y}}_{n-1}, \mathbf{\Omega}_1, \boldsymbol{v}_1) \\ = \ & (X_1, \ldots, X_{n-1}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{n-1}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}, \dot{\boldsymbol{y}}_1, \ldots, \ldots \boldsymbol{y}_{n-1}, \mathbf{\Omega}_1, \boldsymbol{v}_1) \\ = \ & (X, \boldsymbol{y}, \boldsymbol{w}, \dot{\boldsymbol{y}}, \mathbf{\Omega}_1, \boldsymbol{v}_1). \end{aligned} \quad (3.23)$$

As in §3.1 for the $SO(3)^n$ reduction, we have used the notion of a principal connection on a principal bundle. The value of the principal connection is the angular and linear velocity of the first rigid body in its body frame and belongs to $\mathfrak{se}(3)$. Just as in the $SO(3)^n$ case, there are various choices of principal connection, each of which gives rise to its own horizontal and vertical equation, which after change of variables, can be reduced to our choice. For example, one can choose the angular and linear velocity of any of the n rigid bodies in its body frame as a principal connection. The corresponding vertical equation is just the conservation of total angular and linear momentum, expressed in that particular body frame.

We denote by l' the Lagrangian (3.22) on the reduced space such that

$$l'(X, \boldsymbol{y}, \boldsymbol{w}, \dot{\boldsymbol{y}}, \mathbf{\Omega}_1, \boldsymbol{v}_1) = \frac{1}{2}\sum_{i=1}^{n}\Big(\mathbf{\Omega}_i^T I_i \mathbf{\Omega}_i + \boldsymbol{v}_i^T M_i \boldsymbol{v}_i\Big) - V'(X, \boldsymbol{y})$$

where

$$V'(X, \boldsymbol{y}) = \sigma_1 \sum_{i=1}^{n-1} \mathrm{tr}(X_i) + \frac{\sigma_2}{2}\sum_{i=1}^{n-1} \|\boldsymbol{y}_i\|^2.$$

The equations of motion are derived using the variational formulations of mechanics. Horizontal and vertical variations correspond to the identification made in (3.23). The vertical variations correspond to variations in the symmetry group direction, i.e., to variations in $(\mathbf{\Omega}_1, \boldsymbol{v}_1)$. These variations preserve the shape of the system. The horizontal variations correspond to variations in the reduced space $(X, \boldsymbol{y}, \boldsymbol{w}, \dot{\boldsymbol{y}})$ and preserve the total momentum of the system [5]. Let t_0, t_1 be the initial and final time, respectively, of the paths in the variations. The splitting of variations into a vertical variation δ_V and horizontal variation δ_H can be computed to be

$$\begin{aligned} \delta_V(X, \boldsymbol{y}, \boldsymbol{w}, \dot{\boldsymbol{y}}, \mathbf{\Omega}_1, \boldsymbol{v}_1) &= \Big((X, \boldsymbol{y}, \boldsymbol{w}, , \dot{\boldsymbol{y}}, \mathbf{\Omega}_1, \boldsymbol{v}_1), (0, 0, 0, 0, \delta\mathbf{\Omega}_1, \delta\boldsymbol{v}_1)\Big) \\ \delta_H(X, \boldsymbol{y}, \boldsymbol{w}, \dot{\boldsymbol{y}}, \mathbf{\Omega}_1, \boldsymbol{v}_1) &= \Big((X, \boldsymbol{y}, \boldsymbol{w}, \dot{\boldsymbol{y}}, \mathbf{\Omega}_1, \boldsymbol{v}_1), (\widehat{\boldsymbol{\lambda}}X, \delta\boldsymbol{y}, \dot{\boldsymbol{\lambda}} - \boldsymbol{w} \times \boldsymbol{\lambda}, \delta\dot{\boldsymbol{y}}, 0, 0)\Big) \end{aligned}$$

where $\boldsymbol{\eta}, \boldsymbol{\lambda} \in \mathbb{R}^3$, $\boldsymbol{\eta}_1(t_i) = 0, \boldsymbol{\eta}_2(t_i) = 0, \boldsymbol{\lambda}_j(t_i) = 0$ for $i = 0, 1$ and $j = 1, \ldots, n-1$. Here

$$\begin{aligned} \delta\boldsymbol{\Omega}_1 &= \dot{\boldsymbol{\eta}}_1 + \boldsymbol{\Omega}_1 \times \boldsymbol{\eta}_1 \\ \delta\boldsymbol{v}_1 &= \dot{\boldsymbol{\eta}}_2 + \boldsymbol{\Omega}_1 \times \boldsymbol{\eta}_2 + \boldsymbol{v}_1 \times \boldsymbol{\eta}_1 \end{aligned}$$

and $\widehat{\boldsymbol{\lambda}}X$, $\dot{\boldsymbol{\lambda}} - \boldsymbol{w} \times \boldsymbol{\lambda}$ are as defined in §3.2.

The splitting gives rise to a vertical equation and a horizontal equation of motion, respectively, by taking the vertical and horizontal variations of the Lagrangian l'. This follows analogously to the $SO(3)^n$ case. Details can be found in [26]. As before let $\boldsymbol{\Delta}_i, \boldsymbol{\Sigma}_i, \boldsymbol{\Gamma}_i$ be the column vectors of $R_{i+1}^T R_i$. The reduced, closed-loop equations of motion in $T(SE(3)^n)/SE(3)$ are as follows:

$$\begin{aligned} I_i\dot{\boldsymbol{\Omega}}_i &= (I_i\boldsymbol{\Omega}_i) \times \boldsymbol{\Omega}_i + (M_i\boldsymbol{v}_i) \times \boldsymbol{v}_i + \sigma_1(\boldsymbol{u}^{ps}_{\tau,i-1} - \boldsymbol{u}^{ps}_{\tau i}) \\ M_i\dot{\boldsymbol{v}}_i &= (M_i\boldsymbol{v}_i) \times \boldsymbol{\Omega}_i + \sigma_2\left(\boldsymbol{u}^{ps}_{f,i-1} - \boldsymbol{u}^{ps}_{fi}\right) \end{aligned} \tag{3.24}$$

where

$$\boldsymbol{u}^{ps}_{\tau i} = (\boldsymbol{\Delta}_i \times \boldsymbol{e}_1 + \boldsymbol{\Sigma}_i \times \boldsymbol{e}_2 + \boldsymbol{\Gamma}_i \times \boldsymbol{e}_3), \quad i = 1, \ldots, n-1,$$

$$\boldsymbol{u}^{ps}_{\tau 0} = \boldsymbol{u}^{ps}_{\tau n} = 0,$$

$$\boldsymbol{u}^{ps}_{fi} = -R_i^T(\boldsymbol{b}_i - \boldsymbol{b}_{i+1}), \quad i = 1, \ldots, n-1,$$

$$\boldsymbol{u}^{ps}_{f1} = \boldsymbol{u}^{ps}_{fn} = 0.$$

The closed-loop dynamics are the Lagrangian dynamics corresponding to the Lagrangian (3.22). By comparing these dynamics with equation (2.3), the control inputs can be read off as the last term on the right side of (3.24).

4. **Stability of rigid body networks.** In this section we prove stability of the desired relative equilibria defined by (2.2) for the $SO(3)^n$ network and by (2.4) for the $SE(3)^n$ network. Recall that for the $SO(3)^n$ case, the desired equilibrium motion corresponds to the bodies all aligned and rotating about their short axis. For the $SE(3)$ case, the desired equilibrium motion corresponds to the bodies aligned, both in orientation and position, and at the same time rotating about and translating along their short axis. The non-rotating $SO(3)^n$ network stability is proven in [10] using the Energy-Casimir method; the authors in [10] are not able to prove stability when the bodies are rotating as they are not able to find Casimir functions for this case.

Since we are eventually interested in exponentially stabilizing the synchronized steady motion of the networks, we break some of the symmetry that remains after coupling to shrink the symmetry to an Abelian group. The symmetry breaking corresponds to alignment of the short axis of each rigid body (the axis of rotation/translation) with the inertial axis $\boldsymbol{e}_1$ as desired. This allows us to use Routh reduction to prove stability of the network and also to use a result from [3] to prove exponential stability by constructing a Lyapunov function for the desired relative equilibrium manifold.

4.1. **Stability of $SO(3)^n$ network.** The potential function we use for the $SO(3)^n$ network is a modification of $\tilde{V}_1$ given by

$$V_1 = \sigma_1 \mathrm{tr}(\sum_{i=1}^{n-1} R_{i+1}^T R_i) + \sigma_1 \boldsymbol{e}_1^T R_1 \boldsymbol{e}_1. \tag{4.1}$$

The extra term $V_{B1} := \sigma_1 \boldsymbol{e}_1^T R_1 \boldsymbol{e}_1$ in the potential V_1, as compared to $\tilde{V}_1$, reflects the interest in aligning the short axis of each body with the inertial axis $\boldsymbol{e}_1$ (the choice of $\boldsymbol{e}_1$ is arbitrary). Without this term, the symmetry group is $SO(3)$. With this term, the symmetry group is the abelian group S^1, corresponding here to rotation about the $\boldsymbol{e}_1$ axis. The equations of motion for the network are identical to (3.20) except that here

$$\boldsymbol{u}_{\tau 0}^{ps} = -R_1^T((R_1\boldsymbol{e}_1) \times \boldsymbol{e}_1) \tag{4.2}$$

due to the new term in the potential V_1. To see how this term arises, consider the most general variation of V_{B1}. Let $\delta\boldsymbol{\theta}_i$ be the variation in R_i, i.e., $\delta R_i = \widehat{\delta\boldsymbol{\theta}}_i R_i$. We have,

$$\begin{aligned} \delta V_{B1} &= \sigma_1 \boldsymbol{e}_1^T \delta R_1 \boldsymbol{e}_1 \\ &= \sigma_1 \boldsymbol{e}_1^T \widehat{\delta\boldsymbol{\theta}_1} R_1 \boldsymbol{e}_1 \\ &= \sigma_1 \delta\boldsymbol{\theta}_1^T \left((R_1\boldsymbol{e}_1) \times \boldsymbol{e}_1\right). \end{aligned} \tag{4.3}$$

When the variation is vertical, i.e., $\delta\boldsymbol{\theta}_1 \| \boldsymbol{e}_1$ in (4.3), then we have $\delta_V V_{B1} = 0$. Hence, the only additional term that arises is due to the horizontal variation and equal to $\sigma_1\left((R_1\boldsymbol{e}_1) \times \boldsymbol{e}_1\right)$ in the $\delta\boldsymbol{\theta}_1$ direction. After premultiplying by R_1^T to transform the new term from inertial frame to body frame, we define $\boldsymbol{u}_{\tau 0}^{ps}$, consistent with how it appears in (3.20), to get (4.2).

Consider the following function

$$V_1^A = \sigma_1 \mathrm{tr}(\sum_{i=1}^{n-1} R_{i+1}^T R_i) + \sigma_1 \boldsymbol{e}_1^T R_1 \boldsymbol{e}_1 - \frac{1}{2}\sum_{i=1}^{n} \boldsymbol{e}_1^T I_i^l \boldsymbol{e}_1$$

where $I_i^l = R_i I_i R_i^T$ is the moment of inertia of the i^{th} body in inertial space. V_1^A is the amended potential [22] for the $SO(3)^n$ network corresponding to the relative equilibrium given by (2.2). To prove that the relative equilibrium given by (2.2) is stable, we show that the amended potential V_1^A has a positive definite variation in directions away from the relative equilibrium solution (see Proposition 8.9.4 in [23]). That is, we show that $\delta^2 V_1^A \geq 0$ and is equal to zero only when the conditions in (2.2) are met. When $R_i = R_j = R_e$, we compute

$$\begin{aligned} \delta^2 V_1^A &= -2\sigma_1 \sum_{i=1}^{n-1} (\delta\boldsymbol{\theta}_{i+1} - \delta\boldsymbol{\theta}_i)^T(\delta\boldsymbol{\theta}_{i+1} - \delta\boldsymbol{\theta}_i) - \sigma_1(\delta\boldsymbol{\theta}_1 \times \boldsymbol{e}_1)^T(\delta\boldsymbol{\theta}_1 \times \boldsymbol{e}_1) \\ &\quad + \sum_{i=1}^{n} (\delta\boldsymbol{\theta}_i \times \boldsymbol{e}_1)^T (I_{i,11}\mathbb{I}_{3\times 3} - I_i^l)(\delta\boldsymbol{\theta}_i \times \boldsymbol{e}_1). \end{aligned}$$

Details of the computation can be found in [26]. When $\sigma_1 < 0$, then $\delta^2 V_1^A \geq 0$ and is equal to zero only when (2.2) is satisfied. We have proved the following theorem.

Theorem 1. *The steady motion given by* (2.2) *is a stable relative equilibrium for the rigid body $SO(3)^n$ network with controlled equations of motion given by* (3.20) *with* (3.11) *and* (4.2). *This equilibrium corresponds to all n rigid bodies with synchronized* *orientations. Each body is rotating about its short axis which is aligned with the* $\boldsymbol{e}_1$ *axis in the inertial frame.*

4.2. **Stability of $SE(3)^n$ network.** The potential function we use for the $SE(3)^n$ network is a modification of $\tilde{V}_2$ given by

$$V_2 = \sigma_1 \mathrm{tr}(\sum_{i=1}^{n-1} R_{i+1}^T R_i) + \sigma_1 \boldsymbol{e}_1^T R_1 \boldsymbol{e}_1 + \frac{\sigma_2}{2}\|\boldsymbol{b}_{i+1} - \boldsymbol{b}_i\|^2 + \frac{\sigma_2}{2}(b_{12}^2 + b_{13}^2). \qquad (4.4)$$

The extra term $V_{B2} = \sigma_1 \boldsymbol{e}_1^T R_1 \boldsymbol{e}_1 + (\sigma_2/2)(b_{12}^2 + b_{13}^2)$ in the potential V_2, as compared to $\tilde{V}_2$, reflects the interest in aligning the short axis (of rotation and translation) with the inertial $\boldsymbol{e}_1$ axis (the choice of $\boldsymbol{e}_1$ is arbitrary). Without these terms, the symmetry of the $SE(3)^n$ network is $SE(3)$. With these terms, the symmetry group is the abelian group $S^1 \times \mathbb{R}$, corresponding to rotation about and translation along the $\boldsymbol{e}_1$ axis. The equations of motion for the network are identical to (3.24) except that here

$$\begin{aligned} \boldsymbol{u}_{\tau 0}^{ps} &= -R_1^T((R_1\boldsymbol{e}_1) \times \boldsymbol{e}_1) \\ \boldsymbol{u}_{f0}^{ps} &= -R_1^T((\boldsymbol{b}_1 \cdot \boldsymbol{e}_2)\boldsymbol{e}_2 + (\boldsymbol{b}_1 \cdot \boldsymbol{e}_3)\boldsymbol{e}_3). \end{aligned} \qquad (4.5)$$

To see how these terms arise, consider the most general variation of V_{B2}. We have

$$\begin{aligned} \delta V_{B2} &= \sigma_1 \boldsymbol{e}_1^T \delta R_1 \boldsymbol{e}_1 + \sigma_2 (b_{12}\delta b_{12} + b_{13}\delta b_{13}) \\ &= \sigma_1 \boldsymbol{e}_1^T \widehat{\delta\boldsymbol{\theta}_1} R_1 \boldsymbol{e}_1 + \sigma_2 (b_{12}\delta b_{12} + b_{13}\delta b_{13}) \\ &= \sigma_1 \delta\boldsymbol{\theta}_1^T ((R_1\boldsymbol{e}_1) \times \boldsymbol{e}_1) + \sigma_2 (b_{12}\delta b_{12} + b_{13}\delta b_{13}). \end{aligned} \qquad (4.6)$$

When the variation is vertical, i.e., $\delta\boldsymbol{\theta}_1 \| \boldsymbol{e}_1$ and $\delta b_{12} = \delta b_{13} = 0$ in (4.6), then we have $\delta_V V_{B2} = 0$. Hence, the only additional terms that arise are all due to the horizontal variation and equal to $\sigma_1 ((R_1\boldsymbol{e}_1) \times \boldsymbol{e}_1)$ in the $\delta\boldsymbol{\theta}_1$ direction, $\sigma_2 b_{12}$ in the δb_{12} direction which is the $\boldsymbol{e}_2$ direction and $\sigma_2 b_{13}$ in the δb_{13} direction which is the $\boldsymbol{e}_3$ direction. We can substitute the identities: $b_{12} = (\boldsymbol{b}_1 \cdot \boldsymbol{e}_2)$ and $b_{13} = (\boldsymbol{b}_1 \cdot \boldsymbol{e}_3)$. After premultiplying all new terms by R_1^T to transform from inertial to body frame, we define $\boldsymbol{u}_{\tau 0}^{ps}$ and $\boldsymbol{u}_{f0}^{ps}$, consistent with how they appear in (3.24), to get (4.5).

Consider the following function

$$V_2^A = \sigma_1 \mathrm{tr}(\sum_{i=1}^{n-1} R_{i+1}^T R_i) + \sigma_1 \boldsymbol{e}_1^T R_1 \boldsymbol{e}_1 + \sigma_2 \|\boldsymbol{b}_{i+1} - \boldsymbol{b}_i\|^2 + \sigma_2 (b_{12}^2 + b_{13}^2) - \frac{1}{2}\sum_{i=1}^{n} \boldsymbol{e}_1^T \left(I_i^l + M_i^l\right) \boldsymbol{e}_1$$

where $I_i^l = R_i I_i R_i^T$ and $M_i^l = R_i M_i R_i^T$. V_2^A is the amended potential [22] for the $SE(3)^n$ network corresponding to the relative equilibrium given by (2.4). To prove that the relative equilibrium given by (2.4) is stable, we show that the amended potential V_2^A has a positive definite variation in directions away from the relative equilibrium solution. (see Proposition 8.9.4 in [23]) That is, we show that $\delta^2 V_2^A \geq 0$ and is equal to zero only when the conditions in (2.4) are met. When $R_i = R_j = R_e$

and $\boldsymbol{b}_i = \boldsymbol{b}_j \| \boldsymbol{e}_1$, we compute

$$\begin{aligned}\delta^2 V_2^A &= -2\sigma_1 \sum_{i=1}^{n-1} (\delta\boldsymbol{\theta}_{i+1} - \delta\boldsymbol{\theta}_i)^T (\delta\boldsymbol{\theta}_{i+1} - \delta\boldsymbol{\theta}_i) - \sigma_1 (\delta\boldsymbol{\theta}_1 \times \boldsymbol{e}_1)^T (\delta\boldsymbol{\theta}_1 \times \boldsymbol{e}_1) \\ &\quad + \sum_{i=1}^{n} (\delta\boldsymbol{\theta}_i \times \boldsymbol{e}_1)^T (I_{i,11}\mathbb{I}_{3\times 3} - I_i^l)(\delta\boldsymbol{\theta}_i \times \boldsymbol{e}_1) \\ &\quad + 2\sigma_2 \sum_{i=1}^{n-1} (\delta\boldsymbol{b}_{i+1} - \delta\boldsymbol{b}_i)^T (\delta\boldsymbol{b}_{i+1} - \delta\boldsymbol{b}_i) + \sigma_2((\delta b_{12})^2 + (\delta b_{13})^2) \\ &\quad + \sum_{i=1}^{n} (\delta\boldsymbol{\theta}_i \times \boldsymbol{e}_1)^T (M_{i,11}\mathbb{I}_{3\times 3} - M_i^l)(\delta\boldsymbol{\theta}_i \times \boldsymbol{e}_1).\end{aligned}$$

Details of the computation can be found in [26]. When $\sigma_1 < 0$ and $\sigma_2 > 0$, then $\delta^2 V_2^A \geq 0$ and is equal to zero only when (2.4) is satisfied. We have proved the following theorem.

Theorem 2. *The steady motion (2.4) is a stable relative equilibrium for the rigid body $SE(3)^n$ network with controlled equations of motion given by (3.24). This equilibrium corresponds to all n rigid bodies having the same orientation and position vectors with each one rotating about its short axis and translating along the same axis aligned with the $\boldsymbol{e}_1$ axis in the inertial frame.*

5. **Exponential stability.** In this section, we show how to add a dissipative term to the control law designed in §4.1 and §4.2 to achieve exponential stability of the relative equilibria given by (2.2) and (2.4). The idea is to first construct a Lyapunov function for the relative equilibrium manifold, i.e., a function that has its minimum on the relative equilibrium manifold and has a definite second variation in a neighborhood of the relative equilibrium manifold. Once we have made this construction, we can use a result from [3] to appropriately design dissipative controls to prove nonlinear exponential stability of the relative equilibrium manifold. Note that the notion of exponential convergence is not coordinate independent in the nonlinear setting where coordinate transformations can be nonlinear [35]. That is, if the state converges exponentially to a particular value in one set of coordinates, this does not imply that the state converges to that value exponentially in another set of coordinates. However, one can define a coordinate-free exponential convergence (that we call nonlinear exponential stability) by looking at the exponential decay of Lyapunov functions as in [3]. Following [3], nonlinear exponential stability of an equilibrium point $\boldsymbol{x}_0 \in M$ means that there is a Lyapunov function $\Phi : M \to \mathbb{R}$ that is zero at $\boldsymbol{x}_0$ and positive elsewhere in a neighborhood of $\boldsymbol{x}_0$ and there exist positive constants c_1 and c_2 such that along the system dynamics $\Phi(\boldsymbol{x}(t)) \leq c_1 \Phi(\boldsymbol{x}(0)) \exp(-c_2 t)$. For convenience, we recall the nonlinear exponential stability theorem from [3] that will be used in this section.

Theorem 3. [3] *Let M be a smooth m-dimensional manifold and consider the control system on M with coordinates $\boldsymbol{x}$ defined by*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{x}) u_j \tag{5.1}$$

where $\boldsymbol{f}, \boldsymbol{g}_j$ are smooth vector fields and $u_j(t)$ are bounded measurable functions. Let Φ be a smooth function such that $\Phi(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in S$ where S is a smooth, *one-dimensional submanifold of M. Let B_S be a neighborhood of S in M and Φ be such that $\Phi(\boldsymbol{x}) > 0$ whenever $\boldsymbol{x} \in B_S - S$. Let Φ be a first integral of $\boldsymbol{f}$ and u_j be chosen to be $u_j = -\kappa \nabla \Phi \cdot \boldsymbol{g}_j$. If the system (5.1) is linearly controllable at each point in B_S then the submanifold S is asymptotically stable in the sense that $\boldsymbol{x}(t) \to S$ as $t \to \infty$. If additionally $\delta^2 \Phi(\boldsymbol{x}) > 0$ for each $\boldsymbol{x} \in S$, then the submanifold S is (nonlinearly) exponentially stable in the sense that $\Phi(\boldsymbol{x}(t)) \leq c_1 \Phi(\boldsymbol{x}(0)) \exp^{-c_2 t}$ for some positive constants c_1, c_2.*

5.1. **Exponential stability for $SO(3)^n$ network.** Consider the equations of motion defined by (3.20) with (3.11) and (4.2) where we add a dissipative term $\boldsymbol{u}_{\tau i}^{\text{diss}}$ to each control input $\boldsymbol{u}_{\tau i}$, for $i = 1, \dots, n$:

$$I_i \dot{\boldsymbol{\Omega}}_i = (I_j \boldsymbol{\Omega}_i) \times \boldsymbol{\Omega}_i + \sigma_1 (\boldsymbol{u}_{\tau, i-1}^{ps} - \boldsymbol{u}_{\tau i}^{ps}) + \boldsymbol{u}_{\tau i}^{\text{diss}}. \tag{5.2}$$

We design the terms $\boldsymbol{u}_{\tau i}^{\text{diss}}$ and prove nonlinear exponential stability of the relative equilibrium (2.2). Given that we have already proved that (2.2) is Lyapunov stable in §1, we construct a Lyapunov function for the relative equilibrium manifold as follows. Consider the following function:

$$E_1 = \frac{1}{2} \sum_{i=1}^{n} \Big((\boldsymbol{\omega}_i - \boldsymbol{e}_1)^T I_i^l (\boldsymbol{\omega}_i - \boldsymbol{e}_1) \Big) + V_1^A - V_{1e}^A \tag{5.3}$$

where V_{1e}^A is the constant value of V_1^A at the relative equilibrium (2.2) and $\sigma_1 < 0$.

To apply Theorem 3 we let the manifold M be $TSO(3)^n$, the one-dimensional submanifold S be the relative equilibrium (2.2) and the function Φ be E_1. The dimension of M is $m = 6n$. For notational convenience, we choose coordinates for the whole $6n$-dimensional system to be $(R_1, \boldsymbol{\omega}_1, \dots, R_n, \boldsymbol{\omega}_n)$. In these coordinates, the vector field corresponding to the control input for the i^{th} vehicle is given by*

$$\sum_{k=1}^{3} \boldsymbol{g}_{i,k} u_{\tau i,k}^{\text{diss}} = \begin{bmatrix} 0_{3\times 3} \\ \mathbb{I}_{3\times 3} \end{bmatrix} \boldsymbol{u}_{\tau i}^{\text{diss}} = \begin{bmatrix} 0_{3\times 1} \\ \boldsymbol{u}_{\tau i}^{\text{diss}} \end{bmatrix}. \tag{5.4}$$

Since the system is fully actuated, it is also linearly controllable at each point.

By construction, E_1 is zero on the relative equilibrium manifold and positive elsewhere. Note that on the set $E_1 = 0$, it holds that $\boldsymbol{\omega}_i = \boldsymbol{e}_1$, $R_1 = \cdots = R_n = R_e$ and $R_e \boldsymbol{e}_1 = \boldsymbol{e}_1$. Since $R_e \boldsymbol{e}_1 = \boldsymbol{e}_1$, we get $\boldsymbol{\Omega}_i = R_e^T \boldsymbol{\omega}_i = R_e^T \boldsymbol{e}_1 = \boldsymbol{e}_1$. In §4.1 we showed that the second variation of V_1^A is positive semi-definite. The kinetic energy part of E_1 always has a positive definite second variation because of the regularity of the corresponding Lagrangian. Hence, the second variation of E_1 is positive definite throughout. Let the gradient vector of E_1 with respect to the i^{th} body state be denoted by $\nabla_i E_1$. Then (again with abuse of notation)

$$\nabla_i E_1 \odot \left(\sum_{k=1}^{3} \boldsymbol{g}_{i,k} \right) = (0, 0, 0, (I_i^l (\boldsymbol{\omega}_i - \boldsymbol{e}_1))^T)^T \tag{5.5}$$

---

* Note that we are abusing notation in (5.4), which is technically the embedding of the i^{th} body controller in the $6n$ dimensional system, i.e.,

$$\sum_{k=1}^{3} \boldsymbol{g}_{i,k} u_{\tau i,k}^{\text{diss}} = \begin{bmatrix} 0_{(6i-3)\times 1} \\ \boldsymbol{u}_i \\ 0_{6(n-i)\times 1} \end{bmatrix}.$$

where $\boldsymbol{x} \odot \boldsymbol{y}$ is a vector with i^{th} entry $x_i y_i$, i.e., a pointwise product of two vectors. From (5.4) and (5.5), we see that if we choose the dissipation control terms (in body frames) such that

$$R_i \boldsymbol{u}_{\tau i}^{\text{diss}} = -\kappa I_i^l(\boldsymbol{\omega}_i - \boldsymbol{e}_1), \quad \kappa > 0, \tag{5.6}$$

for $i = 1, \ldots, n$, then (5.6) satisfies the requirement in Theorem 3. Note that for this particular form of dissipation, the time derivative of E_1 is non-positive. This can be checked using the computation from Appendix A which yields

$$\dot{E}_1 = \sum_{i=1}^{n} (\boldsymbol{\omega}_i - \boldsymbol{e}_1) \cdot (R_i \boldsymbol{u}_{\tau i}^{\text{diss}}). \tag{5.7}$$

Then $\dot{E}_1 \leq 0$ for $\boldsymbol{u}_{\tau i}^{\text{diss}}$ given by (5.6). Also, E_1 is conserved when $\boldsymbol{u}_{\tau i}^{\text{diss}} = 0$. Thus, all the conditions in Theorem 3 are satisfied and we conclude that the solution goes to the set $E_1 = 0$, i.e., E_1 decays to zero exponentially. By Theorem 3 the solution converges exponentially to the desired relative equilibrium (2.2) and we have proved the following theorem.

Theorem 4. *The steady motion given by* (2.2) *is an exponentially stable relative equilibrium for the rigid body* $SO(3)^n$ *network with equations of motion given by* (5.2) *with* (3.11) *and* (4.2), $\sigma_1 < 0$ *and dissipation chosen as in* (5.6).

Note that our choice of unit angular velocity is arbitrary. If one wants the rigid bodies to be synchronized and rotating at k rad/s, then all one needs to do is replace $\boldsymbol{e}_1$ in the right side of (5.6) with $k\boldsymbol{e}_1$.

Figures 5.1 and 5.2 illustrate the results of a MATLAB simulation for the controlled network of three identical $SO(3)$ systems. The inertia matrix parameters are $I_{i,11} = 8$ kg-m^2, $I_{i,22} = 4$ kg-m^2, $I_{i,33} = 1$ kg-m^2 for $i = 1, 2, 3$. The relative equilibrium velocity is chosen to be $\boldsymbol{\omega}_i = \boldsymbol{e}_1$ rad/s. Orientation R_i is parametrized using quaternions given by

$$\boldsymbol{q}_i = \begin{bmatrix} \cos(\theta_i/2) & \sin(\theta_i/2)\bar{\boldsymbol{q}}_i^T \end{bmatrix}^T$$

where $\boldsymbol{q}_i \in \mathbb{R}^4$, $\bar{\boldsymbol{q}}_i \in \mathbb{R}^3$ denotes the axis of rotation and θ_i denotes the angle of rotation for the i^{th} body. The control gains are $\sigma_1 = -2, \kappa = 2$. The initial conditions are

$$\begin{aligned} \boldsymbol{q}_1(0) &= \begin{pmatrix} 0.88 \\ 0.25 \\ 0.40 \\ 0.05 \end{pmatrix}, \boldsymbol{q}_2(0) = \begin{pmatrix} 0.93 \\ 0.19 \\ 0.24 \\ 0.18 \end{pmatrix}, \boldsymbol{q}_3(0) = \begin{pmatrix} 0.10 \\ 0.01 \\ 0.00 \\ 0.00 \end{pmatrix}, \\ \dot{\boldsymbol{q}}_1(0) &= \begin{pmatrix} -0.41 \\ -0.13 \\ 0.86 \\ 0.79 \end{pmatrix}, \dot{\boldsymbol{q}}_2(0) = \begin{pmatrix} 0.41 \\ -0.49 \\ -0.56 \\ -0.84 \end{pmatrix}, \dot{\boldsymbol{q}}_3(0) = \begin{pmatrix} 0.01 \\ -0.80 \\ -0.67 \\ -0.67 \end{pmatrix}. \end{aligned}$$

Figure 5.1 shows plots of $\boldsymbol{\Omega}_i$ as a function of time and Figure 5.2 shows the orientation of the bodies in inertial space in terms of quartenions $\boldsymbol{q}_i$ as a function of time. Note that in the steady state, the rigid bodies are aligned and the last two components of the quaternions are zero which indicates that the bodies are rotating about the $\boldsymbol{e}_1$ axis with angular velocity 1 rad/s. Also note that our initial conditions are large suggesting a large region of attraction.

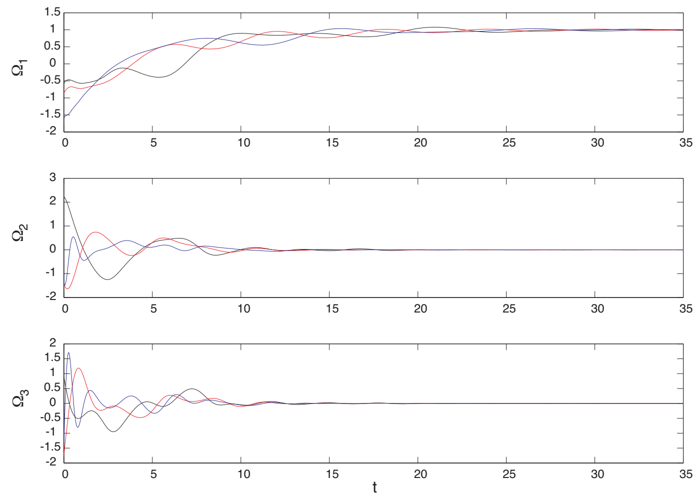

FIGURE 5.1. The angular velocities $\boldsymbol{\Omega}_i$ (rad/s) for three $SO(3)$ systems as a function of time.

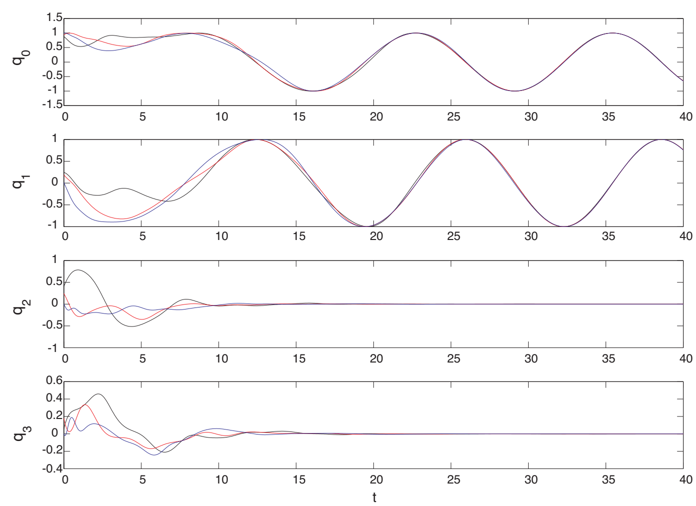

FIGURE 5.2. The orientations in terms of quaternions $\boldsymbol{q}_i$ (rad) for three $SO(3)$ systems as a function of time.

5.2. **Exponential stability for $SE(3)^n$ network.** Consider the equations of motion defined by (3.24) with (4.5) where we add dissipative terms $\boldsymbol{u}_{\tau i}^{\text{diss}}$ and and $\boldsymbol{u}_{fi}^{\text{diss}}$ to each control input $\boldsymbol{u}_{\tau i}$ and $\boldsymbol{u}_{fi}$ respectively, for $i = 1, \ldots, n$:

$$I_i \dot{\boldsymbol{\Omega}}_i = (I_i \boldsymbol{\Omega}_i) \times \boldsymbol{\Omega}_i + (M_i \boldsymbol{v}_i) \times \boldsymbol{v}_i + \sigma_1 (\boldsymbol{u}_{\tau,i-1}^{ps} - \boldsymbol{u}_{\tau i}^{ps}) + \boldsymbol{u}_{\tau i}^{\text{diss}} \tag{5.8}$$

$$M_i \dot{\boldsymbol{v}}_i = (M_i \boldsymbol{v}_i) \times \boldsymbol{\Omega}_i + \sigma_2 \boldsymbol{u}_{fi}^{ps} + \boldsymbol{u}_{fi}^{\text{diss}}. \tag{5.9}$$

We design the terms $\boldsymbol{u}_{\tau i}^{\text{diss}}, \boldsymbol{u}_{fi}^{\text{diss}}$ and prove nonlinear exponential stability of the relative equilibrium (2.4). Given that we have already proved that (2.4) is Lyapunov stable in §2, we construct a Lyapunov function for the relative equilibrium manifold as follows. Consider the following function

$$E_2 = \frac{1}{2}\sum_{i=1}^{n}\Big((\boldsymbol{\omega}_i - \boldsymbol{e}_1)^T I_i^l(\boldsymbol{\omega}_i - \boldsymbol{e}_1)\Big) + \frac{1}{2}\sum_{i=1}^{n}\Big((\boldsymbol{v}_i - \boldsymbol{e}_1)^T M_i^l(\boldsymbol{v}_i - \boldsymbol{e}_1)\Big) + V_2^A - V_{2e}^A \tag{5.10}$$

where V_{2e}^A is the constant value of V_2^A at the relative equilibrium (2.4), $\sigma_1 < 0$ and $\sigma_2 > 0$.

To apply Theorem 3 we let the manifold M be $TSE(3)^n$, the one-dimensional submanifold S be the relative equilibrium (2.4) and the function Φ be E_2. The dimension of M is $m = 12n$. For notational convenience, we choose coordinates for the whole $12n$-dimensional system to be $(R_1, \boldsymbol{b}_1, \boldsymbol{\omega}_1, \dot{\boldsymbol{b}}_1, \ldots, R_n, \boldsymbol{b}_n, \boldsymbol{\omega}_n, \dot{\boldsymbol{b}}_n)$. In these coordinates, the vector field corresponding to the control input for the i^{th} vehicle is given by†

$$\sum_{k=1}^{6} \boldsymbol{g}_{i,k} u_{i,k} = \begin{bmatrix} 0_{6\times 6} \\ \mathbb{I}_{6\times 6} \end{bmatrix} \boldsymbol{u}_i = \begin{bmatrix} 0_{6\times 1} \\ \boldsymbol{u}_i \end{bmatrix}. \tag{5.11}$$

Here $\boldsymbol{u}_i = (\boldsymbol{u}_{\tau i}^T, \boldsymbol{u}_{fi}^T)^T$. Since the system is fully actuated, it is also linearly controllable at each point.

By construction, E_2 is zero on the relative equilibrium manifold and positive elsewhere. Note that on the set $E_2 = 0$, it holds that $\boldsymbol{\omega}_i = \boldsymbol{e}_1$, $\dot{\boldsymbol{b}}_i = \boldsymbol{e}_1$, $R_1 = \cdots = R_n = R_e$ and $R_e\boldsymbol{e}_1 = \boldsymbol{e}_1$, $\boldsymbol{b}_1 = \cdots = \boldsymbol{b}_n || \boldsymbol{e}_1$ and $R_e\boldsymbol{e}_1 = \boldsymbol{e}_1$. Since $R_e\boldsymbol{e}_1 = \boldsymbol{e}_1$, we get $\boldsymbol{\Omega}_i = R_e^T\boldsymbol{\omega}_i = R_e^T\boldsymbol{e}_1 = \boldsymbol{e}_1$ and $\boldsymbol{v}_i = R_e^T\dot{\boldsymbol{b}}_i = R_e^T\boldsymbol{e}_1 = \boldsymbol{e}_1$. . In §4.2 we showed that the second variation of V_2^A is positive semi-definite. The kinetic energy part of E_2 always has a positive definite second variation because of the regularity of the corresponding Lagrangian. Hence, the second variation of E_2 is positive definite throughout. Let the gradient vector of E_2 with respect to the i^{th} body state be $\nabla_i E_2$. Then (again with abuse of notation)

$$\nabla_i E_2 \odot \left(\sum_{k=1}^{6} \boldsymbol{g}_{i,k}\right) = (0,0,0,0,0,0,(I_i^l(\boldsymbol{\omega}_i - \boldsymbol{e}_1))^T, (M_i^l(\boldsymbol{v}_i - \boldsymbol{e}_1))^T)^T. \tag{5.12}$$

From (5.11) and (5.12), we see that if we choose the dissipation control terms (in body frames) such that

$$R_i\boldsymbol{u}_{\tau i}^{\text{diss}} = -\kappa I_i^l(\boldsymbol{\omega}_i - \boldsymbol{e}_1) \tag{5.13}$$

$$R_i\boldsymbol{u}_{fi}^{\text{diss}} = -\kappa M_i^l(\dot{\boldsymbol{b}}_i - \boldsymbol{e}_1), \quad \kappa > 0, \tag{5.14}$$

for $k = 1, \ldots, n$, then (5.13) and (5.14) satisfy the requirement in Theorem 3. Note that for this particular form of dissipation, the time derivative of E_2 is non-positive.

† Note that we are abusing notation in (5.11), which is technically the embedding of the i^{th} body controller in the $12n$ dimensional system, i.e.,

$$\sum_{k=1}^{6} \boldsymbol{g}_{i,k} u_{i,k} = \begin{bmatrix} 0_{(12i-6)\times 1} \\ \boldsymbol{u}_i \\ 0_{12(n-i)\times 1} \end{bmatrix}.$$

This can be checked using a calculation similar to the one for $\dot{E}_1$ in §5.1. We get

$$\frac{d}{dt}E_2 = \sum_{i=1}^{n}(\boldsymbol{\omega}_i - \boldsymbol{e}_1)\cdot(R_i\boldsymbol{u}_{\tau i}^{\text{diss}}) + \sum_{i=1}^{n}(\boldsymbol{v}_i - \boldsymbol{e}_1)\cdot(R_i\boldsymbol{u}_{fi}^{\text{diss}}). \tag{5.15}$$

Then $\dot{E}_2 \leq 0$ for $\boldsymbol{u}_{\tau i}^{\text{diss}}$ and $\boldsymbol{u}_{fi}^{\text{diss}}$ given by (5.13) and (5.14) respectively. Also, E_2 is conserved when $\boldsymbol{u}_{\tau i}^{\text{diss}} = 0$ and $\boldsymbol{u}_{fi}^{\text{diss}} = 0$. Thus, all the conditions in Theorem 3 are satisfied and we conclude that the solution goes to the set $E_2 = 0$, i.e., E_2 decays to zero exponentially. Hence, the solution converges exponentially to the desired relative equilibrium (2.4) and we have proved the following theorem.

Theorem 5. *The steady motion given by* (2.4) *is an exponentiallly stable relative equilibrium for the rigid body $SE(3)^n$ network with equations of motion given by* (5.8) *and* (5.9) *and dissipation chosen as in* (5.13) *and* (5.14) *.*

If one wants the rigid bodies to be synchronized, rotating at k_1 rad/s and translating at k_2 m/s, then all one needs to do is replace $\boldsymbol{e}_1$ in the right side of (5.13) with $k_1\boldsymbol{e}_1$ and replace $\boldsymbol{e}_1$ in the right side of (5.14) with $k_2\boldsymbol{e}_1$.

6. Synchronization of unstable rigid body motions.

In §4 and §5, we proved stability and exponential stability for $SO(3)^n$ and $SE(3)^n$ networks respectively under the assumption that the individual rigid body was rotating about its short axis for the $SO(3)$ case and rotating as well as translating about its short axis for the $SE(3)$ case. We now show how to use kinetic shaping to relax this assumption. We first stabilize the otherwise unstable middle axis motion using kinetic shaping and then superimpose upon this, the controllers derived in §4 and §5. The main idea is to choose controllers such that the closed-loop dynamics are described by rigid body dynamics with modified mass and inertial matrices. This way, we make the middle axis effectively behave like the short axis in the closed-loop dynamics. The particular superposition of the kinetic and potential shaping term in this section is relatively straightforward in comparision with the work in [28] primarily because of the Lie group structure and full actuation.

6.1. Kinetic shaping for $SO(3)^n$.

Synchronized rotation of a network of rigid bodies, each rotating about its middle axis, can be exponentially stabilized by combining the kinetic shaping control law developed in [29] with the potential shaping and dissipative control terms derived in earlier sections. Consider the following controlled equations of motion for a system in $SO(3)$:

$$I\dot{\boldsymbol{\Omega}} = (I\boldsymbol{\Omega})\times\boldsymbol{\Omega} + \boldsymbol{u}_{\tau}^{ks} \tag{6.1}$$

where $\boldsymbol{\Omega}$ and I are moment of inertia matrix and body angular velocity for the single system and superscript "ks" refers to "kinetic shaping". In the above equations, we choose the components of $\boldsymbol{u}_{\tau}^{ks}$ as follows:

$$\begin{aligned} u_{\tau,1}^{ks} &= 0 \\ u_{\tau,2}^{ks} &= \left(I_{33}(\frac{\rho_3}{\rho_2}-1) + I_{11}(1-\frac{1}{\rho_2})\right)\Omega_3\Omega_1 \\ u_{\tau,3}^{ks} &= \left(I_{11}(\frac{1}{\rho_3}-1) + I_{22}(1-\frac{\rho_2}{\rho_3})\right)\Omega_1\Omega_2 \end{aligned} \tag{6.2}$$

where ρ_2 and ρ_3 satisfy the equation

$$\rho_2 I_{22} - \rho_3 I_{33} = I_{22} - I_{33}. \tag{6.3}$$

The closed-loop equations can now be verified to be

$$\bar{I}\dot{\mathbf{\Omega}} = (\bar{I}\mathbf{\Omega}) \times \mathbf{\Omega} \tag{6.4}$$

where $\bar{I}$ is a diagonal matrix with entries $I_{11}, \rho_2 I_{22}, \rho_3 I_{33}$. Therefore, the closed-loop equations correspond to a rigid body with Lagrangian

$$L_c = \frac{1}{2}\mathbf{\Omega}^T \bar{I}\mathbf{\Omega}. \tag{6.5}$$

Since $I_{22} > I_{33}$, we get $\rho_2 I_{22} > \rho_3 I_{33}$. If we now choose

$$\rho_3 > \frac{I_{11}}{I_{33}} \tag{6.6}$$

then we get the following inequality

$$\rho_2 I_{22} > \rho_3 I_{33} > I_{11}.$$

Using kinetic shaping, the open-loop middle axis for the inertia matrix is effectively made the closed-loop short axis in the Lagrangian defining the closed-loop dynamics. We have therefore stabilized the relative equilibrium when the rigid body is rotating about its middle axis.

Now consider the following controlled dynamics for the ith body in the n-body network given by

$$I_i\dot{\mathbf{\Omega}}_i = (I_i\mathbf{\Omega}_i) \times \mathbf{\Omega}_i + \begin{pmatrix} \tilde{u}_{\tau i,1} \\ \tilde{u}_{\tau i,2}/\rho_2 + u^{ks}_{\tau i,2} \\ \tilde{u}_{\tau i,3}/\rho_3 + u^{ks}_{\tau i,3} \end{pmatrix} \tag{6.7}$$

where $\tilde{\boldsymbol{u}}_{\tau i} = (\tilde{u}_{\tau,1}, \tilde{u}_{\tau,2}, \tilde{u}_{\tau,3})^T = \sigma_1(\boldsymbol{u}^{ps}_{\tau,i-1} - \boldsymbol{u}^{ps}_{\tau i}) + \boldsymbol{u}^{\text{diss}}_{\tau i}$ corresponds to the potential shaping and dissipation control terms on the right hand side of (5.2). The kinetic shaping control terms $u^{ks}_{\tau i,2}, u^{ks}_{\tau i,3}$ are as given in (6.2) for the ith body. It can easily be checked that the closed-loop equations now have the form (5.2), but with the original middle axis now the short axis. Hence, we get the following corollary.

Corollary 1. *The steady motion corresponding to n rigid bodies with the same orientation and each rotating about its unstable, middle axis, is an exponentially stable relative equilibrium for the controlled dynamics of (6.7) where the gains ρ_2 and ρ_3 are chosen to satisfy equations* (6.3) *and* (6.6).

6.2. **Kinetic shaping for $SE(3)^n$.** Synchronized rotation and translation of a network of rigid bodies, each rotating about and translating along its middle axis, can be asymptotically stabilized by combining a kinetic shaping control law with the potential shaping and dissipation control terms developed earlier in this paper. The method is analogous to the kinetic shaping controller derived in §6.1 for the $SO(3)^n$ case.

Consider the following controlled equations of motion for a system in $SE(3)$:

$$I\dot{\mathbf{\Omega}} = (I\mathbf{\Omega}) \times \mathbf{\Omega} + (M\boldsymbol{v}) \times \boldsymbol{v} + \boldsymbol{u}^{ks}_{\tau} \tag{6.8}$$

$$M\dot{\boldsymbol{v}} = (M\boldsymbol{v}) \times \mathbf{\Omega} + \boldsymbol{u}^{ks}_{f} \tag{6.9}$$

where I, M are inertia and mass matrices and $\mathbf{\Omega}$, $\boldsymbol{v}$, body angular and linear velocities for the single body. In the above equations, we choose the components of

$\boldsymbol{u}_\tau^{ks}$ and $\boldsymbol{u}_f^{ks}$ as follows:

$$\begin{aligned}
u_{\tau,1}^{ks} &= 0 \\
u_{\tau,2}^{ks} &= \left(I_{33}(\frac{\rho_3}{\rho_2}-1)+I_{11}(1-\frac{1}{\rho_2})\right)\Omega_3\Omega_1+\left(M_{33}(\frac{\bar{\rho}_3}{\rho_2}-1)+M_{11}(1-\frac{1}{\rho_2})\right)v_3v_1 \\
u_{\tau,3}^{ks} &= \left(I_{11}(\frac{1}{\rho_3}-1)+I_{22}(1-\frac{\rho_2}{\rho_3})\right)\Omega_1\Omega_2+\left(M_{11}(\frac{1}{\rho_3}-1)+M_{22}(1-\frac{\bar{\rho}_2}{\rho_3})\right)v_1v_2 \\
u_{f,1}^{ks} &= ((\bar{\rho}_2-1)M_{22}v_2\Omega_3+(1-\bar{\rho}_3)M_{33}v_3\Omega_2) \\
u_{f,2}^{ks} &= \left((\frac{\bar{\rho}_3}{\bar{\rho}_2}-1)M_{33}v_3\Omega_1+(1-\frac{1}{\bar{\rho}_2})M_{11}v_1\Omega_3\right) \\
u_{f,3}^{ks} &= \left((\frac{1}{\bar{\rho}_3}-1)M_{11}v_1\Omega_2+(1-\frac{\bar{\rho}_2}{\bar{\rho}_3})M_{22}v_2\Omega_1\right)
\end{aligned} \tag{6.10}$$

where ρ_2 and ρ_3 satisfy the equation

$$\rho_2 I_{22} - \rho_3 I_{33} = I_{22} - I_{33} \tag{6.11}$$

and $\bar{\rho}_2$ and $\bar{\rho}_3$ satisfy the equation

$$\bar{\rho}_2 M_{22} - \bar{\rho}_3 M_{33} = M_{22} - M_{33}. \tag{6.12}$$

Here, the superscript ks denotes kinetic shaping. The closed-loop equations can be verified to be

$$\begin{aligned}
\bar{I}\dot{\boldsymbol{\Omega}} &= (\bar{I}\boldsymbol{\Omega})\times\boldsymbol{\Omega}+(\bar{M}\boldsymbol{v})\times\boldsymbol{v} \\
\bar{M}\dot{\boldsymbol{v}} &= (\bar{M}\boldsymbol{v})\times\boldsymbol{\Omega}
\end{aligned}$$

where $\bar{I}$ is a diagonal matrix with entries $I_{11}, \rho_2 I_{22}, \rho_3 I_{33}$ and $\bar{M}$ is a diagonal matrix with entries $M_{11}, \bar{\rho}_2 M_{22}, \bar{\rho}_3 M_{33}$. Therefore, the closed-loop equations correspond to a rigid body with Lagrangian

$$L_c = \frac{1}{2}\left(\boldsymbol{\Omega}^T\bar{I}\boldsymbol{\Omega}+\boldsymbol{v}^T\bar{M}\boldsymbol{v}\right). \tag{6.13}$$

Since $I_{22} > I_{33}$ and $M_{22} > M_{33}$, we get $\rho_2 I_{22} > \rho_3 I_{33}$ and $\bar{\rho}_2 M_{22} > \bar{\rho}_3 M_{33}$. If we now choose

$$\rho_3 > \frac{I_{11}}{I_{33}} \tag{6.14}$$

and

$$\bar{\rho}_3 > \frac{M_{11}}{M_{33}}, \tag{6.15}$$

then we get the following two inequalities

$$\begin{aligned}
\rho_2 I_{22} > \rho_3 I_{33} > I_{11} \\
\bar{\rho}_2 M_{22} > \bar{\rho}_3 M_{33} > M_{11}.
\end{aligned}$$

Using kinetic shaping, the open-loop middle axis for both the mass matrix and inertia matrix is effectively made the closed-loop short axis in the Lagrangian defining the closed-loop dynamics. We have therefore stabilized the relative equilibrium when the rigid body is rotating about its middle axis and translating along its middle axis.

Now consider the following controlled dynamics for the i^{th} body given by

$$I_i\dot{\mathbf{\Omega}}_i = (I_i\mathbf{\Omega}_i)\times\mathbf{\Omega}_i + (M_i\boldsymbol{v}_i)\times\boldsymbol{v}_i + \begin{pmatrix} \tilde{u}_{\tau i,1} \\ \tilde{u}_{\tau i,2}/\rho_2 + u^{ks}_{\tau i,2} \\ \tilde{u}_{\tau i,3}/\rho_3 + u^{ks}_{\tau i,3} \end{pmatrix} \tag{6.16}$$

$$M_i\dot{\boldsymbol{v}}_i = (M_i\boldsymbol{v}_i)\times\mathbf{\Omega}_i + \begin{pmatrix} \tilde{u}_{fi,1} + u^{ks}_{fi,1} \\ \tilde{u}_{fi,2}/\bar{\rho}_2 + u^{ks}_{fi,2} \\ \tilde{u}_{fi,3}/\bar{\rho}_3 + u^{ks}_{fi,3} \end{pmatrix} \tag{6.17}$$

where $\tilde{\boldsymbol{u}}_{\tau i} = \sigma_1(\boldsymbol{u}^{ps}_{\tau,i-1} - \boldsymbol{u}^{ps}_{\tau i}) + \boldsymbol{u}^{\text{diss}}_{\tau i}$ and $\tilde{\boldsymbol{u}}_{fi} = \sigma_2\boldsymbol{u}^{ps}_{fj} + \boldsymbol{u}^{\text{diss}}_{fj}$ correspond to the potential shaping and dissipation control terms on the right hand side of (5.8) and (5.9), respectively. The kinetic shaping control terms $\boldsymbol{u}^{ks}_{\tau i}, \boldsymbol{u}^{ks}_{fi}$ are as given in (6.10) for the i^{th} body. It can easily be checked that the closed-loop equations now have the form (5.8) and (5.9) but with the original middle axis now the short axis. Hence, we get the following corollary.

Theorem 6. *The steady motion corresponding to n rigid bodies with the same orientation and position and each rotating about and translating along its otherwise unstable, middle axis, is an exponentially stable relative equilibrium for the controlled dynamics of* (6.16) *and* (6.17) *where the gains $\rho_2, \rho_3, \bar{\rho}_2, \bar{\rho}_3$ are chosen so as to satisfy equations* (6.11)*,* (6.12)*,* (6.14) *and* (6.15)*.*

7. **Conclusions and future directions.** In this paper, we have shown how to exponentially stabilize synchronized motion of a network of n rigid bodies, each with configuration space $SO(3)$ or $SE(3)$. We do this even in the case that the desired steady motion corresponds to an unstable motion for the individual systems. For the network on $SO(3)^n$, synchronized motion corresponds to alignment of orientations of n spinning rigid bodies. For the network on $SE(3)^n$, synchronized motion corresponds to alignment of orientations and positions of n translating and spinning rigid bodies. We design control laws so that the closed-loop dynamics of the network of rigid bodies are Lagrangian dynamics for which nonlinear exponential stability can be proved using known energy methods. For the $SO(3)^n$ network, we illustrate our results with simulations.

There are many directions in which the current work can be extended. One of the immediate tasks is to incorporate collision avoidance into our scheme. See [37, 6], for example, where the authors design collision avoidance schemes for point particles based on gyroscopic forcing. Collision avoidance based on gyroscopic forcing is interesting, both because it preserves energy and because it fits in well with the geometric approach in this paper. It is promising to consider building on these results to design gyroscopic collision avoidance schemes in the rigid body setting.

Another important direction is the extension to time-varying communication topologies. In [32] the authors develop some new results for the $SO(3)^n$ network case using a consensus-based approach. It is also of interest to consider the problem of optimizing our control laws to reduce the control effort. Integrating our treatment with optimization techniques based on mechanical integrators (compare [15] for example) will be a worthwhile effort.

Throughout this paper, we assume a simple diagonal form for the mass and inertial matrices for individual rigid bodies on $SE(3)$ representing underwater vehicles. As was noted in [17], hydrodynamic coupling between bodies plays a crucial role in simple models for aquatic locomotion. It will be interesting to consider the effects of more general interbody hydrodynamic interaction in our setting and to explore if including hydrodynamic interaction leads to control efficiency. See [27] for some efforts along this direction.

Appendix A. **Calculation of $\dot{E}_1$.** We first write down the equations of motion given by (5.2) in inertial coordinates using the fact that $R_i\boldsymbol{\Omega}_i = \boldsymbol{\omega}_i$ and $\dot{R}_i = \widehat{\boldsymbol{\omega}}_i R_i$. The equations of motion in the inertial frame turn out to be

$$\dot{\boldsymbol{\pi}}_i = \sigma_1(\mathfrak{A}_{i-1} - \mathfrak{A}_i) + R_i \boldsymbol{u}_{\tau i}^{diss} \tag{A.1}$$

where $\mathfrak{A}_i = R_i \boldsymbol{u}_{\tau i}^{ps}$ for $i = 1, \dots, n$. Here, $\boldsymbol{u}_{\tau i}^{ps}$ is given as in (5.2) and corresponds to the control on the i^{th} body in the i^{th} body frame and $\mathfrak{A}_i$ is the same control with respect to the inertial frame. Letting $I_i^l = R_i I_i R_i^T$ be the inertia matrix for the i^{th} body in the inertial frame and $\boldsymbol{\pi}_i = I_i^l \boldsymbol{\omega}_i$ the angular momentum of the i^{th} body in the inertial frame, we can write the above equation as

$$I_i^l \dot{\boldsymbol{\omega}}_i + \widehat{\boldsymbol{\omega}}_i I_i^l \boldsymbol{\omega}_i = \sigma_1(\mathfrak{A}_{i-1} - \mathfrak{A}_i) + R_i \boldsymbol{u}_{\tau i}^{diss}. \tag{A.2}$$

Using the expression for E_1 given by (5.3), we can calculate its time derivative to be

$$\begin{aligned} \dot{E}_1 &= \sum_{i=1}^{n} \Big((\boldsymbol{\omega}_i - \boldsymbol{e}_1)^T I_i^l (\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \boldsymbol{e}_1) + \boldsymbol{e}_1^T (I_i^l \widehat{\boldsymbol{\omega}}_i) \boldsymbol{e}_1 \Big) \\ &\quad + \sum_{i=1}^{n-1} \sigma_1 (\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i+1})^T (\mathfrak{A}_i) + \sigma_1 \boldsymbol{\omega}_1^T ((R_1 \boldsymbol{e}_1) \times \boldsymbol{e}_1). \end{aligned} \tag{A.3}$$

Plugging in expression for $I_i^l \dot{\boldsymbol{\omega}}_i$ from (A.2) into (A.3), we get

$$\begin{aligned} &\dot{E}_1 \\ = &\sum_{i=1}^{n} \Big((\boldsymbol{\omega}_i - \boldsymbol{e}_1)^T (-\widehat{\boldsymbol{\omega}}_i I_i^l \boldsymbol{\omega}_i + \sigma_1(\mathfrak{A}_{i-1} - \mathfrak{A}_i) + R_i \boldsymbol{u}_{\tau i}^{diss} + I_i^l \widehat{\boldsymbol{\omega}}_i \boldsymbol{e}_1) + \boldsymbol{e}_1^T (I_i^l \widehat{\boldsymbol{\omega}}_i) \boldsymbol{e}_1 \Big) \\ &+ \sum_{i=1}^{n-1} \sigma_1 (\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i+1})^T (\mathfrak{A}_i) + \sigma_1 \boldsymbol{\omega}_1^T ((R_1 \boldsymbol{e}_1) \times \boldsymbol{e}_1). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \dot{E}_1 &= \sigma_1 \sum_{i=1}^{n} \boldsymbol{\omega}_i^T (\mathfrak{A}_{i-1} - \mathfrak{A}_i) + \sigma_1 \sum_{i=1}^{n-1} (\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i+1})^T (\mathfrak{A}_i) + \sigma_1 \boldsymbol{\omega}_1^T ((R_1 \boldsymbol{e}_1) \times \boldsymbol{e}_1) \\ &\quad + \sum_{i=1}^{n} \Big(\boldsymbol{e}_1 \widehat{\boldsymbol{\omega}}_i I_i^l \boldsymbol{\omega}_i + \boldsymbol{\omega}_i^T I_i^l \widehat{\boldsymbol{\omega}}_i \boldsymbol{e}_1 - \boldsymbol{e}_1^T I_i^l \widehat{\boldsymbol{\omega}}_i \boldsymbol{e}_1 + \boldsymbol{e}_1^T I_i^l \widehat{\boldsymbol{\omega}}_i \boldsymbol{e}_1 \Big) \\ &\quad + \sigma_1 \boldsymbol{e}_1^T \sum_{i=1}^{n} (\mathfrak{A}_{i-1} - \mathfrak{A}_i) + \sum_{i=1}^{n} \Big((\boldsymbol{\omega}_i - \boldsymbol{e}_1)^T R_i \boldsymbol{u}_{\tau i}^{diss} \Big). \end{aligned} \tag{A.4}$$

Rearranging the above equation, we have

$$\begin{aligned} \dot{E}_1 &= \sigma_1 \boldsymbol{\omega}_1^T \mathfrak{A}_0 + \sigma_1 \boldsymbol{e}_1^T \mathfrak{A}_0 + \sigma_1 \boldsymbol{\omega}_1^T ((R_1 \boldsymbol{e}_1) \times \boldsymbol{e}_1) \\ &\quad + \sum_{i=1}^{n} \Big((\boldsymbol{\omega}_i - \boldsymbol{e}_1)^T R_i \boldsymbol{u}_{\tau i}^{diss} \Big). \end{aligned} \tag{A.5}$$

Using the fact that $\mathfrak{A}_0 = -((R_1\boldsymbol{e}_1) \times \boldsymbol{e}_1)$, the first line in (A.5) vanishes. Hence, we get the following expression for $\dot{E}_1$:

$$\dot{E}_1 = \sum_{i=1}^{n}\Big((\boldsymbol{\omega}_i - \boldsymbol{e}_1)^T R_i \boldsymbol{u}_{\tau i}^{diss}\Big). \tag{A.6}$$

REFERENCES

[1] C. Belta and V. Kumar, *Motion generation for groups of robots*, ASME Journal of Mechanical Design, (2003).

[2] A. K. Bondhus, K. Y. Pettersen and J. T. Gravdahl, *Leader/follower synchronization of satellite attitude without angular velocity measurements*, In "Proc. IEEE Conf. Decision and Control and European Control Conf.", (2005), 7270–7277.

[3] F. Bullo, *Stabilization of relative equilibria for underactuated systems on Riemannian manifolds*, Automatica, **36** (2000), 1819–1834.

[4] F. Bullo, *Nonlinear control of mechanical systems: A Riemannian geometry approach*, PhD thesis, California Institute of Technology, (2001).

[5] H. Cendra, J. E. Marsden and T. S. Ratiu, *Lagrangian reduction by stages*, Memoirs of American Mathematical Society, **152** (2001).

[6] D. E. Chang, S. C. Shadden, J. E. Marsden and R. Olfati-Saber, *Collision avoidance for multiple agent systems*, In "Proc. IEEE Conf. Decision and Control", Maui, Hawaii, (2003).

[7] J. P. Desai, J. P. Ostrowski and V. Kumar, *Modeling and control of formations of nonholonomic mobile robots*, IEEE Trans. Robotics and Automation, **17** (2001), 905–908.

[8] J. Fax and R. Murray, *Information flow and cooperative control of vehicle formations*, IEEE Trans. Automatic Control, **49** (2004), 1465–1476.

[9] E. Fiorelli, N. E. Leonard, P. Bhatta, D. Paley, R. Bachmayer and D. M. Frantonti, *Multi-AUV control and adaptive sampling in Monterey Bay*, IEEE Journal of Oceanic Engineering, **31** (2006), 935–948.

[10] H. Hanssmann, N. E. Leonard and T. R. Smith, *Symmetry and reduction for coordinated rigid bodies*, European Journal of Control, **12** (2006), 176–194.

[11] P. Holmes, J. Jenkins and N. E. Leonard, *Dynamics of the Kirchhoff equations I: Coincident centers of gravity and buoyancy*, Physica D, **118** (1998), 311–342.

[12] S. P. Hughes and C. D. Hall, *Optimal configurations of rotating spacecraft formations*, Journal of the Astronautical Sciences, **48** (2000), 225–247.

[13] D. Izzo and L. Pettazzi, *Equilibrium shaping: Distributed motion planning for satellite swarm*, In "Proc. 8th International Symposium on Artificial Intelligence, Robotics and Automation in Space", 2005.

[14] A. Jadbabaie, J. Lin and A. S. Morse, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Trans. Aut. Control, **48** (2003), 988–1001.

[15] O. Junge, J. E. Marsden and S. Ober-Blöbaum, *Discrete mechanics and optimal control*, In "Proc. 16th IFAC Conf. on Decision and Control", Prague, Czech, (2005).

[16] E. Justh and P. S. Krishnaprasad, *Equilibria and steering laws for planar formations*, Systems and Control Letters, **52** (2004), 25–38.

[17] E. Kanso, J. E. Marsden, C. W. Rowley and J. Melli-Huber, *Locomotion of articulated bodies in a perfect fluid*, Journal of Nonlinear Science, **15** (2005), 255–289.

[18] T. Krogstad and J. Gravdahl, *Coordinated attitude control of satellites in formation*, **336** of "Lecture Notes in Control and Information Sciences", chapter 9, 153–170, Springer-Verlag, 2006.

[19] J. Lawton and R. W. Beard, *Synchronized multiple spacecraft rotations*, Automatica, **38** (2002), 1359–1364.

[20] N. E. Leonard, *Stabilization of underwater vehicle dynamics with symmetry-breaking potentials*, Systems and Control Letters, **32** (1997), 35–42.

[21] N. E. Leonard and E. Fiorelli, *Virtual leaders, artificial potentials and coordinated control of groups*, In "Proc. IEEE Conf. Decision and Control", (2001), 2968–2973.

[22] J. E. Marsden, "Lectures on Mechanics," Cambridge University Press, New York, 1992.

[23] J. E. Marsden and T. S. Ratiu, "Introduction to Mechanics and Symmetry," Springer-Verlag, New York, 1994.

[24] Y. Matsumoto, "An introduction to Morse theory," American Mathematical Society, Translations of Mathematical Monographs, **208**, 2001.

[25] C. R. McInnes, *Potential function methods for autonomous spacecraft guidance and control*, In "AAS Paper 95-447, AAS/AIAA Astrodynamics Specialist Conference", Halifax, Nova Scotia, 1995.
[26] S. Nair, *Stabilization and Synchronization of Networked Mechanical Systems*, PhD thesis, Princeton University, Princeton, NJ, 2006.
[27] S. Nair and E. Kanso, *Hydrodynamically coupled rigid bodies*, Journal of Fluid Mechanics, To appear.
[28] S. Nair and N. E. Leonard, *Stable synchronization of mechanical system networks*, SIAM J. Control and Optimization, To appear.
[29] S. Nair and N. E. Leonard, Stabilization of a coordinated network of rotating rigid bodies, In *Proc. IEEE Conf. Decision and Control*, 4690–4695, 2004.
[30] R. Olfati-Saber, *Nonlinear Control of Underactuated Mechanical Systems with Application to Robotics and Aerospace Vehicles*, PhD thesis, Massachusetts Institute of Technology, 2001.
[31] R. Olfati-Saber and R. M. Murray, *Graph rigidity and distributed formation stabilization of multi-vehicle systems*, In "Proc. IEEE Conf. Decision and Control", (2002), 2965–2971.
[32] A. Sarlette, R. Sepulchre and N. E. Leonard, *Autonomous rigid body attitude synchronization*, In "Proc. IEEE Conf. Decision and Control", 2007.
[33] R. Sepulchre, D. Paley and N. E. Leonard, *Stabilization of planar collective motion: All-to-all communication*, IEEE Trans. Automatic Control, **52** (2007), 811–824.
[34] T. R. Smith, H. Hanssmann and N. E. Leonard, *Orientation control of multiple underwater vehicles with symmetry-breaking potentials*, In "Proc. IEEE Conf. Decision and Control", 4598–4603, (2001).
[35] E. D. Sontag, *Comments on integral variants of ISS*, Systems and Control Letters, **34** (1998), 93–100.
[36] M. C. VanDyke and C. D. Hall, *Decentralized coordinated attitude control of a formation of spacecraft*, Journal of Guidance, Control and Dynamics, **29** (2006), 1101–1109.
[37] F. Zhang, E. Justh and P. S. Krishnaprasad, *Boundary following using gyroscopic control*, In "Proc. IEEE Conf. Decision and Control", (2004), 5204–5209.

Received June 2007; revised September 2007.

E-mail address: `nair@alumni.princeton.edu`
E-mail address: `naomi@princeton.edu`