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EQUILIBRIA FOR DATA NETWORKS

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Abstract. This paper investigates equilibrium solutions for data flows on a network. We consider a fluid dynamic model based on conservation laws. The dynamics at nodes is solved by FIFO policy combined with through flux maximization. We first link the dimension of the equilibria space to topological properties of the graph associated to the network. Then we focus on regular plane tilings with square or triangular cells. For various networks, we completely determine the characteristics of periodic equilibria and, in some cases, of all equilibria. The obtained results are expected to play a role both in the analysis of asymptotic behavior of network load and for security issues in case of node failures.

1. Introduction.

In this paper we consider the problem of identifying equilibrium solutions of flows on data networks. Our analysis is based on a fluid dynamic model, introduced in [12], for the data flow, encoded in packets, and is motivated by three main scopes.

- The set of equilibrium solutions play a role in understanding the asymptotic behavior of every solution.
- The analysis of equilibria allows comparisons of the fluid dynamic model with other models.
- To investigate security issues, one method is to study the dynamics stemming from node failures of an equilibrium solution.

The main results we obtain in this paper are the followings.

- We determine the dimension of the equilibria space and link it to the topological properties of the network.
- We give an explicit description of the equilibrium solutions for square and triangular networks.
- For square and triangular networks we give a map assigning a unique equilibrium for each set of inflows of the network.

Looking at intermediate time scales, the model of [12], on a single arc of the network, consists of a single conservation law:

$$\rho_t + f(\rho)_x = 0, \qquad \rho \in [0, \rho_{max}],$$

where ρ represents the packets density, ρ_{max} its maximal value, while the flux f is determined by a loss probability function and is usually assumed to be concave.

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Then, the dynamics at nodes is determined assuming a FIFO policy and maximization of the through flux, see section 2 for details.

Such model was inspired by similar ones, used in car traffic, and is useful for understanding the formation of congestions and their propagation along the network. On the other side the model is limited, in its description capacity, by the absence of stochastic sources and the possibility of strong packets redirections.

There are many alternative models for data flows, in particular for Internet. A complete account of the existing literature is beyond the scope of the paper, however some overview can be found, for instance, in [1, 5, 16].

The study of asymptotic behavior for conservation laws deserved much attention. For instance, Liu [18] studied the decay of solutions to the so called N-waves. On the other side, the scalar case, on a single line, is quite well understood, see [21].

Similar studies appear to be much harder in case of networks and, to our knowledge, this is the first paper addressing the issue of equilibria on networks.

Our definition of equilibrium is that of a solution, on the whole network, which is constant in time. We assume that the flux f admits a unique maximum $\sigma \in [0, \rho_{max}]$ and vanishes at extreme points, i.e. $f(0) = f(\rho_{max}) = 0$. A consequence is that there exist shocks with sonic, i.e. zero, velocity. Thus an equilibrium may well exhibit an infinite number of shocks inside each line. Still we are interested in the flux and density values that equilibria takes at nodes. The former are, in fact, constant on each line and the latter are called the equilibria values.

Each network is represented by a collection of lines, modeled by real intervals, and nodes at which lines intersect. It is then natural to associate to each network a topological graph. Also, by means of well established graph theory results, such topological graphs can be embedded in manifolds in a unique way once we fix a rotation system. The latter consists in a cyclic order of lines at each node.

To determine the set of equilibria values, we consider the space of flux values as variables. Thus we have one unknown for each line, while to be in equilibrium the set of unknowns must satisfy a linear relation at each node. The dimension of the space of equilibria values is readily computed, see proposition 3. Moreover, such dimension can be related to the number of faces of the embedded topological graph, see proposition 4.

To each vector of equilibria values, it corresponds at least one equilibrium solution. However, we are more interested in those solutions, whose density is constant along each line. To have this property, an equilibrium should also respect some rules at nodes. More precisely, one defines *bad* and *good* values for incoming and outgoing lines at a node depending if the density is lower or greater than the value σ (of maximum flux), see definition 2. Then, only some combinations of *bad* and *good* values at a node are possible.

Most of the paper then focuses on the analysis of equilibrium solutions, with constant densities along lines, for graphs giving a regular square or triangular tessellation of the plane. More precisely, we consider a Manhattan type tiling and three different networks: one, called "Oriented", with oriented lines and data flowing always up and right, the second, called "Circular", with oriented lines but flow in any direction, finally the "Full" Manhattan with non-oriented flows (modeled by a couple of lines for each edge of the tiling.) Then we deal with triangular tilings for the Oriented and Circular case.

The analysis of equilibria is carried out in the following way. First, we determine the possible types of equilibria at each node, depending on the combination of *good* and *bad* values at the incident lines. Then, we deduce the types for subnetworks consisting of four or nine nodes, which are considered the building blocks of the whole network. Each building block is connected to another one by horizontal (resp. vertical) superposition of a column (resp. row) of nodes.

Such compositions are represented by means of other graphs, where the vertices are types for lines and rows, while an arc is drawn each time two lines or rows can form a building block. Such representation is advantageous, in fact solutions along horizontal or vertical stripes correspond to paths along these new graphs. Also, periodic solutions correspond to cycles in the new graphs, see theorem 2.

For the Oriented Manhattan, the graphs presents 48 vertices and 65 edges. There are some periodic solutions for horizontal and vertical stripe. However, such stripes do not combine to form a whole equilibrium except trivial cases. More precisely, there exist only two periodic equilibrium types, whose node types are in fact constant on the whole network, see theorem 3.

Then we analyze the Circular Manhattan case. In this case the graphs are even larger (with hundreds of nodes.) However, it is possible to make a direct analysis of building blocks combinations, without explicitly constructing the graphs. As a result, not only we are able to show the existence of only two periodic equilibrium types (with constant node type), but also to prove that all equilibrium types are periodic, see theorem 4. Therefore there exist only two equilibrium types. Finally, we can apply the same strategy to the Full Manhattan case getting an entirely similar result, see theorem 5.

Then triangular tilings are addressed. For the oriented case, we have to construct the two graphs. The set of paths on each graph is reacher than in the Manhattan case (see bold arcs and diamond shape nodes in figures 6 and 7). However, the set of periodic equilibrium types is still comprised of two elements with constant node type, see theorem 6.

Finally, for the Circular Triangular case, we drive the same conclusions as for the Circular Manhattan case, see theorem 7.

The paper is organized as follows. In section 2, we introduce the model and give basic definition and results for equilibrium solutions. Section 3 links the set of equilibria with the topological properties of the graph linked to a network. In section 4 we illustrate the plane tilings we consider in the paper and the main tools to investigate equilibria. Then section 5 deals with the Manhattan tilings and, finally, section 6 deals with the Triangular ones. In section 7 we use the results of sections 5 and 6 to refine the results of section 3 on the space of equilibria solutions.

2. **Basics.** A network is formed by a finite collection of transmission lines and nodes (or routers), each packet is seen as a particle on the network and it is assumed that each packet travels on the network with fixed speed and assigned final destination. Moreover it is assumed that routers receive, process and then forward packets. Packets may be lost but, in this case, they are resent by the router. Then we look at an intermediate time scale and assume conservation of packets and get the following simple model consisting of a single conservation law:

$$\rho_t + f(\rho)_x = 0, \tag{1}$$

where ρ is the packet density, v is the velocity and $f(\rho) = v\rho$ is the flux.

We model a telecommunication network by a finite set of intervals $I_i = [a_i, b_i] \subset \mathbb{R}, i = 1, ..., L, a_i < b_i$, on which we consider the equation (1). Hence the datum is given by a finite set of functions ρ_i defined on $[0, +\infty[\times I_i$.

We assume that the transmission lines are connected by some nodes. Each node J is given by a finite number of incoming transmission lines and a finite number of outgoing transmission lines, thus we identify J with $((i_1, ..., i_m), (j_1, ...j_n))$ where the first m-tuple indicates the set of incoming transmission lines and the second n-tuple indicates the set of outgoing transmission lines. Each transmission line can be incoming transmission line at most for one node and outgoing at most for one node. Hence the complete model is given by a couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I} = \{I_i : i = 1, ..., L\}$ is the collection of transmission lines and $\mathcal{J}$ is the collection of nodes. We set N to be the cardinality of $\mathcal{J}$.

In order to consider complex networks, one needs a way of solving dynamics at nodes in which many lines (backbones) intersect. For this, we follow the strategy proposed by [12], and consider the routing algorithm:

RA Packets are processed by arrival time and are sent to outgoing lines in order to maximize the flux.

A key role is played by Cauchy problems with initial data constant on each transmission line called Riemann problems at the node. In order to determine unique solutions to Riemann problems, some additional parameters are introduced, called respectively priority parameters and traffic distribution parameters. The theory for this model is developed in [12].

On each line I_i the evolution is given by equation (1) and we assume that the flux f is a strictly concave function (with $f(0) = f(\rho_{max}) = 0$), thus there exists a unique $\sigma \in [0, \rho_{max}]$ such that $f'(\sigma) = 0$ and is the maximum of f over $[0, \rho_{max}]$. For notational simplicity, we assume, without loss of generality, that $\rho_{max} = 1$.

Definition 1. We let $\tau : [0, 1] \to [0, 1]$ be the map such that $f(\rho) = f(\tau(\rho))$ and $\tau(\rho) \neq \rho$ if $\rho \neq \sigma$. Thus τ sends ρ to the other density value with the same flux (and $\tau(\sigma) = \sigma$.)

For a simple network formed of a single node with m incoming and n outgoing lines, once the packet quantities flowing from initial to final nodes are assigned, the final equilibrium as function of the traffic distribution (and priority) parameters can be computed as follows.

We have only m priority parameters $p \in \,]0, 1[$ and n traffic distribution parameters $\alpha \in \,]0, 1[$. We denote with $\rho_\varphi(t, x)$, $\varphi = 1, \dots, m$, and $\rho_\psi(t, x)$, $\psi = m + 1, \dots, m + n$, the traffic densities, respectively, on the incoming transmission lines and on the outgoing ones and by $(\rho_{\varphi,0}, \rho_{\psi,0})$ the initial data. Since the speed of waves must be negative on incoming lines and positive on outgoing ones, we want to determine a unique $(m + n)$-tuple $(\hat{\rho}_1, ..., \hat{\rho}_{m+n}) \in [0, 1]^{m+n}$ such that

$$\hat{\rho}_\varphi \in \begin{cases} \{\rho_{\varphi,0}\} \cup \,]\tau(\rho_{\varphi,0}), 1], & \text{if } 0 \leq \rho_{\varphi,0} < \sigma, \\ [\sigma, 1], & \text{if } \sigma \leq \rho_{\varphi,0} \leq 1, \end{cases} \tag{2}$$

$\varphi = 1, ..., m$, and

$$\hat{\rho}_\psi \in \begin{cases} [0, \sigma], & \text{if } 0 \leq \rho_{\psi,0} \leq \sigma, \\ \{\rho_{\psi,0}\} \cup [0, \tau(\rho_{\psi,0})[, & \text{if } \sigma < \rho_{\psi,0} \leq 1, \end{cases} \tag{3}$$

$\psi = m+1, ..., m+n$, and on each incoming line I_φ, $\varphi = 1, ..., m$, the solution consists of the single wave $(\rho_{\varphi,0}, \hat{\rho}_\varphi)$, while on each outgoing line I_ψ, $\psi = m + 1, ..., m + n$, the solution consists of the single wave $(\hat{\rho}_\psi, \rho_{\psi,0})$.

Define $\gamma_\varphi^{\max}$ and $\gamma_\psi^{\max}$ as follows:

$$\gamma_\varphi^{\max} = \begin{cases} f(\rho_{\varphi,0}), & \text{if } \rho_{\varphi,0} \in [0,\sigma[, \\ f(\sigma), & \text{if } \rho_{\varphi,0} \in [\sigma,1], \end{cases} \quad \varphi = 1,\dots,m, \tag{4}$$

and

$$\gamma_\psi^{\max} = \begin{cases} f(\sigma), & \text{if } \rho_{\psi,0} \in [0,\sigma], \\ f(\rho_{\psi,0}), & \text{if } \rho_{\psi,0} \in]\sigma,1], \end{cases} \quad \psi = m+1,\dots,m+n. \tag{5}$$

The quantities $\gamma_\varphi^{\max}$ and $\gamma_\psi^{\max}$ represent the maximum flux that can be obtained by a single wave solution on each transmission line. In order to maximize the number of packets through the node over incoming and outgoing lines we define

$$\Gamma = \min\{\Gamma_{in}, \Gamma_{out}\},$$

where $\Gamma_{in} = \sum_{\varphi=1}^{m} \gamma_\varphi^{\max}$ and $\Gamma_{out} = \sum_{\psi=m+1}^{m+n} \gamma_\psi^{\max}$. One can easily see that, to solve the Riemann problem, it is enough to determine the fluxes $\hat{\gamma}_\varphi = f(\hat{\rho}_\varphi), \varphi = 1,\dots,m$, and $\hat{\gamma}_\psi = f(\hat{\rho}_\psi), \psi = m+1,\dots,m+n$. Let us determine $\hat{\gamma}_\varphi, \varphi = 1,\dots,m$. We have to distinguish two cases:

I: $\Gamma_{in} = \Gamma$,

II: $\Gamma_{in} > \Gamma$.

In the first case we set $\hat{\gamma}_\varphi = \gamma_\varphi^{\max}, \varphi = 1,\dots,m$. Let us analyze the second case in which we use the priority parameters $p_1,\dots,p_m$ where $0 < p_\varphi < 1$ and $\sum_{\varphi=1}^{m} p_\varphi = 1$. Not all packets can enter the node, so let C be the amount of packets that can go through. Then $p_\varphi C$ packets come from the φ–st incoming line. Consider the space $(\gamma_1,\dots,\gamma_m)$ and denote by P the point with coordinates $\gamma_\varphi = p_\varphi \Gamma$. Now the final fluxes should belong to the region:

$$\Omega = \left\{(\gamma_1,\dots,\gamma_m) : 0 \le \gamma_\varphi \le \gamma_\varphi^{\max}, \varphi = 1,\dots,m\right\}.$$

We distinguish two cases:

a) P belongs to Ω,

b) P is outside Ω.

In the first case we set $(\hat{\gamma}_1,\dots,\hat{\gamma}_m) = P$, while in the second case we set $(\hat{\gamma}_1,\dots,\hat{\gamma}_m) = Q$, with $Q = proj(P)$ where $proj$ is some projection on Ω. From the choice of this projection the analysis and the choice of the parameters $p_1,\dots,p_m$ can be very different. The most natural projection to take is the projection on a convex set (see [19]).

Let us now determine $\hat{\gamma}_\psi, \psi = m+1,\dots,m+n$. As for the incoming transmission lines we have to distinguish two cases :

I: $\Gamma_{out} = \Gamma$,

II: $\Gamma_{out} > \Gamma$.

In the first case $\hat{\gamma}_\psi = \gamma_\psi^{\max}, \psi = m+1,\dots,m+n$. Let us determine $\hat{\gamma}_\psi$ in the second case in which we use the traffic distribution parameters $\alpha_{m+1},\dots,\alpha_{m+n}$ where $\alpha_\psi \in]0,1[$ and $\sum_{\psi=m+1}^{n} \alpha_\psi = 1$. Since not all packets can go on the outgoing transmission lines, we let C be the amount that goes through. Then $\alpha_\psi C$ packets go on the outgoing line I_ψ. Consider the space $(\gamma_{m+1},\dots,\gamma_{m+n})$ and denote by P the point with coordinates: $\gamma_\psi = \alpha_\psi \Gamma$.

Now the final fluxes should belong to the region:

$$\Omega = \left\{(\gamma_{m+1},\dots,\gamma_{m+n}) : 0 \le \gamma_\psi \le \gamma_\psi^{\max}, \psi = m+1,\dots,m+n\right\}.$$

We distinguish two cases:

a) P belongs to Ω

b) P is outside Ω.

In the first case we set $(\hat{\gamma}_{m+1},\dots,\hat{\gamma}_{m+n}) = P$, while in the second case we set $(\hat{\gamma}_{m+1},\dots,\hat{\gamma}_{m+n}) = Q$, where $Q = proj(P)$.

The solution to the Riemann Problem $((\hat{\rho}_1,\dots,\hat{\rho}_m),(\hat{\rho}_{m+1},\dots,\hat{\rho}_{m+n}))$ is computed from the equilibria fluxes $((\hat{\gamma}_1,\dots,\hat{\gamma}_m),(\hat{\gamma}_{m+1},\dots,\hat{\gamma}_{m+n}))$ by taking the unique solution of equations $\hat{\gamma}_\varphi = f(\hat{\rho}_\varphi),\ \varphi = 1,\dots,m$ and $\hat{\gamma}_\psi = f(\hat{\rho}_\psi),\ \psi = m+1,\dots,m+n$ such that conditions (2) and (3) are satisfied.

Definition 2. A component of the solution at one node, $\hat{\rho}_\varphi,\ \varphi = 1,\dots,m$, is

bad: if $\hat{\rho}_\varphi \in [0,\sigma[$;
good: if $\hat{\rho}_\varphi \in [\sigma,1]$;

and a component of the solution, $\hat{\rho}_\psi,\ \psi = m+1,\dots,m+n$ is

bad: if $\hat{\rho}_\psi \in]\sigma,1]$;
good: if $\hat{\rho}_\psi \in [0,\sigma]$.

Remark. We notice that if there exists one index $\bar{\varphi}$ for which $\hat{\rho}_{\bar{\varphi}}$ is *good* then, for all $\psi = m+1,\dots,m+n$, it must be $\hat{\rho}_\psi$ *bad*. Indeed $\hat{\rho}_{\bar{\varphi}}$ *good* means that $\Gamma_{in} > \Gamma = \Gamma_{out}$ and $\hat{\gamma}_{\bar{\varphi}} < \gamma_{\bar{\varphi}}^{max}$. Therefore, for all ψ, $\hat{\gamma}_\psi = \gamma_\psi^{max}$ and $\hat{\rho}_\psi$ is *bad*.
The viceversa also holds: if an index $\bar{\psi}$ exists such that $\hat{\rho}_{\bar{\psi}}$ is *good* then, for all $\varphi = 1,\dots,m$, it must be that $\hat{\rho}_\varphi$ *bad*.

We are now interested in solutions over the whole network $(\mathcal{I},\mathcal{J})$, not only on solutions at one single node. More precisely we are interested in equilibrium solutions.

Definition 3. An equilibrium is a solution $\rho(t,x) = (\rho_1,\dots,\rho_L)$ (recall that L is the cardinality of $\mathcal{I}$), which is constant in time. We also assume that $\rho(t,\cdot)$ is BV, thus we can define, for every $i = 1,\dots,L$, the values $\rho_i^- = \lim_{x\to a_i} \rho(t,x)$ and $\rho_i^+ = \lim_{x\to b_i} \rho(t,x)$.
Since ρ is a solution then

$$\sum_{\varphi=1}^{m} f(\rho_{j_\varphi}) = \sum_{\psi=m+1}^{m+n} f(\rho_{j_\psi}), \tag{6}$$

is satisfied at each node $J_j \in \mathcal{J},\ j = 1,\dots,N$. In (6) we have denoted by $\rho_{j_1},\dots,\rho_{j_m},\rho_{j_{m+1}},\dots,\rho_{j_{m+n}}$ the densities along the m incoming lines $I_{j_1},\dots,I_{j_m}$ and the n outgoing lines $I_{j_{m+1}},\dots,I_{j_{m+n}}$ at node J_j.
We distinguish two cases

- **i:** there exists $i = 1,\dots,L$, such that $\rho_i^- \neq \rho_i^+$. In this case, $\rho_i^+ = \tau(\rho_i^-)$ and the fluxes $\gamma_i = f(\rho_i^\pm)$, are anyhow constant in time and along the whole line I_i;
- **ii:** for all $i = 1,\dots,L$, $\rho_i^+ = \rho_i^-$ and we call this value ρ_i.

Definition 4. Let $\rho = (\rho_1,\dots,\rho_L)$ be an equilibrium for the network $(\mathcal{I},\mathcal{J})$, satisfying **ii**. We say that $\rho_1,\dots,\rho_L$ are the *values of the equilibrium.* Moreover, if ρ_i is of type τ_i, with $\tau_i \in \{bad, good\}$, then we say that $T = (\tau_1,\dots,\tau_L)$ is the *equilibrium type.*

In section 3 we give a brief description of the system that one has to solve to find possible equilibria values over the whole network, while, in sections 5 and 6, we describe all admissible equilibrium types.

We then consider an equilibrium at one node, which is a $m+n$–vector, and describe componentwise its type *good* or *bad*. We then have that the possible equilibrium types at one node are one among the followings (we use the short notations b for *bad* and g for *good*):

I**:** $((b,\dots,b),(b,\dots,b))$
$II0$**:** $((b,\dots,b),(g,\dots,g))$
$II1.h$**:** $((b,\dots,b),(g,\dots,b,\dots,g))$ (there is only one *bad* in the $m+h$ position).
$II2.h_1.h_2$**:** $((b,\dots,b),(g,\dots,b,\dots,b,\dots,g))$ (there are two *bads* in the $m+h_1$ and $m+h_2$ positions).

$\vdots$

$II(n-1).h_1.\cdots.h_{n-1}$**:** $((b,\dots,b),(b,\dots,g,\dots,b))$ (there are $n-1$ *bads* in the $m+h_1,\dots,m+h_{n-1}$ positions).
$III0$**:** $((g,\dots,g),(b,\dots,b))$
$III1.h$**:** $((g,\dots,b,\dots,g),(b,\dots,b))$ (there is only one *bad* in the h position).
$III2.h_1.h_2$**:** $((g,\dots,b,\dots,b,\dots,g),(b,\dots,b))$ (there are two *bads* in the h_1 and h_2 positions).

$\vdots$

$III(m-1).h_1.\cdots.h_{m-1}$**:** $((b,\dots,g,\dots,b),(b,\dots,b))$ (there are $m-1$ *bads* in the $h_1,\dots,h_{m-1}$ positions).

Definition 5. We denote by $\mathcal{M} = \{I, II0, II1.h, \dots II(n-1).h_1.\cdots.h_{n-1}, III0, III1.h, \dots III(m-1).h_1.\cdots.h_{m-1}\}$, the set of all possible equilibria types at one node.

The following proposition trivially holds

Proposition 1. *The cardinality of* $\mathcal{M}$ *is*

$$M = 1+2+\binom{n}{1}+\cdots+\binom{n}{n-1}+\binom{m}{1}+\cdots+\binom{m}{m-1} = 2^m+2^n-1.$$

Consider now a network comprised of N nodes $J_1,\dots,J_N$. For simplicity we assume that each node has m incoming and n outgoing lines.

Definition 6. We denote by

$$\widehat{\mathcal{N}} = \{(j_1,\dots,j_N), j_1,\dots,j_N \in \mathcal{M}\} = \mathcal{M}^N,$$

the set of all possible equilibria types over the whole network.

Clearly the cardinality of $\widehat{\mathcal{N}}$ is M^N. However, as it will be more clear in a while, not all the elements of $\widehat{\mathcal{N}}$ may arise.
Indeed, by definition 3, two kinds of equilibria over the whole network may be considered. In case **ii** the following compatibility rule must be satisfied:

H: if a line I_i is incoming for some node J_1 and outgoing for some other node J_2 then the following holds. Whenever $\hat{\rho}_i$ is of type *bad* for J_1 then it must be of type *good* for J_2 and viceversa.

Rule **H** gives rise to a compatibility relation among equilibria at adjacent nodes. Such compatibility relation in turns determines the subset $\mathcal{N} \subset \widehat{\mathcal{N}}$ of admissible equilibrium states for the whole network.

This fact does not hold for case **i.**. Indeed a shock wave along the line I_i may transform the density $\hat{\rho}_i$ into $\tau(\hat{\rho}_i)$ while keeping constant the flux $\hat{\gamma}_i$. Therefore $\hat{\rho}_i$ of type *good* or *bad* as incoming for J_1 does not influence the type of $\hat{\rho}_i$ as outgoing density for J_2. No compatibility relation among adjacent nodes can be deduced hence the set of possible equilibrium states for the whole network is $\mathcal{N} = \widehat{\mathcal{N}}$.

We have shown the following:

Proposition 2. *In case we consider equilibria* **i.** *then* $\mathcal{N} = \widehat{\mathcal{N}}$. *If otherwise we consider equilibria* **ii.** *then* $\mathcal{N} \subsetneq \widehat{\mathcal{N}}$.

In the rest of the paper we will consider the equilibria **ii.** and give a characterization of $\mathcal{N}$.

3. **Equilibria solutions.** In this paper we consider networks $(\mathcal{I}, \mathcal{J})$ with N the cardinality of $\mathcal{J}$ and L the cardinality of $\mathcal{I}$.

The following holds.

Proposition 3. *The set of equilibrium values is a* $N - L$ *dimensional subspace of* $\mathbb{R}^L$.

Proof. At the equilibrium, on each line $I \in \mathcal{I}$ there is a density $\hat{\rho}$ among two adjacent nodes which is constant in time and along the line I. Moreover, at each node $J \in \mathcal{J}$ the constraint (6) is satisfied, i.e. at J the total flux incoming from incoming lines, must be equal to the total flux departing through the outgoing lines. Therefore the equilibrium value $\rho = (\rho_1, \ldots, \rho_L)$ must satisfy one constraint (6) for each node $J \in \mathcal{J}$, for a total of N constraints, i.e. the equilibrium is given by solving a system of N equations in L variables. $\square$

Now, the degree of freedom $L - N$ of the system described above is strictly related to the topological structure of the network.

Indeed a network $(\mathcal{I}, \mathcal{J})$ can be seen as an oriented graph where $\mathcal{I}$ is the set of edges and $\mathcal{J}$ is the set of vertices. Moreover a graph can be endowed with a topological structure consisting of a Hausdorff space X and a closed discrete subspace X^0. A point of X^0 is called a vertex of X. The complementary set $X \setminus X^0$ is a disjoint union of open subsets e_i. Every e_i is homeomorphic to an open interval $I \subset \mathbb{R}$ and is called an edge of X.

An embedding $i : X \to M$ of a graph X into a surface M is a $1-1$ continuous map of the topological space X into the topological space M. Two embeddings i_1 and i_2 of X into a surface M are equivalent if there exists a homeomorphism $h : M \to M$ such that $h \circ i_1 = i_2$ (in other words, h brings the image $i_1(X)$ to the image $i_2(X)$).
Two graphs X_1 and X_2 are equivalent if there exist two embeddings $i_1 : X_1 \to M$, $i_2 : X_2 \to M$ and a homeomorphism $h : M \to M$ such that $h \circ i_1(X_1) = i_2(X_2)$.

An orientable surface is described by a genus $g \geq 0$ which counts the number of handles glued into the plane. Then, the genus of a graph is the minimum number of handles that must be added to the plane to embed the graph without any crossings. A graph is planar if it can be drawn in a plane without graph edges crossing (i.e., it has graph genus 0). A graph with genus g is then a graph that can be drawn on a surface of genus g without edges crossings.

If one takes an embedding $i : X \to M$ of a connected graph X into M, then the set $M \backslash i(X)$ is a union of open regions f_m. Each f_m is called face of X. Clearly, gluing up handles to each f_m, it is possible to obtain embeddings of X into the surfaces of an arbitrary high genus. An embedding $i : X \to M$ is called *2-cell (or cellular)*, if all open regions f_m are homeomorphic to an open disc. Further we consider both cellular embeddings or *not*.

For a graph X of genus g the Euler formula $2-2g = V - E + F$ holds where V, E, and F are respectively the numbers of vertices, edges and faces of the graph X. The number $\chi = 2-2g$ is known as Euler characteristic.

A *local rotation* of a vertex v is an oriented cyclic order (defined up to the cyclic permutations) of all edges incident to v. (Local rotation of 1-valent vertices is uniquely defined and is called trivial.) A *rotation system* R (or, simply, a *rotation*) of a graph X is a union of all local rotations over all vertices of X. Rotations give rise to a certain system of faces ($\equiv$ cycles) on X.

The following *face tracing algorithm* allows to determine all faces of a graph X corresponding to rotation R. Take an arbitrary vertex $v_1 \in V(X)$ and an edge a_{v_1}, incident to v_1. Let v_2 be a vertex of X, connected with v_1 by the edge a_{v_2} and let b_{v_2} be an edge of the vertex v_2, which lies *to the right* * in the cyclic order from a_{v_1}. Moving along the edge b_{v_2} to a vertex v_3, we shall define an edge c_{v_3}, which lies *to the right* from b_{v_2}. Proceeding inductively, we stop the process at an edge z_{v_n} if two forthcoming edges will be again a_{v_1} and b_{v_2}. Hereby a cycle $a_{v_1}, b_{v_2}, ..., z_{v_n}$ of a length n, which defines a face f_1 on X, will be traced. For tracing a next face f_2 one should start with an edge which lies *to the right* of any edge of the face f_1 and such, that a corner between them did not occur in f_1 – and apply the above construction. All faces $f_1, f_2, ..., f_m$ on X will be traced, when it remains no unused corners.

Theorem 1. *(*[14]*) Let X be a finite graph endowed with a rotation system R. Then there exists a 2-cell embedding of X into an orientable surface M such that one of two rotations, induced by this embedding, coincides with R. Moreover, two embeddings are equivalent if and only if they have equivalent rotation systems.*

Each embedding $i : X \to M$ induces a pair of rotation systems R and R^*, where R^* is a mirror image of R (i.e can be obtained from R by reversing of the cyclic order of all local rotations). The corresponding embeddings $i(X)$ and $i^*(X)$ are conjugate by a homeomorphism $h : M \to M$, which is not close to id_M. Then we can always endow $(\mathcal{I}, \mathcal{J})$ with a rotation system and, by theorem 1, identify our network with a cellular graph.

In particular, in this paper, we assume that the network $(\mathcal{I}, \mathcal{J})$ is planar, i.e. has genus 0. Since the number of nodes is N and the number of lines is L, by Euler formula, we get that the number of faces of the network is $F - 2 = L - N$.

Hence the following holds.

Proposition 4. *The set of equilibrium values is a $F-2$ dimensional subspace of $\mathbb{R}^L$ where F is the number of faces of the network.*

Proof. By proposition 3 the set of equilibria values over the whole network, is a $L-N$–dimensional space in $\mathbb{R}^L$, hence, by Euler formula, an $F-2$–dimensional space in $\mathbb{R}^L$. ☐

4. **Plane tilings.** Consider now a plane graph X and an embedding $i : X \to \mathbb{R}^2$. The image $i(X)$ is a tiling of the plane. If a plane tiling is regular then the faces are all equal to either triangles, squares or hexagons. For triangular tilings we have 6 lines incident at each node, for square tilings we have 4 lines incident at each node and, finally for hexagonal tilings we have 3 lines incident at each node. See [15] for a simple introduction to the theory of tilings.

*For the 1-valent vertices v_i with the edge e_i, the edge lying to the right of e_i will be again e_i. In other words, such rotation is trivial.

Definition 7. We will say that a planar network $(\mathcal{I}, \mathcal{J})$ is Triangular, Square or Hexagonal if an embedding on the plane exists such that its image gives rise to a triangular, square or hexagonal tiling respectively of a limited region of the plane.

Recall that the network is actually an oriented graph. In this paper we will treat the cases of Triangular and Square networks with orientations as in the pictures 1 which we call respectively (from left to right and from top to bottom) *Oriented Triangular*, *Circular Triangular*, *Oriented Manhattan*, *Circular Manhattan* and *Full Manhattan*.

Notice that we use the term Manhattan instead of "Square" since it models the urban structure of Manhattan quarter. The term Oriented suggests that informations flows from one initial node to a final one without the possibility of going back to the source. The oriented graph well models the unidirectional flows from a set of sources to a set of destinations, while the circular graph better models the urban traffic network. Finally the Full Manhattan Network represents a graph in which informations can pass from one node to an adjacent one in both directions (namely it is a non–oriented graph). We represent each edge by two edges with opposite orientations. It results in a graph where each node has 4 incoming and 4 outgoing lines.

Our aim is to determine the set $\mathcal{N}$ of equilibria types (in short equilibria) over the whole network and the subset of $\mathcal{N}$ of 'periodic' (in a sense that will be clear later) equilibria. Since for small networks the problem of determining the set of equilibria $\mathcal{N}$ is relatively simple, we will represent the whole network $(\mathcal{I}, \mathcal{J})$ as concatenation of building blocks, i.e. small subnetworks. Then we determine the set of equilibria for the building blocks and we will solve the global problem of finding the set $\mathcal{N}$ of possible equilibria over the whole network by studying the gluing of equilibria over the small building blocks.

Due to the regularity of the networks we are considering, the nodes of the whole network can be seen as if they were the nodes of a square tiling and number them in a matricial way, $J_{11}, \ldots, J_{1t}, \ldots, J_{s1}, \ldots, J_{st}$. Given a $s \times t$ matrix it can be decomposed in submatrices. Each submatrix is associated to one building block whose nodes are numbered as the elements of the corresponding submatrix. Moreover, we can choose our building blocks (seen as graphs) to be all equivalent and corresponding to square submatrices of order 2 or 3 (see figures 2 and 3).

Since the building blocks are all equal it is sufficient to determine the set $\mathcal{N}_0$ of possible equilibria for a single building block. Once we have found $\mathcal{N}_0$ we connect together a fixed number of building blocks to get the set $\mathcal{N}$ of equilibria over the whole network. We can have two kind of connections:

- **LEGO:** connection of the building blocks by overlapping: a building block and its adjacent have either one row or one column of nodes in common;
- **DOMINO:** connection of the building blocks by gluing: there is no intersection among the nodes of two adjacent building blocks.

In both connections two building blocks can be vertically (horizontally) connected if a certain compatibility relation holds. Such compatibility relation is different in the two cases.

- **LEGO:** the equilibria types at the nodes of the last row (column) of the first building block must coincide with the equilibria types at the nodes of the first row (column) of the second building block;

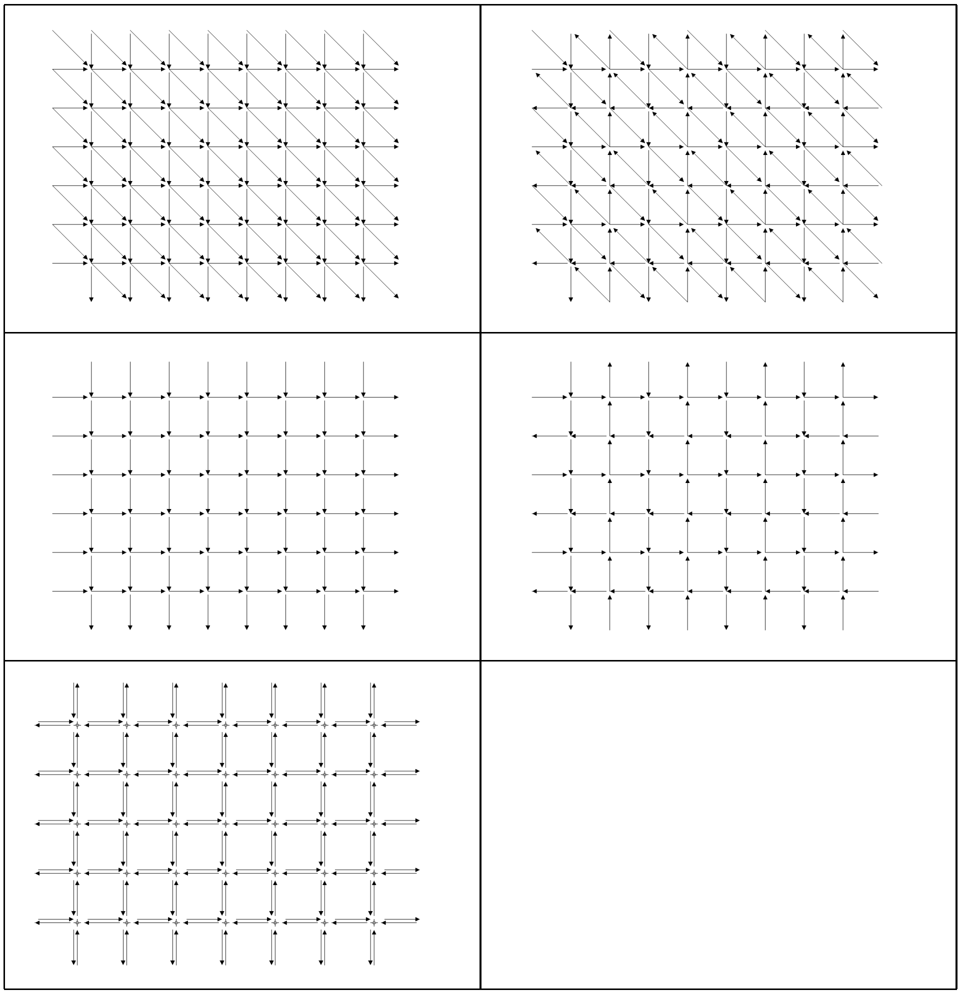

FIGURE 1. From left to right and from top to bottom. Figure1.1. The network with triangular tiling and directions such that the overall flow goes from north to south and from west to east. Figure 1.2. The network with triangular tiling and alternated directions so that circular paths are possible. Figure 1.3. The network with square tiling and directions such that the overall flow goes from north to south and from west to east. Figure 1.4. The network with square tiling and alternated directions so that circular paths are possible. Figure 1.5. The network with square tiling. The flow among two adjacent nodes is possible in both directions.

DOMINO: the last row (column) of the first building block and the first row (column) of the second building block must form another building block.

We will use the LEGO connection since the corresponding compatibility relations can be graphically represented by means of two graphs. Each vertex of the graph represents the equilibrium at the nodes of one row (column). An oriented edge from one vertex to another exists if the two vertices are part of one equilibrium of the

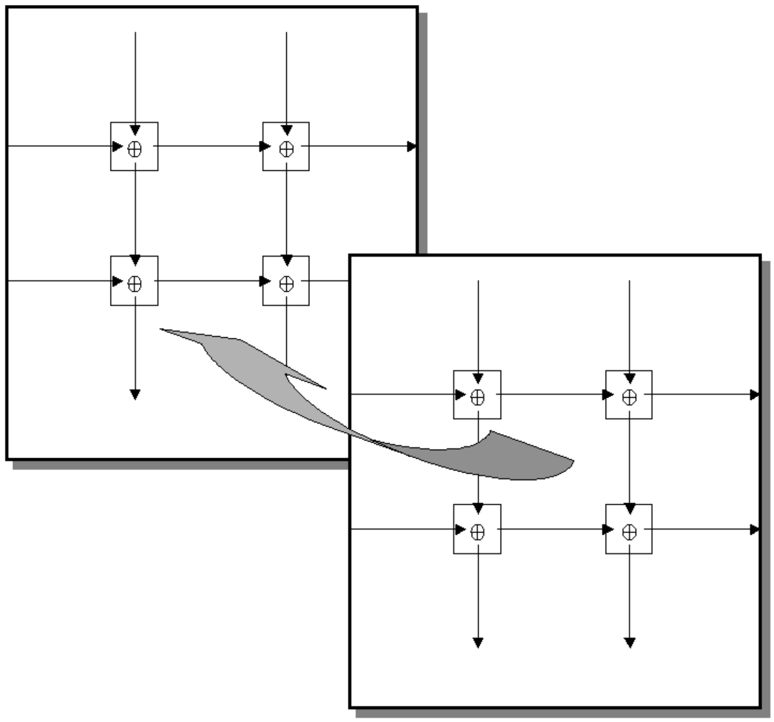

FIGURE 2. Two building blocks of the Oriented Manhattan Network (comprised of 4 nodes each) can be vertically connected by overlapping the first row of the second block over the second row of the first block. Indeed the nodes are all of the same type (one incoming lines from north and one incoming line from west and one outgoing line to south and one outgoing line to east).

building block. A path on the graph represents a sequence of building blocks that can be vertically or horizontally connected and an equilibrium over the so obtained network.

The graphical representation by means of graphs is not only intuitive. Indeed it allows to extract another fundamental information: the periodic equilibria over the whole network. Indeed assume that a closed path of length n exists. This means that n building blocks can be connected vertically (horizontally) and that the last row (column) of the chain is equal to the first row (column), thus the chain of building blocks can be repeated until forming a vertical (horizontal) stripe.

In general we have the following:

Definition 8. Let $(\mathcal{I}, \mathcal{J})$ be a Triangular or Square Network with $N = s \times t$ nodes. Let also $\mathcal{N}$ be the set of admissible equilibria types for $(\mathcal{I}, \mathcal{J})$ which is comprised of vectors of $\mathcal{M}^N$. An equilibrium type $\omega \in \mathcal{N}$ of $(\mathcal{I}, \mathcal{J})$ is said to be periodic of order p if the following holds.
There exists a Triangular or Square subnetwork $(\mathcal{I}_1, \mathcal{J}_1)$ with $N_1 = s_1 \times t_1$ nodes such that:

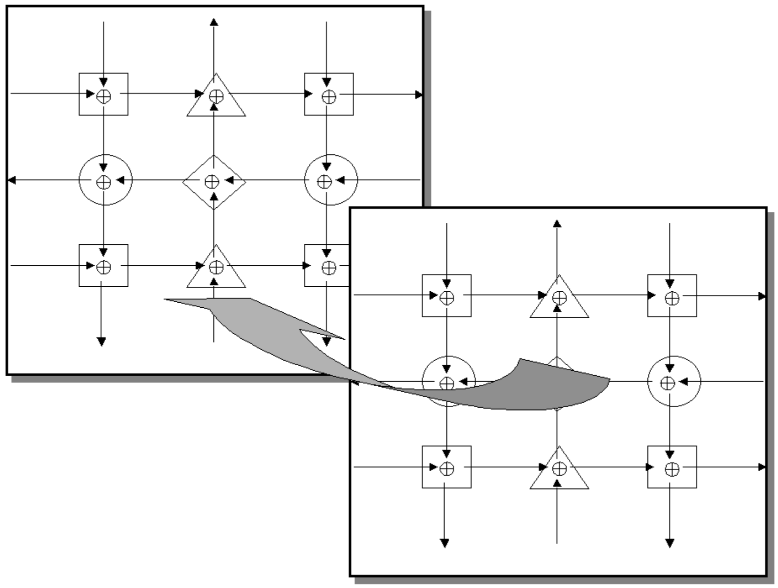

FIGURE 3. Two building blocks of the Circular Manhattan Network (comprised of 9 nodes each) can be vertically connected by overlapping the first row of the second block over the third row of the first block. In this way nodes of the same type glue together (represented by circles, having incoming lines from north and west, and triangles, having incoming lines from south and west, respectively).

- the vertical (domino) connection of $(\mathcal{I}_1, \mathcal{J}_1)$ p_1 times gives a vertical stripe $(\mathcal{I}^v, \mathcal{J}^v)$ of $s = p_1 s_1$ rows and t_1 columns;
- the horizontal (domino) connection of the vertical stripe $(\mathcal{I}^v, \mathcal{J}^v)$ p_2 times gives the network $(\mathcal{I}, \mathcal{J})$ of $s = p_1 s_1$ rows and $t = p_2 t_1$ columns;
- $p = \max\{p_1, p_2\}$.

Moreover there exists an element $\omega_1 \in \mathcal{N}_1$, the set of admissible equilibria for the subnetwork $(\mathcal{I}_1, \mathcal{J}_1)$, such that

- ω_1 can be vertically and horizontally connected to itself and

$$\omega = \underbrace{\begin{bmatrix} \omega_1 & \cdots & \omega_1 \\ \vdots & & \vdots \\ \omega_1 & \cdots & \omega_1 \end{bmatrix}}_{p_2 \ times} \left.\vphantom{\begin{bmatrix} \omega_1 \\ \vdots \\ \omega_1 \end{bmatrix}}\right\} p_1 \ times$$

In the following sections we determine periodic equilibrium types of any order for the Manhattan and the Triangular networks.

5. **Manhattan networks.** The nodes of the network are numbered $J_{11}, \dots, J_{1t}, \dots, J_{s1}, \dots, J_{st}$. For each node J_{ij} there are incoming and outgoing lines. We denote by $incoming^l_{ij}$, $outgoing^l_{ij}$ with $l = 1, \dots, \ell$ the type of the densities along the incoming and outgoing lines. In particular for the Oriented and the Circular

Manhattan Networks we have two incoming and two outgoing lines (thus $\ell = 2$) while for the Full Manhattan Network we have four incoming and four outgoing lines (thus $\ell = 4$). Therefore in the Oriented and Circular Manhattan Network the equilibrium type at one node is described by a vector of length 4 while in the Full Manhattan Network the equilibrium type at one node is described by a vector of length 8. The first half components describe the types of the densities along the incoming lines while the second half components describe the types of the densities along the outgoing ones.

5.1. **Oriented Manhattan Networks.** Consider now the network represented in picture 1.3. Recall that $\mathcal{M}$ is the set of possible equilibria types at one node. Then, since in this case we have 2 incoming and 2 outgoing lines for each node of the network, we have that

$$\mathcal{M} = \{I, II0, II1.1, II1.2, III0, III1.1, III1.2\}.$$

Definition 9. A building block of the Oriented Manhattan Networks is given by a subnetwork $(\mathcal{I}_0, \mathcal{J}_0)$ with

$$\mathcal{J}_0 = \{J_{11}, J_{12}, J_{21}, J_{22}\}$$

and

$$\mathcal{I}_0 = \{incoming^l_{ij}, outgoing^l_{ij},\ i, j = 1, 2, l = h, v\},$$

where h stands for horizontal and v stands for vertical. We can connect building blocks vertically and horizontally to obtain the subnetworks stripes $(\mathcal{I}^v, \mathcal{J}^v)$ and $(\mathcal{I}^h, \mathcal{J}^h)$ with

$$\mathcal{J}^v = \{J_{11}, J_{12}, \dots, J_{s1}, J_{s2}\},$$
$$\mathcal{J}^h = \{J_{11}, J_{21}, \dots, J_{1t}, J_{2t}\}$$

and

$$\mathcal{I}^v = \{incoming^l_{ij}, outgoing^l_{ij},\ i = 1, \dots s, j = 1, 2, l = 1, \dots, \ell\},$$
$$\mathcal{I}^h = \{incoming^l_{ij}, outgoing^l_{ij},\ i = 1, 2, j = 1, \dots s, l = 1, \dots, \ell\}.$$

Finally we can horizontally connect vertical stripes or vertically connect horizontal stripes to obtain the whole network $(\mathcal{I}, \mathcal{J})$.

For each node J_{ij} the vector $((incoming^h_{ij}, incoming^v_{ij}), (outgoing^h_{ij}, outgoing^v_{ij}))$ describes the types of the equilibrium densities along the lines as incident lines at that node and it is an element of $\mathcal{M}$. Now we want to determine $\mathcal{N}_0$ for a building block and this is done by applying rule **H**.

Rule **H** can be rewritten as follows:

- $incoming^h_{ij} = bad$ if and only if $outgoing^h_{i\ j-1} = good$;
- $incoming^v_{ij} = bad$ if and only if $outgoing^v_{i-1\ j} = good$.

We can now determine the set $\mathcal{N}_0$ of admissible configurations of the subnetwork $(\mathcal{I}_0, \mathcal{J}_0)$ as follows.

Algorithm 1. **Step 0:** *Set* $\mathcal{N}_0 = \emptyset$.

Step 1: *For* $j_{11} \in \mathcal{M}$ *we set* J_{11} *to be of type* j_{11}. *Following rule* **H** *we determine all admissible configurations* $\mathcal{M}_{12} \subset \mathcal{M}$ *for node* J_{12} *and* $\mathcal{M}_{21} \subset \mathcal{M}$ *for node* J_{21}.

: *For* $j_{12} \in \mathcal{M}_{12}$ *and for* $j_{21} \in \mathcal{M}_{21}$, *following rule* **H** *we determine all admissible configuration* $\mathcal{M}_{22} \subset \mathcal{M}$ *for node* J_{22}.

: *For* $j_{22} \in \mathcal{M}_{22}$ *set* $\mathcal{N}_0 = \mathcal{N}_0 \cup \{(j_{11}, j_{12}, j_{21}, j_{22})\}$.

Following the above procedure we obtain the set $\mathcal{N}_0$ which is reported in the Appendix (section 8). The cardinality of $\mathcal{N}_0$ is 73 however many among the elements of $\mathcal{N}_0$ are in symmetry relation with respect to the main diagonal of the network. More precisely we consider the following permutation on the network:

- nodes J_{11} and nodes J_{22} remain unvaried while $J_{12} \leftrightarrow J_{21}$ (the two nodes are exchanged);
- incoming and outgoing lines are exchanged as follows:

$$\begin{cases} incoming^h_{ij} \leftrightarrow incoming^v_{ij} \\ outgoing^h_{ij} \leftrightarrow outgoing^v_{ij} \end{cases} \quad i = j \in \{1,2\}$$
$$\begin{cases} incoming^h_{ij} \leftrightarrow incoming^v_{ji} \\ outgoing^h_{ij} \leftrightarrow outgoing^v_{ji} \end{cases} \quad i \neq j \ i,j \in \{1,2\}.$$

This permutation induce the following mapping on $\mathcal{M}$:

$$S : \begin{cases} I \leftrightarrow I \\ II0 \leftrightarrow II0 \\ II1.1 \leftrightarrow II1.2 \\ III0 \leftrightarrow III0 \\ III1.1 \leftrightarrow III1.2 \end{cases}$$

and consequently on $\mathcal{N}_0$ we have the induced action (which we still denote by S):

$$S : (j_1, j_2, j_3, j_4) \mapsto (S(j_1), S(j_3), S(j_2), S(j_4)).$$

Trivially, we have the following:

Proposition 5. *If $j \in \mathcal{N}_0$ then $S(j) \in \mathcal{N}_0$.*

With the symmetry relation S the elements of $\mathcal{N}_0$ reduce to the set $\mathcal{N}_0/S$ which is reported in the Appendix (section 8) and whose cardinality is 42.

Now we consider the vertical and horizontal connections of building blocks which gives a vertical stripe $(\mathcal{I}^v, \mathcal{J}^v)$ and a horizontal stripe $(\mathcal{I}^h, \mathcal{J}^h)$. We next compute what are $\mathcal{N}^v$ and $\mathcal{N}^h$ from vertical and horizontal admissible concatenation of elements of $\mathcal{N}_0$.

Definition 10. A concatenation of 2 elements of $\mathcal{N}_0$, $j^1 = (j^1_{11}, j^1_{12}, j^1_{21}, j^1_{22})$ and $j^2 = (j^2_{11}, j^2_{12}, j^2_{21}, j^2_{22})$, with $j^1, j^2 \in \mathcal{N}_0$ is *vertical* admissible if $(j^1_{21}, j^1_{22}, j^2_{11}, j^2_{12}) \in \mathcal{N}_0$. A concatenation of 2 elements $j^1, j^2 \in \mathcal{N}_0$ is *horizontal* admissible if $(j^1_{12}, j^2_{11}, j^1_{22}, j^2_{21}) \in \mathcal{N}_0$.

We begin with the vertical concatenations. To determine what are the vertical admissible concatenations of elements of $\mathcal{N}_0$ we build the graph G^v as follows.

Enumeration 1. Consider the mapping

$$\beta : \{0, 1, \dots, 6\} \to \mathcal{M},$$

with $\beta(0) = I$, $\beta(1) = II0$, $\beta(2) = II1.1$, $\beta(3) = II1.2$, $\beta(4) = III0$, $\beta(5) = III1.1$, $\beta(6) = III1.2$. Consider also the mapping

$$\delta : \{0, \dots, 48\} \to \{0, 1, \dots, 6\} \times \{0, 1, \dots, 6\},$$

with $\delta(i) = (n, r)$, where n and r are such that $i \equiv r \pmod 7$ and $i = 7n + r$. For notation convenience we denote $n = \delta_1(i)$ and $r = \delta_2(i)$.

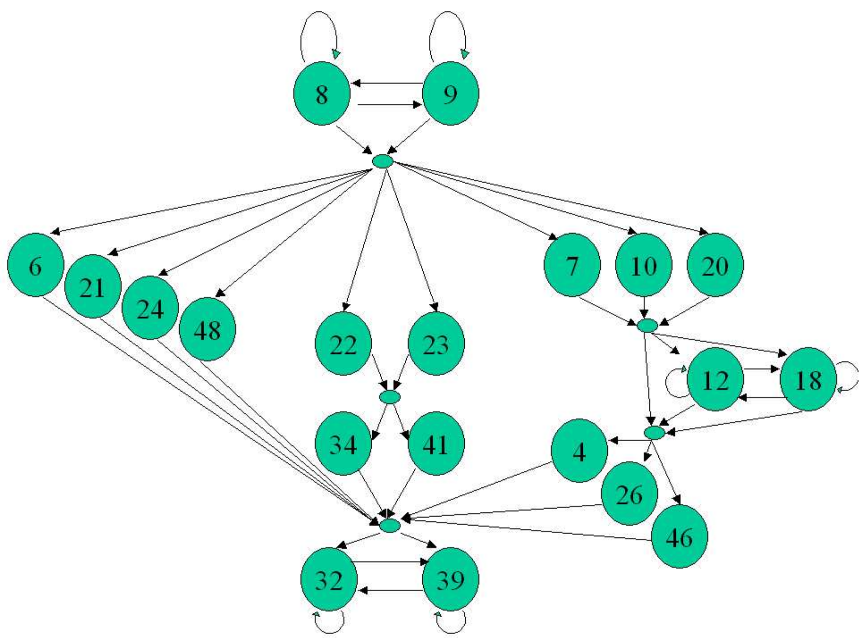

FIGURE 4. The graph shows the vertical admissible connections among building blocks for the Oriented Manhattan Network. Isolated nodes do not appear in the graph.

Definition 11. Let $\mathcal{V} = \{V_0, \dots, V_{48}\}$ with

$$V_i = (\beta(\delta_1(i)), \beta(\delta_2(i))) \in \mathcal{M} \times \mathcal{M}.$$

We define G^v the graph whose set of vertices is $\mathcal{V}$. Two vertices V_i and V_k, corresponding to two pairs $(j_{i_1}, j_{i_2}), (j_{k_1}, j_{k_2}) \in \mathcal{M} \times \mathcal{M}$ respectively, are joined by a directed edge from V_i to V_k if and only if $(j_{i_1}, j_{i_2}, j_{k_1}, j_{k_2}) \in \mathcal{N}_0$.
Based on the graph G^v we define a second graph, G^h, as follows. The set of vertices of G^h is $\mathcal{V}$. In G^h two vertices $V_i = (j_{i_1}, j_{i_2})$ and $V_k = (j_{k_1}, j_{k_2})$, with $i = 7n_i + r_i$ and $k = 7n_k + r_k$, are joined by a directed edge from V_i to V_k if and only if $(j_{i_1}, j_{k_1}, j_{i_2}, j_{k_2}) \in \mathcal{N}_0$, that is, if and only if, in G^v, there is directed edge from $V_{i'} = (j_{i_1}, j_{k_1})$ to $V_{k'} = (j_{i_2}, j_{k_2})$, with $i' = 7n_i + n_k$ and $k' = 7r_i + r_k$.

In figure 4 we give a graphical representation of the graph G^v of vertical admissible connections among building blocks of the Oriented Manhattan Network. For simplifying the picture we avoid to represent isolated nodes. More precisely the graph G^v shows what are the possible equilibria types j_3 and j_4 at the nodes J_{21} and J_{22} once fixed the equilibria types j_1 and j_2 at the nodes J_{11} and J_{12} (so that $j = (j_1, j_2, j_3, j_4) \in \mathcal{N}_0$.) The horizontal admissible concatenations, instead, are shown in the graph G^h. See fig. 5 for the graphical representation of G^h.

By studying the closed loops of the graph G^v and G^h, one can deduce all informations relative to possible periodic equilibria on the network. Indeed we have the following:

Theorem 2. *To each equilibrium solution, there corresponds a set of paths on the graphs G^v and G^h. More precisely to each vertical stripe there corresponds a path in*

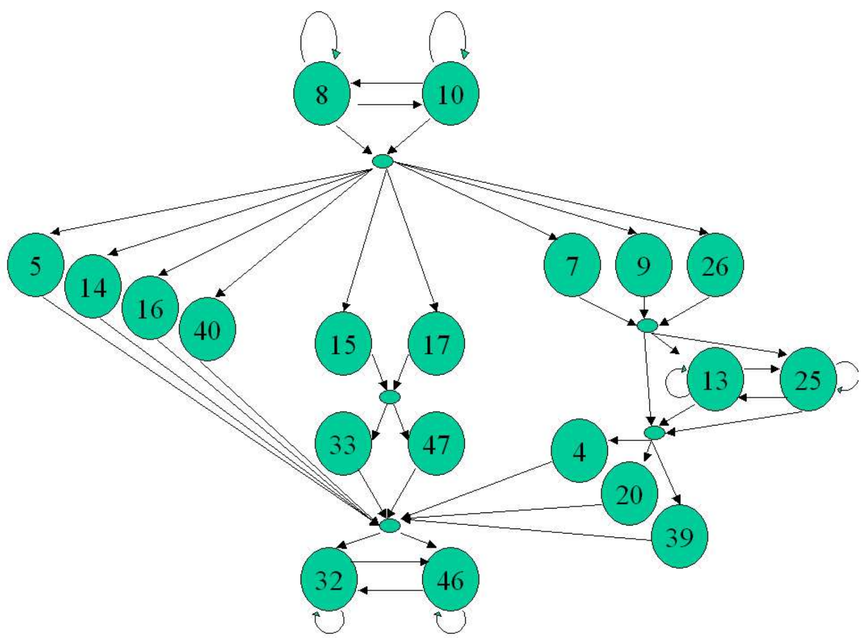

FIGURE 5. The graph shows the horizontal admissible connections among building blocks for the Oriented Manhattan Network. Isolated nodes have been erased from the graph.

G^v and to each horizontal stripe there corresponds a path in G^h. If the equilibrium is periodic then the paths are closed.

Definition 12. We say that a vertical stripe is periodic of order 0 if all nodes are of the same type. It can be obtained as vertical concatenation of an element of type $j = (j_1, j_1, j_1, j_1)$ with itself.
A vertical stripe is periodic of order 1 if all nodes J_{i1} are of the same type and all nodes J_{i2} are of the same type. It can be obtained as vertical concatenation of an elements of type $j = (j_1, j_2, j_1, j_2)$ with itself.
A vertical stripe is periodic of order n if all nodes $J_{r+nk,1}$ and $J_{r+nk,2}$, for $r = 1, \ldots, n$ and $k \in \mathbb{Z}$ are of the same type.

We immediately have the following.

Proposition 6. *Any isolated node of the graph G^v corresponds to a pair of equilibrium types that can not be part of an equilibrium on the building block.*
A loop on a vertex describes an equilibrium with periodicity 1. In particular a loop on a vertex V, corresponding to a pair of type (j, j), describes an equilibrium with periodicity 0. In general, closed paths of length n describe the equilibria with periodicity n.

Then from the graph G^v we see that the vertices $V_8, V_9, V_{12}, V_{18}, V_{32}, V_{39}$ give the periodicity 1 on the network. Since a pair of type (j, j) corresponds to a vertex V_i with $i = 7n+r$, where $n = r$, hence with $i = 8n$, we get that the vertices which give periodicity 0 are V_8 and V_{32}. The closed paths of length 2 are given by the pairs (of vertices) (V_8, V_9), (V_{12}, V_{18}) and (V_{32}, V_{39}). Other closed paths of any length can be built trivially by composing closed paths of length 2 with loops.

From the graph G^h we get that the vertices $V_8, V_{10}, V_{13}, V_{25}, V_{32}, V_{46}$ give periodicity 1 where, in particular the vertices V_8 and V_{32} give the periodicity 0. Closed paths of length 2 are given by the pairs (V_8, V_{10}), (V_{13}, V_{25}) and (V_{32}, V_{46}).

Now we want to see what are the periodic equilibria over the whole network.

Theorem 3. *The periodic structures over the whole network are the following:*

- *equilibrium of type $II0$ for each node of the network;*
- *equilibrium of type $III0$ for each node of the network.*

Proof. To prove the theorem we must analyze which are the periodic vertical structures that can be repeated horizontally.

The 0–periodic vertical structures are given by the loops in G^v on the vertices V_8 and V_{32}. Since $8 = 7 \times 1 + 1$ and $32 = 7 \times 4 + 4$, these vertices correspond to equilibrium type $II0$ and $III0$ at each node. This vertical structures is compatible with the 0–periodic horizontal structure given by the loops in G^h on the vertices V_8 and V_{32}.

Next we see that there are no other periodic vertical structure compatible with a periodic horizontal one.

We begin with the structure given by loop on the vertex V_9 of G^v. Since $9 = 7 \times 1 + 2$, V_9 corresponds to the pair $(II0, II1.1)$. Then we can consider a vertical stripe where each row is comprised of two nodes the equilibrium type of which is described by the elements $II0, II1.1$ of $\mathcal{M}$.

If a horizontal periodic structure was compatible with this vertical one then there would exists a closed path on G^h beginning in V_8 through V_{16} (since $8 = 7 \times 1 + 1$ and $16 = 7 \times 2 + 2$) and going back to V_8. But from G^h it is easy to check that such a closed path does not exists.

The same reasoning applies to the loop on vertex V_{12}. $12 = 7 \times 1 + 5$, therefore a horizontal periodic structure compatible with this vertical one requires that in G^h there exists a closed path beginning at V_8, through V_{40} ($40 = 7 \times 5 + 5$), and going back to V_8. As before, such a path does not exists.

Concerning the loops on vertices V_{18} and V_{39}, having a compatible horizontal periodic structure would mean the existence of closed paths in G^h based at V_{16} and V_{40} respectively, which is false.

Next we consider the closed paths $(8, 9)$, $(12, 18)$ and $(32, 29)$. Horizontal structures compatible with the above vertical ones exist if and only if in G^h there exist closed paths based respectively at V_8, V_9 and V_{33} and passing respectively through V_9, V_{39} and V_{32}. Such paths do not exist, then there are no periodic horizontal structures compatible with the vertical ones except for the trivial ones corresponding to the loops on vertices V_8 and V_{32}. □

By rewriting the vertices back to pairs in $\mathcal{M} \times \mathcal{M}$ we get that the vertical (resp. horizontal) structures with periodicity 0, 1 and 2 are given by vertical (resp. horizontal) concatenation of j with itself where $j \in \mathcal{S}_0$, $j \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, respectively, and

$$\mathcal{S}_0 = \{(II0, II0, II0, II0), (III0, III0, III0, III0)\},$$

$$\begin{aligned} \mathcal{S}_1 = \quad & \{(II0, II1.1, II0, II1.1), (II0, III1.1, II0, III1.1), \\ & (II1.1, III0, II1.1, III0), (III1.1, III0, III1.1, III0)\}, \end{aligned}$$

and

$$\mathcal{S}_2 = \{(II0, II1.1, II0, II0), (II0, III1.1, II1.1, III0), (III0, III0, III1.1, III0), (II0, II0, II0, II1.1), (II1.1, III0, II0, III1.1), (III1.1, III0, III0, III0)\}.$$

5.2. **Circular Manhattan Network.** Consider now the network represented in picture 1.4. Also in this case we have 2 incoming and 2 outgoing lines for each node of the network. Hence, as in the previous case,

$$\mathcal{M} = \{I, II0, II1.1, II1.2, III0, III1.1, III1.2\}.$$

A building block is now given as in the following:

Definition 13. A building block of the Circular Manhattan Network is given by a subnetwork $(\mathcal{I}_0, \mathcal{J}_0)$ with

$$\mathcal{J}_0 = \{J_{11}, J_{12}, J_{13}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{33}\}$$

and

$$\mathcal{I}_0 = \{incoming^l_{ij}, outgoing^l_{ij},\ i, j = 1, 2, 3, l = h, v\},$$

where h stands for horizontal and v stands for vertical. We can connect building blocks vertically and horizontally to obtain the subnetworks stripes $(\mathcal{I}^v, \mathcal{J}^v)$ and $(\mathcal{I}^h, \mathcal{J}^h)$ with

$$\mathcal{J}^v = \{J_{11}, J_{12}, J_{13}, \ldots, J_{s1}, J_{s2}, J_{s3}\},$$

$$\mathcal{J}^h = \{J_{11}, J_{21}, J_{31}, \ldots, J_{1t}, J_{2t}, J_{3t}\}$$

and

$$\mathcal{I}^v = \{incoming^l_{ij}, outgoing^l_{ij},\ i = 1, \ldots s, j = 1, 2, 3, l = 1, \ldots, \ell\},$$

$$\mathcal{I}^h = \{incoming^l_{ij}, outgoing^l_{ij},\ i = 1, 2, 3, j = 1, \ldots s, l = 1, \ldots, \ell\}.$$

Finally we can horizontally connect vertical stripes or vertically connect horizontal stripes to obtain the whole network $(\mathcal{I}, \mathcal{J})$.

For each node J_{ij}, the vector $((incoming^h_{ij}, incoming^v_{ij}), (outgoing^h_{ij}, outgoing^v_{ij}))$ describes the types of the equilibrium densities along the lines as incident lines at that node and it is an element of $\mathcal{M}$. For determining the set $\mathcal{N}_0$ we rewrite rule **H** as follows:

- if i is odd and j is odd then
  $incoming^h_{ij} = bad$ if and only if $outgoing^h_{ij-1} = good$;
  $incoming^v_{ij} = bad$ if and only if $outgoing^v_{i-1\ j} = good$;
- if i is odd and j is even then
  $incoming^h_{ij} = bad$ if and only if $outgoing^h_{ij-1} = good$;
  $incoming^v_{ij} = bad$ if and only if $outgoing^v_{i+1\ j} = good$;
- if i is even and j is odd then
  $incoming^h_{ij} = bad$ if and only if $outgoing^h_{ij+1} = good$;
  $incoming^v_{ij} = bad$ if and only if $outgoing^v_{i-1\ j} = good$;
- if i is even and j is even then
  $incoming^h_{ij} = bad$ if and only if $outgoing^h_{ij+1} = good$;
  $incoming^v_{ij} = bad$ if and only if $outgoing^v_{i+1\ j} = good$.

Following rule **H** we can determine what is the set $\mathcal{N}_0$ of admissible configurations of the subnetwork $(\mathcal{I}_0, \mathcal{J}_0)$ as follows.

a): Consider node J_{23}. It has one incoming line from J_{13} and one outgoing line to J_{22}. These two lines cannot be both *good*, i.e. $outgoing^v_{13}$ and $incoming^h_{22}$ cannot be both *bad*. Thus if $J_{22} \in \{I, II0, II1.1, II1.2, III1.1\}$, then $J_{13} \in \{II0, II1.1\}$. If otherwise $J_{22} \in \{III0, III1.2\}$ then we have no restriction on J_{13} which thus can assume any value in $\mathcal{M}$.

b): The same argument applies to J_{32}. It has one incoming line from J_{31} and one outgoing line to J_{22}. These two lines cannot be both *good*, i.e. $outgoing^h_{31}$ and $incoming^v_{22}$ cannot be both *bad*. Thus if $J_{22} \in \{I, II0, II1.1, II1.2, III1.2\}$, then $J_{31} \in \{II0, II1.2\}$. If otherwise $J_{22} \in \{III0, III1.1\}$ then we have no restriction on J_{31} which thus can assume any value in $\mathcal{M}$.

c): Now assume that $J_{22} \in \{I, II0, II1.1, II1.2, III1.1\}$ and $J_{13} \in \{II0, II1.1\}$. Since J_{13} has at least one *good* outgoing line, it must be $incoming^h_{13} = bad$ and $outgoing^h_{12} = good$ thus implying $J_{12} \in \{II0, II1.2\}$. In turn we get that $incoming^v_{12} = bad$ and $outgoing^v_{22} = good$. Then we get $J_{22} \in \{II0, II1.1\}$.

d): On the other hand, if we assume that $J_{22} \in \{I, II0, II1.1, II1.2, III1.2\}$ and $J_{31} \in \{II0, II1.2\}$, since J_{31} as at least one *good* outgoing line, it must be $incoming^v_{31} = bad$ and $outgoing^v_{21} = good$ thus implying $J_{21} \in \{II0, II1.1\}$. In turn we get that $incoming^h_{21} = bad$ and $outgoing^h_{22} = good$. Then we get $J_{22} \in \{II0, II1.1\}$. Together with c) we obtain that $J_{22} = II0$.

e): For $J_{12} \in \{II0, II1.2\}$ and $J_{21} \in \{II0, II1.1\}$ it must be $outgoing^h_{11} = good$ and $outgoing^v_{11} = good$. Hence $J_{11} = II0$.
For $J_{32} = II0$ and $J_{23} = II0$ it must be $incoming^h_{33} = bad$ and $incoming^v_{33} = bad$, hence $J_{33} \in \{I, II0, II1.1, II1.2\}$.
If $J_{32} = II0$ and $J_{23} = II1.2$ then $incoming^h_{33} = bad$ and $incoming^v_{33} = good$, hence $J_{33} = III1.1$.
If $J_{32} = II1.1$ and $J_{23} = II0$ then $incoming^h_{33} = good$ and $incoming^v_{33} = bad$, hence $J_{33} = III1.2$.
Finally, if $J_{32} = II1.1$ and $J_{23} = II1.2$ then $incoming^h_{33} = good$ and $incoming^v_{33} = good$, hence $J_{33} = III0$.

f): If $J_{22} \in \{III0, III1.2\}$ then $incoming^v_{12} = good$ and $incoming^h_{21} = good$. Hence $J_{12} \in \{III0, III1.1\}$ and $J_{21} \in \{III0, III1.2\}$. Then it must be $incoming^h_{13} = good$ and $incoming^v_{31} = good$, i.e. $J_{13} \in \{III0, III1.2\}$ and $J_{31} \in \{III0, III1.1\}$. In turn this gives $incoming^v_{23} = good$ and $incoming^h_{32} = good$, i.e. $J_{23} \in \{III0, III1.1\}$ and $J_{32} \in \{III0, III1.2\}$. Finally this gives $incoming^v_{33} = good$ and $incoming^h_{33} = good$, $incoming^h_{22} = good$ and $incoming^v_{22} = good$, that is $J_{33} = III0$ and $J_{22} = III0$.
Moreover if $J_{12} = III0$ and $J_{21} = III0$, since $outgoing^h_{11} = bad$ and $outgoing^v_{11} = good$ then $J_{11} \in \{I, III0, III1.1, III1.2\}$.
If otherwise $J_{12} = III1.1$ and $J_{21} = III0$, since $outgoing^h_{11} = good$ and $outgoing^v_{11} = bad$ then $J_{11} = II1.2$.
If $J_{12} = III0$ and $J_{21} = III1.2$, since $outgoing^h_{11} = bad$ and $outgoing^v_{11} = good$ then $J_{11} = II1.1$.
If finally $J_{12} = III1.1$ and $J_{21} = III1.2$, since $outgoing^h_{11} = good$ and $outgoing^v_{11} = good$ then $J_{11} = II0$.

Now we consider the vertical and horizontal concatenations of building blocks which give the whole network $(\mathcal{I}, \mathcal{J})$.

Definition 14. A concatenation of 2 elements of $\mathcal{N}_0$, $j^1 = (j^1_{11}, j^1_{12}, j^1_{13}, j^1_{21}, j^1_{22}, j^1_{23}, j^1_{31}, j^1_{32}, j^1_{33})$ and $j^2 = (j^2_{11}, j^2_{12}, j^2_{13}, j^2_{21}, j^2_{22}, j^2_{23}, j^2_{31}, j^2_{32}, j^2_{33})$, with $j^1, j^2 \in \mathcal{N}_0$ is *vertical* admissible if there exists (j_{21}, j_{22}, j_{23}) such that

$$(j^1_{31}, j^1_{32}, j^1_{33}, j_{21}, j_{22}, j_{23}, j^2_{11}, j^2_{12}, j^2_{13}) \in \mathcal{N}_0.$$

A concatenation of 2 elements $j^1, j^2 \in \mathcal{N}_0$ is *horizontal* admissible if there exists (j_{12}, j_{22}, j_{32}) such that

$$(j^1_{13}, j_{12}, j^2_{11}, j^1_{23}, j_{22}, j^2_{21}, j^1_{33}, j_{32}, j^2_{31}) \in \mathcal{N}_0.$$

Now the set $\mathcal{N}_0$ is very large. Then we follow a different approach here and use the informations on $\mathcal{N}_0$ given in the above description to directly obtain $\mathcal{N}$, i.e. the set of equilibria over the whole network. We get the following.

Theorem 4. *We have*

$$\mathcal{N} = \{\{J_{ij} = II0,\ i = 1\ldots, s, j = 1, \ldots, t\}, \{J_{ij} = III0,\ i = 1\ldots, s, j = 1, \ldots, t\}\}$$

that is $\mathcal{N}$ is comprised of only two equilibria. Moreover, each equilibrium solution of the Circular Manhattan Network is periodic.

Proof. To prove the theorem we write the following two tables:

$II0$	$II0, II1.2$	$II0, II1.1$
$II0, II1.1$	$II0$	$II0, II1.2$
$II0, II1.2$	$II0, II1.1$	J_{33}

J_{11}	$III0, III1.1$	$III0, III1.2$
$III0, III1.2$	$III0$	$III0, III1.1$
$III0, III1.1$	$III0, III1.2$	$III0$

.

which indicate which are the possible equilibria at each node J_{ij}, for $i = 1, 2, 3$ and $j = 1, 2, 3$. The generic J_{33} in the first table and J_{11} in the second mean that the equilibrium at those nodes has to be deduced from the equilibrium at the adjacent nodes along with the descriptions given in **e)** and **f)**.

From the first table it is clear that to have vertical connections among building blocks it must be $J_{31} = J_{11} = II0$, $J_{32} = J_{12} = II0$ and $J_{33} = J_{13} \in \{II0, II1.1\}$ from which, by **e)**, $J_{23} = II0$. To also have horizontal connections it must be $J_{21} = J_{23} = II0$ and $J_{33} = J_{31} = J_{13} = II0$, i.e. $J_{ij} = II0$ for all $i, j = 1, 2, 3$.

Analogously, from the second table, we get that it must be $J_{12} = J_{32} = III0$, $J_{13} = J_{33} = III0$ and $J_{11} = J_{31} \in \{III0, III1.1\}$, from which, by **f)**, $J_{21} = III0$. To also have horizontal connections we get that $J_{31} = J_{33} = III0$, $J_{21} = J_{23} = III0$ and $J_{11} = J_{13} = III0$, i.e. $J_{ij} = III0$ for all $i, j = 1, 2, 3$. □

5.3. **Full Manhattan Network.** Finally we consider the Full Manhattan Network (see fig. 1.5). In this case we have 4 incoming and 4 outgoing lines from each node of the network and we denote by $incoming^l_{ij}, outgoing^l_{ij}$, with $l = 1, 2, 3, 4$, the type of the densities along, respectively, the incoming and the outgoing lines at the node ij. Then

$$\begin{aligned}\mathcal{M} = \ & \{I, II0, II1.1, II1.2, II1.3, II1.4, II2.1.2, II2.1.3, II2.1.4, II2.2.3, II2.2.4, \\ & II2.3.4, II3.1.2.3, II3.1.2.4, II3.1.3.4, II3.2.3.4, \\ & III0, III1.1, III1.2, III1.3, III1.4, III2.1.2, III2.1.3, III2.1.4, III2.2.3, \\ & III2.2.4, III2.3.4, III3.1.2.3, III3.1.2.4, III3.1.3.4, III3.2.3.4\}.\end{aligned}$$

The building blocks can be taken as in definition 9. Rule **H** is rewritten as follows:

- $incoming^1_{ij} = bad$ if and only if $outgoing^3_{ij-1} = good$;
- $incoming^2_{ij} = bad$ if and only if $outgoing^4_{i-1\ j} = good$;
- $incoming^3_{ij} = bad$ if and only if $outgoing^1_{ij+1} = good$;
- $incoming^4_{ij} = bad$ if and only if $outgoing^2_{i+1\ j} = good$.

From rule **H** we get a similar result as for the Circular Manhattan Network.

Theorem 5. *We have*

$$\mathcal{N} = \{\{J_{ij} = II0,\ i = 1\dots, s, j = 1, \dots, t\}, \{J_{ij} = III0,\ i = 1\dots, s, j = 1, \dots, t\}\}$$

that is $\mathcal{N}$ is comprised of only two equilibria. Moreover, each equilibrium solution of the Full Manhattan Network is periodic.

Proof. We first consider a building block $(\mathcal{I}_0, \mathcal{J}_0)$ and see what are the possible equilibria for it.

Assume first that at node J_{11} the equilibrium is of type I. Then at nodes J_{12} and J_{21} the equilibrium is one among the III types, i.e. the outgoing lines from J_{12} and J_{21} are all of type *bad*. In particular there is one outgoing line for J_{12} (for J_{21}) which is incoming for J_{11} which, by rule **H**, must be *good*. But this contradicts the fact that the equilibrium at J_{11} is of type I.

Assume now that the equilibrium at node J_{11} is one among the II types. This means that there is at least one *good* outgoing line from J_{12} and one *good* outgoing line from J_{21}, hence the equilibrium at nodes J_{12} and J_{21} are also of type II. The same conclusion holds for J_{22}, since it must have two outgoing lines which are *good*. From this argument we extract some more information:

- the outgoing lines from J_{11} towards J_{12} and J_{21} are good,
- the outgoing lines from J_{12} towards J_{11} and J_{22} are good,
- the outgoing lines from J_{21} towards J_{11} and J_{22} are good,
- the outgoing lines from J_{22} towards J_{12} and J_{21} are good.

The same argument can be produced in the case where the equilibrium at node J_{11} is of type III. We obtain that the equilibrium at any node of the building block is of type III and

- the incoming lines at J_{11} from J_{12} and J_{21} are good,
- the incoming lines at J_{12} from J_{11} and J_{22} are good,
- the incoming lines at J_{21} from J_{11} and J_{22} are good,
- the incoming lines at J_{22} from J_{12} and J_{21} are good.

By concatenating building blocks among them both horizontally and vertically we get that the only possibility is that given in the thesis. Indeed, assuming that the nodes are all of type II,

- the equilibria type at J_{11} must be equal to the equilibria type of both J_{12} and J_{21};
- the equilibria type at J_{12} must be equal to the equilibria type of both J_{11} and J_{22};
- the equilibria type at J_{21} must be equal to the equilibria type of both J_{11} and J_{22};
- the equilibria type at J_{22} must be equal to the equilibria type of both J_{12} and J_{21}.

Hence all the densities on the outgoing lines from all the nodes $J_{ij},\ i,j = 1,2$, must be of type good. The same argument holds when we assume that the nodes are all of type III. □

6. **Triangular networks.**

By drawing the triangular tiling as in the pictures 1.1 and 1.2 we can look at the nodes as if they were elements of a matrix and number them J_{ij}, with $i = 1,\dots,s$ and $j = 1\dots,t$. For each node there are 3 incoming and 3 outgoing lines and by $incoming^l_{ij}, outgoing^l_{ij}$, with $l = h,d,v$ (for horizontal, diagonal and vertical respectively), we describe the types of the equilibrium densities along the lines incident at the node J_{ij} . The equilibrium type at one node can then be assigned by giving a vector of length 6:

$$((incoming^h_{ij}, incoming^d_{ij}, incoming^v_{ij}), (outgoing^h_{ij}, outgoing^d_{ij}, outgoing^v_{ij})).$$

Hence we have:

$$\begin{aligned}\mathcal{M} = \quad & \{I, II0, II1.1, II1.2, II1.3, II2.1.2, II2.1.3, II2.2.3,\\ & III0, III1.1, III1.2, III1.3, III2.1.2, III2.1.3, III2.2.3\}\end{aligned}$$

6.1. **Oriented Triangular Network.**

A building block $(\mathcal{I}_0, \mathcal{J}_0)$ is as in definition 9: it is comprised of the four nodes $J_{11}, J_{12}, J_{21}, J_{22}$. And the whole network is obtained by connecting building blocks both horizontally and vertically as in the Oriented Manhattan case.

Rule **H** can be rewritten as follows:

- $incoming^h_{ij} = bad$ if and only if $incoming^h_{ij-1} = good$;
- $incoming^d_{ij} = bad$ if and only if $incoming^d_{i-1\ j-1} = good$;
- $incoming^v_{ij} = bad$ if and only if $incoming^v_{i-1\ j} = good$.

Now from rule **H** one finds the set $\mathcal{N}_0$ as described in Algorithm 1. We get that the cardinality of $\mathcal{N}_0$ is 674. Proceeding as in section 5.1 we describe the vertical and horizontal admissible connections by means of the graphs obtained as follows (and drawn in figures 6 and 7).

Enumeration 2. Let

$$\beta : \{0, 1, \dots, 14\} \to \mathcal{M},$$

with $\beta(0) = I$, $\beta(1) = II0$, $\beta(2) = II1.1$, $\beta(3) = II1.2$, $\beta(4) = II1.3$, $\beta(5) = II2.2.3$, $\beta(6) = II2.1.3$, $\beta(7) = II2.1.2$, $\beta(8) = III0$, $\beta(9) = III1.1$, $\beta(10) = III1.2$, $\beta(11) = III1.3$, $\beta(12) = III2.2.3$, $\beta(13) = III2.1.3$, $\beta(14) = II2.1.2$. Let also

$$\delta : \{0, \dots, 224\} \to \mathcal{M}$$

with $\delta(i) = (n, r)$, where n and r are such that $i \equiv r \pmod{15}$ and $i = 15n + r$. For notation convenience we denote $n = \delta_1(i)$ and $r = \delta_2(i)$.

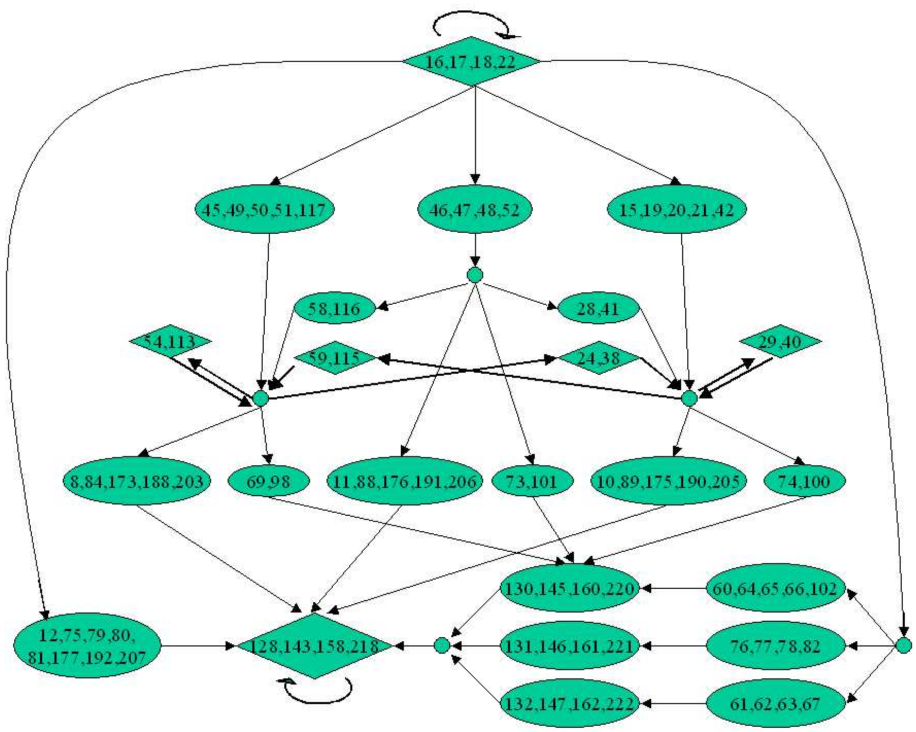

FIGURE 6. The graph shows the vertical admissible connections among building blocks for the Oriented Triangular Network.

Definition 15. Let $\mathcal{V} = \{V_0, \ldots, V_{224}\}$ with

$$V_i = (\beta(\delta_1(i)), \beta(\delta_2(i))) \in \mathcal{M} \times \mathcal{M}.$$

We denote by G^v the graph whose set of vertices is $\mathcal{V}$. Two vertices V_i and V_k, corresponding to two pairs $(j_{i_1}, j_{i_2}), (j_{k_1}, j_{k_2}) \in \mathcal{M} \times \mathcal{M}$ respectively, are joined by a directed edge from V_i to V_k if and only if $(j_{i_1}, j_{i_2}, j_{k_1}, j_{k_2}) \in \mathcal{N}_0$.
From the graph G^v we build the graph G^h. The set of vertices of G^h is $\mathcal{V}$ and two vertices $V_i = (j_{i_1}, j_{i_2})$ and $V_k = (j_{k_1}, j_{k_2})$, with $i = 15n_i + r_i$ and $k = 15n_k + r_k$, are joined by a directed edge from V_i to V_k if and only if $(j_{i_1}, j_{k_1}, j_{i_2}, j_{k_2}) \in \mathcal{N}_0$, that is, if and only if there is directed edge from $V_{i'} = (j_{i_1}, j_{k_1})$ to $V_{k'} = (j_{i_2}, j_{k_2})$, with $i' = 15n_i + n_k$ and $k' = 15r_i + r_k$, in G^v.

In figures 6 and 7 are represented the graphs G^v and G^h respectively. For simplifying the pictures we have:

- avoided the representation of isolated nodes;
- put in the same circle nodes that have same parents and children;
- drawn a loop on a circle (diamond) containing more than one node with the meaning that the nodes inside form a complete subgraph (i.e. all nodes are connected to each other and to themselves).

As for the Oriented Manhattan Network, G^v and G^h show the feasible vertical and horizontal concatenations respectively and give informations about the admissible periodic structures of equilibria.

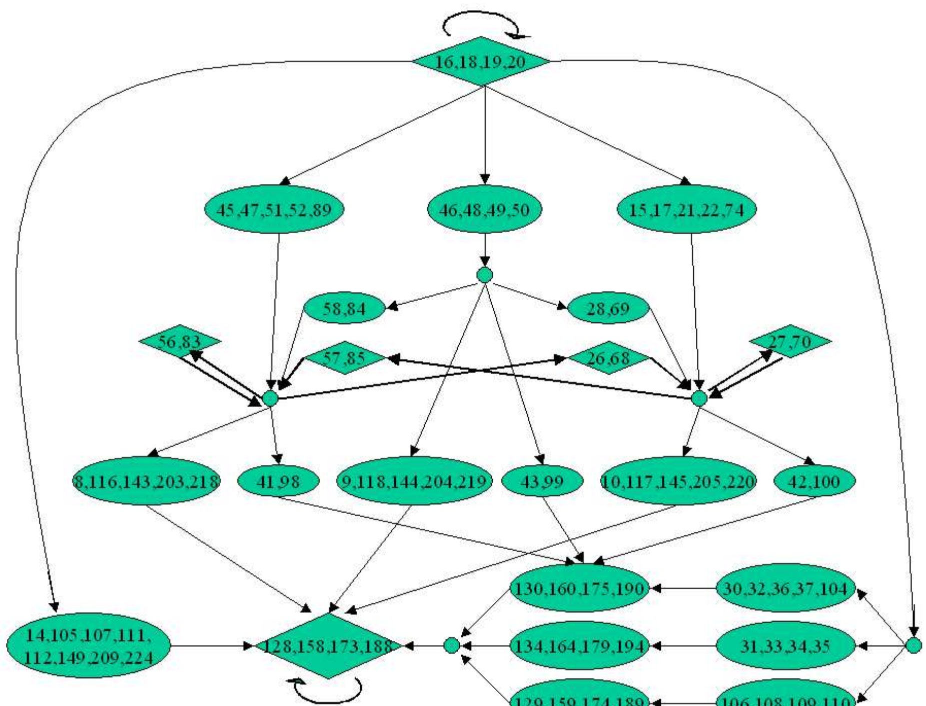

FIGURE 7. The graph shows the horizontal admissible connections among building blocks for the Oriented Triangular Network.

From pictures 6 and 7 it can be seen that graphs G^v and G^h present each:

a: 2 complete subgraphs, i.e. each vertex of the subgraph is connected to itself and to all other vertices of the subgraph, which are represented by the big diamonds on top and at the bottom of the graphs, and

b: a subgraph, in the middle of the picture, comprised of the 4 diamond connected by the thick arrows.

The complete subgraphs described in **a** contain loops and cycles of any order. Therefore we have periodicities of any order. In particular there are periodicities of order 0 corresponding to the loops $(16, 16)$ and $(128, 128)$. Among the cycles of length 2 we have $(16, 18)$ and $(128, 158)$. Also the subgraph described in **b** presents loops and cycles of any order. However, as for the Oriented Manhattan Network, possible periodic structures for the whole network are very few.

Theorem 6. *The periodic structures over the whole network are the following:*

- *equilibrium of type* $II0$ *for each node of the network;*
- *equilibrium of type* $III0$ *for each node of the network.*

Proof. To prove the theorem we will show that the only vertical periodicities compatible with the horizontal ones are those given by the cycles $(16, 16)$ and $(128, 128)$. To see what are the vertical periodicities compatible with the horizontal ones it is sufficient to translate the vertical cycles into horizontal connections and verify if the so obtained horizontal connections either form a closed path or form a subpath of a closed path.

We perform this computation in some cases being the others completely equivalent. Since $16 = 15 \times 1 + 1$, the vertical loop $(16, 16)$ corresponds to the horizontal loop $(16, 16)$.
On the other hand the vertical loops $(17, 17)$, $(18, 18)$ and $(22, 22)$ correspond to the horizontal connections $16 \to 32$, $16 \to 48$ and $16 \to 112$.

The cycles $(16, 17)$ and $(16, 22)$ correspond to the horizontal connections $16 \to 17$ and $16 \to 22$ which are not contained in any closed path in G^h. Analogously, the cycles $(17, 18)$, $(17, 22)$ and $(18, 22)$ correspond to the horizontal connections $16 \to 33$, $16 \to 37$ and $16 \to 49$ which are not contained in any closed path in G^h.

The vertical cycle $(16, 18)$ corresponds to the horizontal cycle $(16, 18)$. However, assume we consider a vertical stripe of type $(16, 18, 16)$. To be horizontally compatible it should be that the vertical connection $18 \to 16$ corresponds to a horizontal closed path (or part of it). Instead the vertical $18 \to 16$ connection corresponds to the horizontal connection $16 \to 46$.

A similar analysis can be performed for the closed paths of the subgraph described in **b** and in the complete graph at the bottom of G^v. □

6.2. **Circular Triangular Network.** A building block for a Circular Triangular Network is comprised of the 9 nodes $J_{ij},\ i = 1, \ldots, 3,\ j = 1, \ldots, 3$, as described in definition 13. The whole network is obtained by connecting the building blocks both vertically and horizontally as it has been done for the Circular Manhattan Network.

Rule **H** can now be rewritten as follows:

- if i is odd and j is odd then
  $incoming^h_{ij}$ is bad if and only if $outgoing^h_{i\ j-1}$ is good;
  $incoming^v_{ij}$ is bad if and only if $outgoing^v_{i-1\ j}$ is good;
  $incoming^d_{ij}$ is bad if and only if $outgoing^d_{i-1\ j-1}$ is good;
- if i is odd and j is even then
  $incoming^h_{ij}$ is bad if and only if $outgoing^h_{i\ j-1}$ is good;
  $incoming^v_{ij}$ is bad if and only if $outgoing^v_{i+1\ j}$ is good;
  $incoming^d_{ij}$ is bad if and only if $outgoing^d_{i+1\ j+1}$ is good;
- if i is even and j is odd then
  $incoming^h_{ij}$ is bad if and only if $outgoing^h_{i\ j+1}$ is good;
  $incoming^v_{ij}$ is bad if and only if $outgoing^v_{i-1\ j}$ is good;
  $incoming^d_{ij}$ is bad if and only if $outgoing^d_{i+1\ j+1}$ is good;
- if i is even and j is even then
  $incoming^h_{ij}$ is bad if and only if $outgoing^h_{i\ j+1}$ is good;
  $incoming^v_{ij}$ is bad if and only if $outgoing^v_{i+1\ j}$ is good;
  $incoming^d_{ij}$ is bad if and only if $outgoing^d_{i-1\ j-1}$ is good.

Using rule **H** we determine the set $\mathcal{N}_0$ of admissible configurations of the building block as follows.

a): Consider node J_{23}. It has one incoming line from J_{13} and one outgoing line to J_{22}. Since these two lines cannot be both *good*, it must be that $outgoing^v_{13}$ and $incoming^h_{22}$ cannot be both *bad*. Thus, if

$$J_{22} \in \{I, II0, II1.1, II1.2, II1.3, II2.2.3, II2.1.3, II2.1.2, III1.1, III2.1.3, III2.1.2\}$$

then $J_{13} \in \{II0, II1.1, II1.2, II2.1.2\}$. If otherwise

$$J_{22} \in \{III0, III1.2, III1.3, III2.2.3\},$$

then J_{13} can be of any type in $\mathcal{M}$.

b): The same argument applies to J_{32}. It has one incoming line from J_{31} and one outgoing line to J_{22}. Since these two lines cannot be both *good*, it must be that $outgoing^h_{31}$ and $incoming^v_{22}$ cannot be both *bad*. Thus, if

$$J_{22} \in \{I, II0, II1.1, II1.2, II1.3, II2.2.3, II2.1.3, II2.1.2, III1.3, III2.2.3, III2.1.3\}$$

then $J_{31} \in \{II0, II1.2, II1.3, II2.2.3\}$. If otherwise

$$J_{22} \in \{III0, III1.1, III1.2, III2.1.2\},$$

then J_{31} can be of any type in $\mathcal{M}$.

c): Now if

$$J_{22} \in \{I, II0, II1.1, II1.2, II1.3, II2.2.3, II2.1.3, II2.1.2, III1.1, III2.1.3, III2.1.2\}$$

then, by **a)**, $J_{13} \in \{II0, II1.1, II1.2, II2.1.2\}$ with $outgoing^v_{13} = good$,
hence $J_{12} \in \{II0, II1.2, II1.3, II2.2.3\}$ with $outgoing^h_{12} = good$,
$J_{22} \in \{II0, II1.1, II1.2, II2.1.2\}$ with $outgoing^v_{22} = good$
and $J_{23} \in \{II0, II1.3\}$ with $outgoing^d_{23} = good$ and $outgoing^h_{23} = good$.

d): On the other hand, if we assume that

$$J_{22} \in \{I, II0, II1.1, II1.2, II1.3, II2.2.3, II2.1.3, II2.1.2, III1.3, III2.2.3, III2.1.3\}$$

then, by **b)**, $J_{31} \in \{II0, II1.2, II1.3, II2.2.3\}$ with $outgoing^h_{31} = good$,
hence $J_{21} \in \{II0, II1.1, II1.2, II2.1.2\}$ with $outgoing^v_{21} = good$,
$J_{22} \in \{II0, II1.2, II1.3, II2.2.3\}$ with $outgoing^h_{22} = good$
and $J_{32} \in \{II0, II1.1\}$ with $outgoing^d_{32} = good$ and $outgoing^v_{32} = good$.

e): From **c)** and **d)** we get that $J_{22} \in \{II0, II1.2\}$ and $J_{11} = II0$.

f): Now, if $J_{22} = II0, J_{23} = II0, J_{32} = II0$ then

$$J_{33} \in \{I, II0, II1.1, II1.2, II1.3, II2.2.3, II2.1.3, II2.1.2\},$$

if $J_{22} = II0, J_{23} = II0, J_{32} = II1.1$ then $J_{33} = III2.2.3$,
if $J_{22} = II0, J_{23} = II1.3, J_{32} = II0$ then $J_{33} = III2.1.2$,
if $J_{22} = II1.2, J_{23} = II0, J_{32} = II0$ then $J_{33} = III2.1.3$,
if $J_{22} = II0, J_{23} = II1.3, J_{32} = II1.1$ then $J_{33} = III1.2$,
if $J_{22} = II1.2, J_{23} = II0, J_{32} = II1.1$ then $J_{33} = III1.3$,
if $J_{22} = II1.2, J_{23} = II1.3, J_{32} = II0$ then $J_{33} = III1.1$,
if $J_{22} = II1.2, J_{23} = II1.3, J_{32} = II1.1$ then $J_{33} = III0$.

g): If otherwise

$$J_{22} \in \{III0, III1.2, III1.3, III2.2.3\},$$

then $incoming^v_{12} = good$, $incoming^d_{33} = good$ and $incoming^h_{21} = good$. It follows that $incoming^h_{13} = good$ and $incoming^v_{23} = good$ from which $incoming^d_{12} = good$ and $incoming^v_{33} = good$, i.e. $J_{12}, J_{33} \in \{III0, III1.1\}$, while $J_{13} \in \{III0, III1.2, III1.3, III2.2.3\}$ and $J_{23} \in \{III0, III1.1, III1.2, III2.1.2\}$. It also follows that $incoming^v_{31} = good$ hence $incoming^h_{32} = good$ from which $incoming^d_{21} = good$, $incoming^v_{22} = good$ and $incoming^h_{33} = good$, i.e. $J_{31} \in \{III0, III1.1, III1.2, III2.1.2\}$, $J_{32} \in \{III0, III1.2, III1.3, III2.2.3\}$, $J_{21} \in$

$\{III0, III1.3\}$, $J_{22} \in \{III0, III1.2\}$ and $J_{33} = III0$.
Finally, if $J_{21} = III0$, $J_{22} = III0$ and $J_{12} = III0$ then

$$J_{11} \in \{III0, III1.1, III1.2, III1.3, III2.2.3, III2.1.3, III2.1.2\},$$

if $J_{21} = III1.3$, $J_{22} = III0$ and $J_{12} = III0$ then $J_{11} = II2.1.2$,
if $J_{21} = III0$, $J_{22} = III1.2$ and $J_{12} = III0$ then $J_{11} = II2.1.3$,
if $J_{21} = III0$, $J_{22} = III0$ and $J_{12} = III1.1$ then $J_{11} = II2.2.3$,
if $J_{21} = III0$, $J_{22} = III1.2$ and $J_{12} = III1.1$ then $J_{11} = II2.3$,
if $J_{21} = III1.3$, $J_{22} = III0$ and $J_{12} = III1.1$ then $J_{11} = II2.2$,
if $J_{21} = III1.3$, $J_{22} = III1.2$ and $J_{12} = III0$ then $J_{11} = II2.1$.

Clearly the set $\mathcal{N}_0$ is very large. As in section 5.2, for the Circular Manhattan case we use the informations on $\mathcal{N}_0$ to directly describe the set $\mathcal{N}$ of equilibria over the whole network. We also obtain a similar result:

Theorem 7.

$$\mathcal{N} = \{\{J_{ij} = II0,\ i = 1\dots,s, j = 1,\dots,t\}, \{J_{ij} = III0,\ i = 1\dots,s, j = 1,\dots,t\}\}$$

that is, $\mathcal{N}$ is comprised of only two equilibria. Moreover, each equilibrium solution of the Circular Triangular Network is periodic.

Proof. To prove the theorem we write the following two tables:

$II0$	$II0, II1.2, II1.3, II2.2.3$	$II0, II1.1, II1.2, II2.1.2$
$II0, II1.1, II1.2, II2.1.2$	$II0, II1.2$	$II0, II1.3$
$II0, II1.2, II1.3, II2.2.3$	$II0, II1.1$	J_{33}

J_{11}	$III0, III1.1$	$III0, III1.2, III1.3, III2.2.3$
$III0, III1.3$	$III0, III1.2$	$III0, III1.1, III1.2, III2.1.2$
$III0, III1.1, III1.2, III2.1.2$	$III0, III1.2, III1.3, III2.2.3$	$III0$

which indicate which are the possible equilibria at each node J_{ij}, for $i = 1, 2, 3$ and $j = 1, 2, 3$. The generic J_{33} in the first table and J_{11} in the second mean that the equilibrium at those nodes has to be deduced from the equilibria at the adjacent nodes along with the descriptions given in **f)** and **g)**.

From the first table it is clear that to have vertical connections among building blocks it must be $J_{31} = J_{11} = II0$, $J_{32} = J_{12} = II0$ and $J_{33} = J_{13} \in \{II0, II1.1, II1.2, II2.1.2\}$, from which, by **f)**, $J_{22} = II0$ and $J_{23} = II0$. To also have horizontal connections it must be $J_{21} = J_{23} = II0$ and $J_{33} = J_{31} = J_{13} = II0$, i.e. $J_{ij} = II0$ for all $i, j = 1, 2, 3$.

Analogously, from the second table, we get that it must be $J_{12} = J_{32} = III0$, $J_{13} = J_{33} = III0$ and $J_{11} = J_{31} \in \{III0, III1.1, III1.2, III2.1.2\}$, from which, by **g)**, $J_{21} = J_{22} = III0$. To also have horizontal connections we get that $J_{31} = J_{33} = III0$, $J_{21} = J_{23} = III0$ and $J_{11} = J_{13} = III0$, i.e. $J_{ij} = III0$ for all $i, j = 1, 2, 3$. □

7. **Qualitative versus quantitative equilibria.** In this section, we add some details to the analysis done in section 3 about the equilibria set. Combining proposition 3 with theorems 3, 4, 5, 6 and 7 we get the followings.

Corollary 1. *The space of periodic equilibria for an Oriented Network is a $(L - nN)$–dimensional space, where $n = 2$ for the Square Networks and $n = 3$ for the Triangular Networks.*

Proof. From theorems 3 and 6 we have that the periodic structures over the oriented (both Square and Triangular) Networks are given by either the equilibrium of type $II0$ for each node or by the equilibrium of type $III0$ for each node. Assume that at each node of the network the equilibrium is of type $II0$. This means that the incoming flows at each node are of type *bad* and the outgoing flows at each node are of type *good*. Therefore the outgoing flows are such that $\gamma_\psi < \gamma_\psi^{max}$ and $\gamma_\psi = \alpha_\psi \Gamma$, where α_ψ are the traffic distribution parameters, with $\sum \alpha_\psi = 1$, and $\Gamma = \Gamma_{in}$. Analogously, if at each node of the network the equilibrium is of type $III0$, the incoming flows are all of type *good* with $\gamma_\varphi < \gamma_\varphi^{max}$. Then we have $\gamma_\varphi = p_\varphi \Gamma$, where p_φ are the priority parameters, with $\sum p_\varphi = 1$ and $\Gamma = \Gamma_{out}$. Finally, in both cases, we get $n-1$ constraints for each node in addition to the constraint (6). Combining this result with proposition 3 we get that the space of equilibria is a $L - nN$ dimensional space. □

Corollary 2. *The equilibria for a Circular Network or for the Full Manhattan Network is a $(L - nN)$–dimensional space, where $n = 2$ for the Circular Manhattan Network, $n = 3$ for the Circular Triangular Network and $n = 4$ for the Full Manhattan Network.*

Proof. From theorems 4, 5 and 7, we have that the only equilibria structures over the Circular (both Square and Triangular) Networks and over the Full Square Network are given by either the equilibrium of type $II0$ at each node or by the equilibrium of type $III0$ at each node. Similarly to the proof of corollary 1 we get a system of nN constraints in L variables and the proof is finalized. □

Now we compute the cardinality L of the set $\mathcal{I}$ of lines of the network. For an Oriented or Circular Manhattan Network with $N = s \times t$ nodes, we have that

$$L = (s + 1)t + (t + 1)s = 2st + s + t = 2N + s + t;$$

for a Full Manhattan Network we have that

$$L = 2(s + 1)t + 2(t + 1)s = 4st + 2s + 2t = 4N + 2s + 2t;$$

for a Triangular Network we have that

$$L = (s+1)t + (t+1)s + st + s + t - 1 = 3st + 2s + 2t - 1 = 3N + 2s + 2t - 1.$$

Definition 16. We call *in–line* a line of the network which is incoming for some node and outgoing for none. We call *out–line* a line of the network which is outgoing for some node and incoming for none. Furthermore we call *in–flow* the flux on an in–line and *out–flow* the flux on an out–line.

A simple computation gives the following.

Proposition 7. *The number w of in–lines for a network is*

- $w = s + t$*, for the Oriented and the Circular Manhattan Networks;*
- $w = 2s + st$*, for the Full Manhattan Network;*
- $w = 2s + 2t - 1$*, for the Oriented and the Circular Triangular Networks.*

Finally, by computing $L - nN$ we get the followings.

Corollary 3. *The set of periodic equilibria for an Oriented Network is a w–dimensional space.*

Corollary 4. *The set of equilibria for a Circular Network or for the Full Manhattan Network is a w–dimensional space.*

In other words, from the above corollaries 3 and 4 we finally get our main result:

Theorem 8. *Consider the initial–boundary value problem for an oriented (resp. circular or full) square or triangular network, with Dirichlet conditions at the in–lines and Neumann conditions at the out–lines. Assume that a set of w constant boundary data for the in–flows are fixed. Then there exists a unique equilibrium (periodic equilibrium) solution on the network.*

8. **Appendix.** The set of admissible equilibria for the building block $(\mathcal{I}_0, \mathcal{J}_0)$ of the Oriented Manhattan Network is:
$\mathcal{N}_0 = \{[I, III0, III0, III0], [I, III0, III1.1, III0], [I, III1.2, III0, III0],$
$[I, III1.2, III1.1, III0],$

$[II0, I, I, III0], [II0, I, II0, III1.1], [II0, I, II1.1, III0],$
$[II0, I, II1.2, III1.1], [II0, I, III1.2, III0],$
$[II0, II0, I, III1.2], [II0, II0, II0, I], [II0, II0, II0, II0],$
$[II0, II0, II0, II1.1], [II0, II0, II0, II1.2], [II0, II0, II1.1, III1.2],$
$[II0, II0, II1.2, I], [II0, II0, II1.2, II0], [II0, II0, II1.2, II1.1],$
$[II0, II0, II1.2, II1.2], [II0, II0, III1.2, III1.2],$
$[II0, II1.1, I, III1.2], [II0, II1.1, II0, I], [II0, II1.1, II0, II0],$
$[II0, II1.1, II0, II1.1], [II0, II1.1, II0, II1.2], [II0, II1.1, II1.1, III1.2],$
$[II0, II1.1, II1.2, I], [II0, II1.1, II1.2, II0], [II0, II1.1, II1.2, II1.1],$
$[II0, II1.1, II1.2, II1.2], [II0, II1.1, III1.2, III1.2],$
$[II0, II1.2, I, III0], [II0, II1.2, II0, III1.1], [II0, II1.2, II1.1, III0],$
$[II0, II1.2, II1.2, III1.1], [II0, II1.2, III1.2, III0],$
$[II0, III1.1, I, III0], [II0, III1.1, II0, III1.1], [II0, III1.1, II1.1, III0],$
$[II0, III1.1, II1.2, III1.1], [II0, III1.1, III1.2, III0],$

$[II1.1, III0, I, III0], [II1.1, III0, II0, III1.1], [II1.1, III0, II1.1, III0],$
$[II1.1, III0, II1.2, III1.1], [II1.1, III0, III1.2, III0],$

$[II1.1, III1.2, I, III0]$, $[II1.1, III1.2, II0, III1.1]$, $[II1.1, III1.2, II1.1, III0]$,
$[II1.1, III1.2, II1.2, III1.1]$, $[II1.1, III1.2, III1.2, III0]$,

$[II1.2, I, III0, III0]$, $[II1.2, I, III1.1, III0]$,
$[II1.2, II0, III0, III1.2]$, $[II1.2, II0, III1.1, III1.2]$,
$[II1.2, II1.1, III0, III1.2]$, $[II1.2, II1.1, III1.1, III1.2]$,
$[II1.2, II1.2, III0, III0]$, $[II1.2, II1.2, III1.1, III0]$,
$[II1.2, III1.1, III0, III0]$, $[II1.2, III1.1, III1.1, III0]$,

$[III0, III0, III0, III0]$, $[III0, III0, III1.1, III0]$,
$[III0, III1.2, III0, III0]$, $[III0, III1.2, III1.1, III0]$,

$[III1.1, III0, III0, III0]$, $[III1.1, III0, III1.1, III0]$,
$[III1.1, III1.2, III0, III0]$, $[III1.1, III1.2, III1.1, III0]$,

$[III1.2, III0, III0, III0]$, $[III1.2, III0, III1.1, III0]$,
$[III1.2, III1.2, III0, III0]$, $[III1.2, III1.2, III1.1, III0]\}$.

The set $\mathcal{N}_0$ of equilibria for the building block of the Oriented Manhattan Network, up to the symmetry relation S is the following:
$\mathcal{N}_0/S = \{[I, III0, III0, III0]$, $[I, III0, III1.1, III0]$,
$[I, III1.2, III1.1, III0]$,

$[II0, I, I, III0]$, $[II0, I, II0, III1.1]$, $[II0, I, II1.1, III0]$,
$[II0, I, II1.2, III1.1]$, $[II0, I, III1.2, III0]$,
$[II0, II0, II0, I]$, $[II0, II0, II0, II0]$,
$[II0, II0, II0, II1.1]$, $[II0, II0, II1.1, III1.2]$,
$[II0, II0, II1.2, I]$, $[II0, II0, II1.2, II0]$, $[II0, II0, II1.2, II1.1]$,
$[II0, II0, II1.2, II1.2]$, $[II0, II0, III1.2, III1.2]$,
$[II0, II1.1, II1.1, III1.2]$,
$[II0, II1.1, II1.2, I]$, $[II0, II1.1, II1.2, II0]$, $[II0, II1.1, II1.2, II1.1]$,
$[II0, II1.1, III1.2, III1.2]$,
$[II0, II1.2, II1.1, III0]$,
$[II0, II1.2, III1.2, III0]$,
$[II0, III1.1, III1.2, III0]$,

$[II1.1, III0, I, III0]$, $[II1.1, III0, II0, III1.1]$, $[II1.1, III0, II1.1, III0]$,
$[II1.1, III0, II1.2, III1.1]$, $[II1.1, III0, III1.2, III0]$,
$[II1.1, III1.2, I, III0]$, $[II1.1, III1.2, II0, III1.1]$, $[II1.1, III1.2, II1.1, III0]$,
$[II1.1, III1.2, II1.2, III1.1]$, $[II1.1, III1.2, III1.2, III0]$,

$[III0, III0, III0, III0]$, $[III0, III0, III1.1, III0]$,
$[III0, III1.2, III1.1, III0]$,

$[III1.1, III0, III0, III0]$, $[III1.1, III0, III1.1, III0]$,
$[III1.1, III1.2, III0, III0]$, $[III1.1, III1.2, III1.1, III0]\}$,

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