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STABILIZATION OF THE WAVE EQUATION ON 1-D NETWORKS WITH A DELAY TERM IN THE NODAL FEEDBACKS

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Abstract. In this paper we consider the wave equation on 1-d networks with a delay term in the boundary and/or transmission conditions. We first show the well posedness of the problem and the decay of an appropriate energy. We give a necessary and sufficient condition that guarantees the decay to zero of the energy. We further give sufficient conditions that lead to exponential or polynomial stability of the solution. Some examples are also given.

1. **Introduction/notations.** Time delay effects arise in many practical problems, see for instance [20, 28, 1] for biological, electrical engineering, or mechanical applications. Furthermore it is well known that they can induce some instabilities [17, 18, 19, 30, 25], or on the contrary improve the performance of the system [28, 1].

Recently, control problems on 1-d networks are paying attention of many authors, see [22, 16] and the references cited there. We here investigate the effect of time delay in boundary and/or transmission stabilization of the wave equation in $1-d$ networks. To our knowledge, the analysis of this effect to $1-d$ networks is not yet done.

Before going on, let us recall some definitions and notations about $1-d$ networks used in the whole paper. We refer to [2, 3, 11, 12, 13, 14, 24, 26] for more details.

Definition 1.1. A $1-d$ network $\mathcal{R}$ is a connected set of $\mathbb{R}^n$, $n \geq 1$ defined by

$$\mathcal{R} = \bigcup_{j=1}^{N} e_j$$

where e_j is a curve that we identify with the interval $(0, l_j)$, $l_j > 0$, and such that for $k \neq j$, $\overline{e_j} \cap \overline{e_k}$ is either empty or a common extremity called a vertex or a node (here $\overline{e_j}$ means the closure of e_j).

For a function $u : \mathcal{R} \longrightarrow \mathbb{R}$, we set $u_j = u_{|e_j}$ the restriction of u to the edge e_j.

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We denote by $\mathcal{E} = \{e_j\,;\, 1 \le j \le N\}$ the set of edges of $\mathcal{R}$ and by $\mathcal{V}$ the set of vertices of $\mathcal{R}$. For a fixed vertex v, let

$$\mathcal{E}_v \quad = \quad \{j \in \{1, ..., N\}\,;\, v \in \overline{e_j}\}$$

be the set of edges having v as vertex. If card $(\mathcal{E}_v) = 1$, v is an exterior node, while if card $(\mathcal{E}_v) \ge 2$, v is an interior node. We set $\mathcal{V}_{ext}$ the set of exterior nodes and $\mathcal{V}_{int}$ the set of interior nodes. For $v \in \mathcal{V}_{ext}$, the single element of $\mathcal{E}_v$ is denoted by j_v.

We now fix a partition of $\mathcal{V}_{ext}$:

$$\mathcal{V}_{ext} = \mathcal{D} \cup \mathcal{N} \cup \mathcal{V}^c_{ext}.$$

Clearly we will impose Dirichlet boundary condition at the nodes of $\mathcal{D}$; Neumann boundary condition at the nodes of $\mathcal{N}$ and finally a feedback boundary condition at the nodes of $\mathcal{V}^c_{ext}$. We further fix a subset $\mathcal{V}^c_{int}$ of $\mathcal{V}_{int}$, where a feedback transmission condition will be imposed. For shortness, we denote by $\mathcal{V}_c$ the set of controlled nodes, namely

$$\mathcal{V}_c = \mathcal{V}^c_{int} \cup \mathcal{V}^c_{ext}.$$

We also suppose that $\mathcal{D} \neq \emptyset$; so that the H^1 semi-norm becomes a norm.

We can now formulate our initial/boundary value problem:

$$\begin{cases} \frac{\partial^2 u_j}{\partial t^2}(x,\,t) - \frac{\partial^2 u_j}{\partial x^2}(x,\,t) = 0 & 0 < x < l_j,\, t > 0, \\ & \quad \forall j \in \{1, ..., N\}, \\ u_j(v,\,t) = u_l(v,\,t) = u(v,t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \\ \sum\limits_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = -(\alpha_1^{(v)} \frac{\partial u}{\partial t}(v,\,t) + \alpha_2^{(v)} \frac{\partial u}{\partial t}(v,\,t - \tau_v)) & \forall v \in \mathcal{V}_c, t > 0, \\ \sum\limits_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = 0 & \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}^c_{int}, t > 0, \\ u_{j_v}(v,t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \frac{\partial u_{j_v}}{\partial n_{j_v}}(v,t) = 0 & \forall v \in \mathcal{N}, t > 0, \\ u(t=0) = u^{(0)},\ \frac{\partial u}{\partial t}(t=0) = u^{(1)}, & \\ \frac{\partial u}{\partial t}(v,\,t - \tau_v) = f^0_v(t - \tau_v) & \forall v \in \mathcal{V}_c,\, 0 < t < \tau_v, \end{cases} \tag{1}$$

where $\alpha_1^{(v)}$, $\alpha_2^{(v)} \ge 0$ are fixed nonnegative real numbers, the delay $\tau_v > 0$ is also supposed to be fixed and $\frac{\partial u_j}{\partial n_j}(v,\,\cdot)$ means the outward normal (space) derivative of u_j at the vertex v.

Note that u_j represents the displacement of the string e_j.

Remark that the condition $\frac{\partial u}{\partial t}(v,\,t - \tau_v) = f^0_v(t - \tau_v)$ for $v \in \mathcal{V}_c$, $0 < t < \tau_v$ denotes an initial value in the past, but is necessary due to the delay equation.

In the absence of delay, i. e., $\alpha_2^{(v)} = 0$ for all $v \in \mathcal{V}_c$, the above problem has been considered by some authors in some particular situations, for instance Ammari and Tucsnak [9], Ammari, Henrot and Tucsnak [4], Ammari and Jellouli [5, 6], Ammari, Jellouli and Khenissi [7] and Xu, Liu and Liu [29]. In these papers, some sufficient conditions are given in order to guarantee some stabilities of the system. On the contrary, if $\alpha_1^{(v)} = 0$ that is if we have only the delay part in the boundary/transmission condition, system (1) may become unstable. See, for instance Datko, Lagnese and Polis [19] for the example of a string. Therefore it is interesting to seek for stabilization results in general 1-d networks when the parameters $\alpha_1^{(v)}$ and $\alpha_2^{(v)}$ are both nonzero. In the special case of one string and a feedback law at one extremity, this problem has been studied by Xu, Yung and

Li [30], where the authors use a spectral analysis. For the wave equation in higher dimensional space domain, we refer to [25].

In accordance with [30, 25], assuming that

$$\alpha_2^{(v)} \leq \alpha_1^{(v)}, \forall v \in \mathcal{V}_c,$$

we show the decay of an appropriate energy. We further give a necessary and sufficient condition for the decay to zero of the energy. If the above condition does not hold, we conjecture that the energy does not decay. We do not investigate this problem in its full generality but study it in a particular case.

Now if

$$\alpha_2^{(v)} < \alpha_1^{(v)}, \forall v \in \mathcal{V}_c,$$

we first give a necessary and sufficient condition for the exponential decay of the energy. We secondly find a sufficient condition for the polynomial decay of the energy.

Our method is based on the use of observability estimates of the problem without damping. Here we have chosen to obtain these observability estimates by a frequency domain method. The use of other techniques like the d'Alembert representation formula [16, 5] may avoid the use of the frequency domain method but give quite often non optimal decay rates for the energy. Note finally that the observability estimate is independent of the delay term.

The paper is organized as follows. After the recall of some definitions and notations, we show in the second section that our problem is well posed. Then in section 3, we prove the decay of an appropriate energy and give a necessary and sufficient condition which guarantees the decay to 0 of the energy. Section 4 is devoted to the proof of a regularity result and an a priori estimate used for the stability results. In section 5 we give a necessary and sufficient condition for the exponential stability of our system. Similarly section 6 is concerned with a sufficient condition for the polynomial stability of our system. Finally we end up with some illustrative examples in section 7.

In the whole paper the notation $a \lesssim b$ means that there exists a positive constant C independent of a and b such that $a \leq C\ b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

2. **Well posedness of the problem.**

We aim to show that problem (1) is well-posed. For that purpose, we use semi-group theory and an idea from [25].

For future uses, we introduce the spatial operator associated with the system similar to (1) but without damping. Introduce

$$L^2(\mathcal{R}) = \{u : \mathcal{R} \to \mathbb{R}; u_j \in L^2(0, l_j), \forall j = 1, \cdots, N\},$$

which is a Hilbert space for the natural inner product. Its associated norm will be denoted by $\| \cdot \|_{L^2(\mathcal{R})}$. Let further V be the Hilbert space

$$V := \{\phi \in \prod_{j=1}^{N} H^1(0,\, l_j) \,:\, \phi_j(v) = \phi_k(v) \,\forall j,\, k \in \mathcal{E}_v,\, \forall v \in \mathcal{V}_{int}\,;\, \phi_{j_v}(v) = 0 \,\forall v \in \mathcal{D}\},$$

equipped with the inner product

$$< \phi,\, \tilde{\phi} >_V = \sum_{j=1}^{N} \int_0^{l_j} \frac{\partial \phi_j}{\partial x} \frac{\partial \tilde{\phi}_j}{\partial x} dx.$$

For shortness for $u \in L^1(\mathcal{R}) = \{u : \mathcal{R} \to \mathbb{R}; u_j \in L^1(0, l_j), \forall j = 1, \cdots, N\}$, we often write

$$\int_{\mathcal{R}} u = \sum_{j=1}^{N} \int_0^{l_j} u_j(x)\, dx.$$

Now we introduce the operator A from $L^2(\mathcal{R})$ into itself by

$$\begin{aligned} D(A) \quad &:= \quad \{u \in V \cap \prod_{j=1}^{N} H^2(0,\, l_j) \,:\, \sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v) = 0,\, \forall v \in \mathcal{V}_{int}\,; \\ &\qquad \frac{\partial u_{j_v}}{\partial n_{j_v}}(v) = 0,\, \forall v \in \mathcal{N} \cup \mathcal{V}_{ext}^c\}, \\ (Au)_j \quad &= \quad -\frac{\partial^2 u_j}{\partial x^2} \quad \forall j = 1, \cdots, N, \forall u \in D(A). \end{aligned}$$

This operator is a positive selfadjoint operator since it is the Friedrichs extension of the triple $(L^2(\mathcal{R}), V, a)$, where the bilinear form a is defined by

$$a(u, v) = \sum_{j=1}^{N} \int_0^{l_j} \frac{\partial u_j}{\partial x} \frac{\partial v_j}{\partial x}\, dx, \forall u, v \in V.$$

Let further set $X = V \cap \prod_{j=1}^{N} H^2(0, l_j)$, which is a Hilbert space with the inner product

$$(u, v)_X = (u, v)_{L^2(\mathcal{R})} + (\Delta u, \Delta v)_{L^2(\mathcal{R})}, \forall u, v \in X,$$

where we have set

$$(\Delta u)_j = \frac{\partial^2 u_j}{\partial x^2} \forall j = 1, \cdots, N, u \in X.$$

Now we come back to our system (1) and transform it as follows. For all $v \in \mathcal{V}_c$ let us introduce the auxiliary variable $z_v(\rho, t) = \frac{\partial u}{\partial t}(v,\, t - \tau_v \rho)$ for $\rho \in (0,\, 1)$ and $t > 0$. In this manner, we eliminate the delay term in (1) and problem (1) is equivalent to

$$\begin{cases} \frac{\partial^2 u_j}{\partial t^2}(x,\, t) - \frac{\partial^2 u_j}{\partial x^2}(x,\, t) = 0 & 0 < x < l_j, \quad t > 0,\, \forall j \in \{1, ..., N\}, \\ \tau_v \frac{\partial z_v}{\partial t}(\rho,\, t) + \frac{\partial z_v}{\partial \rho}(\rho,\, t) = 0 & 0 < \rho < 1,\, t > 0,\, \forall v \in \mathcal{V}_c, \\ u_j(v,\, t) = u_l(v,\, t) = u(v, t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \\ \sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\, t) = -(\alpha_1^{(v)} \frac{\partial u}{\partial t}(v,\, t) + \alpha_2^{(v)} z_v(1,\, t)) & \forall v \in \mathcal{V}_c, t > 0, \\ \sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\, t) = 0 & \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}_{int}^c, t > 0, \\ u_{j_v}(v, t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \frac{\partial u_{j_v}}{\partial n_{j_v}}(v, t) = 0 & \forall v \in \mathcal{N}, t > 0, \\ z_v(0,\, t) = \frac{\partial u}{\partial t}(v,\, t) & \forall v \in \mathcal{V}_c,\, t > 0, \\ u(t = 0) = u^{(0)},\, \frac{\partial u}{\partial t}(t = 0) = u^{(1)}, & \\ z_v(\rho,\, 0) = f_v^0(-\tau_v \rho) & \forall v \in \mathcal{V}_c,\, 0 < \rho < 1. \end{cases} \tag{2}$$

Note that z_v satisfies a transport equation in the t, ρ variables, with an initial datum at $t = 0$ and $\rho = 0$.

If we introduce $z=(z_v)_{v\in\mathcal{V}_c}$ and

$$U := (u,\, \frac{\partial u}{\partial t},\, z)^\top,$$

then U satisfies

$$U' = (\frac{\partial u}{\partial t},\, \frac{\partial^2 u}{\partial t^2},\, \frac{\partial z}{\partial t})^\top = (\frac{\partial u}{\partial t},\, \Delta u,\, -(\frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho})_{v\in\mathcal{V}_c})^\top.$$

Consequently the problem (2) may be rewritten as the first order evolution equation

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (u^0,\, u^1,\, (f^0(-\tau_v\cdot))_v)^\top, \end{cases} \tag{3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ w \\ z \end{pmatrix} := \begin{pmatrix} w \\ \Delta u \\ -(\frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho})_v \end{pmatrix}$$

with domain

$$\begin{aligned} &D(\mathcal{A}) := \{(u,\, w,\, z) \in (V \cap \prod_{j=1}^{N} H^2(0,\, l_j)) \times V \times H^1(0,\, 1)^{V_c} \ : \\ &\sum_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v) = -\,(\alpha_1^{(v)} w(v) + \alpha_2^{(v)} z_v(1))\,\forall v \in \mathcal{V}_c\,; \\ &\sum_{j\in\mathcal{E}_v} \tfrac{\partial u_j}{\partial n_j}(v) = 0\,\forall v\in \mathcal{V}_{int}\backslash\mathcal{V}_{int}^c\,; \\ &\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v) = 0\,\forall v \in \mathcal{N}\,;\ z_v(0) = w(v)\,\forall v\in\mathcal{V}_c\}, \end{aligned}$$

where V_c is the number of nodes of $\mathcal{V}_c$.

Now introduce the Hilbert space

$$H := V \times L^2(\mathcal{R}) \times L^2(0,\, 1)^{V_c},$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ w \\ z \end{pmatrix},\, \begin{pmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{z} \end{pmatrix} \right\rangle = \sum_{j=1}^{N} \int_0^{l_j} (\frac{\partial u_j}{\partial x}\frac{\partial \tilde{u}_j}{\partial x} + w_j \tilde{w}_j)dx + \sum_{v\in\mathcal{V}_c} \int_0^1 z_v(\rho)\tilde{z}_v(\rho)d\rho.$$

Lemma 2.1. *$D(\mathcal{A})$ is dense in H.*

Proof. Let $(f,\, g,\, h)^\top \in H$ be orthogonal to all elements of $D(\mathcal{A})$, namely

$$0 = \left\langle \begin{pmatrix} u \\ w \\ z \end{pmatrix},\, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle = \sum_{j=1}^{N} \int_0^{l_j} (\frac{\partial u_j}{\partial x}\frac{\partial f_j}{\partial x} + w_j g_j)dx + \sum_{v\in\mathcal{V}_c} \int_0^1 z_v(\rho)h_v(\rho)d\rho,$$

for all $(u,\, w,\, z)^\top \in D(\mathcal{A})$.

We first take $u = 0$ and $w = 0$ and $z \in \mathcal{D}(0,\, 1)^{V_c}$. As $(0,\, 0,\, z) \in D(\mathcal{A})$, we get

$$\sum_{v\in\mathcal{V}_c} \int_0^1 z_v(\rho)h_v(\rho)d\rho = 0.$$

Since $\mathcal{D}(0,\, 1)$ is dense in $L^2(0,\, 1)$, we deduce that $h = 0$.

In the same manner as $\prod_{j=1}^{N}\mathcal{D}(0,\, l_j)$ is dense in $\prod_{j=1}^{N}L^2(0,\, l_j)$, by taking $u=0$, $z=0$ and $w\in\prod_{j=1}^{N}\mathcal{D}(0,\, l_j)$ we see that $g=0$.

The above orthogonality condition is then reduced to

$$0=\sum_{j=1}^{N}\int_0^{l_j}\frac{\partial u_j}{\partial x}\frac{\partial f_j}{\partial x}dx,\ \forall(u,\, w,\, z)\in D(\mathcal{A}).$$

By restricting ourselves to $w=0$ and $z=0$, we obtain

$$\sum_{j=1}^{N}\int_0^{l_j}\frac{\partial u_j}{\partial x}\frac{\partial f_j}{\partial x}dx=0,\ \forall(u,\, 0,\, 0)\in D(\mathcal{A}).$$

But we easily check that $(u,\, 0,\, 0)\in D(\mathcal{A})$ if and only if $u\in D(A)$. Since it is well known that $D(A)$ is dense in V (equipped with the inner product $<.,.>_V$), we conclude that $f=0$. □

Let us now suppose that

$$\alpha_2^{(v)}\leq\alpha_1^{(v)},\ \forall v\in\mathcal{V}_c. \tag{4}$$

Under this condition, we will show that the operator $\mathcal{A}$ generates a C_0-semi-group in H.

For that purpose, we choose positive real numbers ξ^v such that

$$\tau_v\alpha_2^{(v)}\leq\xi^{(v)}\leq\tau_v(2\alpha_1^{(v)}-\alpha_2^{(v)}),\ \forall v\in\mathcal{V}_c. \tag{5}$$

These constants exist owing to the condition (4).

We now introduce the following inner product on H

$$\left\langle\begin{pmatrix}u\\w\\z\end{pmatrix},\begin{pmatrix}\tilde u\\\tilde w\\\tilde z\end{pmatrix}\right\rangle_H=\sum_{j=1}^{N}\int_0^{l_j}(\frac{\partial u_j}{\partial x}\frac{\partial\tilde u_j}{\partial x}+w_j\tilde w_j)dx+\sum_{v\in\mathcal{V}_c}\xi^{(v)}(\int_0^1 z_v(\rho)\tilde z_v(\rho)d\rho)$$

This inner product is clearly equivalent to the usual inner product of H.

Theorem 2.2. *For an initial datum $U_0\in H$, there exists a unique solution $U\in C([0,+\infty),\, H)$ to problem (3). Moreover, if $U_0\in D(\mathcal{A})$, then*

$$U\in C([0,+\infty),\, D(\mathcal{A}))\cap C^1([0,+\infty),\, H).$$

Proof. By Lumer-Phillips' theorem, it suffices to show that $\mathcal{A}$ is dissipative and maximal monotone.

We first prove that $\mathcal{A}$ is dissipative. Take $U=(u,\, w,\, z)^\top\in D(\mathcal{A})$. Then

$$\begin{aligned}(\mathcal{A}U,\, U)&=\left\langle\begin{pmatrix}w\\\Delta u\\-(\frac{1}{\tau_v}\frac{\partial z_v}{\partial\rho})_v\end{pmatrix},\begin{pmatrix}u\\w\\z\end{pmatrix}\right\rangle_H\\&=\sum_{j=1}^{N}\int_0^{l_j}(\frac{\partial w_j}{\partial x}\frac{\partial u_j}{\partial x}+\frac{\partial^2u_j}{\partial x^2}w_j)dx+\sum_{v\in\mathcal{V}_c}\xi^{(v)}(\int_0^1-\frac{1}{\tau_v}\frac{\partial z_v}{\partial\rho}(\rho)z_v(\rho)d\rho).\end{aligned}$$

By integrating by parts, we obtain

$$
\begin{aligned}
(\mathcal{A}U,\, U) =&\sum_{j=1}^{N}\int_0^{l_j}(-w_j\frac{\partial^2 u_j}{\partial x^2}\\
&+\frac{\partial^2 u_j}{\partial x^2}w_j)dx+\sum_{j=1}^{N}[w_j\frac{\partial u_j}{\partial x}]_0^{l_j}-\sum_{v\in\mathcal{V}_c}\frac{\xi^{(v)}}{\tau_v}(\int_0^1\frac{\partial z_v}{\partial\rho}(\rho)z_v(\rho)d\rho)\\
=&\sum_{j=1}^{N}[w_j\frac{\partial u_j}{\partial x}]_0^{l_j}-\sum_{v\in\mathcal{V}_c}\frac{\xi^{(v)}}{\tau_v}(\int_0^1\frac{\partial z_v}{\partial\rho}(\rho)z_v(\rho)d\rho).
\end{aligned}
$$

Again an integration by parts leads to

$$
\int_0^1\frac{\partial z_v}{\partial\rho}(\rho)z_v(\rho)d\rho=\frac{1}{2}(z_v^2(1)-z_v^2(0)).
$$

Moreover by the boundary/transmission conditions satisfied by $(u,\, w,\, z)^\top\in D(\mathcal{A})$, we have

$$
\begin{aligned}
\sum_{j=1}^{N}[w_j\tfrac{\partial u_j}{\partial x}]_0^{l_j} &= \sum_{v\in\mathcal{V}}\sum_{j\in\mathcal{E}_v}w_j(v)\tfrac{\partial u_j}{\partial n_j}(v)\\
&= \sum_{v\in\mathcal{V}_c}\sum_{j\in\mathcal{E}_v}w_j(v)\tfrac{\partial u_j}{\partial n_j}(v)+\sum_{v\in\mathcal{D}}w_{j_v}(v)\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v)+\sum_{v\in\mathcal{N}}w_{j_v}(v)\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v)\\
&\quad+\sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c}\sum_{j\in\mathcal{E}_v}w_j(v)\tfrac{\partial u_j}{\partial n_j}(v)\\
&= \sum_{v\in\mathcal{V}_c}(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v))w_j(v)+\sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c}(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v))w_j(v)\\
&= \sum_{v\in\mathcal{V}_c}-(\alpha_1^{(v)}w(v)+\alpha_2^{(v)}z_v(1))z_v(0)\\
&= -\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}z_v(0)^2+\alpha_2^{(v)}z_v(1)z_v(0)).
\end{aligned}
$$

These properties yield

$$
\begin{aligned}
(\mathcal{A}U,\, U) &= -\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}z_v(0)^2+\alpha_2^{(v)}z_v(1)z_v(0))-\sum_{v\in\mathcal{V}_c}\tfrac{\xi^{(v)}}{2\tau_v}(z_v^2(1)-z_v^2(0))\\
&= -\sum_{v\in\mathcal{V}_c}[(\alpha_1^{(v)}-\tfrac{\xi^{(v)}}{2\tau_v})z_v(0)^2+\tfrac{\xi^{(v)}}{2\tau_v}z_v^2(1)+\alpha_2^{(v)}z_v(1)z_v(0)].
\end{aligned}
$$

By Cauchy-Schwarz's inequality we have

$$
-\alpha_2^{(v)}z_v(1)z_v(0)\le\frac{\alpha_2^{(v)}}{2}z_v^2(1)+\frac{\alpha_2^{(v)}}{2}z_v^2(0)
$$

and therefore

$$
(\mathcal{A}U,\, U)\le-\sum_{v\in\mathcal{V}_c}[(\alpha_1^{(v)}-\frac{\xi^{(v)}}{2\tau_v}-\frac{\alpha_2^{(v)}}{2})z_v(0)^2+(\frac{\xi^{(v)}}{2\tau_v}-\frac{\alpha_2^{(v)}}{2})z_v^2(1)]
$$

with $\alpha_1^{(v)}-\frac{\xi^{(v)}}{2\tau_v}-\frac{\alpha_2^{(v)}}{2}\ge 0$ and $\frac{\xi^{(v)}}{2\tau_v}-\frac{\alpha_2^{(v)}}{2}\ge 0$ because $\alpha_1^{(v)}$ and $\alpha_2^{(v)}$ satisfy condition (5). This shows that $(\mathcal{A}U,\, U)\le 0$ and then the dissipativeness of $\mathcal{A}$.

Let us now prove that $\mathcal{A}$ is maximal monotone, i. e., that $\lambda I-\mathcal{A}$ is surjective for some $\lambda>0$.

Let $(f,\, g,\, h)^\top \in H$. We look for $U = (u,\, w,\, z)^\top \in D(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u \\ w \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \tag{6}$$

or equivalently

$$\begin{cases} \lambda u_j - w_j = f_j & \forall j \in \{1, ..., N\}, \\ \lambda w_j - \frac{\partial^2 u_j}{\partial x^2} = g_j & \forall j \in \{1, ..., N\}, \\ \lambda z_v + \frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho} = h_v & \forall v \in \mathcal{V}_c. \end{cases} \tag{7}$$

Suppose that we have found u with the appropriate regularity. Then for all $j \in \{1, ..., N\}$, we have

$$w_j := \lambda u_j - f_j \in H^1(0,\, l_j) \tag{8}$$

with $w_{j_v}(v) = \lambda u_{j_v}(v) - f_{j_v}(v) = 0$ for $v \in \mathcal{D}$.

We can then determine z since $w(v) = z_v(0)$. Indeed, for $v \in \mathcal{V}_c$, z_v satisfies the differential equation

$$\lambda z_v + \frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho} = h_v$$

and the boundary condition

$$z_v(0) = w(v) = \lambda u(v) - f(v).$$

Therefore z_v is explicitly given by

$$z_v(\rho) = \lambda u(v) e^{-\lambda\tau_v\rho} - f(v) e^{-\lambda\tau_v\rho} + \tau_v e^{-\lambda\tau_v\rho}\int_0^\rho e^{\lambda\tau_v\sigma} h_v(\sigma) d\sigma.$$

This means that once u is found with the appropriate properties, we can find z and w. Note that in particular we have

$$\begin{array}{rcl} z_v(1) & = & \lambda u(v) e^{-\lambda\tau_v} - f(v) e^{-\lambda\tau_v} + \tau_v e^{-\lambda\tau_v}\int_0^1 e^{\lambda\tau_v\sigma} h_v(\sigma) d\sigma \\ & = & \lambda u(v) e^{-\lambda\tau_v} + z_v^0(v) \end{array}$$

where $z_v^0(v) = -f(v)e^{-\lambda\tau_v} + \tau_v e^{-\lambda\tau_v}\int_0^1 e^{\lambda\tau_v\sigma} h_v(\sigma) d\sigma$ is a fixed real number depending only on f and h.

It remains to find u. By (7) and (8), u_j must satisfy

$$\lambda^2 u_j - \frac{\partial^2 u_j}{\partial x^2} = g_j + \lambda f_j.$$

Multiplying this identity by a test function ϕ_j, integrating in space and using integration by parts, we obtain

$$\begin{aligned} \sum_{j=1}^N \int_0^{l_j} (\lambda^2 u_j - \frac{\partial^2 u_j}{\partial x^2})\phi_j dx =& \sum_{j=1}^N \int_0^{l_j} (\lambda^2 u_j \phi_j + \frac{\partial u_j}{\partial x}\frac{\partial \phi_j}{\partial x}) dx - \sum_{j=1}^N [\frac{\partial u_j}{\partial x}\phi_j]_0^{l_j} \\ =& \sum_{j=1}^N \int_0^{l_j} (\lambda^2 u_j \phi_j + \frac{\partial u_j}{\partial x}\frac{\partial \phi_j}{\partial x}) dx - \sum_{v\in\mathcal{V}}\sum_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v)\phi_j(v). \end{aligned}$$

But using the fact that $(u,\, w,\, z)^\top$ must belong to $D(\mathcal{A})$, we have

$$\begin{aligned}\sum_{v\in\mathcal{V}}\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\phi_j(v) &= \sum_{v\in\mathcal{V}_c}\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\phi_j(v)+\sum_{v\in\mathcal{D}}\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v)\phi_{j_v}(v)\\ &\quad+\sum_{v\in\mathcal{N}}\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v)\phi_{j_v}(v)+\sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}^c_{int}}\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\phi_j(v)\\ &= \sum_{v\in\mathcal{V}_c}(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v))\phi(v)\\ &= -\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}w_j(v)+\alpha_2^{(v)}z_v(1))\phi(v).\end{aligned}$$

Using the above expression for $z_v(1)$ we arrive at the problem

$$\begin{aligned}\sum_{j=1}^N\int_0^{l_j}(\lambda^2u_j\phi_j+\tfrac{\partial u_j}{\partial x}\tfrac{\partial \phi_j}{\partial x})dx \quad &+ \quad \sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}+\alpha_2^{(v)}e^{-\lambda\tau_v})\lambda u(v)\phi(v)\\ &= \sum_{j=1}^N\int_0^{l_j}(g_j+\lambda f_j)\phi_j dx\\ &\quad+\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}f(v)-\alpha_2^{(v)}z_v^0(v))\phi(v),\ \forall\phi\in V.\end{aligned}\tag{9}$$

This problem has a unique solution $u\in V$ by Lax-Milgram's lemma, because the left-hand side of (9) is coercive on V. If we consider $\phi\in\prod_{j=1}^N\mathcal{D}(0,\, l_j)\subset V$, then u satisfies

$$\lambda^2u_j-\frac{\partial^2u_j}{\partial x^2}=g_j+\lambda f_j \text{ in } \mathcal{D}'(0,\, l_j)\quad \forall j=1,\cdots,N.$$

This directly implies that $u\in\prod_{j=1}^N H^2(0,\, l_j)$ and then $u\in V\cap\prod_{j=1}^N H^2(0,\, l_j)$. Coming back to (9) and by integrating by parts, we find

$$\begin{aligned}&\sum_{v\in\mathcal{V}_c}[\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)+(\alpha_1^{(v)}+\alpha_2^{(v)}e^{-\lambda\tau_v})\lambda u(v)+(\alpha_2^{(v)}z_v^0(v)-\alpha_1^{(v)}f(v))]\phi(v)\\ &\quad=-\sum_{v\in\mathcal{N}}\tfrac{\partial u_j}{\partial n_j}(v)\phi_j(v)-\sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}^c_{int}}(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v))\phi_j(v),\forall\phi\in V.\end{aligned}$$

Consequently, by taking particular test functions ϕ, we obtain

$$\begin{aligned}\frac{\partial u_{j_v}}{\partial x}(v) &= 0 \quad \forall v\in\mathcal{N},\\ \sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v) &= 0 \quad \forall v\in\mathcal{V}_{int}\setminus\mathcal{V}^c_{int}\\ \sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v) &= -(\alpha_1^{(v)}+\alpha_2^{(v)}e^{-\lambda\tau_v})\lambda u(v)-(\alpha_2^{(v)}z_v^0(v)-\alpha_1^{(v)}f(v))\\ &= -\alpha_2^{(v)}z_v(1)-\alpha_1^{(v)}(\lambda u(v)+f(v))\\ &= -(\alpha_2^{(v)}z_v(1)+\alpha_1^{(v)}w(v))\quad\forall v\in\mathcal{V}_c.\end{aligned}$$

In summary we have found $(u,\, w,\, z)^\top\in D(\mathcal{A})$ satisfying (6). □

3. **The energy.** We now restrict the hypothesis (4) to obtain the decay of the energy. Namely we suppose that

$$\alpha_2^{(v)} < \alpha_1^{(v)},\ \forall v \in \mathcal{V}_c. \tag{10}$$

Let us choose the following energy (which corresponds to the inner product on H)

$$E(t) := \frac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}((\frac{\partial u_j}{\partial t})^2 + (\frac{\partial u_j}{\partial x})^2)dx + \sum_{v\in\mathcal{V}_c}\frac{\xi^{(v)}}{2}(\int_0^1(\frac{\partial u}{\partial t}(v,\, t-\tau_v\rho))^2 d\rho) \tag{11}$$

where $\xi^{(v)}$ is a positive constant satisfying (that exists due to (10))

$$\tau_v\alpha_2^{(v)} < \xi^{(v)} < \tau_v(2\alpha_1^{(v)} - \alpha_2^{(v)}),\ \forall v \in \mathcal{V}_c. \tag{12}$$

3.1. **Decay of the energy.**

Proposition 3.1. *For all regular solution of problem (1), the energy is non increasing and there exists two positive constants C_1 and C_2 depending only on $\xi^{(v)}$, $\alpha_1^{(v)}$, $\alpha_2^{(v)}$ and τ_v such that*

$$\begin{aligned}-C_2\sum_{v\in\mathcal{V}_c}((\tfrac{\partial u}{\partial t}(v,\, t))^2 + (\tfrac{\partial u}{\partial t}(v,\, t-\tau_v))^2) &\le E'(t)\\ \le -C_1\sum_{v\in\mathcal{V}_c}((\tfrac{\partial u}{\partial t}(v,\, t))^2 + (\tfrac{\partial u}{\partial t}(v,\, t-\tau_v))^2).&\end{aligned} \tag{13}$$

Proof. Deriving (11) and integrating by parts in space, we obtain

$$\begin{aligned}
E'(t) &= \sum_{j=1}^{N}\int_0^{l_j}(\frac{\partial u_j}{\partial t}\frac{\partial^2 u_j}{\partial t^2} + \frac{\partial u_j}{\partial x}\frac{\partial^2 u_j}{\partial x\partial t})dx\\
&\quad + \sum_{v\in\mathcal{V}_c}\xi^{(v)}(\int_0^1\frac{\partial u}{\partial t}(v,\, t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\, t-\tau_v\rho)d\rho)\\
&= \sum_{j=1}^{N}\int_0^{l_j}(\frac{\partial^2 u_j}{\partial t^2} - \frac{\partial u_j^2}{\partial x^2})\frac{\partial u_j}{\partial t}dx + \sum_{j=1}^{N}[\frac{\partial u_j}{\partial t}\frac{\partial u_j}{\partial x}]_0^{l_j}\\
&\quad + \sum_{v\in\mathcal{V}_c}\xi^{(v)}(\textstyle\int_0^1\frac{\partial u}{\partial t}(v,\, t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\, t-\tau_v\rho)d\rho)\\
&= \sum_{v\in\mathcal{V}}\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\tfrac{\partial u_j}{\partial t}(v) + \sum_{v\in\mathcal{V}_c}\xi^{(v)}\textstyle\int_0^1\frac{\partial u}{\partial t}(v,\, t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\, t-\tau_v\rho)d\rho\\
&= \sum_{v\in\mathcal{V}_c}(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v))\tfrac{\partial u}{\partial t}(v) + \sum_{v\in\mathcal{D}}\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v)\tfrac{\partial u_{j_v}}{\partial t}(v) + \sum_{v\in\mathcal{N}}\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v)\tfrac{\partial u_{j_v}}{\partial t}(v)\\
&\quad + \sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c}(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v))\tfrac{\partial u}{\partial t}(v)\\
&\quad + \sum_{v\in\mathcal{V}_c}\xi^{(v)}\textstyle\int_0^1\frac{\partial u}{\partial t}(v,\, t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\, t-\tau_v\rho)d\rho\\
&= -\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}(\tfrac{\partial u}{\partial t}(v,\, t))^2 + \alpha_2^{(v)}\tfrac{\partial u}{\partial t}(v,\, t-\tau_v)\tfrac{\partial u}{\partial t}(v,\, t))\\
&\quad + \sum_{v\in\mathcal{V}_c}\xi^{(v)}\textstyle\int_0^1\frac{\partial u}{\partial t}(v,\, t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\, t-\tau_v\rho)d\rho.
\end{aligned}$$

Now for all $v \in \mathcal{V}_c$, recalling that $z_v(\rho, t) = \frac{\partial u}{\partial t}(v, t - \tau_v \rho)$, we see that

$$\begin{aligned}\int_0^1 \frac{\partial u}{\partial t}(v, t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v, t-\tau_v\rho)d\rho &= \int_0^1 z_v(\rho,t)\frac{\partial z_v}{\partial t}(\rho,t)d\rho \\ &= -\frac{1}{\tau_v}\int_0^1 z_v(\rho,t)\frac{\partial z_v}{\partial \rho}(\rho,t)d\rho.\end{aligned}$$

By an integration by parts in ρ, we obtain

$$\int_0^1 \frac{\partial u}{\partial t}(v, t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v, t-\tau_v\rho)d\rho = -\frac{1}{2\tau_v}((\frac{\partial u}{\partial t}(v, t-\tau_v))^2 - (\frac{\partial u}{\partial t}(v, t))^2).$$

Therefore, we have

$$\begin{aligned}E'(t) &= -\sum_{v\in\mathcal{V}_c}[\alpha_1^{(v)}(\tfrac{\partial u}{\partial t}(v, t))^2 + \alpha_2^{(v)}\tfrac{\partial u}{\partial t}(v, t-\tau_v)\tfrac{\partial u}{\partial t}(v, t) \\ &\quad +\tfrac{\xi^{(v)}}{2\tau_v}((\tfrac{\partial u}{\partial t}(v, t-\tau_v))^2 - (\tfrac{\partial u}{\partial t}(v, t))^2)] \\ &= -\sum_{v\in\mathcal{V}_c}[(\alpha_1^{(v)} - \tfrac{\xi^{(v)}}{2\tau_v})(\tfrac{\partial u}{\partial t}(v, t))^2 + \alpha_2^{(v)}\tfrac{\partial u}{\partial t}(v, t-\tau_v)\tfrac{\partial u}{\partial t}(v, t) \\ &\quad +\tfrac{\xi^{(v)}}{2\tau_v}(\tfrac{\partial u}{\partial t}(v, t-\tau_v))^2].\end{aligned}$$

Cauchy-Schwarz's inequality yields

$$\begin{aligned}E'(t) &\leq -\sum_{v\in\mathcal{V}_c}[(\alpha_1^{(v)} - \frac{\xi^{(v)}}{2\tau_v} - \frac{\alpha_2^{(v)}}{2})(\frac{\partial u}{\partial t}(v, t))^2 + (\frac{\xi^{(v)}}{2\tau_v} - \frac{\alpha_2^{(v)}}{2})(\frac{\partial u}{\partial t}(v, t-\tau_v))^2] \\ E'(t) &\geq -\sum_{v\in\mathcal{V}_c}[(\alpha_1^{(v)} - \frac{\xi^{(v)}}{2\tau_v} + \frac{\alpha_2^{(v)}}{2})(\frac{\partial u}{\partial t}(v, t))^2 + (\frac{\xi^{(v)}}{2\tau_v} + \frac{\alpha_2^{(v)}}{2})(\frac{\partial u}{\partial t}(v, t-\tau_v))^2].\end{aligned}$$

The first estimate leads to

$$E'(t) \leq -C_1\sum_{v\in\mathcal{V}_c}[(\frac{\partial u}{\partial t}(v, t))^2 + (\frac{\partial u}{\partial t}(v, t-\tau_v))^2]$$

with

$$C_1 = \min\{(\alpha_1^{(v)} - \frac{\xi^{(v)}}{2\tau_v} - \frac{\alpha_2^{(v)}}{2}), (\frac{\xi^{(v)}}{2\tau_v} - \frac{\alpha_2^{(v)}}{2}) \,:\, v \in \mathcal{V}_c\}$$

which is positive according to the assumption (12). The second one yields

$$E'(t) \geq -C_2\sum_{v\in\mathcal{V}_c}[(\frac{\partial u}{\partial t}(v, t))^2 + (\frac{\partial u}{\partial t}(v, t-\tau_v))^2]$$

with

$$C_2 = \max\{(\alpha_1^{(v)} - \frac{\xi^{(v)}}{2\tau_v} + \frac{\alpha_2^{(v)}}{2}), (\frac{\xi^{(v)}}{2\tau_v} + \frac{\alpha_2^{(v)}}{2}) \,:\, v \in \mathcal{V}_c\}$$

which is also positive due to (12). □

We have just shown that under the assumption (10), the energy decays. But we would like to obtain stability of the system, in other words, the decay to 0 of the energy. This is the goal of the remainder of this section. But before going on, let us make the next remark that will be useful later on.

Remark 3.2. Integrating the expression (13) between 0 and T, we obtain

$$\sum_{v\in\mathcal{V}_c}\int_0^T[(\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2]dt \lesssim E(0)-E(T)\lesssim E(0)$$

and therefore

$$\int_0^T[(\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2]dt \lesssim E(0), \forall v\in\mathcal{V}_c.$$

This estimate implies that $\alpha_1^{(v)}\frac{\partial u}{\partial t}(v,\,.)+\alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\,.-\tau_v)$ belongs to $L^2(0,\,T)$ for all $v\in\mathcal{V}_c$, with the estimate

$$\begin{aligned}&\left\|\alpha_1^{(v)}\tfrac{\partial u}{\partial t}(v,\,.)+\alpha_2^{(v)}\tfrac{\partial u}{\partial t}(v,\,.-\tau_v)\right\|^2_{L^2(0,\,T)}\\ &=\int_0^T(\alpha_1^{(v)}\tfrac{\partial u}{\partial t}(v,\,t)+\alpha_2^{(v)}\tfrac{\partial u}{\partial t}(v,\,t-\tau_v))^2dt\\ &\le 2\max_{v\in\mathcal{V}_c}\{(\alpha_1^{(v)})^2\}\int_0^T(\tfrac{\partial u}{\partial t}(v,\,t)^2+\tfrac{\partial u}{\partial t}(v,\,t-\tau_v)^2)dt\\ &\lesssim E(0)<+\infty.\end{aligned}$$

3.2. **Problem without damping.** In the sequel we need to consider the problem without damping

$$\left\{\begin{array}{ll}\frac{\partial^2\phi_j}{\partial t^2}-\frac{\partial^2\phi_j}{\partial x^2}=0 & 0<x<l_j,\,t>0,\,\forall j\in\{1,...,N\}\\ \phi_j(v,\,t)=\phi_l(v,\,t)=\phi(v,t) & \forall j,\,l\in\mathcal{E}_v,\,v\in\mathcal{V}_{int},t>0,\\ \sum\limits_{j\in\mathcal{E}_v}\frac{\partial\phi_j}{\partial n_j}(v,\,t)=0 & \forall v\in\mathcal{V}_{int},t>0,\\ \phi_{j_v}(v,t)=0 & \forall v\in\mathcal{D},t>0,\\ \frac{\partial\phi_{j_v}}{\partial n_{j_v}}(v,t)=0 & \forall v\in\mathcal{N}\cup\mathcal{V}_{ext}^c,t>0,\\ \phi(t=0)=u^{(0)},\,\frac{\partial\phi}{\partial t}(t=0)=u^{(1)}. & \end{array}\right. \tag{14}$$

It is well knwon that this problem is well posed in the natural energy space (see for instance [3]).

Lemma 3.3. *Suppose that* $(u^{(0)},\,u^{(1)})\in V\times\prod_{j=1}^N L^2(0,\,l_j)$. *Then problem (14) admits a unique solution*

$$\phi\in C(0,\,T;\,V)\cap C^1(0,\,T;\,\prod_{j=1}^N L^2(0,\,l_j)).$$

This problem is obviously conservative, its energy is constant.

3.3. **Decay of the energy to 0.** We look at the spectral problem associated with problem (14), in other words

$$\left\{\begin{array}{ll}-\lambda^2\phi_j-\frac{\partial^2\phi_j}{\partial x^2}=0 & 0<x<l_j,\,\forall j\in\{1,...,N\},\\ \phi_j(v)=\phi_l(v)=\phi(v) & \forall j,\,l\in\mathcal{E}_v,\,v\in\mathcal{V}_{int},\\ \sum\limits_{j\in\mathcal{E}_v}\frac{\partial\phi_j}{\partial n_j}(v)=0 & \forall v\in\mathcal{V}_{int},\\ \phi_{j_v}(v)=0 & \forall v\in\mathcal{D},\\ \frac{\partial\phi_{j_v}}{\partial n_{j_v}}(v)=0 & \forall v\in\mathcal{N}\cup\mathcal{V}_{ext}^c.\end{array}\right.$$

This system corresponds to an eigenvalue problem of the positive selfadjoint operator A defined above. Let us then denote by $\{\lambda_k^2\}_{k\ge 1}$ the set of eigenvalues counted without their multiplicities, i.e., $\lambda_k \neq \lambda_l$, $\forall k \neq l$, where without any restriction, we may suppose that $\lambda_k > 0$. For all $k \in \mathbb{N}^*$, let l_k be the multiplicity of the eigenvalue λ_k^2 (remark that $l_k \leq 2N$, $\forall k \in \mathbb{N}^*$) and let $\{\varphi_{k,\,i}\}_{1\leq i\leq l_k}$ be the orthonormal eigenvectors associated with the eigenvalue λ_k^2.

Definition 3.4. For $k \geq 1$ and $v \in \mathcal{V}_c$, we denote by $\mathcal{M}_v(\lambda_k^2)$ the following matrix of size l_k

$$\mathcal{M}_v(\lambda_k^2) := \begin{pmatrix} \varphi_{k,\,1}^2(v) & \varphi_{k,\,1}(v)\varphi_{k,\,2}(v) & \dots & \varphi_{k,\,1}(v)\varphi_{k,\,l_k}(v) \\ \varphi_{k,\,1}(v)\varphi_{k,\,2}(v) & \varphi_{k,\,2}^2(v) & \cdots & \varphi_{k,\,2}(v)\varphi_{k,\,l_k}(v) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{k,\,1}(v)\varphi_{k,\,l_k}(v) & \varphi_{k,\,2}(v)\varphi_{k,\,l_k}(v) & \cdots & \varphi_{k,\,l_k}^2(v) \end{pmatrix}.$$

Moreover, let $\mathcal{M}(\lambda_k^2)$ be the matrix of size l_k

$$\mathcal{M}(\lambda_k^2) := \sum_{v\in\mathcal{V}_c} \mathcal{M}_v(\lambda_k^2).$$

Now we recall that the following generalized gap condition holds, namely from Proposition 6.2 of [16], we have

$$\exists\gamma > 0,\, \forall k \geq 1,\, \lambda_{k+N+1} - \lambda_k \geq (N+1)\gamma. \tag{15}$$

From this property we will deduce an inequality of Ingham's type. Namely fix a positive real number $\gamma' \leq \gamma$ and denote by $A_k, k = 1, \cdots, N+1$ the set of natural numbers m satisfying (see for instance [10])

$$\begin{cases} \lambda_m - \lambda_{m-1} \geq \gamma' & \\ \lambda_n - \lambda_{n-1} < \gamma' & \text{for } m+1 \leq n \leq m+k-1, \\ \lambda_{m+k} - \lambda_{m+k-1} \geq \gamma'. & \end{cases}$$

Then one easily checks that the sets $A_{k+j}, j = 0, \cdots, k-1, k = 1, \cdots, N+1$, form a partition of $\mathbb{N}^*$.

Now for $m \in A_k$, we recall that the finite differences $e_{m+j}(t)$, $j = 0, \cdots, k-1$, corresponding to the exponential functions $e^{i\lambda_{m+j}t}, j = 0, \cdots, k-1$ are given by

$$e_{m+j}(t) = \sum_{p=m}^{m+j} \prod_{\substack{q=m \\ q\neq p}}^{m+j} (\lambda_p - \lambda_q)^{-1} e^{i\lambda_p t}.$$

Write for shortness, $e_{-n}(t)$ the same finite differences functions corresponding to $-\lambda_n$.

Now we are ready to recall the next inequality of Ingham's type, see for instance Theorem 1.5 of [10]:

Theorem 3.5. *If the sequence $(\lambda_n)_{n\geq 1}$ satisfies (15), then for all sequence $(a_n)_{n\in\mathbb{Z}^*}$ (where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$), the function*

$$f(t) = \sum_{n\in\mathbb{Z}^*} a_n e_n(t),$$

satisfies the estimates

$$\int_0^T |f(t)|^2 \sim \sum_{n\in\mathbb{Z}^*} |a_n|^2, \tag{16}$$

for $T > 2\pi\gamma$.

Going back to the original functions $e^{i\lambda_n t}$, the above equivalence (16) means that, for $T > 2\pi\gamma$, the function (from now on $\lambda_{-n} = -\lambda_n$)

$$f(t) = \sum_{n\in\mathbb{Z}^*} \alpha_n e^{i\lambda_n t},$$

satisfies the estimates

$$\int_0^T |f(t)|^2 \sim \sum_{k=1}^{N+1} \sum_{|n|\in A_k} \|B_n^{-1} C_n\|_2^2, \tag{17}$$

where $\|\cdot\|_2$ means the Euclidean norm of the vector, for $n \in A_k$, the vector C_n is given by

$$C_n = (\alpha_n, \cdots, \alpha_{n+k-1})^\top,$$

and the $k \times k$ matrix B_n allows to pass from the coefficients a_n to α_n, namely

$$C_n = B_n \cdot (a_n, \cdots, a_{n+k-1})^\top,$$

and is given by $B_n = (B_{n,ij})_{1\le i,j\le k}$ with

$$B_{n,ij} = \begin{cases} \prod\limits_{\substack{q=n\\ q\neq n+i-1}}^{n+j-1} (\lambda_{n+i-1} - \lambda_q)^{-1} & \text{if } i \le j,\ (i,j) \neq (1,1),\\ 1 & \text{if } i = j = 1,\\ 0 & \text{if } i > j. \end{cases}$$

We proceed similarly for $n \le -1$, but the indices being decreasing from n to $n-k+1$.

Remark 3.6. Notice that if the standard gap condition

$$\exists\gamma > 0,\ \forall k \ge 1,\ \lambda_{k+1} - \lambda_k \ge \gamma \tag{18}$$

holds, then $A_1 = \mathbb{Z}^*$ and $B_1 = 1$ and in that case the next equivalence holds (see [21]):

$$\int_0^T |f(t)|^2 \sim \sum_{n\in\mathbb{Z}^*} |\alpha_n|^2.$$

We are now ready to give a necessary and sufficient condition that guarantees the decay to 0 of the energy.

Proposition 3.7. *For all initial data in H, we have*

$$\lim_{t\to\infty} E(t) = 0 \tag{19}$$

if and only if the operator A satisfies

$$\lambda_{min}(\mathcal{M}(\lambda_k^2)) > 0, \forall k \in \mathbb{N}^*, \tag{20}$$

where $\lambda_{min}(\mathcal{M})$ denotes the smallest eigenvalue of the matrix $\mathcal{M}$.

Proof. $\boxed{\Leftarrow}$ Let us show that (20) implies (19): Let $S(t)$ be the semi-group of contractions generated by the operator $\mathcal{A}$.

It suffices to show that

$$\lim_{t\to\infty} S(t) \begin{pmatrix} u^{(0)} \\ u^{(1)} \\ (f^0(-\tau_v.))_v \end{pmatrix} = 0,\ \forall \begin{pmatrix} u^{(0)} \\ u^{(1)} \\ (f^0(-\tau_v.))_v \end{pmatrix} \in D(\mathcal{A}).$$

Let us fix $U_0 = \begin{pmatrix} u^{(0)} \\ u^{(1)} \\ (f^0(-\tau_v .))_v \end{pmatrix} \in D(\mathcal{A})$. As $D(\mathcal{A})$ is compactly embedded into H, the set

$$orb(U_0) = \bigcup_{t \geq 0} S(t)U_0$$

is precompact in H. Indeed, for any sequence $(t_n)_n$, as $U_0 \in D(\mathcal{A})$, one has $S(t_n)U_0 \in D(\mathcal{A})$ and

$$\begin{aligned} \|S(t_n)U_0\|_{D(\mathcal{A})} &= \|S(t_n)U_0\|_H + \|\mathcal{A}S(t_n)U_0\|_H = \|S(t_n)U_0\|_H + \|S(t_n)\mathcal{A}U_0\| \\ &\leq \|U_0\|_H + \|\mathcal{A}U_0\|_H = cste. \end{aligned}$$

Therefore the sequence $S(t_n)U_0$ is bounded in D($\mathcal{A}$) and by the compact embedding of $D(\mathcal{A})$ into H, there exists a subsequence, still denote by $S(t_n)U_0$ which converges in H. In this case, the ω-limit of U_0 defined by

$$\omega(U_0) = \{U \in H \,:\, \exists (t_n),\, t_n \to \infty,\, S(t_n)U_0 \to U,\, t \to \infty\}$$

is non empty.

On the other hand, if $\Phi \in \omega(U_0)$, then

$$S(t)\Phi \in \omega(U_0).$$

Note further that one readily checks that $S(t)\Phi$ is of the form

$$S(t)\Phi = \begin{pmatrix} \phi(.,\, t) \\ \frac{\partial \phi}{\partial t}(.,\, t) \\ \psi \end{pmatrix},$$

for some $\phi \in C([0,\infty); V) \cap C^1([0,\infty); L^2(\mathcal{R}))$ and $\psi \in C([0,\infty); L^2(0,1)^{\mathcal{V}_c})$.

We can now apply LaSalle's invariance principle [15] with the relatively compact set $\bigcup_{t\geq 0} S(t)U_0$ and the Liapounov functional $\phi = \|.\|_H$. As $\begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}$ and $S(t)\begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} = \begin{pmatrix} \phi(.,\, t) \\ \frac{\partial \phi}{\partial t}(.,\, t) \\ \psi \end{pmatrix}$ belong to $\omega(U_0)$, we find that

$$\left\| \begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} \right\|_H = \left\| \begin{pmatrix} \phi(.,\, t) \\ \frac{\partial \phi}{\partial t}(.,\, t) \\ \psi \end{pmatrix} \right\|_H = L \quad \forall t \geq 0.$$

Therefore ϕ satisfies problem (2) with initial conditions $\phi(\cdot,\, 0) = \phi^{(0)}$ and $\frac{\partial \phi}{\partial t}(\cdot,\, 0) = \phi^{(1)}$. Moreover by (13) we have

$$\begin{aligned} 0 = L - L &= \left\| \begin{pmatrix} \phi(.,\, t) \\ \frac{\partial \phi}{\partial t}(.,\, t) \\ \psi \end{pmatrix} \right\|_H - \left\| \begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} \right\|_H \\ &\leq -C_1 \sum_{v \in \mathcal{V}_c} \int_0^t (\psi_v(0,\, t)^2 + \psi_v(1,\, t)^2) dt \leq 0. \end{aligned}$$

In other words, it holds

$$\sum_{v \in \mathcal{V}_c} \int_0^t (\psi_v(0,\, t)^2 + \psi_v(1,\, t)^2) dt = 0$$

which implies that

$$\psi_v(0,\,t) = \psi_v(1,\,t) = 0 \quad \forall t \geq 0, \forall v \in \mathcal{V}_c. \tag{21}$$

In particular, this implies that ϕ is solution of problem (14) with initial data $\phi(\cdot,\,0) = \phi^{(0)}$ and $\frac{\partial\phi}{\partial t}(\cdot,\,0) = \phi^{(1)}$ because $0 = \psi_v(0,\,t) = \frac{\partial\phi}{\partial t}(v,\,t)$ and $0 = \psi_v(1,\,t) = \frac{\partial\phi}{\partial t}(v,\,t-\tau_v)$, which means that in (2) the damping terms disappear.

Let us now write

$$\begin{aligned}\phi^{(0)} &= \sum_{k\geq 1}\sum_{i=1}^{l_k} a_{k,\,i}\varphi_{k,\,i},\\ \phi^{(1)} &= \sum_{k\geq 1}\sum_{i=1}^{l_k} b_{k,\,i}\varphi_{k,\,i},\end{aligned}$$

where $(\lambda_{k,\,i}a_{k,\,i})_{i,\,k}$, $(b_{k,\,i})_{i,\,k} \in l^2(\mathbb{N}^*)$. Then ϕ is given by

$$\phi(\cdot,\,t) = \sum_{k\geq 1}\sum_{i=1}^{l_k}(a_{k,\,i}\cos(\lambda_k t) + \frac{b_{k,\,i}}{\lambda_k}\sin(\lambda_k t))\varphi_{k,\,i}.$$

Consequently by (21) for $v \in \mathcal{V}_c$ and $j \in \mathcal{E}_v$

$$0 = \frac{\partial\phi_j}{\partial t}(v,\,t) = \sum_{k\geq 1}\sum_{i=1}^{l_k}(-a_{k,\,i}\lambda_k\sin(\lambda_k t) + b_{k,\,i}\cos(\lambda_k t))\varphi_{k,\,i}(v).$$

By grouping the terms corresponding to the same eigenvalue, we get

$$\begin{aligned}0 = \frac{\partial\phi_j}{\partial t}(v,\,t) &= \sum_{k\geq 1}[\sum_{i=1}^{l_k} - a_{k,\,i}\varphi_{k,\,i}(v)]\lambda_k\sin(\lambda_k t) + \sum_{k\geq 1}[\sum_{i=1}^{l_k} b_{k,\,i}\varphi_{k,\,i}(v)]\cos(\lambda_k t)\\ &= \sum_{n\in\mathbb{Z}^*}\alpha_n(v)e^{i\lambda_n t},\end{aligned}$$

where

$$\begin{aligned}\alpha_k(v) &= \frac{1}{2}\left([\sum_{i=1}^{l_k} b_{k,\,i}\varphi_{k,\,i}(v)] + i[\sum_{i=1}^{l_k} a_{k,\,i}\varphi_{k,\,i}(v)]\lambda_k\right), \forall k \geq 1,\\ \alpha_{-k}(v) &= \frac{1}{2}\left([\sum_{i=1}^{l_k} b_{k,\,i}\varphi_{k,\,i}(v)] - i[\sum_{i=1}^{l_k} a_{k,\,i}\varphi_{k,\,i}(v)]\lambda_k\right), \forall k \geq 1.\end{aligned}$$

Integrating this identity between 0 and $T > 0$ sufficiently large and using Ingham's inequality (17), we obtain (with the notations introduced above)

$$0 = \int_0^T (\frac{\partial\phi_j}{\partial t}(v,\,t))^2 dt \gtrsim \sum_{k=1}^{N+1}\sum_{|n|\in A_k}\|B_n^{-1}C_n(v)\|_2^2,$$

where $C_n(v)$ is defined as C_n using $\alpha_n(v)$ instead of α_n. Summing on $v \in \mathcal{V}_c$:

$$0 \gtrsim \sum_{v\in\mathcal{V}_c}\sum_{k=1}^{N+1}\sum_{|n|\in A_k}\|B_n^{-1}C_n(v)\|_2^2 \geq 0.$$

This implies that for all $k = 1, \cdots, N+1$ and all $|n| \in A_k$, we have

$$\sum_{v\in\mathcal{V}_c}\|B_n^{-1}C_n(v)\|_2^2 = 0.$$

As B_n^{-1} is invertible, there exits $\gamma_n > 0$ such that

$$\|B_n^{-1}C_n(v)\|_2 \geq \gamma_n \|C_n(v)\|_2.$$

From this estimate we deduce that for all $k = 1, \cdots, N+1$ and all $|n| \in A_k$, we have

$$\sum_{v\in\mathcal{V}_c} \|C_n(v)\|_2^2 = 0.$$

As $\lambda_k \neq 0$, we necessarily have

$$\forall k \geq 1, \sum_{v\in\mathcal{V}_c}\left(\sum_{i=1}^{l_k} a_{k,\,i}\varphi_{k,\,i}(v)\right)^2 = 0 \text{ and } \sum_{v\in\mathcal{V}_c}\left(\sum_{i=1}^{l_k} b_{k,\,i}\varphi_{k,\,i}(v)\right)^2 = 0.$$

For a fixed $k \geq 1$, if we set $b = \begin{pmatrix} b_{k,\,1} \\ \vdots \\ b_{k,\,l_k}\end{pmatrix}$; then

$$\sum_{v\in\mathcal{V}_c}\left(\sum_{i=1}^{l_k} b_{k,\,i}\varphi_{k,\,i}(v)\right)^2 \quad = \quad {}^t b\mathcal{M}(\lambda_k^2)b.$$

As a consequence if $\lambda_{min}(\mathcal{M}(\lambda_k^2)) > 0$, we obtain that $b_{k,\,1} = ... = b_{k,\,l_k} = 0$. In the same manner we have $a_{k,\,1} = ... = a_{k,\,l_k} = 0$.

We have proved that $\phi^{(0)} = 0 = \phi^{(1)}$.

Moreover $\phi^{(2)} = 0$ because ψ_v satisfies the transport equation

$$\begin{cases} \frac{\partial \psi_v}{\partial t} = -\frac{1}{\tau_v}\frac{\partial \psi_v}{\partial \rho}, \\ \psi_v(0,\, t) = \psi_v(1,\, t) = 0, \\ \psi_v(\rho,\, 0) = \phi^{(2)}, \end{cases}$$

for all $v \in \mathcal{V}_c$.

We have shown that for all $\left(\phi^{(0)}, \phi^{(1)}, \phi^{(2)}\right)^\top \in \omega(U_0)$, we have $\phi^{(0)} = 0 = \phi^{(1)} = \phi^{(2)}$. Consequently $\lim_{t\to\infty} S(t)U_0 = 0$ and then $\lim_{t\to\infty} E(t) = 0$.

$\boxed{\Rightarrow}$ Let us show that (19) implies (20). For that purpose we use a contradiction argument. Suppose that there exists $k > 0$ such that $\lambda_{min}(\mathcal{M}(\lambda_k^2)) = 0$. This means that there exists $a = (a_{k,\,1}, \cdots, a_{k,\,l_k})^\top \neq 0$, such that

$$\sum_{v\in\mathcal{V}_c}\left(\sum_{i=1}^{l_k} a_{k,\,i}\varphi_{k,\,i}(v)\right)^2 = 0.$$

Let us set

$$u(.,\, t) = \left(\sum_{i=1}^{l_k} a_{k,\,i}\varphi_{k,\,i}\right)\cos(\lambda_k t).$$

Then u is solution of (1) and satisfies

$$\begin{aligned} E(t) &= \tfrac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}((\frac{\partial u_j}{\partial t})^2 + (\frac{\partial u_j}{\partial x})^2)dx + \sum_{v\in\mathcal{V}_c}\frac{\xi^{(v)}}{2}(\int_0^1(\frac{\partial u_j}{\partial t}(v,\, t-\tau_v\rho))^2 d\rho) \\ &= \tfrac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}((\frac{\partial u_j}{\partial t})^2 + (\frac{\partial u_j}{\partial x})^2)dx \\ &= E(u(0)) \end{aligned}$$

because

$$\forall v \in \mathcal{V}_c,\, \forall t,\, \frac{\partial u_j}{\partial t}(v,\, t) = \left(\sum_{i=1}^{l_k} a_{k,\, i}\varphi_{k,\, i}(v)\right)(-\lambda_k)\sin(\lambda_k t) = 0.$$

This means that we have obtained a solution of problem (1) with a constant energy and contradicts (19). □

Remark 3.8. 1. Notice that the condition (20) is independent of the choice of the orthonormal basis of eigenvectors associated with the eigenvalue λ_k^2. Indeed if $\{\tilde{\varphi}_{k,\, i}\}_{i=1}^{l_k}$ is another orthonormal basis of eigenvectors associated with the eigenvalue λ_k^2, then there exists an orthogonal matrix $O \in \mathbb{R}^{l_k \times l_k}$ such that

$$\begin{pmatrix} \tilde{\varphi}_{k,\, 1} \\ \vdots \\ \tilde{\varphi}_{k,\, l_k} \end{pmatrix} = O \begin{pmatrix} \varphi_{k,\, 1} \\ \vdots \\ \varphi_{k,\, l_k} \end{pmatrix}.$$

Consequently the matrix $\tilde{\mathcal{M}}(\lambda_k^2)$ build as $\mathcal{M}(\lambda_k^2)$ by using $\{\tilde{\varphi}_{k,\, i}\}_{i=1}^{l_k}$ instead of $\{\varphi_{k,\, i}\}_{i=1}^{l_k}$ is given by

$$\tilde{\mathcal{M}}(\lambda_k^2) = O\mathcal{M}(\lambda_k^2)O^\top,$$

and therefore $\lambda_{min}(\tilde{\mathcal{M}}(\lambda_k^2)) = \lambda_{min}(\mathcal{M}(\lambda_k^2))$.
2. If $l_k = 1$, then the condition (20) is reduced to

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 > 0$$

because $\mathcal{M}(\lambda_k^2) = \sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2$.

3.4. **Counterexample to the stability of the system.** In this section (and in the remainder of the paper) we have made the hypothesis (10). As in [30], we may expect non-stability results if this condition fails.

In [30], the authors consider the wave equation on a string of length π and used a boundary control. They show that if (10) does not hold then non-stabilities appear. Since their problem enters in our framework, this is a first counterexample. As a second counterexample, we consider the wave equation on a string of length π but with an interior control. Namely we consider the problem

$$\left\{ \begin{array}{ll} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & 0 < x < \pi,\, t > 0, \\ \frac{\partial u}{\partial x}(\xi_-,\, t) - \frac{\partial u}{\partial x}(\xi_+,\, t) = -(\alpha_1 \frac{\partial u}{\partial t}(\xi,\, t) + \alpha_2 \frac{\partial u}{\partial t}(\xi,\, t-\tau)) & t > 0, \\ u(\xi_-,\, t) = u(\xi_+,\, t) & t > 0, \\ u(0,\, t) = 0 & t > 0, \\ \frac{\partial u}{\partial x}(\pi,\, t) = 0 & t > 0, \\ u(t=0) = u^{(0)},\, \frac{\partial u}{\partial t}(t=0) = u^{(1)} & 0 < x < \pi, \\ \frac{\partial u}{\partial t}(\xi,\, t-\tau) = f^0(t-\tau) & 0 < t < \tau. \end{array} \right. \tag{22}$$

Lemma 3.9. *A complex number $\lambda \in \mathbb{C}$ is called an eigenvalue associated with system (22) if and only if λ satisfies*

$$(\alpha_1 + \alpha_2 e^{-\lambda\tau})\cosh(\lambda(\xi - \pi))\sinh(\lambda\xi) + \cosh(\lambda\pi) = 0. \tag{23}$$

Proof. Setting $u(t, .) = e^{\lambda t}\varphi$, we see that u is solution of (22) if and only if φ satisfies

$$\begin{cases} \lambda^2\varphi - \frac{\partial^2\varphi}{\partial x^2} = 0 & 0 < x < \pi, \\ \frac{\partial\varphi}{\partial x}(\xi_-) - \frac{\partial\varphi}{\partial x}(\xi_+) = -(\alpha_1 + \alpha_2 e^{-\lambda\tau})\lambda\varphi(\xi), & \\ \varphi(\xi_-) = \varphi(\xi_+), & \\ \varphi(0) = 0, & \\ \frac{\partial\varphi}{\partial x}(\pi) = 0. & \end{cases}$$

We then obtain

$$\varphi(x) = \begin{cases} A\sinh(\lambda x) & \text{in } (0,\, \xi) \\ A_1\cosh(\lambda(x-\pi)) & \text{in } (\xi,\, \pi) \end{cases}$$

where A, A_1 are real constants. The continuity $\varphi(\xi_-) = \varphi(\xi_+)$ and $\frac{\partial\varphi}{\partial x}(\xi_-) - \frac{\partial\varphi}{\partial x}(\xi_+) = -(\alpha_1 + \alpha_2 e^{-\lambda\tau})\lambda\varphi(\xi)$ imply that

$$\begin{pmatrix} \sinh(\lambda\xi) & -\cosh(\lambda(\xi-\pi)) \\ \cosh(\lambda\xi) + (\alpha_1 + \alpha_2 e^{-\lambda\tau})\sinh(\lambda\xi) & -\sinh(\lambda(\xi-\pi)) \end{pmatrix} \begin{pmatrix} A \\ A_1 \end{pmatrix} = 0.$$

Therefore a non trivial solution exists if and only if

$$\det\begin{pmatrix} \sinh(\lambda\xi) & -\cosh(\lambda(\xi-\pi)) \\ \cosh(\lambda\xi) + (\alpha_1 + \alpha_2 e^{-\lambda\tau})\sinh(\lambda\xi) & -\sinh(\lambda(\xi-\pi)) \end{pmatrix} = 0,$$

and simple calculations lead to the characteristic equation (23). □

The characteristic equation (23) is equivalent to

$$\begin{aligned} \Delta(\lambda) :=& (\alpha_1 + \alpha_2 e^{-\lambda\tau})e^{2\lambda(\xi-\pi)} - (\alpha_1 + \alpha_2 e^{-\lambda\tau} - 2)e^{-2\lambda\pi} \\ &- (\alpha_1 + \alpha_2 e^{-\lambda\tau})e^{-2\lambda\xi} + \alpha_2 e^{-\lambda\tau} + 2 + \alpha_1 = 0. \end{aligned}$$

Take an interior control ξ and a delay τ such that

$$\frac{\xi}{\pi} = \frac{2m+1}{2n+1},\ \tau = \frac{2(2k+1)}{2n+1}\pi$$

where n, m, $k \in \mathbb{Z}$. Now we look for λ in the form

$$\lambda = \eta + i\,\frac{2n+1}{2},\ \eta \in \mathbb{R}.$$

For such a λ, we have

$$\begin{aligned} \Delta(\lambda) \quad &= \quad (\alpha_1 - \alpha_2 e^{-\eta\tau})e^{2\eta(\xi-\pi)} + (\alpha_1 - \alpha_2 e^{-\eta\tau} - 2)e^{-2\eta\pi} \\ &\quad\quad + (\alpha_1 - \alpha_2 e^{-\eta\tau})e^{-2\eta\xi} - \alpha_2 e^{-\eta\tau} + 2 + \alpha_1 \\ &=: \quad \Delta_0(\eta). \end{aligned}$$

If we suppose that

$$\alpha_2 \geq \alpha_1 \geq 0.$$

Then

$$\Delta_0(0) = (\alpha_1 - \alpha_2) + (\alpha_1 - \alpha_2 - 2) + (\alpha_1 - \alpha_2) - \alpha_2 + 2 + \alpha_1 = 4(\alpha_1 - \alpha_2) \leq 0$$

and

$$\lim_{\eta \longrightarrow +\infty} \Delta_0(\eta) = \alpha_1 + 2 > 0.$$

By the mean value theorem, there exists $\eta \geq 0$ such that $\Delta_0(\eta) = 0$. Therefore $\lambda = \eta + \frac{2n+1}{2}i$, $n \in \mathbb{Z}$, is an eigenvalue of system (22) with $Re(\lambda) = \eta \geq 0$. The system is then unstable for the countable set of delays τ and of control points ξ in the above form.

4. **A regularity result and an a priori estimate.** We consider the following problem with non-homogeneous transmission conditions

$$\left\{\begin{array}{ll} \frac{\partial^2 w_j}{\partial t^2} - \frac{\partial^2 w_j}{\partial x^2} = 0 & 0 < x_j < l_j,\, t > 0,\, \forall j \in \{1, ..., N\} \\ w_j(v,\, t) = w_l(v,\, t) = w(v, t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \\ \sum\limits_{j \in \mathcal{E}_v} \frac{\partial w_j}{\partial n_j}(v,\, t) = k_v(t) & \forall v \in \mathcal{V}_c, t > 0, \\ \sum\limits_{j \in \mathcal{E}_v} \frac{\partial w_j}{\partial n_j}(v,\, t) = 0 & \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}_{int}^c, t > 0, \\ w_{j_v}(v, t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \frac{\partial w_{j_v}}{\partial n_{j_v}}(v, t) = 0 & \forall v \in \mathcal{N}, t > 0, \\ w(t = 0) = 0,\, \frac{\partial w}{\partial t}(t = 0) = 0 & \end{array}\right. \tag{24}$$

This system modelizes the vibrations of a network of strings with local forces at the nodes of $\mathcal{V}_c$. The next proposition gives existence and regularity results for the solution of problem (24).

Proposition 4.1. *Let $T > 0$ be fixed. Suppose that $k_v \in L^2(0, T)$ for all $v \in \mathcal{V}_c$. Then the problem (24) admits a unique solution $w \in \prod_{j=1}^{N} H^1((0,\, l_j) \times (0,\, T))$. Moreover $w(v,\, .) \in H^1(0,\, T)$ for all $v \in \mathcal{V}_c$ and*

$$\sum_{v \in \mathcal{V}_c} \|w(v,\, .)\|_{H^1(0,T)} \lesssim \sum_{v \in \mathcal{V}_c} \|k_v\|_{L^2(0,T)} \quad \forall v \in \mathcal{V}_c. \tag{25}$$

The proof of this proposition is relatively technical and requires some preliminary results.

We first consider the following problem

$$\left\{\begin{array}{cc} w_{tt}(x,\, t) - w_{xx}(x,\, t) = 0 & \text{in } (0,\, 1) \times (0,\, T), \\ w(1,\, t) = 0 & \text{on } (0,\, T), \\ w_x(0,\, t) = k(t) & \text{on } (0,\, T), \\ w(x,\, 0) = 0,\; w_t(x,\, 0) = 0 & \text{on } (0,\, 1). \end{array}\right. \tag{26}$$

Lemma 4.2. *Assume that $k \in L^2(0, T)$. Then problem (26) has a unique solution $w \in H^1((0,\, 1) \times (0,\, T))$ which satisfies*

$$\|w\|_{H^1((0,1)\times(0,T))} \lesssim \|k\|_{L^2(0,T)}\,.$$

Moreover $w(0,\, .) \in H^1(0,\, T)$ and satisfies

$$\|w(0,\, .)\|_{H^1(0,T)} \lesssim \|k\|_{L^2(0,T)}\,.$$

Proof. We extend k by 0 on $\mathbb{R} \setminus [0,\, T]$ because (26) is reversible in time.

Let $\hat{w}(x,\, \lambda)$ where $\lambda = \gamma + i\eta$, $\gamma > 0$, $\eta \in \mathbb{R}$, be the Laplace transform of w with respect to t. Then $\hat{w}$ satisfies

$$\left\{\begin{array}{cc} \lambda^2 \hat{w}(x,\, \lambda) - \frac{\partial^2 \hat{w}}{\partial x^2}(x,\, \lambda) = 0 & \text{in } (0,\, 1), \\ \hat{w}(1,\, \lambda) = 0, & \\ \frac{\partial \hat{w}}{\partial x}(0,\, \lambda) = \hat{k}(\lambda), & \end{array}\right.$$

where $\Re \lambda > 0$. Consequently $\hat{w}(x,\, \lambda) = a \cosh(\lambda(x - 1)) + b \sinh(\lambda(x - 1))$, with two complex numbers a and b.

As $\hat{w}(1,\,\lambda)=a=0$ and $\frac{\partial \hat{w}}{\partial x}(0,\,\lambda)=b\lambda\cosh(\lambda)=\hat{k}(\lambda)$, we deduce that

$$\hat{w}(x,\,\lambda)=\frac{\sinh(\lambda(x-1))}{\lambda\cosh(\lambda)}\hat{k}(\lambda).$$

because $\cosh(\lambda)\neq 0$.

• Existence of w: $w(x,\,t)=\mathcal{L}^{-1}(\frac{\sinh(\lambda(x-1))}{\lambda\cosh(\lambda)})\star k$ where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

• Uniqueness of w: if $w_1,\,w_2$ are two solutions of (26), then $w=w_1-w_2$ satisfies the wave equation in $(0,1)$ with homogeneous boundary and initial conditions. Therefore $w_1-w_2=0$, which proves the uniqueness of the solution of (26).

• Regularity: Let $\gamma>0$ be fixed and set $C_\gamma:=\{\lambda\in\mathbb{C}\,;\,\Re\lambda=\gamma\}$. Define $H(x,\,\lambda)=\frac{\sinh(\lambda(x-1))}{\cosh(\lambda)}$, for $\lambda\in C_\gamma$. We clearly have

$$\|H\|_{L^\infty((0,\,1)\times C_\gamma)}\leq\coth(\gamma).$$

Therefore

$$\|\hat{w}(x,\,\lambda)\|_{L^2((0,\,1)\times\mathbb{R}_\eta)}=\left\|\frac{H(x,\,\lambda)}{\lambda}\hat{k}(\lambda)\right\|_{L^2((0,\,1)\times\mathbb{R}_\eta)}\leq\frac{\coth(\gamma)}{\gamma}\left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}, \tag{27}$$

which implies that $w\in L^2((0,\,1)\times(0,\,T))$ with

$$\|w\|_{L^2((0,\,1)\times(0,\,T))}\lesssim\|k\|_{L^2(0,\,T)}\,. \tag{28}$$

Indeed, we have

$$\begin{aligned}\hat{w}(x,\,\lambda)=&\int_0^\infty e^{-(\gamma+i\eta)t}w(x,\,t)dt=\int_{-\infty}^{+\infty}e^{-\gamma t}e^{-i\eta t}w(x,\,t)dt\\ =&\mathcal{F}(e^{-\gamma\cdot}w)(\eta)=\mathcal{F}(w_1)(\eta)\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform and where we have set $w_1=e^{-\gamma\cdot}w$. Therefore

$$\begin{array}{rcll}\|w\|_{L^2((0,\,1)\times(0,\,T))} & \sim & \|w_1\|_{L^2((0,\,1)\times(0,\,T))} & \text{as } e^{-\gamma T}\leq e^{-\gamma t}\leq 1 \text{ on } (0,\,T)\\ & \leq & \|w_1\|_{L^2((0,\,1)\times\mathbb{R})} & \\ & \sim & \|\mathcal{F}(w_1)(\eta)\|_{L^2((0,\,1)\times\mathbb{R}_\eta)} & \text{by Plancherel's formula}\\ & \sim & \|\hat{w}(x,\,\lambda)\|_{L^2((0,\,1)\times\mathbb{R}_\eta)} & \end{array}$$

while

$$\begin{array}{rcl}\|k\|_{L^2(0,\,T)} & = & \|k\|_{L^2(\mathbb{R})}\sim\|ke^{-\gamma\cdot}\|_{L^2(\mathbb{R})}\\ & \sim & \|\mathcal{F}(e^{-\gamma\cdot}k)(\eta)\|_{L^2(\mathbb{R}_\eta)}\\ & = & \left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}.\end{array}$$

These two equivalences and the estimate (27) lead to (28).

As

$$\|\lambda\hat{w}(x,\,\lambda)\|_{L^2((0,\,1)\times\mathbb{R}_\eta)}\leq\coth(\gamma)\left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}$$

we deduce that $w\in H^1(0,\,T\,;\,L^2(0,\,1))$ and

$$\|w\|_{H^1(0,\,T\,;\,L^2(0,\,1))}\lesssim\|k\|_{L^2(0,\,T)}\,.$$

Indeed

$$\begin{array}{rcll} \left\|\frac{\partial w_1}{\partial t}\right\|_{L^2((0,1)\times(0,T))} & \leq & \left\|\frac{\partial w_1}{\partial t}\right\|_{L^2((0,1)\times\mathbb{R})} & \\ & \sim & \left\|\mathcal{F}(\frac{\partial w_1}{\partial t})(\eta)\right\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \text{by Plancherel's formula} \\ & \sim & \|(i\eta)\mathcal{F}(w_1)(\eta)\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \\ & \sim & \|(i\eta)\hat{w}(x,\lambda)\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \end{array}$$

and

$$\begin{array}{rcl} \|\lambda\hat{w}(x,\lambda)\|^2_{L^2((0,1)\times\mathbb{R}_\eta)} & = & \int_0^1\int_{\mathbb{R}_\eta}|\lambda|^2\,\hat{w}(x,\lambda)^2 d\lambda dx \\ & = & \int_0^1\int_{\mathbb{R}_\eta}(\gamma^2+\eta^2)\hat{w}(x,\lambda)^2 d\lambda dx \\ & \geq & \|(i\eta)\hat{w}(x,\lambda)\|^2_{L^2((0,1)\times\mathbb{R}_\eta)}. \end{array}$$

Therefore

$$\begin{array}{rcl} \left\|\frac{\partial w_1}{\partial t}\right\|_{L^2((0,1)\times(0,T))} & \lesssim & \|\lambda\hat{w}(x,\lambda)\|_{L^2((0,1)\times\mathbb{R}_\eta)} \\ & \lesssim & \left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)} \lesssim \|k\|_{L^2(0,T)}. \end{array}$$

We finally conclude that

$$\begin{array}{rcl} \left\|\frac{\partial w}{\partial t}\right\|_{L^2((0,1)\times(0,T))} & = & \left\|e^{\gamma\cdot}\frac{\partial w_1}{\partial t}+\gamma w\right\|_{L^2((0,1)\times(0,T))} \\ & \lesssim & \left\|\frac{\partial w_1}{\partial t}\right\|_{L^2((0,1)\times(0,T))}+\|w\|_{L^2((0,1)\times(0,T))} \\ & \lesssim & \|k\|_{L^2(0,T)}. \end{array}$$

In a similar manner we have

$$\|\lambda\hat{w}(0,\lambda)\|_{L^2(\mathbb{R}_\eta)} \leq \coth(\gamma)\left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}$$

which implies that $w(0,.)\in H^1(0,T)$ and satisfies

$$\|w(0,.)\|_{H^1(0,T)} \lesssim \|k\|_{L^2(0,T)}.$$

Finally $\frac{\partial\hat{w}}{\partial x}(x,\lambda) = \frac{\cosh(\lambda(x-1))}{\cosh(\lambda)}\hat{k}(\lambda)$. But the standard estimate $|\cosh z| \leq \cosh(Re\,z), \forall z\in\mathbb{C}$ implies that

$$\left|\frac{\cosh(\lambda(x-1))}{\cosh(\lambda)}\right| \leq \frac{\cosh(\gamma(1-x))}{\sinh(\gamma)} \leq \frac{\cosh(\gamma)}{\sinh(\gamma)} = \coth\gamma;$$

therefore

$$\left\|\frac{\partial\hat{w}(x,\lambda)}{\partial x}\right\|_{L^2((0,1)\times\mathbb{R}_\eta)} \leq \coth(\gamma)\left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}$$

which leads to $\frac{\partial w}{\partial x}\in L^2((0,1)\times(0,T))$ with

$$\left\|\frac{\partial w}{\partial x}\right\|_{L^2((0,1)\times(0,T))} \lesssim \|k\|_{L^2(0,T)}.$$

Indeed

$$\begin{array}{rcll} \left\|\frac{\partial w}{\partial x}\right\|_{L^2((0,1)\times(0,T))} & \sim & \left\|\frac{\partial w_1}{\partial x}\right\|_{L^2((0,1)\times(0,T))} & \leq \left\|\frac{\partial w_1}{\partial x}\right\|_{L^2((0,1)\times\mathbb{R})} \\ & \sim & \left\|\mathcal{F}(\frac{\partial w_1}{\partial x})(\eta)\right\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \text{by Plancherel's formula} \\ & \sim & \left\|\frac{\partial\hat{w}}{\partial x}\right\|_{L^2((0,1)\times\mathbb{R}_\eta)}. & \end{array}$$

It remains to check the initial conditions. We remark that

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{\sinh(\lambda(x-1))}{\lambda\cosh(\lambda)}\right) =& \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{(1)^n}{2n-1}\sin\left(\frac{(2n-1)\pi(x-1)}{2}\right)\sin\left(\frac{(2n-1)\pi t}{2}\right) \\ =& F(x,t). \end{aligned}$$

Note further that $F \in L^\infty([0, T]\,;\, L^2(0, 1))$. Therefore

$$w(x, t) = \int_0^t F(x, t-s)k(s)ds.$$

Consequently we directly see that

$$w(x, 0) = 0$$

(the trace having a meaning because $w \in H^1(0, 1)$). Moreover

$$\frac{\partial w}{\partial t}(x, t) = F(x, 0)k(t) + \int_0^t \frac{\partial F}{\partial t}(x, t-s)k(s)ds = \int_0^t \frac{\partial F}{\partial t}(x, t-s)k(s)ds.$$

Consequently

$$\begin{array}{rcl} \left\| \frac{\partial w}{\partial t}(., t)\right\|_{H^{-1}(0,1)} & = & \left\| \int_0^t \frac{\partial F}{\partial t}(x, t-s)k(s)ds\right\|_{H^{-1}(0,1)} \\ & \leq & \int_0^t \left\| \frac{\partial F}{\partial t}(x, t-s)\right\|_{H^{-1}(0,1)} |k(s)|\, ds. \end{array}$$

But we may write

$$\begin{array}{rcl} \frac{\partial F}{\partial t}(x, t) & = & \frac{4}{\pi}\frac{\pi}{2}\sum_{n=1}^{+\infty}(-1)^n \sin(\frac{(2n-1)\pi(x-1)}{2})\cos(\frac{(2n-1)\pi t}{2}) \\ & = & \sum_{n=1}^{+\infty} a_n(t)\sin(\frac{(2n-1)\pi(x-1)}{2}), \end{array}$$

where $a_n(t) = 2(-1)^n \cos(\frac{(2n-1)\pi t}{2})$ and $\{\sin(\frac{(2n-1)\pi(x-1)}{2})\}_n$ is an orthogonal basis of $L^2(0,1)$. This implies that

$$\left\| \frac{\partial F}{\partial t}(x, t)\right\|_{H^{-1}(0,1)} \sim \sum_{n=1}^{+\infty} a_n^2(t) n^{-2},$$

with $a_n^2(t) \leq 4$. Therefore

$$\left\| \frac{\partial F}{\partial t}(x, t)\right\|_{H^{-1}(0,1)} \lesssim 1$$

and consequently

$$\begin{array}{rcl} \left\| \frac{\partial w}{\partial t}(\cdot, t)\right\|_{H^{-1}(0,1)} & \lesssim & \int_0^t |k(s)|\, ds \\ & \lesssim & (\int_0^t ds)^{\frac{1}{2}} (\int_0^t |k(s)|^2\, ds)^{\frac{1}{2}} \\ & \lesssim & \sqrt{t}\, \|k\|_{L^2(0,1)}. \end{array}$$

This shows that

$$\frac{\partial w}{\partial t} \in C([0, T]\,;\, H^{-1}(0, 1))$$

and moreover

$$\frac{\partial w}{\partial t}(x, 0) = 0.$$

This ends the proof of the Lemma. □

We now make a local construction. Namely for a fixed $v \in \mathcal{V}_c$, we consider $\tilde{w}^{(v)}$ solution of

$$\left\{\begin{array}{ll} \frac{\partial^2 \tilde{w}_j^{(v)}}{\partial t^2} - \frac{\partial^2 \tilde{w}_j^{(v)}}{\partial x^2} = 0 & 0 < x < l_v,\, t > 0,\, \forall j \in \mathcal{E}_v, \\ \tilde{w}_j^{(v)}(0,\, t) = \tilde{w}_l^{(v)}(0,\, t) & \forall j,\, l \in \mathcal{E}_v,\, t > 0, \\ \sum_{j \in \mathcal{E}_v} \frac{\partial \tilde{w}_j^{(v)}}{\partial n_j}(v,\, t) = k_v(t) & t > 0, \\ \tilde{w}_j^{(v)}(l_v, t) = 0 & \forall j \in \mathcal{E}_v,\, t > 0, \\ \tilde{w}^{(v)}(t = 0) = 0,\, \frac{\partial \tilde{w}^{(v)}}{\partial t}(t = 0) = 0, & \end{array}\right. \tag{29}$$

where $l_v = \min_{j \in \mathcal{E}_v} l_j$ (without loss of generality we may identify v with the extremity 0 for all edges of $\mathcal{E}_v$).

The unique solution of this system is simply $\tilde{w}_j^{(v)}(x,t) = w(x/l_v, t/l_v)$, where w is solution of problem (26) with $k(t) = -\frac{l_v}{E_v} k_v(l_v t)$, when E_v is the cardinal of $\mathcal{E}_v$. By Lemma 4.2, we directly obtain the

Lemma 4.3. *The system (29) admits a unique solution $\tilde{w}_j^{(v)} \in H^1((0,\, l_v) \times (0,\, T))$, $j \in \mathcal{E}_v$ such that*

$$\left\| \tilde{w}_j^{(v)} \right\|_{H^1((0,\, l_v) \times (0,\, T))} \lesssim \|k_v\|_{L^2(0,\, T)}\, .$$

Moreover $\tilde{w}_j^{(v)}(v,\, .) \in H^1(0,\, T)$ with the estimate

$$\left\| \tilde{w}_j^{(v)}(v,\, .) \right\|_{H^1(0,\, T)} \lesssim \|k_v\|_{L^2(0,\, T)}\, .$$

Let us now set (assuming for the moment that w exists)

$$\omega(x,\, t) := w(x,t) - \sum_{v \in \mathcal{V}_c} \eta^{(v)}(x) \tilde{w}^{(v)}(x,\, t)$$

where $\tilde{w}^{(v)}$ is solution of problem (29) and $\eta^{(v)}$ is a cut-off function such that supp $\eta^{(v)} = \mathcal{E}_v$, $\eta^{(v)}$ is equal to 1 in a neighborhood of v and is 0 outside a larger neighborhood of v. Then we easily see that ω is solution of the following system

$$\left\{\begin{array}{ll} \frac{\partial^2 \omega_j}{\partial t^2} - \frac{\partial^2 \omega_j}{\partial x^2} = h_j(x,\, t) & 0 < x_j < l_j,\, t > 0,\, \forall j \in \{1, ..., N\}, \\ \omega_j(v,\, t) = \omega_l(v,\, t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int},\, t > 0, \\ \sum_{j \in \mathcal{E}_v} \frac{\partial \omega_j}{\partial n_j}(v,\, t) = 0 & \forall v \in \mathcal{V}_c \cup \mathcal{V}_{int}^c,\, t > 0, \\ \omega_{j_v}(v,\, t) = 0 & \forall v \in \mathcal{D},\, t > 0, \\ \frac{\partial \omega_{j_v}}{\partial n_{j_v}}(v,\, t) = 0 & \forall v \in \mathcal{N},\, t > 0, \\ \omega(t = 0) = 0,\, \frac{\partial \omega}{\partial t}(t = 0) = 0, & \end{array}\right. \tag{30}$$

where h_j is given by

$$h_j(x,\, t) := \sum_{v \in \mathcal{V}_c\, :\, j \in \mathcal{E}_v} (\frac{\partial^2 \eta_j^{(v)}}{\partial x^2}(x) \tilde{w_j}^{(v)}(x,\, t) + 2 \frac{\partial \eta_j^{(v)}}{\partial x}(x) \frac{\partial \tilde{w}_j^{(v)}}{\partial x}(x,\, t)).$$

We have then transformed the system with nonhomogeneous transmission conditions to a system with nonhomogeneous right-hand sides. This system can be written in the form

$$\frac{\partial^2 \omega}{\partial t^2} + A\omega = h,$$

where the operator A was defined before. As A is a positive selfadjoint operator on $L^2(\mathcal{R})$, we directly obtain the

Lemma 4.4. *The solution ω of (30) has the regularity*

$$\omega \in C(0, T\,;\, V) \cap C^1(0, T\,;\, L^2(\mathcal{R}))$$

and is given by the so-called constant variation formula

$$\omega(\cdot, t) = \sum_{k\geq 1}\sum_{i=1}^{l_k} \frac{1}{\lambda_k}\Big(\int_0^t \sin((t-s)\lambda_k)(h(\cdot, s), \varphi_{k,i})_{L^2(\mathcal{R})} ds\Big)\varphi_{k,i}.$$

Lemmas 4.3 and 4.4 guarantee the existence of a unique solution $w \in \prod_{j=1}^{N} H^1((0, l_j) \times (0, T))$ of problem (24) given by

$$w(x, t) = \omega(x, t) + \sum_{v\in\mathcal{V}_c} \eta^{(v)}(x)\tilde{w}^{(v)}(x, t),$$

and satisfying

$$\sum_{j=1}^{N} \|w_j\|_{H^1((0, l_j)\times(0, T))} \lesssim \sum_{v\in\mathcal{V}_c} \|k_v\|_{L^2(0,T)}. \tag{31}$$

It remains to show the regularity at the nodes of $\mathcal{V}_c$. Fix one vertex $v \in \mathcal{V}_c$ and a cut-off function $\chi^{(v)}$ such that

$$\chi_j^{(v)} \equiv 1 \text{ on } [0, l_v/3], \quad \chi_j^{(v)} \equiv 0 \text{ on } [2l_v/3, l_j], \forall j \in \mathcal{E}_v,$$

where we have identified v to 0. Let us now set

$$W = \chi^{(v)} w.$$

This function W is solution of the following wave equation:

$$\begin{cases} \frac{\partial^2 W_j}{\partial t^2} - \frac{\partial^2 W_j}{\partial x^2} = \tilde{h}_j(x,t) & 0 < x < l_v,\ t > 0,\ \forall j \in \mathcal{E}_v, \\ W_j(0, t) = W_l(0, t) & \forall j, l \in \mathcal{E}_v,\ t > 0, \\ \sum_{j\in\mathcal{E}_v} \frac{\partial W_j}{\partial x}(0, t) = k_v(t) & t > 0, \\ W_j(l_v, t) = 0 & \forall j \in \mathcal{E}_v,\ t > 0, \\ W(t=0) = 0,\ \frac{\partial W}{\partial t}(t=0) = 0, & \end{cases} \tag{32}$$

where $\tilde{h}_j$ is given by

$$\tilde{h}_j(x, t) := -\sum_{v\in\mathcal{V}_c\,:\, j\in\mathcal{E}_v} \Big(\frac{\partial^2 \chi_j^{(v)}}{\partial x^2}(x) w_j(x, t) + 2\frac{\partial \chi_j^{(v)}}{\partial x}(x)\frac{\partial w_j}{\partial x}(x, t)\Big).$$

According to the estimate (31) satisfied by w, we have

$$\|\tilde{h}_j\|_{L^2((0,l_v)\times(0,T))} \lesssim \sum_{v'\in\mathcal{V}_c} \|k_{v'}\|_{L^2(0,T)}.$$

Recalling that E_v is the cardinal of $\mathcal{E}_v$, we then may write

$$\mathcal{E}_v = \{j_i\}_{i=1,\cdots,E_v}.$$

Introduce

$$\begin{aligned} V_1 &= \sum_{i=1}^{E_v} W_{j_i}, \\ V_i &= W_{j_i} - W_{j_1}, \forall i = 2, \cdots, E_v. \end{aligned}$$

We remark that $V_i, i = 2, \cdots, E_v$ is solution of the wave equation with Dirichlet boundary condition at 0 and l_v, while V_1 is solution of the wave equation with Dirichlet boundary condition at l_v and Neumann boundary condition at 0, namely

$$\left\{\begin{array}{ll}\frac{\partial^2 V_1}{\partial t^2} - \frac{\partial^2 V_1}{\partial x^2} = g(x,t) & 0 < x < l_v,\, t > 0, \\ \frac{\partial V_1}{\partial x}(0,\, t) = k_v(t) & t > 0, \\ V_1(l_v,\, t) = 0 & t > 0, \\ V_1(t=0) = 0,\, \frac{\partial V_1}{\partial t}(t=0) = 0, & \end{array}\right. \tag{33}$$

where $g = \sum_{j \in \mathcal{E}_v} \tilde{h}_j$ and then satisfies

$$\|g\|_{L^2((0,l_v)\times(0,T))} \lesssim \sum_{v' \in \mathcal{V}_c} \|k_{v'}\|_{L^2(0,T)}. \tag{34}$$

But Lemma 4.5 below shows that

$$\|V_1(0,\cdot)\|_{H^1(0,T)} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)}.$$

As

$$\begin{aligned} W_{j_1} &= \frac{1}{E_v}(V_1 - \sum_{i=2}^{E_v} V_i), \\ W_{j_i} &= V_i + W_{j_1}, \forall i = 2, \cdots, E_v, \end{aligned}$$

we conclude that

$$\|W_{j_i}(0,\cdot)\|_{H^1(0,T)} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)}, \forall i = 1, \cdots, E_v.$$

Using (34), we obtain the estimate (25) from Proposition 4.1.

Lemma 4.5. *Let $V_1 \in H^1((0,l_v)\times(0,T))$ be a solution of (33) with $g \in L^2((0,l_v)\times(0,T))$ and $k_v \in L^2(0,T)$. Then*

$$\|\frac{\partial V_1}{\partial t}(0,\cdot)\|_{L^2(0,T)} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)}.$$

Proof. Let us denote by V the solution of

$$\left\{\begin{array}{ll}\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = \tilde{g}(x,t) & 0 < x < l_v,\, t > 0, \\ \frac{\partial V}{\partial x}(0,\, t) = \tilde{k}_v(t) & t > 0, \\ V(l_v, t) = 0 & t > 0, \\ V(\cdot, t=0) = 0,\, \frac{\partial V}{\partial t}(\cdot, t=0) = 0, & \end{array}\right. \tag{35}$$

where $\tilde{g}$ (resp. $\tilde{k}_v$) means the extension of g (resp. k_v) by zero outside $(0,T)$. As $V = V_1$ on $(0,l_v)\times(0,T)$, it suffices to show that

$$\|\frac{\partial V}{\partial t}(0,\cdot)\|_{L^2(0,T)} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)}. \tag{36}$$

For that purpose we use a multiplier technique. Namely we multiply the wave equation satisfied by V by $(l_v - x)(2T - t)V_x(x,t)$ (here and below for shortness we write $V_x = \frac{\partial V}{\partial x}$) and integrate the result on $(0,l_v)\times(0,2T)$. This yields

$$\int_0^{l_v} \int_0^{2T} \tilde{g}(x,t)(l_v - x)(2T - t)V_x(x,t)\, dxdt = I_1 - I_2, \tag{37}$$

where

$$
\begin{aligned}
I_1 &= \int_0^{l_v}\int_0^{2T} V_{tt}(l_v - x)(2T-t)V_x\,dxdt,\\
I_2 &= \int_0^{l_v}\int_0^{2T} V_{xx}(l_v - x)(2T-t)V_x\,dxdt.
\end{aligned}
$$

For the first term I_1 an integration by parts in time yields (no boundary terms occur because $V_t(t=0)=0$)

$$
I_1 = \int_0^{l_v}\int_0^{2T} V_t(l_v - x)V_x(x,t)\,dxdt - I_3, \tag{38}
$$

where

$$
I_3 = \int_0^{l_v}\int_0^{2T} V_t(l_v - x)(2T-t)V_{xt}(x,t)\,dxdt.
$$

For I_3, an integration by parts in space and Leibniz's rule lead to

$$
\begin{aligned}
I_3 &= -\int_0^{l_v}\int_0^{2T} \frac{\partial}{\partial x}(V_t(l_v - x))(2T-t)V_t\,dxdt\\
&\quad + \left[\int_0^{2T}(l_v - x))(2T-t)|V_t|^2\,dt\right]_0^{l_v}\\
&= \int_0^{l_v}\int_0^{2T} |V_t|^2(2T-t)\,dxdt - I_3\\
&\quad - \int_0^{2T}(2T-t)l_v|V_t(0,t)|^2\,dt.
\end{aligned}
$$

This shows that

$$
I_3 = \frac{1}{2}\int_0^{l_v}\int_0^{2T} |V_t|^2(2T-t)\,dxdt - \frac{l_v}{2}\int_0^{2T}(2T-t)|V_t(0,t)|^2\,dt.
$$

Inserting this expression in (38) we find that

$$
\begin{aligned}
I_1 &= \int_0^{l_v}\int_0^{2T} V_t(l_v - x)V_x(x,t)\,dxdt\\
&\quad -\frac{1}{2}\int_0^{l_v}\int_0^{2T} |V_t|^2(2T-t)\,dxdt\\
&\quad +\frac{l_v}{2}\int_0^{2T}(2T-t)|V_t(0,t)|^2\,dt.
\end{aligned} \tag{39}
$$

Similarly for the second term I_2 an integration by parts in space yields

$$
\begin{aligned}
I_2 &= -\int_0^{l_v}\int_0^{2T} V_x(2T-t)\frac{\partial}{\partial x}((l_v - x)V_x)\,dxdt\\
&\quad + \left[\int_0^{2T}(2T-t)(l_v - x)|V_x|^2\,dt\right]_0^{l_v}\\
&= -I_2 + \int_0^{l_v}\int_0^{2T} |V_x|^2(2T-t)\,dxdt\\
&\quad -l_v\int_0^{2T}(2T-t)|V_x(0,t)|^2\,dt.
\end{aligned}
$$

This means that

$$I_2 = \frac{1}{2}\int_0^{l_v}\int_0^{2T} |V_x|^2(2T-t)\,dxdt - \frac{l_v}{2}\int_0^{2T}(2T-t)|V_x(0,t)|^2\,dt. \tag{40}$$

Inserting (39) and (40) into the identity (37), we have obtained that

$$\begin{aligned}\int_0^{l_v}\int_0^{2T}\tilde{g}(x,t)(l_v-x)(2T-t)V_x(x,t)\,dxdt &= \int_0^{l_v}\int_0^{2T} V_t(l_v-x)V_x(x,t)\,dxdt\\ &-\frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_t|^2(2T-t)\,dxdt\\ &+\frac{l_v}{2}\int_0^{2T}(2T-t)|V_t(0,t)|^2\,dt.\\ &-\frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_x|^2(2T-t)\,dxdt\\ &+\frac{l_v}{2}\int_0^{2T}(2T-t)|V_x(0,t)|^2\,dt.\end{aligned}$$

Reminding the Neumann boundary condition satisfied by V we get

$$\begin{aligned}\frac{l_v}{2}\int_0^{2T}(2T-t)|V_t(0,t)|^2\,dt &= \int_0^{l_v}\int_0^{2T}\tilde{g}(x,t)(l_v-x)(2T-t)V_x(x,t)\,dxdt\\ &-\int_0^{l_v}\int_0^{2T} V_t(l_v-x)V_x(x,t)\,dxdt\\ &+\frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_t|^2(2T-t)\,dxdt\\ &+\frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_x|^2(2T-t)\,dxdt\\ &-\frac{l_v}{2}\int_0^{T}(2T-t)|k_v(t)|^2\,dt.\end{aligned}$$

By Cauchy-Schwarz's inequality we obtain finally

$$\int_0^{2T}(2T-t)|V_t(0,t)|^2\,dt \lesssim \|g\|^2_{L^2((0,l_v)\times(0,T))} + \|V\|^2_{H^1((0,l_v)\times(0,2T))}.$$

This leads to the conclusion owing to the estimate (34) and the estimate (consequence of our previous considerations, see (31))

$$\|V\|_{H^1((0,l_v)\times(0,2T))} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)},$$

and since $2T-t \geq T$ on $(0,T)$. □

Now we prove an a priori estimate which uses the trace regularity result of Proposition 4.1 and that will be useful to prove our stability results for problem (1).

Let $u \in C(0,T;V)\cap C^1(0,T;L^2(\mathcal{R}))$ be the solution of (1). Then it can be splitted up in the form

$$u = \phi + \psi$$

where ϕ is solution of problem without damping (14), and ψ satisfies

$$\left\{\begin{array}{lll} \frac{\partial^2\psi_j}{\partial t^2} - \frac{\partial^2\psi_j}{\partial x^2} = 0 & 0 < x < l_j, & t > 0,\, \forall j \in \{1, ..., N\}, \\ \psi_j(v,\, t) = \psi_l(v,\, t) & & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int},\, t > 0, \\ \sum_{j\in\mathcal{E}_v} \frac{\partial\psi_j}{\partial n_j}(v,\, t) = -(\alpha_1^{(v)} \frac{\partial u}{\partial t}(v,\, t) + \alpha_2^{(v)} \frac{\partial u}{\partial t}(v,\, t-\tau_v)) & & \forall v \in \mathcal{V}_c,\, t > 0, \\ \sum_{j\in\mathcal{E}_v} \frac{\partial\psi_j}{\partial n_j}(v,\, t) = 0 & & \forall v \in \mathcal{V}_{int}\backslash\mathcal{V}_{int}^c,\, t > 0, \\ \psi_{j_v}(v,\, t) = 0 & & \forall v \in \mathcal{D},\, t > 0, \\ \frac{\partial\psi_{j_v}}{\partial n_{j_v}}(v,\, t) = 0 & & \forall v \in \mathcal{N},\, t > 0, \\ \psi(t=0) = 0,\, \frac{\partial\psi}{\partial t}(t=0) = 0. & & \end{array}\right. \tag{41}$$

In other words ψ is solution of problem (24) with

$$k_v(t) = -(\alpha_1^{(v)} \frac{\partial u}{\partial t}(v,\, t) + \alpha_2^{(v)} \frac{\partial u}{\partial t}(v,\, t-\tau_v)). \tag{42}$$

For all $v \in \mathcal{V}_c$, we may write

$$\left\| \frac{\partial\phi}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)} \leq \left\| \frac{\partial u}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)} + \left\| \frac{\partial\psi}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)}.$$

Now by Remark 3.2, $k_v = -(\alpha_1^{(v)} \frac{\partial u}{\partial t}(v, \cdot) + \alpha_2^{(v)} \frac{\partial u}{\partial t}(v, \cdot - \tau_v)) \in L^2(0, T)$ for all $v \in \mathcal{V}_c$. Then we can apply Proposition 4.1 to ψ and obtain

$$\begin{array}{lll} \sum_{v\in\mathcal{V}_c} \left\| \frac{\partial\psi}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)} & \lesssim & \sum_{v\in\mathcal{V}_c} \left\| \alpha_1^{(v)} \frac{\partial u}{\partial t}(v,\, .) + \alpha_2^{(v)} \frac{\partial u}{\partial t}(v,\, .-\tau_v) \right\|_{L^2(0,\,T)} \\ & \lesssim & \sum_{v\in\mathcal{V}_c} (\left\| \frac{\partial u}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)} + \left\| \frac{\partial u}{\partial t}(v,\, .-\tau_v) \right\|_{L^2(0,\,T)}). \end{array}$$

The two above estimates yield

$$\begin{array}{lll} \sum_{v\in\mathcal{V}_c} \left\| \frac{\partial\phi}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)} & \leq & \sum_{v\in\mathcal{V}_c} \left\| \frac{\partial u}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)} + \sum_{v\in\mathcal{V}_c} \left\| \frac{\partial\psi}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)} \\ & \lesssim & \sum_{v\in\mathcal{V}_c} (\left\| \frac{\partial u}{\partial t}(v,\, .) \right\|_{L^2(0,\,T)} + \left\| \frac{\partial u}{\partial t}(v,\, .-\tau_v) \right\|_{L^2(0,\,T)}). \end{array}$$

This directly leads to the next a priori bound:

Lemma 4.6. *Suppose that $(u^{(0)}, u^{(1)}, (f^0(-\tau_v\cdot))_{v\in\mathcal{V}_c}) \in H$. Then the solutions u of (1) with initial data $(u^{(0)}, u^{(1)}, (f^0(-\tau_v\cdot))_{v\in\mathcal{V}_c})$ and ϕ of (14) with initial data $(u^{(0)}, u^{(1)})$ (which belongs to $V \times L^2(\mathcal{R})$) satisfy*

$$\sum_{v\in\mathcal{V}_c} \int_0^T (\frac{\partial\phi}{\partial t}(v,\, t))^2 dt \lesssim \sum_{v\in\mathcal{V}_c} \int_0^T ((\frac{\partial u}{\partial t}(v,\, t))^2 + (\frac{\partial u}{\partial t}(v,\, t-\tau_v))^2) dt.$$

5. **The exponential stability.** Our approach is based (as for the polynomial stability) on a trace regularity result (Proposition 4.1 and Lemma 4.6) and on an observability inequality of problem without damping (14).

5.1. **An observability inequality.**

Proposition 5.1. *Let* $(\varphi_{k,\, i})_{1\leq i\leq l_k,\, k\geq 1}$ *be the orthonormal basis of the operator* A. *Let* ϕ *be the solution of (14) with* $(u^{(0)},\, u^{(1)}) \in V \times L^2(\mathcal{R})$. *Then there exists a time* $T > 0$ *and a constant* $C > 0$ *(depending on* T*) such that*

$$\left\| u^{(0)} \right\|_V^2 + \left\| u^{(1)} \right\|_{L^2(\mathcal{R})}^2 \leq C \sum_{v\in\mathcal{V}_c} \int_0^T (\frac{\partial \phi}{\partial t}(v,\, t))^2 dt \tag{43}$$

if and only if

$$\exists \alpha > 0 : \lambda_{min}(\tilde{\mathcal{M}}^n) \geq \alpha,\, \forall n \in A_k, k = 1, \cdots, N+1, \tag{44}$$

where the matrix $\tilde{\mathcal{M}}^n$ *is defined by*

$$\tilde{\mathcal{M}}^n = \sum_{v\in\mathcal{V}_c} \Phi_n(v)^\top B_n^{-\top} B_n^{-1} \Phi_n(v),$$

the matrix $\Phi_n(v)$ *of size* $k \times L_n$, *where* $L_n = \sum_{i=1}^k l_{n+i-1}$, *is given as follows: for all* $i = 1, \cdots, k$, *we set*

$$(\Phi_n(v))_{ij} = \begin{cases} \varphi_{n+i-1, j-L_{n,i-1}}(v) & \text{if } L_{n,i-1} < j \leq L_{n,i}, \\ 0 & \text{else,} \end{cases}$$

where $L_{n,0} = 0$ *and for* $i \geq 1$, $L_{n,i} = \sum_{i'=1}^i l_{n+i'-1}$.

Proof. We first show that (44)⇒(43). Writting $u^{(0)} = \sum_{k\geq 1}\sum_{i=1}^{l_k} a_{k,\, i}\varphi_{k,\, i}$ and $u^{(1)} = \sum_{k\geq 1}\sum_{i=1}^{l_k} b_{k,\, i}\varphi_{k,\, i}$ where $(\lambda_k a_{k,\, i})_{i,\, k},\, (b_{k,\, i})_{i,\, k} \in l^2(\mathbb{N}^*)$, then the solution ϕ of problem (14) is given by

$$\phi(\cdot,\, t) = \sum_{k\geq 1}\sum_{i=1}^{l_k} (a_{k,\, i} \cos(\lambda_k t) + \frac{b_{k,\, i}}{\lambda_k} \sin(\lambda_k t))\varphi_{k,\, i}.$$

Consequently for any $v \in \mathcal{V}_c$, we get

$$\frac{\partial \phi}{\partial t}(v,\, t) = \sum_{k\geq 1}\sum_{i=1}^{l_k} (-a_{k,\, i}\lambda_k \sin(\lambda_k t) + b_{k,\, i} \cos(\lambda_k t))\varphi_{k,\, i}(v).$$

Putting together the terms corresponding to the same eigenvalue, we obtain

$$\frac{\partial \phi}{\partial t}(v,\, t) = \sum_{k\geq 1} (\sum_{i=1}^{l_k} - a_{k,\, i}\varphi_{k,\, i}(v))\lambda_k \sin(\lambda_k t) + \sum_{k\geq 1} (\sum_{i=1}^{l_k} b_{k,\, i}\varphi_{k,\, i}(v)) \cos(\lambda_k t).$$

Using the notations introduced in Proposition 3.7, this is equivalent to

$$\frac{\partial \phi}{\partial t}(v,\, t) = \sum_{n\in\mathbb{Z}^*} \alpha_n(v) e^{i\lambda_n t}.$$

Integrating the square of this identity between 0 and $T > 0$ and using Ingham's inequality (17) for T large enough, and summing on $v \in \mathcal{V}_c$, we get

$$\sum_{v\in\mathcal{V}_c} \int_0^T (\frac{\partial \phi}{\partial t}(v,\, t))^2 dt \gtrsim \sum_{k=1}^{N+1} \sum_{|n|\in A_k} \sum_{v\in\mathcal{V}_c} \|B_n^{-1} C_n(v)\|_2^2.$$

But for all $n \in A_k$, setting

$$\begin{aligned}\tilde{A}_n &= \big(\lambda_n a_{n,1}, \cdots, \lambda_n a_{n,l_n}, \lambda_{n+1} a_{n+1,1}, \cdots, \lambda_{n+1} a_{n+1,l_{n+1}}, \cdots,\\ &\qquad \lambda_{n+k-1} a_{n+k-1,1}, \cdots, \lambda_{n+k-1} a_{n+k-1,l_{n+k-1}}\big)^\top,\\ \tilde{B}_n &= \big(b_{n,1}, \cdots, b_{n,l_n}, b_{n+1,1}, \cdots, b_{n+1,l_{n+1}}, \cdots, b_{n+k-1,1}, \cdots, b_{n+k-1,l_{n+k-1}}\big)^\top,\end{aligned}$$

we readily check that

$$\sum_{v\in\mathcal{V}_c} \|B_n^{-1} C_n(v)\|_2^2 = \frac{1}{4}(\tilde{B}_n^\top \tilde{\mathcal{M}}^n \tilde{B}_n + \tilde{A}_n^\top \tilde{\mathcal{M}}^n \tilde{A}_n).$$

Hence the assumption (44) yields (because $\tilde{\mathcal{M}}^n$ is a symmetric matrix)

$$\sum_{v\in\mathcal{V}_c} \|B_n^{-1} C_n(v)\|_2^2 \gtrsim \|\tilde{B}_n\|_2^2 + \|\tilde{A}_n\|_2^2.$$

Therefore under our hypothesis, we have

$$\begin{aligned}\sum_{v\in\mathcal{V}_c} \int_0^T (\tfrac{\partial\phi}{\partial t}(v,t))^2 dt &\gtrsim \sum_{k\ge 1}\sum_{i=1}^{l_k} (a_{k,i}^2 \lambda_k^2 + b_{k,i}^2)\\ &\gtrsim \left\|u^{(0)}\right\|_V^2 + \left\|u^{(1)}\right\|_{L^2(\mathcal{R})}^2\end{aligned}$$

because $(\varphi_{k,i})_{k,i}$ is an orthonormal basis associated with the operator A.

It remains to show that (43)⇒(44).

Let $k = 1, \cdots, N+1$ and $n \in A_k$ be fixed. Take $u^{(0)} = \sum_{m=n}^{n+k-1} \sum_{i=1}^{l_m} a_{m,i}\varphi_{m,i}$ and $u^{(1)} = \sum_{m=n}^{n+k-1} \sum_{i=1}^{l_m} b_{m,i}\varphi_{m,i}$. Then the solution ϕ of problem (14) is given by

$$\phi(\cdot, t) = \sum_{m=n}^{n+k-1} \sum_{i=1}^{l_m} (a_{m,i}\cos(\lambda_m t) + \frac{b_{m,i}}{\lambda_m}\sin(\lambda_m t))\varphi_{m,i}.$$

Then for $v \in \mathcal{V}_c$

$$\frac{\partial\phi}{\partial t}(v, t) = \sum_{m=n}^{n+k-1} \sum_{i=1}^{l_m} (-a_{m,i}\lambda_m \sin(\lambda_m t) + b_{m,i}\cos(\lambda_m t))\varphi_{m,i}(v).$$

Applying again Ingham's inequality, we get for T large enough

$$\sum_{v\in\mathcal{V}_c} \int_0^T (\frac{\partial\phi}{\partial t}(v,t))^2 dt \sim \tilde{B}_n^\top \tilde{\mathcal{M}}^n \tilde{B}_n + \tilde{A}_n^\top \tilde{\mathcal{M}}^n \tilde{A}_n.$$

By (43), we obtain

$$\tilde{B}_n^\top \tilde{\mathcal{M}}^n \tilde{B}_n + \tilde{A}_n^\top \tilde{\mathcal{M}}^n \tilde{A}_n \ge C \sum_{m=n}^{n+k-1} \sum_{i=1}^{l_m} (a_{m,i}^2 \lambda_n^2 + b_{m,i}^2),$$

for some $C > 0$. Hence we conclude that

$$\lambda_{min}(\tilde{\mathcal{M}}^n) \ge C.$$

This ends the proof. □

Remark 5.2. If the standard gap condition (18) holds, then the condition (44) reduces to

$$\exists \alpha > 0 : \lambda_{min}(\mathcal{M}(\lambda_k^2)) \geq \alpha, \forall k \geq 1. \tag{45}$$

In particular if $l_k = 1$, then the condition (45) becomes

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 \geq \alpha.$$

5.2. **The stability result.** We are now ready to give a necessary and sufficient condition that guarantees the exponential stability of (1).

Theorem 5.3. *The system (1) is exponentially stable in the energy space if and only if (43) holds.*

Proof. We first show that (43) implies that the system (1) is exponentially stable.

Let u be a solution of (1) with initial data $(u^{(0)}, u^{(1)}, (f^0(-\tau_v \cdot))_{v\in\mathcal{V}_c}) \in H$.

Integrating the inequality (13) of Proposition 3.1 between 0 and T where T is sufficiently large, we obtain

$$\begin{aligned}
&E(0) - E(T) \gtrsim \sum_{v\in\mathcal{V}_c} \int_0^T [(\tfrac{\partial u}{\partial t}(v, t))^2 + (\tfrac{\partial u}{\partial t}(v, t-\tau_v))^2]dt \\
&\gtrsim \tfrac{1}{2} \sum_{v\in\mathcal{V}_c} \int_0^T [(\tfrac{\partial u}{\partial t}(v, t))^2 + (\tfrac{\partial u}{\partial t}(v, t-\tau_v))^2]dt + \tfrac{1}{2} \sum_{v\in\mathcal{V}_c} \int_0^T (\tfrac{\partial u}{\partial t}(v, t-\tau_v))^2 dt \\
&\gtrsim \sum_{v\in\mathcal{V}_c} \int_0^T (\tfrac{\partial \phi}{\partial t}(v, t))^2 dt + \sum_{v\in\mathcal{V}_c} \int_0^T (\tfrac{\partial u}{\partial t}(v, t-\tau_v))^2 dt \text{ by Lemma 4.6} \\
&\gtrsim \left\|u^{(0)}\right\|_V^2 + \left\|u^{(1)}\right\|_{L^2(\mathcal{R})}^2 + \sum_{v\in\mathcal{V}_c} \int_0^T (\tfrac{\partial u}{\partial t}(v, t-\tau_v))^2 dt \\
&\qquad \text{by assumption} \\
&\gtrsim \left\|u^{(0)}\right\|_V^2 + \left\|u^{(1)}\right\|_{L^2(\mathcal{R})}^2 + \sum_{v\in\mathcal{V}_c} \tau_v \int_0^1 (\tfrac{\partial u}{\partial t}(v, -\tau_v\rho))^2 d\rho.
\end{aligned}$$

Indeed, for $T > \tau_v$, by changes of variables, we have

$$\begin{aligned}
\int_0^T (\tfrac{\partial u}{\partial t}(v, t-\tau_v))^2 dt &= \int_{-\tau_v}^{T-\tau_v} (\tfrac{\partial u}{\partial t}(v, t))^2 dt \\
&\geq \int_{-\tau_v}^0 (\tfrac{\partial u}{\partial t}(v, t))^2 dt = \tau_v \int_0^1 (\tfrac{\partial u}{\partial t}(v, -\tau_v\rho))^2 d\rho.
\end{aligned} \tag{46}$$

The previous inequality directly implies that

$$E(0) - E(T) \gtrsim \frac{1}{2}\left(\|u^{(0)}\|_V^2 + \|u^{(1)}\|_{L^2(\mathcal{R})}^2 + \sum_{v\in\mathcal{V}_c} \frac{\xi^{(v)}}{2} \int_0^1 (\frac{\partial u}{\partial t}(v, -\tau_v\rho))^2 d\rho \right).$$

This means that for T large enough

$$E(0) - E(T) \geq CE(0)$$

for some $C > 0$. As our system is invariant by translation and the energy is decreasing, it is well known that the above estimate implies the existence of $C_1 > 0$ and $C_2 > 0$ such that

$$E(t) \leq C_1 E(0) e^{-C_2 t}, \forall t \geq 0.$$

Hence the energy decays exponentially.

Let us now show the inverse implication. Proposition 3.1 implies that

$$E(0) - E(T) \lesssim \sum_{v\in\mathcal{V}_c} \int_0^T [(\frac{\partial u}{\partial t}(v, t))^2 + (\frac{\partial u}{\partial t}(v, t-\tau_v))^2]dt.$$

Now by hypothesis, we have

$$E(T) \leq Ce^{-\omega T}E(0),$$

and then for T large enough ($T \geq \frac{\ln(\frac{4C}{3})}{\omega}$), we have

$$E(T) \leq \frac{3}{4}E(0).$$

The two above estimates yield

$$\frac{1}{4}E(0) \lesssim \sum_{v\in\mathcal{V}_c} \int_0^T [(\frac{\partial u}{\partial t}(v,\, t))^2 + (\frac{\partial u}{\partial t}(v,\, t-\tau_v))^2]dt. \tag{47}$$

Now we split up u as in section 4, i. e.,

$$u = \phi + \psi$$

where ϕ is solution of problem without damping (14) and ψ is solution of system (41). We define the energy $E(\psi,\, t)$ of system (41) by

$$E(\psi,\, t) = \frac{1}{2}\sum_{j=1}^N \int_0^{l_j} ((\frac{\partial \psi_j}{\partial t})^2 + (\frac{\partial \psi_j}{\partial x})^2)dx + \sum_{v\in\mathcal{V}_c} \frac{\xi^{(v)}}{2}(\int_0^1 (\frac{\partial \psi}{\partial t}(v,\, t-\tau_v\rho))^2 d\rho).$$

Then the same calculations as in the proof of Proposition 3.1 lead to

$$\begin{aligned}
E'(\psi,\, t) &= \sum_{j=1}^N \int_0^{l_j} (\frac{\partial \psi_j}{\partial t}\frac{\partial^2 \psi_j}{\partial t^2} + \frac{\partial \psi_j}{\partial x}\frac{\partial^2 \psi_j}{\partial x \partial t})dx \\
&\quad + \sum_{v\in\mathcal{V}_c} \xi^{(v)}(\int_0^1 \frac{\partial \psi}{\partial t}(v,\, t-\tau_v\rho)\frac{\partial^2 \psi}{\partial t^2}(v,\, t-\tau_v\rho)d\rho) \\
&= \sum_{j=1}^N \int_0^{l_j} (\frac{\partial \psi_j}{\partial t}\frac{\partial^2 \psi_j}{\partial t^2} - \frac{\partial \psi_j}{\partial t}\frac{\partial^2 \psi_j}{\partial x^2})dx + \sum_{j=1}^N [\frac{\partial \psi_j}{\partial x}\frac{\partial \psi_j}{\partial t}]_0^{l_j} \\
&\quad + \sum_{v\in\mathcal{V}_c} \xi^{(v)}(\textstyle\int_0^1 \frac{\partial \psi}{\partial t}(v,\, t-\tau_v\rho)\frac{\partial^2 \psi}{\partial t^2}(v,\, t-\tau_v\rho)d\rho) \\
&= -\sum_{v\in\mathcal{V}_c} (\alpha_1^{(v)}\tfrac{\partial u}{\partial t}(v,\, t) + \alpha_2^{(v)}\tfrac{\partial u}{\partial t}(v,\, t-\tau_v))\tfrac{\partial \psi}{\partial t}(v,\, t) \\
&\quad - \sum_{v\in\mathcal{V}_c} \tfrac{\xi^{(v)}}{\tau_v^3}(\textstyle\int_0^1 \frac{\partial \psi}{\partial \rho}(v,\, t-\tau_v\rho)\frac{\partial^2 \psi}{\partial \rho^2}(v,\, t-\tau_v\rho)d\rho) \\
&= -\sum_{v\in\mathcal{V}_c} (\alpha_1^{(v)}\tfrac{\partial u}{\partial t}(v,\, t) + \alpha_2^{(v)}\tfrac{\partial u}{\partial t}(v,\, t-\tau_v))\tfrac{\partial \psi}{\partial t}(v,\, t) \\
&\quad - \sum_{v\in\mathcal{V}_c} \tfrac{\xi^{(v)}}{2\tau_v}(\tfrac{\partial \psi}{\partial t}(v,\, t-\tau_v)^2 - \tfrac{\partial \psi}{\partial t}(v,\, t)^2).
\end{aligned}$$

As $u = \phi + \psi$ we get

$$\begin{aligned}
E'(\psi,\, t) &= -\sum_{v\in\mathcal{V}_c} (\alpha_1^{(v)}\tfrac{\partial \psi}{\partial t}(v,\, t) + \alpha_2^{(v)}\tfrac{\partial \psi}{\partial t}(v,\, t-\tau_v))\tfrac{\partial \psi}{\partial t}(v,\, t) \\
&\quad - \sum_{v\in\mathcal{V}_c} \tfrac{\xi^{(v)}}{2\tau_v}(\tfrac{\partial \psi}{\partial t}(v,\, t-\tau_v)^2 - \tfrac{\partial \psi}{\partial t}(v,\, t)^2) \\
&\quad - \sum_{v\in\mathcal{V}_c} (\alpha_1^{(v)}\tfrac{\partial \phi}{\partial t}(v,\, t) + \alpha_2^{(v)}\tfrac{\partial \phi}{\partial t}(v,\, t-\tau_v))\tfrac{\partial \psi}{\partial t}(v,\, t).
\end{aligned}$$

The same arguments as in the proof of Proposition 3.1 lead to

$$\begin{aligned}E'(\psi,\,t)\leq &-C\sum_{v\in\mathcal{V}_c}((\frac{\partial\psi}{\partial t}(v,\,t))^2+(\frac{\partial\psi}{\partial t}(v,\,t-\tau_v))^2)\\ &-\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}\frac{\partial\phi}{\partial t}(v,\,t)+\alpha_2^{(v)}\frac{\partial\phi}{\partial t}(v,\,t-\tau_v))\frac{\partial\psi}{\partial t}(v,\,t),\end{aligned}$$

for some $C>0$. Integrating this estimate between 0 and T, we deduce that

$$\begin{array}{rcl}0\leq E(\psi,\,T) & = & \int_0^T E'(\psi,\,t)dt\\ & \leq & -C\sum\limits_{v\in\mathcal{V}_c}\int_0^T((\frac{\partial\psi}{\partial t}(v,\,t))^2+(\frac{\partial\psi}{\partial t}(v,\,t-\tau_v))^2)dt\\ & & -\sum\limits_{v\in\mathcal{V}_c}\int_0^T(\alpha_1^{(v)}\frac{\partial\phi}{\partial t}(v,\,t)+\alpha_2^{(v)}\frac{\partial\phi}{\partial t}(v,\,t-\tau_v))\frac{\partial\psi}{\partial t}(v,\,t)dt.\end{array}$$

By continuous and discrete Cauchy-Schwarz's inequalities we obtain

$$\begin{aligned}&\sum_{v\in\mathcal{V}_c}\int_0^T((\frac{\partial\psi}{\partial t}(v,\,t))^2+(\frac{\partial\psi}{\partial t}(v,\,t-\tau_v))^2)dt\\ &\leq -C^{-1}\sum_{v\in\mathcal{V}_c}\int_0^T(\alpha_1^{(v)}\frac{\partial\phi}{\partial t}(v,\,t)+\alpha_2^{(v)}\frac{\partial\phi}{\partial t}(v,\,t-\tau_v))\frac{\partial\psi}{\partial t}(v,\,t)dt\\ &\lesssim\sum_{v\in\mathcal{V}_c}(\int_0^T(\frac{\partial\phi}{\partial t}(v,\,t)+\frac{\partial\phi}{\partial t}(v,\,t-\tau_v))^2dt)^{\frac{1}{2}}\cdot(\int_0^T\frac{\partial\psi}{\partial t}(v,\,t)^2dt)^{\frac{1}{2}}\\ &\lesssim(\sum_{v\in\mathcal{V}_c}\int_0^T(\frac{\partial\phi}{\partial t}(v,\,t)+\frac{\partial\phi}{\partial t}(v,\,t-\tau_v))^2dt)^{\frac{1}{2}}\cdot(\sum_{v\in\mathcal{V}_c}\int_0^T\frac{\partial\psi}{\partial t}(v,\,t)^2dt)^{\frac{1}{2}}\\ &\lesssim(\sum_{v\in\mathcal{V}_c}\int_0^T(\frac{\partial\phi}{\partial t}(v,\,t)+\frac{\partial\phi}{\partial t}(v,\,t-\tau_v))^2dt)^{\frac{1}{2}}\\ &\quad\cdot(\sum_{v\in\mathcal{V}_c}\int_0^T(\frac{\partial\psi}{\partial t}(v,\,t)^2+\frac{\partial\psi}{\partial t}(v,\,t-\tau_v)^2)dt)^{\frac{1}{2}}.\end{aligned}$$

This directly leads to

$$\sum_{v\in\mathcal{V}_c}\int_0^T((\frac{\partial\psi}{\partial t}(v,\,t))^2+(\frac{\partial\psi}{\partial t}(v,\,t-\tau_v))^2)dt\lesssim\sum_{v\in\mathcal{V}_c}\int_0^T(\frac{\partial\phi}{\partial t}(v,\,t)^2+\frac{\partial\phi}{\partial t}(v,\,t-\tau_v)^2)dt.$$

Finally as $u=\phi+\psi$, the above estimate implies that

$$\sum_{v\in\mathcal{V}_c}\int_0^T((\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2)dt\lesssim\sum_{v\in\mathcal{V}_c}\int_0^T((\frac{\partial\phi}{\partial t}(v,\,t))^2+(\frac{\partial\phi}{\partial t}(v,\,t-\tau_v))^2)dt. \tag{48}$$

Using this estimate (48) in (47), we get

$$\begin{array}{rcl}E(0) & \lesssim & \sum\limits_{v\in\mathcal{V}_c}\int_0^T((\frac{\partial\phi}{\partial t}(v,\,t))^2+(\frac{\partial\phi}{\partial t}(v,\,t-\tau_v))^2)dt\\ & \lesssim & \sum\limits_{v\in\mathcal{V}_c}\int_0^T(\frac{\partial\phi}{\partial t}(v,\,t))^2dt+C\sum\limits_{v\in\mathcal{V}_c}\int_{-\tau_v}^{T-\tau_v}(\frac{\partial\phi}{\partial t}(v,\,t))^2dt\\ & \lesssim & \sum\limits_{v\in\mathcal{V}_c}\int_{-\tau_{max}}^T(\frac{\partial\phi}{\partial t}(v,\,t))^2dt,\end{array}$$

when $\tau_{max} = \max_{v \in \mathcal{V}_c} \tau_v$. As the system (14) is conservative, we finally obtain

$$E(\phi(-\tau_{max})) = E(\phi(0)) \leq E(0) \lesssim \sum_{v \in \mathcal{V}_c} \int_{-\tau_{max}}^{T} (\frac{\partial \phi}{\partial t}(v, t))^2 dt.$$

Setting $\tilde{\phi}(., t) = \phi(., t - \tau_{max})$, we see that $\tilde{\phi}$ satisfies

$$\begin{cases} \tilde{\phi}_{tt} + A\tilde{\phi} = 0, \\ \tilde{\phi}(., t = 0) = \phi(., -\tau_{max}), \\ \tilde{\phi}_t(., t = 0) = \phi_t(., -\tau_{max}), \end{cases}$$

and the same boundary conditions than ϕ. For $\tilde{\phi}$, we have

$$\begin{array}{rcl} E(\tilde{\phi}(0)) & = & E(\phi(-\tau_{max})) \\ & \lesssim & \sum_{v \in \mathcal{V}_c} \int_{-\tau_{max}}^{T} (\frac{\partial \phi}{\partial t}(v, t))^2 dt \\ & \lesssim & \sum_{v \in \mathcal{V}_c} \int_{0}^{T+\tau_{max}} (\frac{\partial \phi}{\partial t}(v, t - \tau_{max}))^2 dt, \end{array}$$

in other words

$$E(\tilde{\phi}(0)) \lesssim \sum_{v \in \mathcal{V}_c} \int_{0}^{T+\tau_{max}} (\frac{\partial \tilde{\phi}}{\partial t}(v, t))^2 dt,$$

which means that $\tilde{\phi}$ satisfies the observability estimate (43). □

6. **The polynomial stability.**

6.1. **An observability estimate.**

Proposition 6.1. *Let $(\varphi_{k,\,i})_{1 \leq i \leq l_k,\, k \geq 1}$ be an orthonormal basis of eigenvectors of the operator A. Let $m \in \mathbb{N}^*$. Let ϕ be the solution of (14) with $(u^{(0)}, u^{(1)}) \in V \times L^2(\mathcal{R})$. Then there exists a time $T > 0$ and a constant $C > 0$ such that*

$$\sum_{v \in \mathcal{V}_c} \int_{0}^{T} (\frac{\partial \phi}{\partial t}(v, t))^2 dt \geq C(\sum_{k \geq 1} \frac{1}{k^{2m}} \sum_{i=1}^{l_k} (a_{k,\,i}^2 \lambda_k^2 + b_{k,\,i}^2)) \tag{49}$$

where $u^{(0)} = \sum_{k \geq 1} \sum_{i=1}^{l_k} a_{k,\,i} \varphi_{k,\,i}$ and $u^{(1)} = \sum_{k \geq 1} \sum_{i=1}^{l_k} b_{k,\,i} \varphi_{k,\,i}$ if and only if

$$\exists m \in \mathbb{N}^*, \exists \alpha > 0 : \lambda_{min}(\tilde{\mathcal{M}}^n) \geq \frac{\alpha}{k^{2m}}, \quad \forall n \in A_k, k = 1, \cdots, N+1. \tag{50}$$

Proof. The proof is similar to the one of Proposition 5.1 and is therefore omitted. □

Remark 6.2. As before, if the standard gap condition (18) holds, then the condition (50) reduces to

$$\exists m \in \mathbb{N}^*, \exists \alpha > 0 : \lambda_{min}(\mathcal{M}(\lambda_k^2)) \geq \frac{\alpha}{k^{2m}}, \forall k \geq 1, \tag{51}$$

and if $l_k = 1$, then condition (51) is nothing else than

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 \geq \frac{\alpha}{k^{2m}}.$$

By the so-called Weyl's formula (see for instance [11, 24]), we have

$$\lambda_k \sim \frac{k\pi}{L}$$

where $L = \sum_{j=1}^{N} l_j$. This implies that

$$\begin{aligned}
\sum_{k\geq 1} \tfrac{1}{k^{2m}} \sum_{i=1}^{l_k} (a_{k,\,i}^2 \lambda_k^2 + b_{k,\,i}^2) \quad &\sim \quad \sum_{k\geq 1}\sum_{i=1}^{l_k} (a_{k,\,i}^2 \lambda_k^{2(1-m)} + b_{k,\,i}^2 \lambda_k^{-2m}) \\
&\sim \quad \left\|u^{(0)}\right\|^2_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|^2_{D(A^{-\frac{m}{2}})}
\end{aligned}$$

because, for $u = \sum_{k\geq 1}\sum_{i=1}^{l_k} u_{k,i}\varphi_{k,i}$, we have $\|u\|^2_{D(A^s)} \sim \sum_{k\geq 1}\sum_{i=1}^{l_k} \lambda_k^{4s} u_{k,i}^2$ for all $s \in \mathbb{R}$. Therefore the observability estimate (49) is equivalent to

$$\sum_{v\in\mathcal{V}_c} \int_0^T (\frac{\partial \phi}{\partial t}(v,\,t))^2 dt \gtrsim \left\|u^{(0)}\right\|^2_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|^2_{D(A^{-\frac{m}{2}})}.$$

Now using Lemma 4.6, we obtain

$$\left\|u^{(0)}\right\|^2_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|^2_{D(A^{-\frac{m}{2}})} \lesssim \sum_{v\in\mathcal{V}_c} \int_0^T ((\frac{\partial u}{\partial t}(v,\,t))^2 + (\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2) dt.$$

On the other hand integrating the inequality (13) of Proposition 3.1 between 0 and T where T is sufficiently large, we have

$$\begin{aligned}
E(0) - E(T) \quad &\gtrsim \quad \sum_{v\in\mathcal{V}_c} \textstyle\int_0^T ((\frac{\partial u}{\partial t}(v,\,t))^2 + (\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2) dt \\
&\gtrsim \quad \tfrac{1}{2} \sum_{v\in\mathcal{V}_c} \textstyle\int_0^T ((\frac{\partial u}{\partial t}(v,\,t))^2 + (\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2) dt \\
&\qquad + \tfrac{1}{2} \sum_{v\in\mathcal{V}_c} \textstyle\int_0^T (\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2 dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
E(0) - E(T) \quad &\gtrsim \quad \left\|u^{(0)}\right\|^2_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|^2_{D(A^{-\frac{m}{2}})} + \sum_{v\in\mathcal{V}_c} \textstyle\int_0^T (\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2 dt \\
&\gtrsim \quad \left\|(u^{(0)},\,u^{(1)})\right\|^2_{X_{-m}} + \sum_{v\in\mathcal{V}_c} \tau_v \textstyle\int_0^1 (\frac{\partial u}{\partial t}(v,\,-\tau_v\rho))^2 d\rho,
\end{aligned}$$

where $X_{-m} = D(A^{\frac{1-m}{2}}) \times D(A^{-\frac{m}{2}})$ and owing to (46). This finally shows that the observability estimate (49) implies that

$$E(T) \leq E(0) - K_1(\left\|(u^{(0)},\,u^{(1)})\right\|^2_{X_{-m}} + \sum_{v\in\mathcal{V}_c} \int_0^1 (\frac{\partial u}{\partial t}(v,\,-\tau_v\rho))^2 d\rho), \tag{52}$$

for some $K_1 > 0$.

Now we recall the following interpolation result.

Lemma 6.3. *For* $(u^{(0)}, u^{(1)}) \in D(A) \times V$, *we have*

$$\left\|u^{(0)}\right\|_{D(A^{\frac{1}{2}})}^{m+1} \lesssim \left\|u^{(0)}\right\|_{D(A)}^{m} \left\|u^{(0)}\right\|_{D(A^{\frac{1-m}{2}})},$$
$$\left\|u^{(1)}\right\|_{D(A^{0})}^{m+1} \lesssim \left\|u^{(1)}\right\|_{D(A^{\frac{1}{2}})}^{m} \left\|u^{(1)}\right\|_{D(A^{\frac{-m}{2}})}.$$

Proof. Direct consequence of the equivalence $\|u\|_{D(A^s)}^2 \sim \sum_{k\geq 1}\sum_{i=1}^{l_k} |u_{k,i}|^2 \lambda_k^{4s}$ for all $s \in \mathbb{R}$, when $u = \sum_{k\geq 1}\sum_{i=1}^{l_k} u_{k,i}\varphi_{k,i}$ and of Hölder's inequality. □

Corollary 6.4. *For all* $u \in X$, *it holds*

$$\|u\|_V^{m+1} \lesssim \|u\|_X^m \|u\|_{D(A^{\frac{1-m}{2}})}. \tag{53}$$

Proof. We fix a sequence of cut-off functions $\eta_v, v \in \mathcal{V}$ satisfying

$$\sum_{v\in\mathcal{V}} \eta_v = 1 \text{ on } \mathcal{R}, \quad \eta_v \equiv 1 \text{ near } v,$$

and such that the support of η_v is included into $\mathcal{S}_v$, where $\mathcal{S}_v$ is the star-shaped network made of the edges $e_j, j \in \mathcal{E}_v$.

Denote by A_v the Laplace operator defined on $\mathcal{S}_v$ with Dirichlet boundary condition on all nodes of $\mathcal{S}_v$ except at v, where we impose Neumann or transmission conditions. Let us show that if (53) holds on $\mathcal{S}_v$ for $\eta_v u$, then it holds on $\mathcal{R}$. Indeed a convex inequality yields

$$\|u\|_V^{m+1} \lesssim \sum_{v\in\mathcal{V}} \|\eta_v u\|_V^{m+1}.$$

Now using (53) on $\mathcal{S}_v$ for $\eta_v u$, we get

$$\|u\|_V^{m+1} \lesssim \sum_{v\in\mathcal{V}} \|\eta_v u\|_{X_v}^m \|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})}, \tag{54}$$

the norm X_v being defined as the norm X but on $\mathcal{S}_v$. By Leibniz's rule, we directly have

$$\|\eta_v u\|_{X_v} \lesssim \|u\|_X.$$

Therefore it remains to estimate $\|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})}$. For that purpose, we use a duality argument, namely

$$\begin{aligned}\|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})} &= \sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\int_{\mathcal{S}_v} \eta_v u w}{\|w\|_{D(A_v^{\frac{m-1}{2}})}} \\ &= \sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\int_{\mathcal{S}_v} u \eta_v w}{\|w\|_{D(A_v^{\frac{m-1}{2}})}} \\ &= \sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\int_{\mathcal{R}} u \eta_v w}{\|w\|_{D(A_v^{\frac{m-1}{2}})}},\end{aligned}$$

by extending $\eta_v w$ by zero outside $\mathcal{S}_v$. Using the duality in $\mathcal{R}$, we obtain

$$\|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})} \le \|u\|_{D(A^{\frac{1-m}{2}})} \sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\|\eta_v w\|_{D(A^{\frac{m-1}{2}})}}{\|w\|_{D(A_v^{\frac{m-1}{2}})}}.$$

But again Leibniz's rule yields

$$\sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\|\eta_v w\|_{D(A^{\frac{m-1}{2}})}}{\|w\|_{D(A_v^{\frac{m-1}{2}})}} \lesssim 1,$$

which shows that

$$\|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})} \lesssim \|u\|_{D(A^{\frac{1-m}{2}})}.$$

This estimate in (54) shows that (53) holds.

We are reduced to show that (53) holds on $\mathcal{S}_v$ for $\eta_v u$. For that purpose, without loss of generality we may assume that v is identified with 0 for all edges of $\mathcal{E}_v$ and that supp $\eta_v \subset \cup_{j\in\mathcal{E}_v}[0, 2l_j/3]$. For shortness write $U = \eta_v u$. Now for any $j \in \mathcal{E}_v$, we introduced the following extension of U_j:

$$E_jU_j(x) = \begin{cases} U_j(x) & \text{if } x \in (0, l_j),\\ \sum_{i=0}^{n-1} \nu_i U_j(-2^i x) & \text{if } x \in (-2^{n-1}l_j, 0),\end{cases}$$

where U is extended by zero outside its support and the real numbers ν_i are the unique solution of the system

$$\begin{cases} \sum_{i=0}^{n-1} \nu_i = 1\\ -\sum_{i=0}^{n-1} 2^i\nu_i = 1\\ \sum_{i=0}^{n-1} 2^{2i}\nu_i = 1\\ \sum_{i=0}^{n-1} 2^{-2ki}\nu_i = 1, \forall k = 1, \cdots, n-3,\end{cases}$$

and finally $n = \frac{m+5}{2}$ if m is odd and $n = \frac{m+4}{2}$ if m is even.

With the help of these extension operators, we obtain an extension of $U \in X_v$ to a function EU, which belongs to $D(\tilde{A}_v)$ (due to the three first properties of the ν_i), where $\tilde{A}_v$ is the positive Laplace operator on the star shaped network $\tilde{\mathcal{S}}_v = \cup_{j\in\mathcal{E}_v}(0, l_j) \bigcup \cup_{j\in\mathcal{E}_v}(-2^{n-1}l_j, 0)$, with interior vertex v and Dirichlet boundary conditions at all other vertices. Applying Lemma 6.3 to EU on the network $\tilde{\mathcal{S}}_v$, we may write

$$\|EU\|^{m+1}_{D(\tilde{A}_v^{\frac{1}{2}})} \lesssim \|EU\|^m_{D(\tilde{A}_v)} \|EU\|_{D(\tilde{A}_v^{\frac{1-m}{2}})}.$$

Now using the fact that the norm in $D(\tilde{A}_v^{\frac{1}{2}})$ is equivalent to the H^1 semi-norm and since EU is equal to U on $\mathcal{S}_v$, we obtain

$$\|U\|^{m+1}_{D(A_v^{\frac{1}{2}})} \lesssim \|EU\|^m_{D(\tilde{A}_v)} \|EU\|_{D(\tilde{A}_v^{\frac{1-m}{2}})}. \tag{55}$$

This means that it remains to estimate the right-hand side of this estimate. First by construction, we easily check that

$$\|EU\|_{D(\tilde{A}_v)} \lesssim \|U\|_{X_v}. \tag{56}$$

As before to estimate the second factor, we use a duality argument. First we remark that for $w \in D(\tilde{A}_v^{\frac{m-1}{2}})$, we have

$$\int_{\tilde{\mathcal{S}}_v} EUw = \sum_{j\in\mathcal{E}_v} \int_0^{l_j} U_j(x)w_j(x)\,dx + \sum_{j\in\mathcal{E}_v} \int_{-2^{n-1}l_j}^0 (Eu)_{-j}(x)w_{-j}(x)\,dx,$$

where for shortness we write w_{-j} the restriction of w to the edge $(-2^{n-1}l_j, 0)$. By changes of variables, we obtain

$$\int_{\tilde{\mathcal{S}}_v} Euw = \sum_{j\in\mathcal{E}_v} \int_0^{l_j} U_j(x)(Fw)_j(x)\,dx,$$

where

$$(Fw)_j(x) = w_j(x) + \chi_j(x)\sum_{i=0}^{n-1} \nu_i 2^{-i} U_j(-2^{-i}x), \forall x \in (0, l_j),$$

the cut-off function χ_j being fixed such that $\chi_j \equiv 1$ on $[0, 2l_j/3]$ and $\chi_j \equiv 0$ on $[5l_j/6, l_j]$ (reminding that $U_j(x) \equiv 0$ for $x > 2l_j/3$). Now we notice that the conditions on ν_i guarantees that Fw belongs to $D(A_v^{\frac{m-1}{2}})$ and by Leibniz's rule we have

$$\|Fw\|_{D(A_v^{\frac{m-1}{2}})} \lesssim \|w\|_{D(\tilde{A}_v^{\frac{m-1}{2}})}.$$

By duality we conclude that

$$\|EU\|_{D(\tilde{A}_v^{\frac{1-m}{2}})} \lesssim \|U\|_{D(A_v^{\frac{1-m}{2}})}.$$

This estimate and (56) in (55) show the requested estimate (53) on $\mathcal{S}_v$ for $\eta_v u$. □

Now let $(u^{(0)}, u^{(1)}, z) \in D(\mathcal{A})$ be fixed (and different from 0). By a convexity inequality and since $V \times L^2(\mathcal{R}) = D(A^{\frac{1}{2}}) \times D(A^0)$, we have

$$\left\|(u^{(0)}, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^{m+1} \lesssim \left\|u^{(0)}\right\|_{D(A^{\frac{1}{2}})}^{m+1} + \left\|u^{(1)}\right\|_{D(A^0)}^{m+1}.$$

Using Lemma 6.3 and Corollary 6.4, we get

$$\begin{aligned}
&\|(u^{(0)}, u^{(1)})\|_{V\times L^2(\mathcal{R})}^{m+1} \lesssim \|u^{(0)}\|_X^m \|u^{(0)}\|_{D(A^{\frac{1-m}{2}})} + \|u^{(1)}\|_{D(A^{\frac{1}{2}})}^m \|u^{(0)}\|_{D(A^{\frac{-m}{2}})} \\
&\lesssim (\|u^{(0)}\|_X^m + \|u^{(1)}\|_{D(A^{\frac{1}{2}})}^m)(\|u^{(0)}\|_{D(A^{\frac{1-m}{2}})} + \|u^{(1)}\|_{D(A^{\frac{-m}{2}})}) \\
&\lesssim (\|u^{(0)}\|_X + \|u^{(1)}\|_{D(A^{\frac{1}{2}})})^m(\|u^{(0)}\|_{D(A^{\frac{1-m}{2}})} + \|u^{(1)}\|_{D(A^{\frac{-m}{2}})}) \\
&\lesssim \|(u^{(0)}, u^{(1)})\|_{X\times D(A^{\frac{1}{2}})}^m \|(u^{(0)}, u^{(1)})\|_{D(A^{\frac{1-m}{2}})\times D(A^{-\frac{m}{2}})} \\
&\lesssim \|(u^{(0)}, u^{(1)})\|_{X\times D(A^{\frac{1}{2}})}^m \|(u^{(0)}, u^{(1)})\|_{X_{-m}}.
\end{aligned}$$

This inequality is equivalent to

$$\left\|(u^{(0)}, u^{(1)})\right\|_{X_{-m}}^2 \gtrsim \frac{\|(u^{(0)}, u^{(1)})\|_{V\times L^2(\mathcal{R})}^{2m+2}}{\|(u^{(0)}, u^{(1)})\|_{X\times D(A^{\frac{1}{2}})}^{2m}}.$$

Using the trivial inequality

$$\begin{aligned}
\|(u^{(0)}, u^{(1)})\|_{X\times D(A^{\frac{1}{2}})} &\leq \|(u^{(0)}, u^{(1)}, z)\|_{X\times D(A^{\frac{1}{2}})\times H^1(0,1)^{V_c}} \\
&\lesssim \|(u^{(0)}, u^{(1)}, z)\|_{D(\mathcal{A})},
\end{aligned}$$

we finally obtain

$$\left\|(u^{(0)}, u^{(1)})\right\|_{X_{-m}}^2 \gtrsim \frac{\|(u^{(0)}, u^{(1)})\|_{V\times L^2(\mathcal{R})}^{2m+2}}{\|(u^{(0)}, u^{(1)}, z)\|_{D(\mathcal{A})}^{2m}}. \tag{57}$$

6.2. **Polynomial decay of the energy.** The proof of our stability result requires the next technical Lemma proved in Lemma 5.2 of [8].

Lemma 6.5. *Let $(\varepsilon_k)_k$ be a sequence of positive real numbers satisfying*

$$\varepsilon_{k+1} \leq \varepsilon_k - C\varepsilon_{k+1}^{2+\alpha}, \forall k \geq 0, \tag{58}$$

where $C > 0$ and $\alpha > -1$. Then there exists a positive constant M (depending on α and C) such that

$$\varepsilon_k \leq \frac{M}{(1+k)^{\frac{1}{1+\alpha}}}, \forall k \geq 0.$$

Theorem 6.6. *Let u be a solution of (1) with $(u^{(0)}, u^{(1)}, (f^0(-\tau_v \cdot))) \in D(\mathcal{A})$. If (49) holds, then the energy decays polynomially, i.e.,*

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{m}}} \left\| (u^{(0)}, u^{(1)}, (f^0(-\tau_v \cdot))) \right\|_{D(\mathcal{A})}^2, \forall t \geq 0, \tag{59}$$

for some $C > 0$.

Proof. Introduce the modified energy

$$\tilde{E}(t) = \frac{1}{2}\|U(t)\|_{D(\mathcal{A})}^2 = \frac{1}{2}(\|U(t)\|_H^2 + \|\mathcal{A}U(t)\|_H^2).$$

As in Proposition 3.1, this energy is decaying.

Combining the estimates (52) and (57), we obtain

$$E(T) \leq E(0) - K_2 \left(\frac{\left\| (u^{(0)}, u^{(1)}) \right\|_{V\times L^2(\mathcal{R})}^{2m+2}}{\tilde{E}(0)^m} + \sum_{v\in\mathcal{V}_c} \int_0^1 (\frac{\partial u}{\partial t}(v, -\tau_v \rho))^2 d\rho \right),$$

for some $K_2 > 0$, or equivalently

$$E(T) \leq E(0) - K_2 \left(\frac{\left\| (u^{(0)}, u^{(1)}) \right\|_{V\times L^2(\mathcal{R})}^{2m+2}}{\tilde{E}(0)^m} + \left\| (f_v^0(-\tau_v .)) \right\|_{L^2(0,1)^{V_c}}^2 \right).$$

Using the trivial estimate

$$\begin{aligned} \left\| (f_v^0(-\tau_v .)) \right\|_{L^2(0,1)^{V_c}}^{2m+2} &\leq \left\| (f_v^0(-\tau_v .)) \right\|_{L^2(0,1)^{V_c}}^2 \left\| (f_v^0(-\tau_v .)) \right\|_{H^1(0,1)^{V_c}}^{2m} \\ &\lesssim \left\| (f_v^0(-\tau_v .)) \right\|_{L^2(0,1)^{V_c}}^2 \ \tilde{E}(0)^m, \end{aligned}$$

the above inequality becomes

$$\begin{aligned} E(T) &\leq E(0) - K_2 \frac{\left\| (u^{(0)}, u^{(1)}) \right\|_{V\times L^2(\mathcal{R})}^{2m+2} + \left\| (f_v^0(-\tau_v .)) \right\|_{L^2(0,1)^{V_c}}^{2m+2}}{\tilde{E}(0)^m} \\ &\leq E(0) - K_3 \frac{E(0)^{m+1}}{\tilde{E}(0)^m}, \end{aligned}$$

with $K_3 > 0$. Since the energy of our system is decaying, we obtain

$$E(T) \leq E(0) - K_3 \frac{E(T)^{m+1}}{\tilde{E}(0)^m}. \tag{60}$$

We now follow the method used in [8]. The estimate (60) being valid on the intervals $[kT, (k+1)T]$, for any $k \geq 0$, we have

$$E((k+1)T) \leq E(kT) - K_3 \frac{E((k+1)T)^{m+1}}{\tilde{E}(kT)^m}. \tag{61}$$

Setting

$$\varepsilon_k = \frac{E(kT)}{\tilde{E}(0)}.$$

By dividing (61) by $\tilde{E}(0)$, we obtain

$$\varepsilon_{k+1} \le \varepsilon_k - K_3 \varepsilon_{k+1}^{m+1}, \tag{62}$$

because $\tilde{E}(kT) \le \tilde{E}(0)$. By Lemma 6.5 with $\alpha = m - 1 > -1$ (as $m > 0$), there exists a constant $M > 0$ such that

$$\varepsilon_k \le \frac{M}{(1+k)^{\frac{1}{m}}}, \ \forall k \ge 0,$$

or equivalently

$$E(kT) \le \frac{M}{(1+k)^{\frac{1}{m}}} \tilde{E}(0).$$

This estimate and again the decay of the energy lead to the estimate (59). □

7. **Examples.** Our aim is to give some concrete examples that illustrate our stability results.

7.1. **One string with an interior damping.** We consider a homogeneous string of length π with an interior damping at ξ. Two types of boundary conditions will be considered.

7.1.1. *Mixed boundary conditions.* We study the following problem (see Fig. 1)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & 0 < x < \pi, \, t > 0, \\ \frac{\partial u}{\partial x}(\xi_-, t) - \frac{\partial u}{\partial x}(\xi_+, t) = -(\alpha_1 \frac{\partial u}{\partial t}(\xi, t) + \alpha_2 \frac{\partial u}{\partial t}(\xi, t - \tau)) & t > 0, \\ u(0, t) = 0, \, \frac{\partial u}{\partial x}(\pi, t) = 0 & t > 0, \\ u(t = 0) = u^{(0)}, \, \frac{\partial u}{\partial t}(t = 0) = u^{(1)} & 0 < x < \pi, \\ \frac{\partial u}{\partial t}(\xi, t - \tau) = f^0(t - \tau) & 0 < t < \tau. \end{cases} \tag{63}$$

Here contrary to subsection 3.4, we suppose that $\alpha_2 < \alpha_1$.

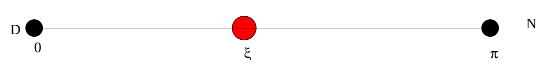

Figure 1.

It is well known that the eigenvectors associated with problem (63) without damping are $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin((k + \frac{1}{2})x)$ of eigenvalue $(k + \frac{1}{2})^2$, $k \ge 0$ of multiplicity 1. We then have to look at

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 = \frac{2}{\pi} \sin^2((k + \frac{1}{2})\xi).$$

For the exponential decay we will use the next result proved in Lemma 2.9 of [27].

Lemma 7.1. *s is a rational number with an irreducible fraction*

$$s = \frac{p}{q}, \quad \textit{where } p \textit{ is odd} \tag{64}$$

if and only if there exists $\alpha > 0$ such that

$$\left| \sin((\frac{\pi}{2} + k\pi)s) \right| > \alpha, \forall k \in \mathbb{N}. \tag{65}$$

Applying Proposition 3.7 and Theorem 5.3 (and the above Lemma), we obtain

Theorem 7.2. *1) The energy of system (63) tends to 0 for all initial data in H if and only if*

$$\frac{\xi}{\pi} \neq \frac{2p}{2q+1}, \ \forall p,\, q \in \mathbb{N}.$$

2) The system (63) is exponentially stable in the energy space if and only if $\frac{\xi}{\pi}$ is a rational number with an irreducible fraction

$$\frac{\xi}{\pi} = \frac{p}{q}, \quad \text{where } p \text{ is odd.}$$

If we consider the system (63) without delay (i. e. $\alpha_2 = 0$), we find the results of [4] obtained by a similar method.

7.1.2. *Dirichlet boundary conditions.* We here consider the problem (see Fig. 2)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & 0 < x < \pi,\, t > 0, \\ \frac{\partial u}{\partial x}(\xi_-,\, t) - \frac{\partial u}{\partial x}(\xi_+,\, t) = -(\alpha_1 \frac{\partial u}{\partial t}(\xi,\, t) + \alpha_2 \frac{\partial u}{\partial t}(\xi,\, t-\tau)) & t > 0, \\ u(0,\, t) = 0,\, u(\pi,\, t) = 0 & t > 0, \\ u(t=0) = u^{(0)},\, \frac{\partial u}{\partial t}(t=0) = u^{(1)} & 0 < x < \pi, \\ \frac{\partial u}{\partial t}(\xi,\, t-\tau) = f^0(t-\tau) & 0 < t < \tau. \end{cases} \tag{66}$$

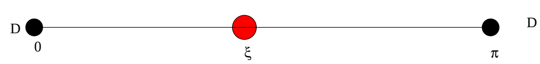

FIGURE 2.

The eigenvectors of problem without damping associated with problem (66) are $\varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$ of eigenvalue k^2, $k \geq 1$ of multiplicity 1. We then have to consider

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 = \frac{2}{\pi}\sin^2(k\xi).$$

Denote by $\mathcal{S}$ the set of all real numbers ρ such that $\rho \notin \mathbb{Q}$ and if $[0,\, a_1, ..., \, a_n, ...]$ is the expansion of ρ as a continued fraction, then the sequence (a_n) is bounded. It is well known that $\mathcal{S}$ is uncountable and that its Lebesgue measure is zero. Roughly speaking, the set $\mathcal{S}$ contains all irrationnel numbers which are badly approximated by rational numbers. In particular, by the Euler-Lagrange theorem, $\mathcal{S}$ contains all irrational quadratic numbers (i. e. the roots of a second order equation with rational coefficients). By a classical result, we have the

Lemma 7.3. *If $s \in \mathcal{S}$, then there exists a positive constant C such that*

$$|\sin(k\pi s)| \geq \frac{C}{k}, \ \forall k \geq 1.$$

By Proposition 3.7 and Theorem 6.6, we then obtain

Theorem 7.4. *1) The energy of system (66) decays to 0 for all initial data in H if and only if*

$$\frac{\xi}{\pi} \notin \mathbb{Q}.$$

2) If $\frac{\xi}{\pi} \in \mathcal{S}$, then for all initial data in $D(\mathcal{A})$, the energy of system (66) decays polynomially like $1/t$.
3) The system (66) is not exponentially stable in the energy space.

Without delay, we find the results of [9].

To prove 3), it suffices to notice that for all ξ, there exists a sequence of natural numbers (q_m) such that $q_m \to \infty$ and

$$|\sin(q_m \xi)| \leq \frac{c}{q_m}, \, \forall m \geq 1.$$

Consequently, there does not exist a positive constant C such that

$$|\sin(k\xi)| > C, \, \forall k \geq 1.$$

Remark 7.5. These two examples show that the boundary conditions influence the stability of the system because we do not have the same hypotheses for the decay to 0 of the energy; moreover for mixed boundary conditions, we may have an exponential stability, while for Dirichlet boundary conditions, we cannot have an exponential stability but have a polynomial stability.

7.2. **A star shaped network.**

7.2.1. *Dirichlet boundary conditions at all extremities.* We take Dirichlet boundary conditions at all extremities and put a damping at the interior node (see Fig. 3).

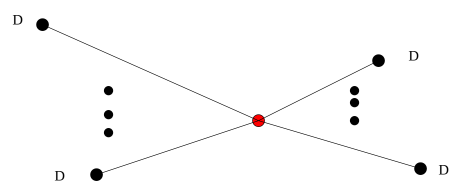

FIGURE 3.

In appropriated coordinates, any eigenvector of the problem without damping is of the form

$$\varphi_j(x) = a_j \sin(\lambda x), \quad 1 \leq j \leq N,$$

for some constants a_j. The transmission conditions at the interior node then lead to

$$a_1 \sin(\lambda l_1) = ... = a_N \sin(\lambda l_N), \tag{67}$$

$$\sum_{j=1}^{N} a_j \cos(\lambda l_j) = 0. \tag{68}$$

We now suppose that

$$\forall i, \, j \in \{1, ..., N\}, \, i \neq j, \, \frac{l_i}{l_j} \notin \mathbb{Q}. \tag{69}$$

In that case we cannot have $\sin(\lambda l_i) = \sin(\lambda l_j) = 0$ for $i \neq j$. Indeed if $\lambda l_i = p\pi$, $p \in \mathbb{Q}$ and $\lambda l_j = q\pi$, $q \in \mathbb{Q}$, then $\frac{l_i}{l_j} = \frac{p}{q} \in \mathbb{Q}$, which contradicts our assumption.

Therefore, there exists $j \in \{1, ..., N\}$, say $j = 1$ such that $\sin(\lambda l_j) \neq 0$. But then $a_1 \neq 0$. Indeed if $a_1 = 0$, then

$$a_2 \sin(\lambda l_2) = ... = a_N \sin(\lambda l_N) = 0.$$

As we cannot have $\sin(\lambda l_i) = \sin(\lambda l_j) = 0$ for $i \neq j$, all a_j are equal to zero except one, say a_N for example and then $\sin(\lambda l_N) = 0$. By the last transmission condition (68) we would have $a_N \cos(\lambda l_N) = 0$ and then $a_N = 0$, which is impossible.

Under the assumption (69), we have

$$\sin(\lambda l_j) \neq 0, \forall j = 1, \cdots, N,$$

and the transmission condition (68) yields the characteristic equation

$$\sum_{j=1}^{N} \cot(\lambda l_j) = 0. \tag{70}$$

From this equation, we deduce that all eigenvalues are simple. Hence

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 = a_1^2 \sin^2(\lambda l_1) > 0,$$

when φ_k is the eigenvector associated with λ and by Proposition 3.7 we directly conclude the

Proposition 7.6. *If (69) holds, then the energy of our system tends to 0 for all initial data in H.*

Without delay we find the result of Proposition 2.1 of [5].

After calculations and normalization, we find that

$$a_1^2 \sin^2(\lambda l_1) = \frac{2}{\sum\limits_{j=1}^{N} \frac{l_j}{\sin^2(\lambda l_j)}}.$$

In its full generality, it is difficult to find the behavior of $a_1^2 \sin^2(\lambda l_1)$ from the characteristic equation (70). We therefore restrict ourselves to some particular cases.

We first suppose that the lengths of the edges are equal to 1. Then easy calculations allow to show that the set of eigenvalues of the operator A is made of two sequences: First $\lambda_k^2 = (\frac{\pi}{2} + k\pi)^2, k \in \mathbb{N}$ is an eigenvalue of multiplicity 1 with associated eigenvector

$$(\varphi_k)_j(x) = \sqrt{\frac{1}{2}} \sin((\frac{\pi}{2} + k\pi)x), \, \forall j \in \{1, ..., N\}.$$

Secondly, $\lambda_k^2 = k^2\pi^2, k \in \mathbb{N}^*$ is of multiplicity $N-1$ with orthonormal eigenvectors given by

$$\varphi_{k,1} = \sin(k\pi x) \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \, \varphi_{k,2} = \frac{2}{\sqrt{3}} \sin(k\pi x) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \cdots,$$

$$\varphi_{k,\,i} = \frac{\sqrt{2i}}{\sqrt{1+i}} \sin(k\pi x) \begin{pmatrix} \frac{1}{i} \\ \frac{1}{i} \\ \vdots \\ \frac{1}{i} \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \varphi_{k,\,N-1} = \frac{\sqrt{2(N-1)}}{\sqrt{N}} \sin(k\pi x) \begin{pmatrix} \frac{1}{N-1} \\ \frac{1}{N-1} \\ \vdots \\ \frac{1}{N-1} \\ -1 \end{pmatrix}.$$

where for shortness we write $\varphi_{k,1}$ as a vector with N components, the j^{th} component corresponding to the restriction of $\varphi_{k,1}$ to the edge j.

If the feedback law is applied only at the interior node, as for the eigenvalue $\lambda_k^2 = k^2\pi^2$, $\mathcal{M}_{v_{int}}(k^2\pi^2) = 0$ (because $\sin(k\pi) = 0$), the eigenvalues of $\mathcal{M}_{v_{int}}(k^2\pi^2)$ are 0. Therefore Proposition 3.7 yields

Proposition 7.7. *If the lengths of the star shaped network are one and the feedback law is applied at the interior node, the energy does not decay to 0.*

We then need to add some interior controls to stabilize the system. On each edge e_j except one (for instance the first one), we put a control at ξ_j (see Fig. 4).

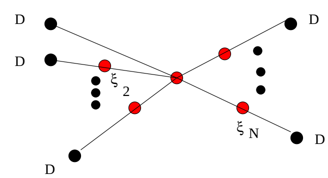

FIGURE 4.

Then for the eigenvalue $\lambda_k^2 = k^2\pi^2$, we readily check that

$$\mathcal{M}(k^2\pi^2) = \mathcal{M}_{v_{int}}(k^2\pi^2) + \sum_{j=2}^{N} \mathcal{M}_{\xi_j}(k^2\pi^2)$$

satifies

$$\eta^\top \mathcal{M}(k^2\pi^2)\eta = \sum_{j=2}^{N} \sin^2(k\pi\xi_j)(\chi_j^\top \eta)^2,$$

where the vectors $\chi_j = ((\chi_j)_i)_{i=2}^N$, $j = 2, \ldots, N$ are non zero vectors, which are independent of k and of $\xi_j, j = 2, \ldots, N$ and satisfy

$$(\chi_j)_i = 0, \forall i < j.$$

Consequently if for all $j \in \{2, ..., N\}$, $\xi_j \in \mathcal{S}$, then $\sin^2(k\pi\xi_j) \gtrsim \frac{1}{k^2}$ and therefore

$$\eta^\top \mathcal{M}(k^2\pi^2)\eta \gtrsim \frac{1}{k^2} \sum_{j=2}^{N} (\chi_j^\top \eta)^2.$$

The above properties of χ_j imply that $(\sum_{j=2}^N (\chi_j^\top \eta)^2)^{1/2}$ is a norm on $\mathbb{R}^{N-1}$ and therefore

$$\eta^\top \mathcal{M}(k^2\pi^2)\eta \gtrsim \frac{1}{k^2} \|\eta\|_2^2,$$

or equivalently $\lambda_{min}(\mathcal{M}(k^2\pi^2)) \gtrsim \frac{1}{k^2}$.

On the other hand for $\lambda_k = \frac{\pi}{2} + k\pi$, we see that

$$\sum_{v\in\mathcal{V}_c} |\varphi_k(v)|^2 = \frac{1}{2}\sin^2(\frac{\pi}{2}+k\pi) + \frac{1}{2}\sum_{j=2}^{N}\sin^2((\frac{\pi}{2}+k\pi)\xi_j) \geq \frac{1}{2}.$$

We then conclude the

Proposition 7.8. *If the lengths of the star shaped network are one and the feedback law is applied at the interior node, and at $N-1$ interior points ξ_j and if*

$$\xi_j \in \mathcal{S},\ \forall j \in \{2, ..., N\},$$

then for any $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$, the energy of the system decays like $\frac{1}{t}$.

Now we assume that $N = 2n$ is even and that

$$l_i = l, \forall i = 1, \cdots, n,\ l_i = l', \forall i = n+1, \cdots, 2n \text{ and } \frac{l}{l'} \notin \mathbb{Q}. \tag{71}$$

Under this assumption, (70) is equivalent to

$$\cot(\lambda l) + \cot(\lambda l') = 0,$$

and then to

$$\sin(\lambda(l + l')) = 0.$$

This means that a first set of eigenvalues is given by

$$\lambda_k = \frac{k\pi}{l+l'}, k \in \mathbb{N}^*.$$

A second set is made of the roots of $\sin(\lambda l) = 0$. Since $\frac{l}{l'} \notin \mathbb{Q}$, we deduce that $a_i = 0$, for all $i = n+1, \cdots, 2n$. Consequently for all $k \in \mathbb{N}^*$, $\frac{k^2\pi^2}{l^2}$ is of multiplicity $n-1$, its associated orthonormal eigenvectors being given by:

$$\begin{aligned}
\varphi_{k,1}(x) &= \frac{1}{\sqrt{l}}\sin(\frac{k\pi x}{l})\,(1,-1,0,\cdots,0,0,\cdots,0)^\top,\\
\varphi_{k,2}(x) &= \frac{2}{\sqrt{3l}}\sin(\frac{k\pi x}{l})\left(\frac{1}{2},\frac{1}{2},-1,0,\cdots,0,0,\cdots,0\right)^\top,\\
&\vdots\\
\varphi_{k,n-1}(x) &= \frac{\sqrt{2(n-1)}}{\sqrt{nl}}\sin(\frac{k\pi x}{l})\left(\frac{1}{n-1},\frac{1}{n-1},\cdots,\frac{1}{n-1},-1,0,\cdots,0\right)^\top.
\end{aligned}$$

By symmetry a third set of eigenvalues is made of the numbers $\frac{k^2\pi^2}{(l')^2}$ of multiplicity $n-1$ for all $k \in \mathbb{N}^*$.

Note that for this example, the standard gap condition holds.

Since the eigenvectors associated with the eigenvalues of the second and third sets are zero at the interior node, if we impose a damping only at this interior node, the energy will not tend to zero. Therefore some interior control points have to be added. More precisely we impose a feedback law at some points ξ_i of the edge e_i, for $i = 2, \cdots, n$ and for $i = n+2, \cdots, 2n$. By direct calculations, the matrix

$$\mathcal{M}(\frac{k^2\pi^2}{l^2}) = \mathcal{M}_{v_{int}}(\frac{k^2\pi^2}{l^2}) + \sum_{j=2}^{n}\mathcal{M}_{\xi_j}(\frac{k^2\pi^2}{l^2}) + \sum_{j=n+2}^{2n}\mathcal{M}_{\xi_j}(\frac{k^2\pi^2}{l^2})$$

satisfies

$$\eta^\top \mathcal{M}(\frac{k^2\pi^2}{l^2})\eta = l^{-1}\sum_{j=2}^{n}\sin^2(\frac{k\pi\xi_j}{l})(\chi_j^\top\eta)^2,$$

where the vectors $\chi_j = ((\chi_j)_i)_{i=2}^n$, $j = 2, \ldots, n$ satisfy the same property than before. Hence if for all $j \in \{2, ..., n\}$, $\frac{\xi_j}{l} \in \mathcal{S}$, then $\sin^2(\frac{k\pi\xi_j}{l}) \gtrsim \frac{1}{k^2}$ and therefore $\lambda_{min}(\mathcal{M}(\frac{k^2\pi^2}{l^2})) \gtrsim \frac{1}{k^2}$.

By symmetry if for all $j \in \{n+2, ..., 2n\}$, $\frac{\xi_j}{l'} \in \mathcal{S}$, then the matrix $\mathcal{M}(\frac{k^2\pi^2}{(l')^2})$ satisfies $\lambda_{min}(\mathcal{M}(\frac{k^2\pi^2}{(l')^2})) \gtrsim \frac{1}{k^2}$.

Finally for the eigenvalue $\lambda_k^2 = \frac{k^2\pi^2}{(l+l')^2}, k \in \mathbb{N}^*$, as we have $\sin(\lambda_k l') = (-1)^{k+1}$ $\sin(\lambda_k l)$, we deduce that

$$a_1^2\sin^2(\lambda_k l_1) = \frac{2\sin^2(\lambda_k l)}{n(l+l')} = \frac{2}{n(l+l')}\sin^2(\frac{k\pi l}{l+l'}).$$

By Lemma 7.3 if $\frac{l}{l+l'} \in \mathcal{S}$, then its associated matrix $\mathcal{M}(\frac{k^2\pi^2}{(l+l')^2})$ satisfies $\lambda_{min}(\mathcal{M}$ $(\frac{k^2\pi^2}{(l+l')^2})) \gtrsim \frac{1}{k^2}$.

Theorem 6.6 then leads to the

Theorem 7.9. *Consider a star shaped network with Dirichlet boundary condition satisfying (71) and feedback laws at the interior node as well as at the point ξ_i of the edge e_i, for $i = 2, \cdots, n$ and for $i = n+2, \cdots, 2n$. If $\frac{\xi_i}{l} \in \mathcal{S}$, for all $i = 2, \cdots, n$, $\frac{\xi_i}{l'} \in \mathcal{S}$, for all $i = n+2, \cdots, 2n$ and $\frac{l}{l+l'} \in \mathcal{S}$, then for all $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$, the energy of our system decays polynomially like $\frac{1}{t}$.*

Note that the two previous examples do not enter into the setting of [5].

7.2.2. *Mixed boundary conditions.* We suppose that the star shaped network is made of 4 edges, with e_1 and e_4 of same length l, and e_2 and e_3 of same length l'. At the extremities of e_1 and e_2, we impose Dirichlet condition and at the extremities of e_3 and e_4, Neumann condition. We control at the interior node. In this case the eigenvalues of the problem without damping are $\frac{k^2\pi^2}{4(l+l')^2}$ of multiplicity 1.

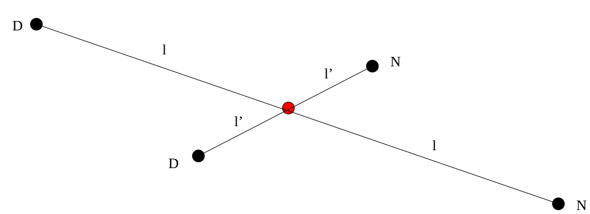

FIGURE 5.

Similar considerations as above allow to show the following results:

Proposition 7.10. *1) If $\frac{l}{l'} \notin \mathbb{Q}$, then the energy of our system tends to 0.*
2) Suppose that $\frac{l}{l'} \notin \mathbb{Q}$ and that $\frac{l}{l+l'} \in \mathcal{S}$ and $\frac{l'}{l+l'} \in \mathcal{S}$, then for all $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$, the energy decays as $\frac{1}{\sqrt{t}}$.

Remark 7.11. For instance if $l = \sqrt{2}$ and $l' = 1 - \sqrt{2}$, then $\frac{l}{l'} \notin \mathbb{Q}$, $\frac{l}{l+l'} \in \mathcal{S}$ and $\frac{l'}{l+l'} \in \mathcal{S}$.

7.3. **More complex networks.** In this subsection, we assume that $\mathcal{R}$ is a network whose edges are all of same length, i.e. $l_j = 1$, for all $j \in \{1, ..., N\}$. In that case, the spectrum of the Laplace operator is explicitly know via the algebraic properties of the network [11, 23]. More precisely, introduce the adjacency matrix $\mathcal{C}$ of the vertices of the network

$$\mathcal{C} = (c_{s,\,t})_{s,\,t\in\mathcal{V}\setminus\mathcal{D}},$$

where

$$\forall s,\, t \in \mathcal{V}\setminus\mathcal{D},\; c_{s,\,t} = \begin{cases} \frac{card(\mathcal{E}_s\cap\mathcal{E}_t)}{\sqrt{card(\mathcal{E}_s)}\sqrt{card(\mathcal{E}_t)}} & \text{if } \mathcal{E}_s\cap\mathcal{E}_t \neq \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

We may now recall the following result proved in [23]:

Theorem 7.12. *Under the above assumption, we have*

$$Sp(A) = S_1 \cup S_2, \text{ where}$$

$S_1 = \{k^2\pi^2$ *with multiplicity* $N - \#(\mathcal{V}\setminus\mathcal{D})$, $k \in \mathbb{N}^*\}$ *of associated eigenvector* $\varphi_{k,\,j}(x) = c_j \sin(k\pi x)$, *for some constants* c_j, *and*
$S_2 = \{\lambda^2 : \cos\lambda \in Sp(\mathcal{C})\cap]-1, 1[\}$. *Moreover* φ *is a eigenvector of* A *associated with* λ^2 *if and only if* $(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}$ *is an eigenvector of the matrix* $\mathcal{C}$ *of eigenvalue* $\cos\lambda$.

Note that this Theorem implies that the standard gap condition holds for networks with edges which are of length one (or more generally rational numbers).

Before going on, let us make a more precise relation between the orthonormal eigenvectors of an eigenvalue λ^2 from S_2 and the eigenvectors of the matrix $\mathcal{C}$ of eigenvalue $\cos\lambda$:

Lemma 7.13. *Let* $\lambda^2 \in S_2$. *There exists a positive definite symmetric matrix* $\mathcal{E}(\lambda) \in \mathbb{R}^{m\times m}$, *where* m *is the cardinal of* $\mathcal{V}\setminus\mathcal{D}$, *such that for all eigenvectors* φ, φ' *associated with* λ^2, *we have*

$$(\varphi, \varphi')_{L^2(\mathcal{R})} = (\varphi'(v))^\top_{v\in\mathcal{V}\setminus\mathcal{D}}\mathcal{E}(\lambda)(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}. \tag{72}$$

Moreover this matrix $\mathcal{E}(\lambda)$ *is uniformly positive definite and uniformly bounded, in the sense that there exists a positive constant* C *(independent of* λ*) such that*

$$\lambda_{min}(\mathcal{E}(\lambda)) \geq C, \quad \|\mathcal{E}(\lambda)\|_2 \leq C^{-1}.$$

Proof. For all $j = 1, \cdots, N$ we may write

$$\varphi_j(x) = a_j \sin(\lambda x) + b_j \sin(\lambda(1-x)), \forall x \in (0,1), \tag{73}$$

where

$$a_j = \frac{\varphi(v)}{\sin\lambda}, b_j = \frac{\varphi(v')}{\sin\lambda}, \tag{74}$$

when v (resp. v') is the vertex corresponding to the extremity 1 (resp. 0) of the edge e_j. We use the same relations for φ' using a'_j and b'_j.

Now by direct calculations, we see that

$$(\varphi, \varphi')_{L^2(\mathcal{R})} \;=\; \frac{1}{2}\sum_{j=1}^{N}(a_j \quad b_j)(\mathcal{B}(\lambda) + \mathcal{B}_r(\lambda)) \begin{pmatrix} a'_j \\ b'_j \end{pmatrix},$$

where the 2×2 matrices $\mathcal{B}(\lambda)$ and $\mathcal{B}_r(\lambda)$ are given by

$$\mathcal{B}(\lambda) = \begin{pmatrix} 1 & -\cos\lambda \\ -\cos\lambda & 1 \end{pmatrix}, \quad \mathcal{B}_r(\lambda) = \frac{1}{2\lambda}\begin{pmatrix} \sin(2\lambda) & 2\sin\lambda \\ 2\sin\lambda & \sin(2\lambda) \end{pmatrix}.$$

This proves the identity (72) due to the relation (74).

Let us now remark that $\mathcal{B}(\lambda)$ depends only on $\cos\lambda$ and then on $\cos\lambda_0$ with $\lambda_0 \in (0,\pi)$, when $\lambda = \lambda_0 + 2k\pi$ or $\lambda = -\lambda_0 + 2(k+1)\pi$, for some $k \in \mathbb{N}$. The eigenvalues of $\mathcal{B}(\lambda)$ being $1 \pm \cos\lambda$, this matrix is uniformly positive definite, i. e., there exists a positive constant C (independent of λ) such that

$$(a \quad b)\mathcal{B}(\lambda)\begin{pmatrix} a \\ b \end{pmatrix} \geq C(a^2+b^2), \forall a, b \in \mathbb{R}.$$

We further remark that the matrix $\mathcal{B}_r(\lambda)$ is a remainder since

$$\|\mathcal{B}_r(\lambda)\|_2 \lesssim \frac{1}{\lambda}.$$

Therefore we introduce the matrix $\mathcal{F}(\lambda_0) \in \mathbb{R}^{m\times m}$ as follows:

$$(\xi_v)^\top_{v\in\mathcal{V}\setminus\mathcal{D}}\mathcal{F}(\lambda_0)(\xi'_v)_{v\in\mathcal{V}\setminus\mathcal{D}} = \frac{1}{2}\sum_{j=1}^{N}(a_j \quad b_j)\mathcal{B}(\lambda)\begin{pmatrix} a'_j \\ b'_j \end{pmatrix}, \quad \forall(\xi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}, (\xi'_v)_{v\in\mathcal{V}\setminus\mathcal{D}} \in \mathbb{R}^m,$$

with the relation

$$a_j = \frac{\xi_v}{\sin\lambda}, b_j = \frac{\xi_{v'}}{\sin\lambda}. \tag{75}$$

From the uniform positiveness of $\mathcal{B}(\lambda)$ and the above relations between a_j, b_j and ξ_v, we directly deduce that $\mathcal{F}(\lambda_0)$ is uniformly positive definite.

The previous considerations clearly show that

$$\mathcal{E}(\lambda) = \mathcal{F}(\lambda_0) + \mathcal{F}_r(\lambda),$$

with

$$\|\mathcal{F}_r(\lambda)\|_2 \lesssim \frac{1}{\lambda}.$$

Therefore there exists $\Lambda > 0$ such that $\mathcal{E}(\lambda)$ is uniformly positive definite, for all $\lambda > \Lambda$.

It remains to consider the case $0 < \lambda \leq \Lambda$. But in that case we see that

$$(\xi_v)^\top_{v\in\mathcal{V}\setminus\mathcal{D}}\mathcal{E}(\lambda)(\xi'_v)_{v\in\mathcal{V}\setminus\mathcal{D}} = (\varphi,\varphi')_{L^2(\mathcal{R})}, \forall(\xi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}, (\xi'_v)_{v\in\mathcal{V}\setminus\mathcal{D}} \in \mathbb{R}^m,$$

when φ is given by (73) when a_j, b_j are defined by (75) (and similarly for φ'). As the above right-hand side is an inner product on $L^2(\mathcal{R})$, the left-hand side is also an inner product in $\mathbb{R}^m$. Hence the matrix $\mathcal{E}(\lambda)$ is positive definite. The uniformness follows from the fact that the interval $(0,\Lambda]$ contains a finite number of λ such that $\lambda^2 \in S_2$.

The uniform boundedness of $\mathcal{E}(\lambda)$ is proved in the same manner. □

Corollary 7.14. *Assume that $\mathcal{V}_c = \mathcal{V}\setminus\mathcal{D}$. Then for any $\lambda^2 \in S_2$, its associated matrix $\mathcal{M}(\lambda^2)$ is uniformly positive definite, i.e.,*

$$\lambda_{min}(\mathcal{M}(\lambda^2)) \gtrsim 1.$$

Proof. Assume that λ^2 is of multiplicity l and denote by $\varphi_i, i = 1,\cdots,l$ the associated orthonormal eigenvectors. Now we introduce the vectors

$$(\tilde{\varphi}_i(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = \mathcal{E}(\lambda)^{1/2}(\varphi_i(v))_{v\in\mathcal{V}\setminus\mathcal{D}}, \forall i = 1,\cdots,l.$$

According to the relation (72) these vectors are orthonormal:

$$(\tilde{\varphi}_i(v))^\top_{v\in\mathcal{V}\setminus\mathcal{D}}(\tilde{\varphi}_j(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = \delta_{ij}, \forall i, j = 1,\cdots,l.$$

Now we simply remark that

$$(\xi_i)^\top \mathcal{M}(\lambda^2)(\xi_i) = \sum_{i,j} \xi_i \xi_j (\varphi_j(v))_v^\top (\varphi_i(v))_v,$$

and consequently

$$\begin{aligned}(\xi_i)^\top \mathcal{M}(\lambda^2)(\xi_i) &= \sum_{i,j} \xi_i \xi_j (\tilde{\varphi}_j(v))_v^\top \, \mathcal{E}(\lambda)^{-1} (\tilde{\varphi}_i(v))_v \\ &= (\tilde{\varphi}(v))_v^\top \mathcal{E}(\lambda)^{-1} (\tilde{\varphi}(v))_v,\end{aligned}$$

where $(\tilde{\varphi}(v))_v = \sum_i \xi_i (\tilde{\varphi}_i(v))_v$. By the uniform boundedness of $\mathcal{E}(\lambda)$, we deduce that

$$\begin{aligned}(\xi_i)^\top \mathcal{M}(\lambda^2)(\xi_i) &\gtrsim (\tilde{\varphi}(v))_v^\top (\tilde{\varphi}(v))_v \\ &\gtrsim \sum_{i,j} \xi_i \xi_j (\tilde{\varphi}_j(v))_v^\top (\tilde{\varphi}_i(v))_v = \sum_i \xi_i^2,\end{aligned}$$

this last identity following from the orthonormality of the vectors $(\tilde{\varphi}_i(v))_v$. □

From this Corollary we see that if we control at least at all nodes of $\mathcal{V} \setminus \mathcal{D}$ then the assumption for the exponential stability (and obviously polynomial stability) holds for all eigenvalues of S_2. In that case we only need to manage the eigenvalues of S_1. Note further that if S_1 is not empty, then some additional interior controls are necessary since the eigenvectors associated with such eigenvalues are zero at the nodes.

Now if we want to control on a subset of $\mathcal{V} \setminus \mathcal{D}$ then the assumption for the exponential stability does not necessary hold for the eigenvalues of S_2. Let us then describe how we proceed in that case. For a fixed $\lambda^2 \in S_2$, we denote by $(\varphi_i^{app}(v))_{v \in \mathcal{V} \setminus \mathcal{D}} \in \mathbb{R}^m$, $i = 1, \cdots, l$, the eigenvectors of $\mathcal{C}$ of eigenvalue $\cos\lambda$ such that their corresponding eigenvectors on the network are approximated orthonormal, namely

$$(\varphi_i^{app}(v))_{v \in \mathcal{V} \setminus \mathcal{D}}^\top \mathcal{F}(\lambda_0) (\varphi_j^{app}(v))_{v \in \mathcal{V} \setminus \mathcal{D}} = \delta_{ij},$$

This basis can be computed for all λ since it depends only on λ_0 (which form a finite set). Denote by $\mathcal{M}^{app}(\lambda_0^2)$, the matrix build as $\mathcal{M}(\lambda^2)$ by replacing $(\varphi_i(v))_{v \in \mathcal{V}_c}$ by $(\varphi_i^{app}(v))_{v \in \mathcal{V}_c}$, namely

$$\mathcal{M}^{app}(\lambda_0^2) = \begin{pmatrix} (\varphi_1^{app}(v))_{v \in \mathcal{V}_c}^\top \\ \vdots \\ (\varphi_l^{app}(v))_{v \in \mathcal{V}_c}^\top \end{pmatrix} \cdot \left((\varphi_1^{app}(v))_{v \in \mathcal{V}_c} \quad \dots \quad (\varphi_l^{app}(v))_{v \in \mathcal{V}_c} \right).$$

Now we can state the

Lemma 7.15. *Let $\lambda^2 \in S_2$ be fixed. If $\mathcal{M}^{app}(\lambda_0^2)$ is positive definite and if $\mathcal{C}$ has no eigenvector associated with $\cos\lambda_0$ identically equal to zero at the nodes of $\mathcal{V}_c$, then $\mathcal{M}(\lambda^2)$ is uniformly positive definite.*

Proof. As before we show that

$$\mathcal{M}(\lambda^2) = \mathcal{M}^{app}(\lambda_0^2) + \mathcal{M}_r(\lambda^2),$$

where

$$\|\mathcal{M}_r(\lambda^2)\|_2 \lesssim \frac{1}{\lambda}.$$

Consequently for λ large enough, the uniform positive definiteness follows from the positive definiteness of $\mathcal{M}^{app}(\lambda_0^2)$. On the contrary for small λ, it suffices to remark that $\mathcal{M}(\lambda)$ has an eigenvalue equal to zero if and only if $\mathcal{C}$ has an eigenvector associated with $\cos\lambda = \cos\lambda_0$ identically equal to zero at the nodes of $\mathcal{V}_c$. Since we have assumed that such eigenvectors do not exist, $\mathcal{M}(\lambda^2)$ is positive definite and the conclusion follows by finiteness. □

Note that our above Lemma has only to be used for multiple eigenvalues. Indeed assume that $\lambda^2 \in S_2$ is simple, then denote by $(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}$ its eigenvector such that

$$(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}^\top \mathcal{E}(\lambda)(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = 1.$$

Now consider any computed eigenvector $(\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}$ of $\mathcal{C}$ that depends only on $\cos\lambda$ and then on λ_0. This eigenvector then satisfies

$$(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = \mu(\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}},$$

with $\mu \in \mathbb{R}$ such that $\mu^2 = \frac{1}{c(\lambda)}$, when

$$c(\lambda) = (\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}^\top \mathcal{E}(\lambda)(\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}.$$

As $c(\lambda)$ remains uniformly bounded from below and from above, it is equivalent to check the uniform definite positiveness of $\mathcal{M}(\lambda^2)$ using $(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}$ or using $(\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}$.

7.3.1. *A first example.* Consider the tree described by figure 6.

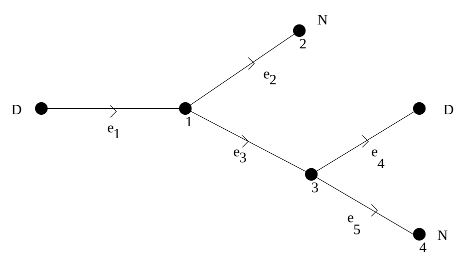

FIGURE 6.

By Theorem 7.12, the eigenvalues of A are $k^2\pi^2$ of multiplicity $5-4=1$, and λ^2 such that $\cos(\lambda) \in Sp(\mathcal{C}) \cap]-1, 1[$.
<u>1st case</u>: We easily check that the eigenvector associated with $k^2\pi^2$ is given by

$$\varphi_k(x) = \sqrt{\frac{2}{3}}\sin(k\pi x)\begin{pmatrix} 1 \\ 0 \\ (-1)^k \\ 1 \\ 0 \end{pmatrix}.$$

As the eigenvector is zero at all nodes of the network, feedbacks at the nodes are not sufficient to stabilize the system. But if we take a control at ξ_1 on the edge e_1 for instance, we have

$$\varphi_1^2(\xi_1) = \frac{2}{3}\sin^2(k\pi\xi_1) \gtrsim \frac{1}{k^2} \text{ for } \xi_1 \in \mathcal{S}.$$

2d case: The eigenvalues of the matrix

$$\mathcal{C} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{3} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

are $-\frac{1}{6}+\frac{1}{6}\sqrt{13}$, $-\frac{1}{6}-\frac{1}{6}\sqrt{13}$, $\frac{1}{6}+\frac{1}{6}\sqrt{13}$, $\frac{1}{6}-\frac{1}{6}\sqrt{13} \in]-1, 1[$ (of multiplicity 1) of associated eigenvectors

$$(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}} \simeq \begin{pmatrix} -0.75 \\ -1 \\ 0.75 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.3 \\ -1 \\ -1.3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.3 \\ 1 \\ 1.3 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.75 \\ 1 \\ -0.75 \\ 1 \end{pmatrix},$$

respectively. Then we can consider a feedback at any node of $\mathcal{V}\setminus\mathcal{D}$, for instance at the node number 4 (see Fig. 7).

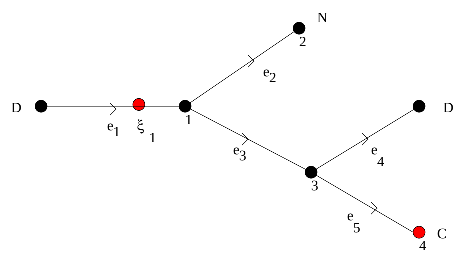

FIGURE 7.

We then have the next result.

Proposition 7.16. *If we control at $\xi_1 \in \mathcal{S}$ on the edge e_1 and at one of the vertices of $\mathcal{V}\setminus\mathcal{D}$, for all $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$, the energy decays like $\frac{1}{t}$. Moreover the system is not exponentially stable in the energy space.*

7.3.2. *A second example.* Consider the tree as described in figure 8.

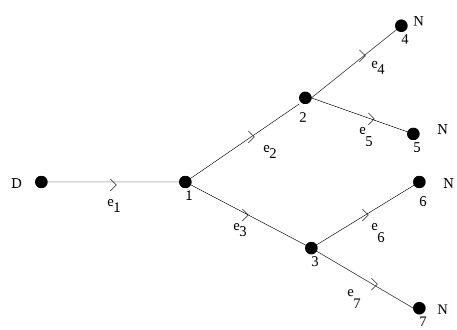

FIGURE 8.

By Theorem 7.12, the operator A has only the eigenvalues λ^2 such that $\cos\lambda \in Sp(\mathcal{C})\cap]-1, 1[$. Therefore by Corollary 7.14, if we control at all nodes except the

Dirichlet one, then we obtain an exponential decay. Without delay we find the result of [29].

Note that it suffices to control at the nodes 5, 6 and 7. Indeed the eigenvalues of the matrix

$$\mathcal{C} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

are approximatively 0.9, -0.9, 0.8, -0.8, 0, 0, $0 \in [-1, 1]$ of eigenvector

$$(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}} \simeq \begin{pmatrix} 1.15 \\ 1.6 \\ 1.6 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.15 \\ -1.6 \\ -1.6 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1.4 \\ 1.4 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.4 \\ -1.4 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1.7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -1.7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

respectively. Therefore if we control at the nodes 5, 6 and 7 the assumption for the exponential stability holds for the simple eigenvalues corresponding to 0.9, -0.9, 0.8, -0.8. Now for $\cos\lambda = 0$, i. e., $\lambda = \frac{\pi}{2} + k\pi$, we see that $\mathcal{C}$ has no eigenvectors which are zero at the nodes 5, 6 and 7. Furthermore we easily check that $\mathcal{F}(\frac{\pi^2}{4})$ is the diagonal matrix with entries $(3/2, 3/2, 3/2, 1/2, 1/2, 1/2, 1/2)$ and then

$$\varphi_1^{app} = \sqrt{2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \varphi_2^{app} = \sqrt{2}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}, \varphi_3^{app} = \sqrt{2}\begin{pmatrix} -2/9 \\ 0 \\ 0 \\ 1/9 \\ 1/9 \\ -7/9 \\ 1 \end{pmatrix}.$$

By direct calculations we obtain

$$\mathcal{M}^{app}(\frac{\pi^2}{4}) = 2\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{9} \\ \frac{1}{2} & \frac{5}{4} & -\frac{13}{18} \\ \frac{1}{9} & -\frac{13}{18} & -\frac{131}{81} \end{pmatrix}.$$

Since this matrix is positive definite, by Lemma 7.15 we deduce the uniform positive definiteness of $\mathcal{M}((\frac{\pi}{2} + k\pi)^2)$.

In other words, we have proved the

Proposition 7.17. *If the feedback law is at the vertices 5, 6 and 7, then the energy decays exponentially in the energy space.*

Note that if we impose Dirichlet boundary conditions at the nodes 4 and 6, then the system is no more exponentially stable.

7.3.3. *A network with a circuit.* Similar considerations than before allow to prove the following result:

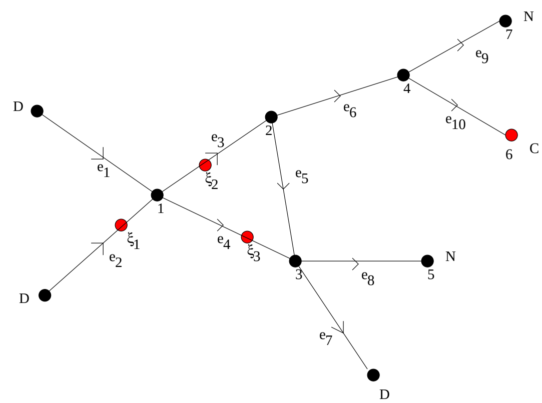

FIGURE 9.

Proposition 7.18. *Consider the network described by figure 9. If the feedback law is at $\xi_1 \in \mathcal{S}$ on the edge e_2, at $\xi_2 \in \mathcal{S}$ on the edge e_3, at $\xi_3 \in \mathcal{S}$ on the edge e_4 and at the vertex 6 or 7, then for all $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$, the energy decays like $\frac{1}{t}$. Moreover the system is not exponentially stable in the energy space.*

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