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THE GREEN'S FUNCTIONS FOR THE BROADWELL MODEL IN A HALF SPACE PROBLEM

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Abstract. We study an initial boundary value problem for the Broadwell model with a supersonic physical boundary. The Green's function for an initial value problem is constructed and its detailed pointwise structure is obtained through the novel decompositions introduced in [8]. With the Green's function for initial value problem and energy estimates together, a new approach to convert a priori L^2-boundary data into L^∞ boundary data is established for the Broadwell model. The Green's function for an initial boundary value problem is obtained. Finally, a nonlinearly time-asymptotic stability of an equilibrium state is proved.

1. **Introduction.** The Boltzmann equation, $f_t + \xi \cdot \nabla_x f = Q(f)$, is a fundamental equation in the rarefied gas theory. Qualitative and quantitative studies on the rarefied gas theory will have many important impacts on the current scientific and industrial communities. For the scientific community, a better understanding on the connection between statistical mechanics and continuum fluid mechanics can be obtained. Since all the semi-conductor device manufactures require high rarefied gas environment, a good research on the rarefied gas may increase the efficiency of production procedure and may be possible to improve technologies also. This would be a good impact on the industrial community.

Most of the interesting phenomena in the rarefied gas are related to the presence of a physical boundary, on which different physical characteristics such as the roughness, the temperature distribution and the geometry will show up. Thus, we are interested in developing quantitative and qualitative theories on initial boundary value problem of the fundamental equation for the rarefied gas, the Boltzmann equation. In particular, one should focus on the structure of the Green's function

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in the sense of "wave propagation". There has been some essential progress in developing the Green's function for initial value problem in [8, 9], but regarding the initial boundary value problem there is no result on the Green's function yet. There are many works on boundary value problems for the Boltzmann equation through formal asymptotic expansions and numerical computations by Sone et. al. [11]. Many interesting phenomena such as thermal transpiration flow, vaporization-condensation problems and the ghost effect have been studied. Analytical studies on the Boltzmann boundary layers have been studied in [1, 4, 5, 12, 14].

We are interested in developing a fundamental approach to study an initial boundary value problem (IBVP) of the kinetic equations in general. Two key components of an IBVP are the construction of the full boundary datum from the imposed datum and the Green's function for the initial value problem. The construction of the Green's function is rather simple due to [8]. The main issue becomes how do we obtain the global boundary datum and find the result of estimates of the global solutions with structures; and the approach should be general so that it can be extended to the Boltzmann equation.

We should start from studying a model equation before considering the initial boundary value problem for the Boltzmann equation. Thus, we consider a simple model equation for Boltzmann equation: the Broadwell model

$$\begin{cases} \partial_t \mathsf{f}_+ + \partial_x \mathsf{f}_+ = \frac{1}{4}\mathsf{f}_0^2 - \mathsf{f}_+\mathsf{f}_-, \\ \partial_t \mathsf{f}_0 = -(\frac{1}{4}\mathsf{f}_0^2 - \mathsf{f}_+\mathsf{f}_-), \\ \partial_t \mathsf{f}_- - \partial_x \mathsf{f}_- = \frac{1}{4}\mathsf{f}_0^2 - \mathsf{f}_+\mathsf{f}_-. \end{cases} \tag{1}$$

This is a model equation for a planar wave solution of Boltzmann equation $\partial_t f + \xi \cdot \nabla_x \mathsf{f} = Q(f)$, [2, 3]. The Boltzmann equation is a kinetic equation for all possible particle velocities $\xi \in \mathbb{R}^3$ with a bilinear integral collision operator, but the Broadwell model only admits three discrete particle velocities which are +1, 0, -1 in x-direction. The state, $(\mathsf{f}_+(x,t), 2\mathsf{f}_0(x,t), \mathsf{f}_-(x,t)) \in \mathbb{R}^3$, gives the velocity density function for particles with velocities +1, 0, -1 at (x,t). Note that in the Boltzmann equation the state $f(x,t,\xi)$ can be identified as a vector in $L^3(\mathbb{R}^3)$, and the multiplication operator ξ can be identified as a diagonal operator in $L^2(\mathbb{R}^3)$. For the Broadwell model, let

$$\mathsf{F}(x,t) \equiv \begin{pmatrix} \mathsf{f}_+(x,t) \\ \mathsf{f}_0(x,t) \\ \mathsf{f}_-(x,t) \end{pmatrix}, \quad \mathsf{V} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q(\mathsf{F}) \equiv \begin{pmatrix} \frac{1}{4}\mathsf{f}_0^2 - \mathsf{f}_+\mathsf{f}_- \\ -(\frac{1}{4}\mathsf{f}_0^2 - \mathsf{f}_+\mathsf{f}_-) \\ \frac{1}{4}\mathsf{f}_0^2 - \mathsf{f}_+\mathsf{f}_- \end{pmatrix}.$$

Then the Broadwell model can be written in a form similar to Boltzmann equation:

$$\partial_t \mathsf{F} + \mathsf{V}\partial_x \mathsf{F} = Q(\mathsf{F}). \tag{2}$$

The equilibrium states for the system (2) are defined by the states F satisfying $Q(\mathsf{F}) \equiv 0$. An absolute equilibrium state is defined by

$$\mathsf{M} \equiv (1/6, 1/3, 1/6)^t.$$

We are interested in the structure of solutions close to the absolute equilibrium state M.

Similar to the collision operator in Boltzmann equation, the Broadwell model also has two collision invariants:

$$\{\psi_0, \psi_1\} \equiv \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}, \text{ i.e. } \psi_i \cdot Q = 0 \text{ for } i = 0, 1. \tag{3}$$

The linearized equation around the absolute equilibrium state M is

$$\partial_t \mathsf{F} + \mathsf{V}\partial_x \mathsf{F} = L\mathsf{F}, \tag{4}$$

with the linearized collision operator L:

$$L \equiv -\frac{1}{6}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Resembling the linear collision operator of the Boltzmann equation, we may decompose L into the form

$$L = -\nu + \mathsf{K} \text{ where } \nu \equiv \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{K} \equiv \begin{pmatrix} 0 & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}. \tag{5}$$

Note that $\{\xi, L\}$ for the Boltzmann equation and $\{\mathsf{V}, L\}$ for the Broadwell model are symmetric operators with respect to the standard Euclidean norm in $L^2(\mathbb{R}^3)$ and $\mathbb{R}^3$. This is an important property to obtain a priori estimates of the linearized equations.

The collision invariants ψ_i span the kernel of the linearized collision operator L:

$$ker\ L = span\{\psi_0, \psi_1\}.$$

Since L is symmetric,

$$ker(L) \perp Range(L) \text{ and } \mathbb{R}^3 = ker(L) \oplus Range(L).$$

Thus, we have the following macro-micro decomposition $(\mathsf{P}_0, \mathsf{P}_1)$ for $\mathbb{R}^3$ similar to the macro-micro decomposition introduced in [8]:

$$\begin{cases} I = \mathsf{P}_0 + \mathsf{P}_1, \quad \mathsf{P}_0 \perp \mathsf{P}_1, \quad \mathsf{P}_0|_{Range(\mathsf{P}_0)} = I|_{Range(\mathsf{P}_0)}, \\[2ex] I \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{P}_0 \equiv \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad \mathsf{P}_1 \equiv \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \end{cases} \tag{6}$$

This macro-micro decomposition $(\mathsf{P}_0, \mathsf{P}_1)$ satisfies that

$$\begin{cases} \mathsf{P}_0 L = L\mathsf{P}_0 = 0, \\ \mathsf{P}_1 L = L = L\mathsf{P}_1. \end{cases} \tag{7}$$

The matrix $\mathsf{P}_0\mathsf{V}\mathsf{P}_0|_{Range(\mathsf{P}_0)}$ has two eigenvalues

$$\{\lambda_1, \lambda_2\} = \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\} \tag{8}$$

which correspond to the speeds of sound in the fluid obtained by Chapman-Enskog expansion.

This paper is interested in studying the initial boundary value problem for (4) with a supersonic physical boundary at $x = bt$ i.e. the speed of the boundary is faster than sound speed $1/\sqrt{3}$; we also assume the speed of boundary is slower than the fastest particle, that is,

$$1 > b > \frac{1}{\sqrt{3}}. \tag{9}$$

The Broadwell model is a hyperbolic system. The assumption $1 > b$ is a natural condition, otherwise no boundary condition can be imposed. The number of imposed boundary conditions is subject to the number of characteristic curves entering the interior domain. Due to (9), one can impose only one boundary condition.

The initial boundary value problem is posed as follows:

$$\begin{cases} \partial_t \mathsf{F} + \mathsf{V}\partial_x \mathsf{F} = L\mathsf{F} \text{ for } x > bt,\ t > 0, \\ \mathsf{f}_+(bt,t) = \alpha_0 \mathsf{f}_0(bt,t) + \alpha_- \mathsf{f}_-(bt,t), \\ \mathsf{F}(x,0) = \mathsf{F}_0(x), \end{cases} \tag{10}$$

where

$$\begin{cases} 2\alpha_0 + \alpha_- = 1, \\ \alpha_0, \alpha_- \geq 0. \end{cases}$$

Remark 1. Here, the condition $2\alpha_0 + \alpha_- = 1$ is imposed so that the equilibrium state M satisfies the boundary condition. The condition $\alpha_0, \alpha_- > 0$ is imposed so that $\mathsf{f}_+(bt,t)$ is positively proportional to $\mathsf{f}_0(bt,t)$ and $\mathsf{f}_-(bt,t)$. This condition is physically reasonable.

Remark 2. When $b > 1/\sqrt{3}$ ($b < -1/\sqrt{3}$), we have a supersonic (subsonic, respectively) physical boundary. In both cases, we can compute a priori estimates of the full boundary data by the weighted energy estimate, thereby obtaining the Green's function. The technique, however, can not be applied to the case $-1/\sqrt{3} < b < 1/\sqrt{3}$ directly. Further investigation is needed.

Remark 3. Regarding the setting of the speed of the boundary $b > 1/\sqrt{3}$, this is a natural condition in the Boltzmann equation as well as in the kinetic equation. Such a setting can be regarded as a supersonic condensation in the condensation-vaporization problem. The analysis in this problem could be applicable to the BGK model and other kinetic modelings.

The Green's function $\mathbb{G}(x,t)$ for the system (4) is a 3×3 matrix valued function which satisfies

$$\begin{cases} \partial_t \mathbb{G} + \mathsf{V}\partial_x \mathbb{G} = L\mathbb{G} \text{ for } x \in \mathbb{R},\quad t > 0 \\ \mathbb{G}(x,0) \equiv \delta(x) I. \end{cases} \tag{11}$$

The Green's function $\mathbb{G}_B(x,t;y,s)$ for the system (10) is also a 3×3 matrix valued function satisfying the forward equation:

$$\begin{cases} \partial_t \mathbb{G}_B(x,t;y,s) + \mathsf{V}\partial_x \mathbb{G}_B(x,t;y,s) - L\mathbb{G}_B(x,t;y,s) = 0 \text{ for } x > bt,\ y > bs,\ t > s, \\ \mathbb{G}_B(x,s;y,s) = \delta(x-y) I, \\ (1, -\alpha_0, -\alpha_-)\mathbb{G}_B(bt,t;y,s) = (0,0,0). \end{cases} \tag{12}$$

Theorem 1. *There exists a positive constant $C > 0$ such that*

$$\left\| \mathbb{G}(x,t) - e^{-t/6} \begin{pmatrix} \delta(x-t) & 0 & 0 \\ 0 & \delta(x) & 0 \\ 0 & 0 & \delta(x+t) \end{pmatrix} \right\| \leq C \left(\frac{e^{-\frac{|x - \frac{1}{\sqrt{3}} t|^2}{C\,(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x + \frac{1}{\sqrt{3}} t|^2}{C\,(1+t)}}}{\sqrt{1+t}} \right) + Ce^{-(|x|+t)/C} \text{ for all } x \in \mathbb{R},\ t > 0. \tag{13}$$

Theorem 2. *There exists a positive constant $C > 0$ such that*

$$\|\mathbb{G}_B(x,t;y,s) - e^{-\frac{t-s}{6}} \begin{pmatrix} \delta(x-y-t+s) & \alpha_0\frac{(1-b)}{b}\,\delta(x+\frac{1-b}{b}y-t) & \alpha_-\frac{(1-b)}{(1+b)}\,\delta(x+\frac{1-b}{1+b}(y+s)-t) \\ 0 & \delta(x-y) & 0 \\ 0 & 0 & \delta(x-y+t-s) \end{pmatrix}\|$$

$$\leq C\left(\frac{e^{-\frac{|x-y-\frac{1}{\sqrt{3}}(t-s)|^2}{C\,(1+t-s)}}}{\sqrt{1+t-s}} + \frac{e^{-\frac{|x-y+\frac{1}{\sqrt{3}}(t-s)|^2}{C\,(1+t-s)}}}{\sqrt{1+t-s}}\right) + Ce^{-\frac{|x-y|+(t-s)}{C}} + C^3e^{-|x-bt|/C}$$

$$\cdot\left(e^{-(|y-bs|+(t-s))/C} + \frac{e^{-\frac{(y-bs-(b-\frac{1}{\sqrt{3}})(t-s))^2}{C(t-s+1)}} + e^{-\frac{(y-bs-(b+\frac{1}{\sqrt{3}})(t-s))^2}{C(t-s+1)}}}{\sqrt{1+t-s}}\right)$$

$$\textit{for } x-bt>0,\, y-bs>0,\, t-s>0. \tag{14}$$

In Section 2, we implement the development of the Green's function in [8] to construct the Green's function for the Broadwell model. The methodology for constructing the Green's function for the Boltzmann equation consists of Long Wave-Short Wave decomposition and Particle-Wave decomposition. It applies to the Broadwell model also. Though there exist works on the Green's function for Broadwell model in terms of special functions, the approach in [8] is conceptually general. It is not only applicable to the Boltzmann equation but also to many other problems such as the Broadwell model and the compressible Navier-Stokes equations. It gives explicitly pointwise estimates in term of elementary functions. Its analysis also reveals some essential physical difference between Broadwell model and the Boltzmann equation. For example, the Boltzmann equation has stronger effective viscosity than the Broadwell model in the sense of gaining regularity.

In Section 3, we use the full boundary data and the Green's function for initial value problem to get a representation of the solution of an initial boundary value problem. Though the out-going boundary data are not imposed by the problem, the global existence of the out-going boundary data can be obtained in the L^2-sense by the weighted energy estimates. Finally, by combining the L^2 estimates of the boundary data and the Green's function, one can obtain strong estimates of the full boundary data in L^∞-sense. After obtaining the strong estimates of the boundary data, one can conclude the strong pointwise estimate of the Green's function in the interior domain for the initial boundary value problem. The problem of the nonlinearly time-asymptotic stability of an equilibrium state becomes a simple corollary. The global L^2 estimates of the boundary data with weight were first introduced in [14] for studying nonlinearly time-asymptotic stability of a Boltzmann boundary layer.

Such an approach to combining the Green's function for initial value problems with energy estimates for boundary data is possible for the Boltzmann equation.

The Green's functions had been effectively used to study nonlinear viscous system [6, 10, 13]. For those viscous problems, the advantage of the Green's function seems to just make the analysis more economic than a priori energy estimates. This is because the solutions of viscous systems would become smooth immediately so that one can use higher energy estimates to close nonlinearity. However, for the Boltzmann initial boundary value problem, the solution may not remain continuous even if the initial data and boundary data are smooth. This can be observed from the numerical studies in [11]. It becomes clear that a priori energy estimates are not suitable for the Boltzmann equation. The Green's functions do not have such limitation on regularity of solutions for nonlinear problem. It turns out that the Green's function is the most hopeful option for studying nonlinear problem for the Boltzmann equation.

2. **Preliminaries.** In this section, we implement the analysis in [8] to construct the Green's function for the Broadwell model.

2.1. **Macro-Micro decomposition and continuum fluid equation.** First, we formally derive the continuum fluid equation from (4) through the macro-micro decomposition given in (6). We can rewrite (4) as follows

$$\begin{cases} \partial_t \mathsf{F}_0 + \mathsf{P}_0 \mathsf{V} \partial_x (\mathsf{F}_0 + \mathsf{F}_1) = 0, \\ \partial_t \mathsf{F}_1 + \mathsf{P}_1 \mathsf{V} \partial_x (\mathsf{F}_0 + \mathsf{F}_1) = L\mathsf{F}_1, \end{cases} \tag{15}$$

where $\mathsf{F} = \mathsf{F}_0 + \mathsf{F}_1$, $(\mathsf{F}_0 \equiv \mathsf{P}_0\mathsf{F},\ \mathsf{F}_1 \equiv \mathsf{P}_1\mathsf{F})$. By substituting $\mathsf{F}_1 = L^{-1}\mathsf{P}_1(\partial_t \mathsf{F}_1 + \mathsf{V}\partial_x(\mathsf{F}_0 + \mathsf{F}_1))$ into the above equations, one can rewrite the system (15) in the form

$$\begin{cases} \partial_t \mathsf{F}_0 + \mathsf{P}_0 \mathsf{V} \partial_x \mathsf{F}_0 + \partial_x^2 \mathsf{P}_0 \mathsf{V} L^{-1} \mathsf{P}_1 \mathsf{V} \mathsf{F}_0 \\ \qquad = -\partial_x (\mathsf{P}_0 \mathsf{V} L^{-1} (\partial_t \mathsf{F}_1 + \mathsf{P}_1 \mathsf{V} \partial_x \mathsf{F}_1)), \\ \partial_t \mathsf{F}_1 + \mathsf{P}_1 \mathsf{V} \partial_x (\mathsf{F}_0 + \mathsf{F}_1) = L\mathsf{F}_1. \end{cases} \tag{16}$$

The first equation in (16) can be treated as the linearized Navier-Stokes equations with an inhomogeneous source $-\partial_x(\mathsf{P}_0\mathsf{V}L^{-1}(\partial_t\mathsf{F}_1 + \mathsf{P}_1\mathsf{V}\partial_x\mathsf{F}_1))$:

$$\begin{cases} a_t + b_x = 0, \\ b_t + \dfrac{1}{3} a_x - \dfrac{4}{3}\partial_x^2 b = \partial_x \psi_1 \cdot (\mathsf{P}_0 \mathsf{V} L^{-1}(\partial_t \mathsf{F}_1 + \mathsf{P}_1 \mathsf{V} \partial_x \mathsf{F}_1)), \end{cases}$$

where $a = \psi_0 \cdot \mathsf{F}$ and $b = \psi_1 \cdot \mathsf{F}$. The hyperbolic system, $\partial_t + \begin{pmatrix} 0 & 1 \\ 1/3 & 0 \end{pmatrix} \partial_x$, in the above defines the speeds of sound $\pm 1/\sqrt{3}$.

2.2. **Spectrum Property.** We consider the Fourier transformation of (4) in the x-variable

$$\partial_t \hat{\mathsf{F}} + i\eta \mathsf{V} \hat{\mathsf{F}} = L\hat{\mathsf{F}}. \tag{17}$$

This results in that

$$\hat{\mathsf{F}}(\eta, t) = e^{(-i\eta\mathsf{V} + L)t} \hat{\mathsf{F}}(\eta, 0). \tag{18}$$

Then, the operator $e^{(-i\mathsf{V}\eta + L)t}$ can be expressed as

$$e^{(-i\mathsf{V}\eta + L)t} = \sum_{j=1}^{3} e^{\sigma_j(\eta)t} \mathsf{e}_j(\eta) \otimes \mathsf{e}_j(\eta),$$

where $\{\sigma_1(\eta), \sigma_2(\eta), \sigma_3(\eta)\}$ is the spectrum of the operator $-i\eta\mathsf{V} + L$. The eigenvalues $\mathsf{e}_j(\eta)$, $j = 1, 2, 3$, are the zeros of

$$0 = det\left[-i\eta\mathsf{V} + L - \sigma I\right] \equiv -\left[\sigma^3 + \frac{1}{2}\sigma^2 + \eta^2\sigma + \frac{1}{6}\eta^2\right]. \tag{19}$$

Lemma 1. *There exist $\kappa_0, \kappa_1 > 0$ such that for any $|\eta| > \kappa_0$*

$$Re(\sigma_j(\eta)) < -\kappa_1 \ for\ j = 1, 2, 3. \tag{20}$$

Furthermore, the eigenvalues $\sigma_j(\eta)$, $j = 1,2,3$, *are analytic functions for* $|\eta| \le \kappa_0$ *and satisfy the following asymptotics for* $|\eta| \le \kappa_0$*:*

$$\begin{cases} \sigma_1(\eta) = \frac{i}{\sqrt{3}}\eta - \frac{2}{3}\eta^2 + O(\eta^3), \\ \sigma_2(\eta) = -\frac{i}{\sqrt{3}}\eta - \frac{2}{3}\eta^2 + O(\eta^3), \\ \sigma_3(\eta) = -\frac{1}{2} + \frac{4}{3}\eta^2 + O(\eta^3); \end{cases} \tag{21}$$

there are corresponding analytic eigenvectors $\mathsf{e}_j(\eta)$ *satisfying the asymptotics for* $|\eta| \le \kappa_0$*:*

$$\mathsf{e}_1(\eta) = \begin{pmatrix} \frac{2-\sqrt{3}}{\sqrt{6(2-\sqrt{3})}} \\ \frac{1-\sqrt{3}}{\sqrt{6(2-\sqrt{3})}} \\ \frac{-1}{\sqrt{6(2-\sqrt{3})}} \end{pmatrix} + O(\eta),\ \mathsf{e}_2(\eta) = \begin{pmatrix} \frac{2+\sqrt{3}}{\sqrt{6(2+\sqrt{3})}} \\ \frac{1+\sqrt{3}}{\sqrt{6(2+\sqrt{3})}} \\ \frac{-1}{\sqrt{6(2+\sqrt{3})}} \end{pmatrix} + O(\eta),\ \mathsf{e}_3(\eta) = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} + O(\eta) \tag{22}$$

This lemma is a simple algebraic calculation. The proof is omitted.

2.3. **Long Wave-Short Wave decomposition.** As in [8], one can implement the Long Wave-Short Wave decomposition to the solution of (4) as follows:

$$\mathsf{F}(x,t) = \mathsf{F}_L(x,t) + \mathsf{F}_S(x,t), \tag{23}$$
$$\hat{\mathsf{F}}_L(x,t) = \chi\left(\frac{|\eta|}{\kappa}\right)\hat{\mathsf{F}}(\eta,t),\ \ \hat{\mathsf{F}}_S(x,t) = \left(1 - \chi\left(\frac{|\eta|}{\kappa}\right)\right)\hat{\mathsf{F}}(\eta,t),$$

where $\chi(y)$ is a characteristic function

$$\chi(y) = \begin{cases} 1 \text{ if } |y| \le 1, \\ 0 \text{ else,} \end{cases}$$

and κ is a given positive constant.

Due to Lemma 1, one has the lemma

Lemma 2. *There exists a constant* $C(\kappa) > 0$ *such that*

$$\|\mathsf{F}_L(\cdot,t)\|_{H^1} \le C(\kappa)\|\mathsf{F}(\cdot,0)\|_{L^2}, \tag{24}$$
$$\|\mathsf{F}_S(\cdot,t)\|_{L^2} \le C(\kappa)e^{-\kappa_1 t}\|\mathsf{F}(\cdot,0)\|_{L^2}. \tag{25}$$

Lemma 3. *For a sufficiently small* $\kappa_0 > 0$*, there exists* $C_0(\kappa_0) > 1$ *such that for any* $|x| \le C_0(\kappa_0)(1+t)/\sqrt{3}$ *one has that*

$$\left\| \int_{|\eta|\le\kappa_0} e^{i\eta x + (-i\eta \mathsf{V} + L)t} \right\| \le C_0 \left(\frac{e^{-\frac{|x - \frac{1}{\sqrt{3}}t|^2}{C_0\,(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x + \frac{1}{\sqrt{3}}t|^2}{C_0\,(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C_0} \right). \tag{26}$$

This lemma is just a consequence of the analytic properties of the spectrum $\{\sigma_j(\eta)\}$ for η around 0 stated in Lemma 1. To derive this lemma one just needs to repeat the method of complex analysis given in [8]. Thus, the details are omitted.

2.4. **Particle-Wave decomposition.** Parallel to the Particle-Wave decomposition in [8], we can have a Particle-Wave decomposition for (4). Using (5) we can rewrite (4) as follows

$$\begin{cases} \partial_t \mathsf{F} + \mathsf{V}\partial_x \mathsf{F} + \nu \mathsf{F} = \mathsf{K}\mathsf{F}, \\ \mathsf{F}(x,0) \equiv \mathsf{F}_0(x). \end{cases} \tag{27}$$

Then, one can have the Picard's iteration for the system (27) as follows

$$\begin{aligned}\mathsf{F}(x,t) &= e^{(-\mathsf{V}\partial_x-\nu)t}\mathsf{F}_0 + e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t}\mathsf{F}_0 \\ &+ e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t}\mathsf{F}_0 \\ &+ e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t}\mathsf{F}_0 + \cdots .\end{aligned} \tag{28}$$

Remark 4. In the above expression, both V and ν are diagonal matrices. Thus, the operator $e^{-(\mathsf{V}\partial_x+\nu)t}$ can be expressed explicitly in terms of characteristic curves.

We take only a finite number of iterations and define the n-th order Particle-Wave decomposition for the system (4).

$$\begin{cases}\mathsf{F} = \mathbb{P}_n + \mathbb{W}_n, \\ \mathbb{W}_n \equiv \mathsf{F} - \mathbb{P}_n, \text{ (The wave component)} \\ \mathbb{P}_n \equiv e^{(-\mathsf{V}\partial_x-\nu)t}\mathsf{F}_0 + e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t}\mathsf{F}_0 \\ +e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t} * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t}\mathsf{F}_0 \\ \quad + \cdots + e^{(-\mathsf{V}\partial_x-\nu)t} * \underbrace{\mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t} * \cdots * \mathsf{K}e^{(-\mathsf{V}\partial_x-\nu)t}}_{n \text{ factors of } \mathsf{K}}\mathsf{F}_0.\end{cases} \tag{29}$$

2.5. **A model problem.** We consider a model initial value problem

$$\begin{cases}\partial_t\mathsf{F} + \mathsf{V}\partial_x\mathsf{F} = L\mathsf{F}, \\ \mathsf{F}(x,0) = \mathsf{F}_0(x), \ \mathit{supp}(\mathsf{F}_0) \subset [-1,1], \ \ \|\mathsf{F}_0\|_{L^\infty} \le 1.\end{cases} \tag{30}$$

Remark 5. There is no regularity condition imposed on $\mathsf{F}_0(x)$. Furthermore, $\mathsf{F}_0(x)$ is allowed to be discontinuous.

Now, we apply the Long Wave-Short Wave decomposition to the solution $\mathsf{F}(x,t)$ of (30). Then, we have that

$$\begin{cases}\mathsf{F} = \mathsf{F}_L + \mathsf{F}_S, \\ \mathsf{F}_L(x,t) = \left(\displaystyle\int_{|\eta|<\kappa} e^{ix\eta+(-i\eta\mathsf{V}+L)t}d\eta\right) * \mathsf{F}_0(x), \\ \|\mathsf{F}_S(\cdot,t)\|_{L^2_x} \le Ce^{-t/C}.\end{cases} \tag{31}$$

From Lemma 3, (31), and $\mathit{supp}(\mathsf{F}_0) \subset [-1,1]$, there exists $C_0 > 1$ such that for $|x| < C_0(1+t)/\sqrt{3}$

$$\|\mathsf{F}_L(x,t)\| \le C\left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C_0\,(1+t)}}}{\sqrt{1+t}} + \frac{e^{-\frac{|x+\frac{1}{\sqrt{3}}t|^2}{C_0\,(1+t)}}}{\sqrt{1+t}} + e^{-(|x|+t)/C_0}\right). \tag{32}$$

By a straightforward calculation, one has the following proposition.

Proposition 1. *Let* $\mathsf{F}(x,t) = \mathbb{P}_1 + \mathbb{W}_1$ *be a 1st order Particle-Wave decomposition in* (29) *for the solution* $\mathsf{F}(x,t)$ *of* (30). *There exist some positive constants* C_1 *and* C_2 *such that*

$$\|(\partial_t + \mathsf{V}\partial_x - L)\mathbb{P}_1\|_{H^1_x} \le O(1)e^{-t/C_1}, \tag{33}$$

$$\|e^{-x/C_2}(\partial_t + \mathsf{V}\partial_x - L)\mathbb{P}_1\|_{L^2_x} \le O(1)e^{-t/C_1}, \tag{34}$$

$$\|e^{x/C_2}(\partial_t + \mathsf{V}\partial_x - L)\mathbb{P}_1\|_{L^2_x} \le O(1)e^{-t/C_1}, \tag{35}$$

$$\|\mathbb{P}_1(x,t)\| \le C_1e^{-(|x|+t)/C_1}. \tag{36}$$

Remark 6. In contrast to the Boltzmann equation, by increasing the order of the Particle-Wave decomposition one can improve that $\|(\partial_t+\xi\cdot\nabla_x-L)\mathbb{W}_{2n}\|_{H^n_x(L^2_\xi)}<O(1)e^{-t/C}$, but there is no such analogy for the Broadwell model.

By the spectrum property in Lemma 1, the operator $e^{(-\mathsf{V}\partial_x+L)t}$ is a uniformly bounded L^2_x operator. This, Duhamel's principle, and (33) yield that

$$\|\mathbb{W}_1(\cdot,t)\|_{H^1_x}=\left\|\int_0^t e^{(-\mathsf{V}\partial_x+L)(t-s)}(\partial_s+\mathsf{V}\partial_x-L)\mathbb{P}_1(\cdot,s)ds\right\|_{H^1_x}\le O(1). \tag{37}$$

Then, from (31), (36) and (37) one has that

$$\begin{cases}\|\mathbb{P}_1-\mathsf{F}_S\|_{H^1_x}=\|\mathsf{F}_L-\mathbb{W}_1\|_{H^1_x}=O(1),\\ \|\mathbb{P}_1-\mathsf{F}_S\|_{L^2_x}\le Ce^{-t/C}\end{cases} \tag{38}$$

for some $C>0$. This and (36) result in

$$\|\mathsf{F}_S(\cdot,t)\|_{L^\infty_x}\le O(1)e^{-t/C}\text{ for some }C>0. \tag{39}$$

From (39) and (32), one has the following proposition.

Proposition 2. *There exists $C>2$ such that for any $|x|\le C(1+t)/\sqrt{3}$ the solution $\mathsf{F}(x,t)$ of (30) satisfies*

$$\|\mathsf{F}(x,t)\|\le C\left(\frac{e^{-\frac{|x-\frac{1}{\sqrt{3}}t|^2}{C\,(1+t)}}}{\sqrt{1+t}}+\frac{e^{-\frac{|x+\frac{1}{\sqrt{3}}t|^2}{C\,(1+t)}}}{\sqrt{1+t}}+e^{-(|x|+t)/C}\right). \tag{40}$$

Proposition 3. *There exists $C>1$ such that the solution $\mathsf{F}(x,t)$ of the model problem (30) satisfies*

$$\|\mathsf{F}(x,t)\|\le Ce^{-(|x|+t)/C}\ \textit{for}\ |x|\ge\frac{3}{2\sqrt{3}}(1+t). \tag{41}$$

Proof. In the region $|x|\le 2(1+t)/\sqrt{3}$, (40) already asserts the proposition. For a symmetric reasoning, we just need to consider the case $x>\frac{3}{2}(1+t)/\sqrt{3}$ only.

We consider the weighted energy estimate

$$0=\int_{\mathbb{R}}e^{\beta(x-\frac{3t}{2\sqrt{3}})}\mathsf{F}\cdot(\partial_t\mathsf{F}+\mathsf{V}\partial_x\mathsf{F}-L\mathsf{F})dx\text{ for some }\beta>0.$$

This yields that

$$\frac{1}{2}\partial_t\int_{\mathbb{R}}e^{\beta(x-\frac{3t}{2\sqrt{3}})}\mathsf{F}\cdot\mathsf{F}dx+\int_{\mathbb{R}}e^{\beta(x-\frac{3t}{2\sqrt{3}})}\left\{\frac{1}{2}\beta\mathsf{F}\cdot\left(\frac{3}{2\sqrt{3}}-\mathsf{V}\right)\mathsf{F}-\mathsf{F}\cdot L\mathsf{F}\right\}dx=0. \tag{42}$$

One can check that $(\mathsf{P}_0\mathsf{F})\cdot\left(\frac{3}{2\sqrt{3}}-\mathsf{V}\right)(\mathsf{P}_0\mathsf{F})\ge\frac{1}{2\sqrt{3}}\|\mathsf{P}_0\mathsf{F}\|^2$ (due to (8)) and $\mathsf{F}\cdot L\mathsf{F}\le-\frac{1}{2}\|\mathsf{P}_1\mathsf{F}\|^2$. This, (42), and Schwartz inequality give that

$$\frac{1}{2}\partial_t\int_{\mathbb{R}}e^{\beta(x-\frac{3t}{2\sqrt{3}})}\mathsf{F}\cdot\mathsf{F}dx+\frac{\beta}{8\sqrt{3}}\int_{\mathbb{R}}e^{\beta(x-\frac{3t}{2\sqrt{3}})}\mathsf{F}\cdot\mathsf{F}dx\le 0\text{ when }\beta\ll 1. \tag{43}$$

Then

$$\begin{aligned}\int_{\mathbb{R}}e^{\beta(x-\frac{3}{2}t/\sqrt{3})}\|\mathsf{F}(x,t)\|^2dx&\le e^{-\frac{\beta}{4\sqrt{3}}t}\left(\int_{\mathbb{R}}e^{\beta(x-\frac{3}{2}t/\sqrt{3})}\|\mathsf{F}(x,t)\|^2dx\right)_{t=0}\\&\le O(1)e^{-\frac{\beta}{4\sqrt{3}}t}.\end{aligned} \tag{44}$$

This and (36) yield that

$$\int_{\mathbb{R}} e^{\beta(x-\frac{3}{2}t/\sqrt{3})}\|\mathbb{W}_1(x,t)\|^2 dx \le O(1)e^{-\frac{\beta}{4\sqrt{3}}t} \text{ for } \beta \ll 1, \tag{45}$$

which and (37) give that

$$\sup_{x\in\mathbb{R}} \|e^{\beta(x-\frac{3}{2}t/\sqrt{3})/4}\mathbb{W}_1(x,t)\| \le O(1)e^{-\frac{\beta}{8\sqrt{3}}t}. \tag{46}$$

This and (36) conclude this proposition for $x > \frac{3}{2}(1+t)/\sqrt{3}$.

□

Proof of Theorem 1. We apply a 1st order Particle-Wave decomposition in (29) to the problem in (11). Then, it follows

$$\mathbb{G}(x,t) \equiv \mathbb{P}_1(x,t) + \mathbb{W}_1(x,t). \tag{47}$$

There exists $C > 0$ such that

$$\left\|\mathbb{P}_1(x,t) - e^{(-\mathsf{V}\partial_x - \nu)t}\delta(x)I\right\| \le Ce^{-(|x|+t)/C}, \tag{48}$$

$$\|(\partial_t + \mathsf{V}\partial_x - L)\mathbb{P}_1(x,t)\| \le Ce^{-(|x|+t)/C}. \tag{49}$$

The equation for $\mathbb{W}_1(x,t)$ is

$$\begin{cases} (\partial_t + \mathsf{V}\partial_x - L)\mathbb{W}_1 = -(\partial_t + \mathsf{V}\partial_x - L)\mathbb{P}_1, \\ \mathbb{W}_1(x,0) \equiv 0. \end{cases} \tag{50}$$

Then, the solution $\mathbb{W}_1(x,t)$ can be written as

$$\begin{aligned} \mathbb{W}_1(x,t) &= -\int_0^t e^{(-\mathsf{V}\partial_x + L)(t-s)}(\partial_s + \mathsf{V}\partial_x - L)\mathbb{P}_1 ds \\ &= -\int_0^t e^{(-\mathsf{V}\partial_x + L)(t-s)} \sum_{j=-\infty}^{\infty} \chi(x-2j+1)\left[(\partial_s + \mathsf{V}\partial_x - L)\mathbb{P}_1\right] ds, \end{aligned} \tag{51}$$

where χ is a characteristic function for the interval $[-1,1)$ i.e. $\chi(x) = 1$ if $x \in [-1,1)$ else $\chi(x) = 0$. Due to (49), the estimates (40) and (41) for the model problem can be applied to $e^{(-\mathsf{V}\partial_x + L)(t-s)}[\chi(x-2j+1)(\partial_s + \mathsf{V}\partial_x - L)\mathbb{P}_1]$. Then one has that

$$\begin{aligned} &\|\sum_{j=-\infty}^{\infty} e^{(-\mathsf{V}\partial_x + L)(t-s)}\chi(x-2j+1)(\partial_s + \mathsf{V}\partial_x - L)\mathbb{P}_1\| \\ &\le C\sum_{j=-\infty}^{\infty}\left(\frac{e^{-\frac{(x-j-(t-s)/\sqrt{3})^2}{C(t-s)}}}{\sqrt{t-s}} + \frac{e^{-\frac{(x-j+(t-s)/\sqrt{3})^2}{C(t-s)}}}{\sqrt{t-s}} + e^{-(|x-j|+(t-s)|/C}\right)e^{-(|j|+s)/C} \\ &\quad \le C_1 e^{-s/C_1}\left(\frac{e^{-\frac{(x-t/\sqrt{3})^2}{C_1(t+1)}}}{\sqrt{t+1}} + \frac{e^{-\frac{(x+t/\sqrt{3})^2}{C_1(t+1)}}}{\sqrt{t+1}} + e^{-(|x|+t)|/C_1}\right) \text{ for some } C_1 > C. \end{aligned} \tag{52}$$

Thus, from (51) and (52) there exists $C > 0$ such that

$$\|\mathbb{W}_1(x,t)\| \le C\left(\frac{e^{-\frac{(x-t/\sqrt{3})^2}{C(t+1)}}}{\sqrt{t+1}} + \frac{e^{-\frac{(x+t/\sqrt{3})^2}{C(t+1)}}}{\sqrt{t+1}} + e^{-(|x|+t)|/C}\right). \tag{53}$$

(48) and (53) conclude Theorem 1. □

3. **Construction of the Green's function for an initial boundary value problem.** Since both V and L are symmetric, the Green's function $\mathbb{G}(x-y,t-s)$ satisfies a backward equation:

$$-\partial_s\mathbb{G}^t-\partial_y\mathbb{G}^t\mathsf{V}-\mathbb{G}^tL=(\partial_t\mathbb{G}+\mathsf{V}\partial_x\mathbb{G}-L\mathbb{G})^t=0. \tag{54}$$

For convenience of presentation, we set the new coordinate system:

$$\begin{cases}\bar{x}=x-bt,\\ \bar{y}=y-bs.\end{cases}$$

Then, problem (10) becomes

$$\begin{cases}[\partial_t+(\mathsf{V}-bI)\partial_{\bar{x}}-L]\mathsf{F}(\bar{x},t)=0 \text{ for } \bar{x}>0,\ t>0,\\ \\ (1,-\alpha_0,-\alpha_-)^t\cdot\mathsf{F}(0,t)=0.\end{cases} \tag{55}$$

3.1. **Structure of model problem for an initial boundary problem.** Similar to the model problem (30), we begin from a model problem for (55):

$$\begin{cases}[\partial_t+(\mathsf{V}-bI)\partial_{\bar{x}}-L]\mathsf{F}(\bar{x},t)=0 \text{ for } \bar{x}>0,\ t>0,\\ (1,-\alpha_0,-\alpha_-)^t\cdot\mathsf{F}(0,t)=0,\\ \mathsf{F}(\bar{x},0)=\mathsf{F}_0(\bar{x}),\end{cases} \tag{56}$$

where

$$\mathsf{F}_0(\bar{x})=\begin{cases}0 & \text{for } |\bar{x}-\bar{y}|>1\\ O(1) & \text{for } |\bar{x}-\bar{y}|\le 1,\end{cases}$$

and $\bar{y}$ is a parameter.

Denote

$$\begin{cases}\mathsf{A}_0(\bar{x},t)\equiv\displaystyle\int_0^\infty\mathbb{G}(\bar{x}+bt-\bar{z},t)\mathsf{F}_0(\bar{z})d\bar{z},\\ \mathsf{E}_1(\bar{x},t)\equiv\mathsf{F}(\bar{x},t)-\mathsf{A}_0(\bar{x},t).\end{cases}$$

Then, the function $\mathsf{A}_0(\bar{x},t)$ satisfies that

$$\|\mathsf{A}_0(\bar{x},t)\|\le C\left(e^{-(|\bar{x}-\bar{y}|+t)/C}+\frac{e^{-\frac{(\bar{x}-\bar{y}+(b-\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}+e^{-\frac{(\bar{x}-\bar{y}+(b+\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}}{\sqrt{1+t}}\right) \tag{57}$$

for some $C>0$. The function $\mathsf{E}_1(x,t)$ satisfies

$$\begin{cases}(\partial_t+(\mathsf{V}-bI)\partial_{\bar{x}}-L)\mathsf{E}_1=0,\\ \mathsf{E}_1(x,0)\equiv 0,\\ \omega(t)\equiv(1,-\alpha_0,-\alpha_-)\cdot\mathsf{E}_1(0,t),\\ |\omega(t)|\le C\left(e^{-(|\bar{y}|+t)/C}+\dfrac{e^{-\frac{(\bar{y}-(b-\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}+e^{-\frac{(\bar{y}-(b+\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}}{\sqrt{1+t}}\right)\end{cases} \tag{58}$$

for some $C>1$ which is independent of $\bar{y}$.

Denote

$$(e_1(t),e_2(t),e_3(t))^t\equiv\mathsf{E}_1(0,t).$$

From (58), we have that

$$\begin{aligned} 0 &= \int_0^\infty e^{\beta\bar{x}} \mathsf{E}_1 \cdot (\partial_t + (\mathsf{V} - bI)\partial_{\bar{x}} - L)\mathsf{E}_1 d\bar{x} \\ &= \frac{1}{2}\partial_t \int_0^\infty e^{\beta\bar{x}} \mathsf{E}_1 \cdot \mathsf{E}_1 d\bar{x} + \int_0^\infty e^{\beta\bar{x}} \left[\frac{-\beta}{2}(\mathsf{E}_1 \cdot (\mathsf{V} - bI)\mathsf{E}_1) - \mathsf{E}_1 \cdot L\mathsf{E}_1\right] d\bar{x} \\ &\quad - \frac{1}{2}\mathsf{E}_1(0,t) \cdot (\mathsf{V} - b)\mathsf{E}_1(0,t). \end{aligned} \tag{59}$$

Since $b \in (1/\sqrt{3}, 1)$, $\alpha_0, \alpha_- \geq 0$, and $2\alpha_0 + \alpha_- = 1$, then from Schwartz inequality one has that

$$\begin{aligned} (1+b)e_3^2 + be_2^2 \geq \frac{(1-b)(\alpha_- e_3 + \alpha_0 e_2)^2}{(1-b)\left(\frac{\alpha_-^2}{1+b} + \frac{\alpha_0^2}{b}\right)} &\geq \frac{b}{1-b}(1-b)(\alpha_- e_3 + \alpha_0 e_2)^2 \\ &\geq \frac{\sqrt{3}+1}{2}(1-b)(\alpha_- e_3 + \alpha_0 e_2)^2. \end{aligned} \tag{60}$$

From (60), the boundary term $-\mathsf{E}_1(0,t) \cdot (\mathsf{V} - bI)\mathsf{E}_1(0,t)$ satisfies

$$\begin{aligned} -\mathsf{E}_1(0,t) \cdot (\mathsf{V} - bI)\mathsf{E}_1(0,t) &= -(e_1, e_2, e_3) \begin{pmatrix} 1-b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -1-b \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ &= (1+b)(e_3)^2 + b(e_2)^2 - (1-b)(e_1)^2 \\ &= (1+b)(e_3)^2 + b(e_2)^2 - (1-b)(\alpha_0 e_2 + \alpha_- e_3 + \omega(t))^2 \\ &\geq \left[1 - \frac{2}{\sqrt{3}+1}\right]((1+b)e_3^2 + be_2^2) + (1-b) \\ &\qquad \cdot (\alpha_0 e_2 + \alpha_- e_3)^2 - (1-b)(\alpha_0 e_2 + \alpha_- e_3 + \omega(t))^2 \\ &\geq \frac{\sqrt{3}-1}{\sqrt{3}+1}((1+b)e_3^2 + be_2^2) \\ &\qquad - 2(1-b)(\alpha_0 e_2 + \alpha_- e_3)\omega(t) - (1-b)\omega(t)^2. \end{aligned} \tag{61}$$

Under the assumption that b is supersonic together with (61), there exist β, γ, and $D > 0$ such that

$$\begin{cases} -\dfrac{\beta}{2}\mathsf{E}_1 \cdot (\mathsf{V} - bI)\mathsf{E}_1 - \mathsf{E}_1 \cdot L\mathsf{E}_1 \geq \dfrac{1}{2}\gamma \mathsf{E}_1 \cdot \mathsf{E}_1, \\ -\mathsf{E}_1(0,t) \cdot (\mathsf{V} - bI)\mathsf{E}_1(0,t) \geq \left(\dfrac{1}{D}(|e_2|^2 + |e_3|^2) - D\omega(t)^2\right). \end{cases} \tag{62}$$

(59) and (62) give that for any $t > 0$

$$\partial_t \int_0^\infty e^{\beta\bar{x}} \mathsf{E}_1 \cdot \mathsf{E}_1 d\bar{x} + \gamma \int_0^\infty e^{\beta\bar{x}} \mathsf{E}_1 \cdot \mathsf{E}_1 d\bar{x} + \frac{1}{D}(|e_2|^2 + |e_3|^2)|_{\bar{x}=0} \leq D\omega(t)^2. \tag{63}$$

This inequality gives the estimate for any $t > 0$

$$\int_0^\infty e^{\beta\bar{x}} \|\mathsf{E}_1(\bar{x},t)\|^2 d\bar{x} + \frac{1}{D}\int_0^t e^{-\gamma\,(t-s)}(|e_2|^2 + |e_3|^2)ds \leq D\int_0^t e^{-\gamma\,(t-s)}\omega(s)^2 ds. \tag{64}$$

Next, we consider

$$\vec{0} = \int_0^t \int_0^\infty \mathbb{G}(\bar{x} - \bar{y} + b(t-s), t-s)(\partial_s + (\mathsf{V} - bI)\partial_y - L)\mathsf{E}_1(\bar{y}, s) d\bar{y} ds. \tag{65}$$

From this, (54), and $\mathsf{E}_1(\bar{x}, 0) \equiv 0$,

$$\mathsf{E}_1(\bar{x}, t) = \int_0^t \mathbb{G}(\bar{x} + b(t-s), t-s)(\mathsf{V} - bI)\mathsf{E}_1(0, s) ds. \tag{66}$$

Thus, by letting $\bar{x} \to 0+$ one has

$$\begin{pmatrix} e_2(t) \\ e_3(t) \end{pmatrix} = \int_0^t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{G}(\bar{x} + b(t-s), t-s) \cdot (\mathsf{V} - bI) \begin{pmatrix} \alpha_0 e_2(s) + \alpha_- e_3(s) + \omega(s) \\ e_2(s) \\ e_3(s) \end{pmatrix} ds|_{\bar{x}=0+}. \tag{67}$$

Notice that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{G}(\bar{x} + b(t-s), t-s)(\mathsf{V} - bI)|_{\bar{x}=0+}$ is a 2×3 matrix whose entries are L^∞-function in $t - s$ due to $b > 0$. Therefore

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{G}(\bar{x}+b(t-s), t-s)(\mathsf{V}-bI)|_{\bar{x}=0+} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{W}_0(b(t-s), t-s)(\mathsf{V}-bI),$$

where $\mathbb{P}_0 + \mathbb{W}_0$ is the 0th order Particle-Wave decomposition for $\mathbb{G}$:

$$\mathbb{G}(x,t) = e^{-\frac{t}{6}} \begin{pmatrix} \delta(x-t) & 0 & 0 \\ 0 & \delta(x) & 0 \\ 0 & 0 & \delta(x+t) \end{pmatrix} + \mathbb{W}_0(x,t).$$

$$\begin{pmatrix} e_2(t) \\ e_3(t) \end{pmatrix} = \int_0^t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{W}_0(b(t-s), t-s)(\mathsf{V}-bI) \begin{pmatrix} \alpha_0 e_2(s) + \alpha_- e_3(s) + \omega(s) \\ e_2(s) \\ e_3(s) \end{pmatrix} ds. \tag{68}$$

One can use Theorem 1 to estimate $\|\mathbb{W}_0(b(t-s), t-s)\|$. Then one can use (58) and (68) to obtain that

$$\begin{aligned} \|(e_2(t), e_3(t))\| &\le O(1) \int_0^t \|\mathbb{W}_0(b(t-s), t-s)\| \cdot (\|(e_2(s), e_3(s))\| + |\omega(s)|) ds \\ &\le O(1) \int_0^t e^{-(t-s)/C} \cdot (\|(e_2(s), e_3(s))\| + |\omega(s)|) ds. \end{aligned} \tag{69}$$

Note that the constants C and γ are independent. Thus, we may choose $\gamma < 1/C$ so that from (58), (69), and (64) together with Schwartz inequality one has that for some $C > 0$

$$\|(e_2(t), e_3(t))\| \le C^2 \left(e^{-(|\bar{y}|+t)/C} + \frac{e^{-\frac{(\bar{y}-(b-\frac{1}{\sqrt{3}})t)^2}{C(t+1)}} + e^{-\frac{(\bar{y}-(b+\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}}{\sqrt{1+t}} \right). \tag{70}$$

With $e_1(t) = \alpha_0 e_2(t) + \alpha_- e_3(t) + \omega(t)$, one obtains the estimate for the boundary data $\mathsf{E}_1(0,t)$

$$\|\mathsf{E}_1(0,t)\| \le C^2 \left(e^{-(|\bar{y}|+t)/C} + \frac{e^{-\frac{(\bar{y}-(b-\frac{1}{\sqrt{3}})t)^2}{C(t+1)}} + e^{-\frac{(\bar{y}-(b+\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}}{\sqrt{1+t}} \right) \text{ for some } C > 1. \tag{71}$$

Then, by substituting this into the representation of $\mathsf{E}_1(\bar{x},t)$ in (66), we have

$$\begin{aligned} &\|\mathsf{E}_1(\bar{x},t)\| \\ &\le C^3 \int_0^t \left(\frac{e^{-\frac{|\bar{x}+b(t-s)-\frac{1}{\sqrt{3}}(t-s+1)|^2}{C\ (1+t-s)}}}{\sqrt{1+t-s}} + \frac{e^{-\frac{|\bar{x}+b(t-s)+\frac{1}{\sqrt{3}}(t-s+1)|^2}{C\ (1+t-s)}}}{\sqrt{1+t-s}} + e^{-(|\bar{x}+b(t-s)|+t-s)/C} \right) \\ &\qquad \cdot \left(e^{-(|\bar{y}|+s)/C} + \frac{e^{-\frac{(\bar{y}-(b-\frac{1}{\sqrt{3}})s)^2}{C(s+1)}} + e^{-\frac{(\bar{y}-(b+\frac{1}{\sqrt{3}})s)^2}{C(s+1)}}}{\sqrt{1+s}} \right) ds \\ &\le C^3 e^{-|\bar{x}|/C'} \left(e^{-(|\bar{y}|+t)/C'} + \frac{e^{-\frac{(\bar{y}-(b-\frac{1}{\sqrt{3}})t)^2}{C'(t+1)}} + e^{-\frac{(\bar{y}-(b+\frac{1}{\sqrt{3}})t)^2}{C'(t+1)}}}{\sqrt{1+t}} \right) \text{ for some } C' > C. \end{aligned} \tag{72}$$

From (57) and (72), the function $\|\mathsf{E}_1(\bar{x},t)\|$ can be bounded by a uniformly multiple of $\|\mathsf{A}_0(\bar{x},t)\|$. Then, we have the following lemma.

Lemma 4. *For the model problem* (56), *there exists* $C > 0$ *such that*

$$\|\mathsf{F}(\bar{x},t)\| \le C^3 \left(e^{-(|\bar{x}-\bar{y}|+t)/C} + \frac{e^{-\frac{(\bar{x}-\bar{y}+(b-\frac{1}{\sqrt{3}})t)^2}{C(t+1)}} + e^{-\frac{(\bar{x}-\bar{y}+(b+\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}}{\sqrt{1+t}} \right). \tag{73}$$

3.2. **Construction of the Green's function.**

Proof of Theorem 2. We only need to impose the boundary condition in (12) for the Particle-Wave decomposition.

Let $\mathbb{P}_1(\bar{x},t) = \Gamma_0(\bar{x},t) + \Gamma_1(\bar{x},t)$ be the Particle-Wave decomposition. The functions Γ_0 and Γ_1 are given by

$$\begin{cases} (\partial_t + (\mathsf{V} - bI)\partial_{\bar{x}} + \nu)\Gamma_0 = 0, \\ \Gamma_0(\bar{x},0) = \delta(\bar{x}-\bar{y})I, \\ (1, -\alpha_0, -\alpha_-)^t \cdot \Gamma_0(0,t) = 0, \end{cases} \tag{74}$$

$$\begin{cases} (\partial_t + (\mathsf{V} - bI)\partial_{\bar{x}} + \nu)\Gamma_1 = \mathsf{K}\Gamma_0, \\ \Gamma_1(\bar{x},0) = 0, \\ (1, -\alpha_0, -\alpha_-)^t \cdot \Gamma_1(0,t) = 0. \end{cases} \tag{75}$$

The equation for $\mathbb{W}_1(x,t)$ is

$$\begin{cases} (\partial_t + (\mathsf{V} - bI)\partial_{\bar{x}} - L)\mathbb{W}_1 = \mathsf{K}\Gamma_1, \\ \mathbb{W}_1(\bar{x},0) = 0, \\ (1, -\alpha_0, -\alpha_-)^t \cdot \mathbb{W}_1(0,t) = 0. \end{cases} \tag{76}$$

One has that

$$\Gamma_0(\bar{x},t) = e^{-\frac{t}{6}} \cdot \begin{pmatrix} \delta(\bar{x}+bt-t-\bar{y}) & \alpha_0 \frac{(1-b)}{b}\, \delta(\bar{x}+\frac{(1-b)\bar{y}}{b}-(1-b)t) & \alpha_- \frac{(1-b)}{(1+b)}\, \delta(\bar{x}+\frac{(1-b)\bar{y}}{1+b}-(1-b)t) \\ 0 & \delta(\bar{x}+bt-\bar{y}) & 0 \\ 0 & 0 & \delta(\bar{x}+bt+t-\bar{y}) \end{pmatrix} \tag{77}$$

for $\bar{x} \geq 0$, and for some $C > 0$

$$\|\Gamma_1(\bar{x},t)\| \leq Ce^{-t/C}e^{-|\bar{x}-\bar{y}|/C} \text{ for } \bar{x} \geq 0. \tag{78}$$

The source term in (76) is pointwisely bounded by a function $\mathsf{K}\Gamma_1(\bar{x},t)$. Thus, the same argument in the proof of Theorem 1 can be applied as follows.

First, one can treat $e^{(-\mathsf{V}\partial_x+L)(t-s)}\chi(x-2j+1)(\partial_s+\mathsf{V}\partial_x-L)\Gamma_1)$ in (51) as a solution of the model problem with initial data $\chi(x-2j+1)(\partial_s+\mathsf{V}\partial_x-L)\Gamma_1(\bar{x},s)$ at initial time equal to s. Then, from Lemma 4 one has that

$$\begin{aligned} &\|e^{(-(\mathsf{V}-b)\partial_{\bar{x}}+L)(t-s)}\chi(\bar{x}-2j+1)\mathsf{K}\Gamma_1(\bar{x},s)\| \\ &\quad\leq C^4e^{-(|j-\bar{y}|+s)/C)} \\ &\quad\cdot\left(e^{-(|\bar{x}-j|+(t-s))/C} + \frac{e^{-\frac{(\bar{x}-j+(b-\frac{1}{\sqrt{3}})(t-s))^2}{C(t-s+1)}} + e^{-\frac{(\bar{x}-j+(b+\frac{1}{\sqrt{3}})(t-s))^2}{C(t-s+1)}}}{\sqrt{1+t-s}}\right). \end{aligned} \tag{79}$$

By the semi-group representation,

$$\mathbb{W}_1(\bar{x},t) = \int_0^t e^{(-(\mathsf{V}-bI)\partial_{\bar{x}}+L)(t-s)}\sum_{j=1}^{\infty}\chi(\bar{x}-2j+1)\mathsf{K}\Gamma_1(\bar{x},s)ds. \tag{80}$$

This representation and (79) yield that there exists $C > 0$ such that

$$\|\mathbb{W}_1(\bar{x},t)\| \leq C^4\left(e^{-(|\bar{x}-\bar{y}|+t)/C} + \frac{e^{-\frac{(\bar{x}-\bar{y}+(b-\frac{1}{\sqrt{3}})t)^2}{C(t+1)}} + e^{-\frac{(\bar{x}-\bar{y}+(b+\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}}{\sqrt{1+t}}\right). \tag{81}$$

Since the Green's function $\mathbb{G}_B(x,t;y,s)$ is given by

$$\mathbb{G}_B(x,t;y,s) = \Gamma_0(\bar{x},t-s)+\Gamma_1(\bar{x},t-s)+\mathbb{W}_1(\bar{x},t-s) \text{ with } \bar{x}=x-bt,\ \bar{y}=y-bs. \tag{82}$$

Then, (77), (78), and (81) imply the theorem. □

3.3. **Nonlinear Stability.** In this subsection we consider the nonlinearly time-asymptotic stability of the equilibrium state M. We consider the following initial boundary value problem:

$$\begin{cases} \partial_t\mathsf{F} + (\mathsf{V}-bI)\partial_{\bar{x}}\mathsf{F} - Q(\mathsf{F}) = 0, \\ (1,-\alpha_0,-\alpha_-)^t\cdot\mathsf{F}(0,t) = 0, \\ \mathsf{F}(\bar{x},0) = \mathsf{M} + \mathsf{v}_0(\bar{x}), \end{cases} \tag{83}$$

where $\mathsf{v}_0(\bar{x})$ is a small perturbation of the equilibrium state M.

Corollary 1 (Nonlinear Stability)**.** *There exist $\sigma > 0$ and $\epsilon_0 > 0$ such that whenever $\|\mathsf{v}_0(\bar{x})\| \leq \epsilon_0 e^{-\sigma|\bar{x}|}$ for all $\bar{x} > 0$ the solution $\mathsf{F}(\bar{x},t)$ of (83) satisfies that*

$$\lim_{t\to\infty}\sup_{\bar{x}>0}\|\mathsf{F}(\bar{x},t)-\mathsf{M}\| = 0. \tag{84}$$

Proof. Rewrite $\mathsf{F}(\bar{x},t)=\mathsf{M}+\mathsf{v}(\bar{x},t)$. The equation for $\mathsf{v}(\bar{x},t)$ is

$$\begin{cases}\partial_t\mathsf{v}+(\mathsf{V}-bI)\partial_{\bar{x}}\mathsf{v}-L\mathsf{v}=Q(\mathsf{v}),\\ (1,-\alpha_0,-\alpha_-)^t\cdot\mathsf{v}(0,t)=0,\\ \mathsf{v}(\bar{x},0)=\mathsf{v}_0(\bar{x}).\end{cases}\tag{85}$$

Then, the semi-group representation in (80) for $\mathbb{W}_1$ can be applied to result in that

$$\begin{aligned}\mathsf{v}(\bar{x},t)=&\,e^{(-(\mathsf{V}-bI)\partial_{\bar{x}}+L)t}\sum_{j=1}^{\infty}\chi(\bar{x}-2j+1)\mathsf{v}_0(\bar{x})\\ &+\int_0^t e^{(-(\mathsf{V}-bI)\partial_{\bar{x}}+L)(t-s)}\sum_{j=1}^{\infty}\chi(\bar{x}-2j+1)Q(\mathsf{v})(\bar{x},s)ds.\end{aligned}\tag{86}$$

Suppose that

$$\|\mathsf{v}_0(\bar{x})\|\le\epsilon e^{-\sigma|\bar{x}|}\text{ for some }\sigma>0\text{ and }\epsilon>0.\tag{87}$$

Then from (81) one has that

$$\begin{aligned}&\left\|e^{(-(\mathsf{V}-bI)\partial_{\bar{x}}+L)t}\sum_{j=1}^{\infty}\chi(\bar{x}-2j+1)\mathsf{v}_0(\bar{x},0)\right\|\\ &\le\epsilon C^4\int_0^\infty\left(\frac{e^{-\frac{(\bar{x}-\bar{y}+(b-\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}}{\sqrt{t+1}}+\frac{e^{-\frac{(\bar{x}-\bar{y}+(b+\frac{1}{\sqrt{3}})t)^2}{C(t+1)}}}{\sqrt{t+1}}+e^{-(|\bar{x}-\bar{y}|+t)/C}\right)e^{-\sigma|\bar{y}|}d\bar{y}.\end{aligned}\tag{88}$$

Under the assumption that the constant σ satisfies

$$\sigma<\frac{(b-\frac{1}{\sqrt{3}})}{C},\tag{89}$$

we have

$$\left\|e^{(-(\mathsf{V}-bI)\partial_{\bar{x}}+L)t}\sum_{j=1}^{\infty}\chi(\bar{x}-2j+1)\mathsf{v}_0(\bar{x},0)\right\|\le K_0C^5\epsilon e^{-\sigma|\bar{x}|-\sigma(b-\frac{1}{\sqrt{3}})t/2}\tag{90}$$

for some constant K_0 independent of σ, ϵ, and C. From (90), we make an ansatz assumption for the solution $\mathsf{v}(\bar{x},t)$ as follows:

$$\|\mathsf{v}(\bar{x},t)\|\le 2K_0C^5\epsilon e^{-\sigma|\bar{x}|-\frac{2}{3}\sigma(b-\frac{1}{\sqrt{3}})t/2}\text{ for all }\bar{x},\ t>0.\tag{91}$$

To justify this ansatz, we just need to verify that

$$\begin{aligned}&\epsilon^2C^{10}K_0^2\int_0^t\int_0^\infty\left(\frac{e^{-\frac{(\bar{x}-\bar{y}+(b-\frac{1}{\sqrt{3}})(t-s))^2}{C(t-s+1)}}}{\sqrt{t-s+1}}+\frac{e^{-\frac{(\bar{x}-\bar{y}+(b+\frac{1}{\sqrt{3}})(t-s))^2}{C(t-s+1)}}}{\sqrt{t-s+1}}+e^{-(|\bar{x}-\bar{y}|+(t-s))/C}\right)\\ &\qquad\cdot\epsilon e^{-2\sigma|\bar{y}|-\frac{4}{3}\sigma(b-\frac{1}{\sqrt{3}})s/2}d\bar{y}ds\\ &\ll\frac{1}{2}K_0C^5\epsilon e^{-\sigma|\bar{x}|-\frac{2}{3}\sigma(b-\frac{1}{\sqrt{3}})t/2},\end{aligned}\tag{92}$$

when both $\epsilon\ll 1$ and (87) hold. We omit the routine calculation for (92).

Now, by substituting the ansatz assumption (91) into the term $Q(\mathsf{v})$ in (86), the estimates (92) and (90) yield that

$$\|\mathsf{v}(\bar{x},t)\| \leq \frac{3}{2}K_0C^5\epsilon e^{-\sigma|\bar{x}|-\frac{2}{3}\sigma(b-\frac{1}{\sqrt{3}})t/2} \text{ for all } \bar{x},\ t>0 \tag{93}$$

when both $\epsilon \ll 1$ and (89) hold. The estimate (93) is stronger than the ansatz assumption (91). Thus, the ansatz assumption (91) is true when both $\epsilon \ll 1$ and (89) hold. The corollary follows.

□

REFERENCES

[1] Bardos, C.; Caflisch, R.E.; Nicolaenko, B. The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas, *Comm. Pure Appl. Math.* **49** (1986), 323-352.

[2] Caflisch, R. E. *The fluid dynamic limit of the nonlinear Boltzmann equation.* Comm. Pure Appl. Math. 33 (1980), no. 5, 651–666.

[3] Caflisch, R. E.; Papanicolaou, G. C. *The fluid dynamical limit of a nonlinear model Boltzmann equation.* Comm. Pure Appl. Math. **32** (1979), no. 5, 589–616.

[4] Coron, F., Golse, F., Sulem, C. *A classification of well-posed kinetic layer problems.* Commun. Pure Appl. Math. **41**, 409435 (1988)

[5] Golse, F., Perthame, B., Sulem, C. *On a boundary layer problem for the nonlinear Boltzmann equation.* Arch. Rational Mech. Anal. **103** (1), 8196 (1988)

[6] Liu, T.-P. Pointwise Convergence to Shock Waves for Viscous Conservation Laws. *Comm. Pure and Appl. Math.* **50** (1997), no. 11, 1113-1182.

[7] Liu, T.-P.; Yu, S.-H. Boltzmann Equation: Micro-Macro Decompositions and Positivity of Shock Profiles. *Comm. Math. Phys.*, **246** (2004), no. 1, 133-179

[8] Liu, T.-P.; Yu, S.-H. The Green's Function and large-Time Behavior of Solutions for One-Dimensional Boltzmann Equation. *Comm. Pure Appl. Math.* **57** (2004), 1543-1608.

[9] Liu, T.-P.; Yu, S.-H. *Green's Function and large-Time Behavior of Solutions of Boltzmann Equation, 3-D waves.* Bulletin, Inst. Math. Academia Sinica (N.S.) Vol. 1 (2006) No. 1 (to appear.)

[10] Liu, T.-P.; Zeng, Y. *Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws.* Mem. Amer. Math. Soc. **125** (1997), no. 599,

[11] Sone, Y. Kinetic Theory and Fluid Dynamics. *Birkhauser* 2002.

[12] Ukai, S.; Yang, T.; Yu, S.-H. Nonlinear Boundary Layers of the Boltzmann Equation: I. Existence. *Commun. Math. Phys.* **236** (2003), 373-393.

[13] Zeng, Y. L^1 asymptotic behavior of compressible, isentropic, viscous 1-D flow. *Comm. Pure Appl. Math.* **47** (1994), no. 8, 1053–1082.

[14] Ukai, S.; Yang, T.; Yu, S.-H. *Nonlinear stability of boundary layers of the Boltzmann equation. I. The case* $M_\infty < -1$. Comm. Math. Phys. **244** (2004), no. 1, 99–109.

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