

## ON A MARKOV CHAIN APPROXIMATION METHOD FOR OPTION PRICING WITH REGIME SWITCHING

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**ABSTRACT.** In this paper, we discuss a Markov chain approximation method to price European options, American options and barrier options in a Markovian regime-switching environment. The model parameters are modulated by a continuous-time, finite-state, observable Markov chain, whose states represent the states of an economy. After selecting an equivalent martingale measure by the regime-switching Esscher transform, we construct a discrete-time, inhomogeneous Markov chain to approximate the dynamics of the logarithmic stock price process. Numerical examples and empirical analysis are used to illustrate the practical implementation of the method.

**1. Introduction.** To extend the classical Black-Scholes-Merton pricing formula and incorporate some stylized features exhibited in financial time series, many models have been proposed in the literature and have been used in practice. Regime-switching models are one of the most popular and practically useful extensions. The history of the regime-switching models may be traced back to the early works of Quandt [17], Goldfeld and Quandt [12] and Tong [24, 25]. This class of models was then popularized into economics and finance by Hamilton [13]. A key feature of these models is that the impacts of changing economic conditions, which may be attributed to changes in economic fundamentals or financial crises, may be incorporated. Specifically, the model dynamics are allowed to change over time according to the states of an underlying Markov chain, which represent the states of an economy. Regime-switching models have been applied to discuss a number of important research problems in finance, including the option valuation problem.

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There has been a considerable interest to develop and investigate various numerical methods to price and hedge options under regime-switching models. Several methods have been proposed, including the numerical partial differential equation (PDE) approach (see, for example, Boyle and Draviam [1]), the fast Fourier transform approach (see, for example, Liu et al. [14], Shen and Siu [18], Shen et al. [19], Fan et al. [10, 11]), the trinomial tree method (Yuen and Yang [28]) and amongst others. It seems that there is a relatively little attention, if none at all, to apply a Markov chain approximation approach to price options under regime-switching diffusion models. Compared with the methods mentioned above, the Markov chain approximation method has several advantages. Firstly, it provides the flexibility that the discretization number of asset prices and the number of time steps can be chosen independently. In other words, the grids used in the computation can be adjusted so that they are tailor-made to a particular practical situation. Secondly, it seems intuitive, easy to implement and understand, which may be attractive to market practitioners. Furthermore, it converges fast and provides accurate numerical results. More specifically, compared with the fast Fourier transform method and the Monte Carlo method, the Markov chain approximation method seems more flexible to deal with the early-exercise feature of American options and the path-dependent feature of barrier options. Furthermore, the Markov chain approximation method is more computationally efficient than the Monte Carlo method since the former may require considerably less computational time than the latter. The trinomial tree method with regime switches in Yuen and Yang [28] also has similar advantages over the fast Fourier transform method and the Monte Carlo method when pricing American options and barrier options under regime-switching models. However, the number of nodes of the underlying asset price will not increase when the time steps increase in the Markov chain approximation method, while that in the trinomial tree method will increase when the time steps increase. Duan and Simonato [4] proposed a Markov chain approximation method to compute European option prices and American option prices under GARCH models and gave a proof for the convergence of this method. Duan et al. [3] considered the valuation of discretely monitored barrier options using the Markov chain approximation method. Simonato [20] investigated the valuation of American options under a lognormal jump-diffusion model with a Markov chain approximation. Instead of considering the Markov chain approximation method to approximate diffusion-type processes and their variants, option valuation in a continuous-time Markov chain market, where the price process of the underlying asset is modeled directly by a continuous-time Markov chain, has been studied in the literature. Some examples along this line are Pliska [16], Norberg [15], Elliott et al. [8], van der Hoek and Elliott [26, 27], Siu [22], Elliott and Siu [9], amongst others. In Song et al. [23], a multivariate Markov chain model was used for pricing options. For an introduction to Markov chain and its applications, one may refer to Ching et al. [2].

In this paper, we discuss a Markov chain approximation method to price European options, American options and barrier options under a Markovian regime-switching model. More specifically, the model parameters, including the risk-free interest rate, the appreciation rate and the volatility, are modulated by a continuous-time, finite-state, observable Markov chain. It is also called the modulating Markov chain (MMC). We first apply the regime-switching Esscher transform in Elliott et al. [6] to select an equivalent martingale measure. Then we use the Markov chain approximation method to compute the option prices, based on the approximations

of the logarithmic stock price process by another discrete-time, finite-state, inhomogeneous Markov chain. It is called the approximating Markov chain (AMC) and the nodes of the underlying stock price are regarded as the states of the approximating Markov chain. We provide numerical examples and empirical studies to illustrate the practical usefulness of the Markov chain approximation method. Through these numerical examples, we also discuss numerically an important issue regarding the pricing of regime-switching risk, which has been considered in some previous works, for example, Elliott et al. [7] and Siu [21].

The rest of the paper is organized as follows. The next section presents the model dynamics. Section 3 discusses the construction and implementation of the Markov chain approximation method. In section 4, we give numerical examples and empirical studies to illustrate the valuation of European options, American options and barrier options via the Markov chain approximation method. Section 5 concludes the paper.

**2. The model dynamics.** The model setting is based on the work of Elliott et al. [6]. A brief introduction will be given in this section. Consider a complete probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ , under which all sources of randomness are defined. We equip the probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  with a filtration  $\mathbb{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$  satisfying the usual conditions of right-continuity and  $\mathcal{P}$ -completeness. Suppose that  $\mathcal{P}$  is the real-world probability measure. Let  $\mathcal{T}$  denote the time parameter set  $[0, T]$  of the model, where  $T < \infty$ . Let  $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$  be a continuous-time, finite-state, observable Markov chain on  $(\Omega, \mathcal{F}, \mathcal{P})$  with the canonical state space representation  $\mathcal{E}_X := \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\} \subset \mathfrak{R}^N$ , where the  $j^{th}$  component of  $\mathbf{e}_i$  is the Kronecker delta  $\delta_{ij}$  for each  $i, j = 1, 2, \dots, N$ . The canonical state space representation of the Markov chain was used, for example, in Elliott et al. [5]. The chain  $\mathbf{X}$  is the MMC.

Let  $\mathbf{A} := [a_{ij}]_{i,j=1,2,\dots,N}$  be the generator matrix of the MMC  $\mathbf{X}$  under  $\mathcal{P}$ , where  $a_{ij}$  is a constant transition intensity of the chain  $\mathbf{X}$  from state  $\mathbf{e}_j$  to state  $\mathbf{e}_i$ . In the later discussions,  $\mathbf{y}'$  denotes the transpose of a vector or a matrix  $\mathbf{y}$  and  $\langle \cdot, \cdot \rangle$  represents the inner product in  $\mathfrak{R}^N$ .

A continuous-time financial market with two investment securities, namely, a bond  $B$  and a stock  $S$ , is considered. The instantaneous market interest rate is given by

$$r(t) := \langle \mathbf{r}, \mathbf{X}(t) \rangle ,$$

where  $\mathbf{r} := (r_1, r_2, \dots, r_N)' \in \mathfrak{R}^N$  with  $r_j > 0$  for each  $j = 1, 2, \dots, N$ . Then the dynamics of the bond price process  $B := \{B(t) | t \in \mathcal{T}\}$  is given by

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1 .$$

Similarly, the appreciation rate  $\mu(t)$  and the volatility  $\sigma(t)$  of the stock are also modulated by the Markov chain  $\mathbf{X}$  as follows:

$$\mu(t) := \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle, \quad \sigma(t) := \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle, \quad t \in \mathcal{T} ,$$

where  $\boldsymbol{\mu} := (\mu_1, \mu_2, \dots, \mu_N)' \in \mathfrak{R}^N$  and  $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \dots, \sigma_N)' \in \mathfrak{R}^N$  with  $\sigma_j > 0$  for each  $j = 1, 2, \dots, N$ .

Let  $W := \{W(t) | t \in \mathcal{T}\}$  be a standard Brownian motion on  $(\Omega, \mathcal{F}, \mathcal{P})$ . To simplify our discussion,  $W$  and  $\mathbf{X}$  are supposed to be stochastically independent under  $\mathcal{P}$ . The dynamics of the stock price under the real-world probability measure  $\mathcal{P}$  is given by

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad S(0) = S_0 > 0 .$$

Let  $Z(t)$  denote the logarithmic return of the underlying stock over the time interval  $[0, t]$ , i.e.,  $Z(t) := \ln(S(t)/S_0)$ , for  $t \in \mathcal{T}$ . Then, the dynamics of the stock price can be written as

$$S(t) = S(u) \exp(Z(t) - Z(u)) ,$$

where

$$Z(t) = \int_0^t \left( \mu(s) - \frac{1}{2} \sigma^2(s) \right) ds + \int_0^t \sigma(s) dW(s) .$$

Write  $Z := \{Z(t) | t \in \mathcal{T}\}$ . Let  $\mathbb{F}^S = \{\mathcal{F}^S(t) | t \in \mathcal{T}\}$ ,  $\mathbb{F}^Z = \{\mathcal{F}^Z(t) | t \in \mathcal{T}\}$  and  $\mathbb{F}^{\mathbf{X}} = \{\mathcal{F}^{\mathbf{X}}(t) | t \in \mathcal{T}\}$  be the right-continuous,  $\mathcal{P}$ -complete, natural filtrations generated by the processes  $S$ ,  $Z$  and  $\mathbf{X}$ , respectively. Define the filtration  $\mathbb{G} = \{\mathcal{G}(t) | t \in \mathcal{T}\}$  by letting  $\mathcal{G}(t) := \mathcal{F}^Z(t) \vee \mathcal{F}^{\mathbf{X}}(t)$ , the minimal  $\sigma$ -field containing  $\mathcal{F}^Z(t)$  and  $\mathcal{F}^{\mathbf{X}}(t)$ .

As in Elliott et al. [6], the regime-switching Esscher transform is adopted to select an equivalent martingale measure. Define the regime-switching Esscher parameter  $\theta(t) := \theta(t, \mathbf{X}(t))$  as follows:

$$\theta(t, \mathbf{X}(t)) = \langle \boldsymbol{\theta}, \mathbf{X}(t) \rangle ,$$

where  $\boldsymbol{\theta} := (\theta_1, \theta_2, \dots, \theta_N)' \in \mathfrak{R}^N$ .

Then, the regime-switching Esscher transform  $\mathcal{Q}_\theta \sim \mathcal{P}$  on  $\mathcal{G}(t)$  with a family of parameters  $\{\theta(s)\}_{s \in [0, t]}$  is given by

$$\begin{aligned} \left. \frac{d\mathcal{Q}_\theta}{d\mathcal{P}} \right|_{\mathcal{G}(t)} &:= \frac{\exp \left( \int_0^t \theta(s) dZ(s) \right)}{\mathbb{E} \left[ \exp \left( \int_0^t \theta(s) dZ(s) \right) \middle| \mathcal{F}^{\mathbf{X}}(t) \right]} \\ &= \exp \left( \int_0^t \theta(s) \sigma(s) dW(s) - \frac{1}{2} \int_0^t \theta^2(s) \sigma^2(s) ds \right), \quad t \in \mathcal{T} . \end{aligned}$$

By the martingale condition, an equivalent martingale measure  $\mathcal{Q}_{\theta^*}$  is determined, where  $\theta^*(t) = \frac{r(t) - \mu(t)}{\sigma^2(t)}$  for each  $t \in \mathcal{T}$ . The Radon-Nikodym derivative of  $\mathcal{Q}_{\theta^*}$  with respect to  $\mathcal{P}$  on  $\mathcal{G}(t)$  is given by:

$$\left. \frac{d\mathcal{Q}_{\theta^*}}{d\mathcal{P}} \right|_{\mathcal{G}(t)} = \exp \left( \int_0^t \left( \frac{r(s) - \mu(s)}{\sigma(s)} \right) dW(s) - \frac{1}{2} \int_0^t \left( \frac{r(s) - \mu(s)}{\sigma(s)} \right)^2 ds \right) .$$

Then, the dynamics of the stock price and the logarithmic return under  $\mathcal{Q}_{\theta^*}$  are as follows:

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sigma(t)S(t)d\tilde{W}(t) , \\ dZ(t) &= \left( r(t) - \frac{1}{2}\sigma^2(t) \right) dt + \sigma(t)d\tilde{W}(t) , \end{aligned}$$

where  $\tilde{W}(t) = W(t) + \int_0^t \left( \frac{r(s) - \mu(s)}{\sigma(s)} \right) ds$  is a standard Brownian motion with respect to the enlarged filtration  $\mathbb{G}$  under  $\mathcal{Q}_{\theta^*}$ .

**3. A Markov chain approach.** In this section, we apply the Markov chain approximation method to investigate the option valuation problem in the financial market defined in the earlier section. We first present the construction of the proposed Markov chain approximation method. Following Duan and Simonato [4] and Simonato [20], a discrete-time Markov chain is adopted to approximate the option prices, which is also called the approximating Markov chain (AMC). However, unlike

Duan and Simonato [4] and Simonato [20], the AMC is time-inhomogeneous (i.e., the transition matrix of the AMC is modulated by the MMC), since the original dynamics of the stock price follows a Markovian regime-switching diffusion model.

To construct the AMC, let  $[\ln S_0 - I_p, \ln S_0 + I_p]$  be the interval covering the logarithmic stock prices and  $\mathbf{p} := (p_1, p_2, \dots, p_M)'$  be the  $M$  distinct cells determined by discretizing the interval. The way to calculate  $I_p$  is crucial. A typical way is to multiply the standard deviation of the return process with a scaling factor. However, in a multi-state regime-switching model, the risk free interest rate  $r$  and the volatility  $\sigma$  are modulated by the Markov chain  $\mathbf{X}$ , i.e., the risk free interest rate and the volatility of the underlying stock price can take values  $r_1, r_2, \dots, r_N$  and  $\sigma_1, \sigma_2, \dots, \sigma_N$ , respectively. To find an interval for all regimes, the standard deviation of the return process used to calculate  $I_p$  has to be larger than the maximum standard deviation over all regimes, i.e.,  $\sigma > \max_{1 \leq i \leq N} \sigma_i$ . For the ease of comparison, we use the following value adopted by Yuen and Yang [28]:

$$\sigma = \max_{1 \leq i \leq N} \sigma_i + (\sqrt{1.5} - 1)\bar{\sigma} ,$$

where  $\bar{\sigma}$  is the arithmetic mean of  $\sigma_i$ . Then,  $I_p$  can be calculated as:

$$I_p = \delta(M) \times \sigma \times \sqrt{T} .$$

As in Duan and Simonato [4] and Simonato [20], the scaling factor  $\delta(M)$  needs to be an increasing function of  $M$ , while the increasing rate of  $\delta(M)$  is smaller than that of  $M$ . Consequently, the following  $M$  values for possible logarithmic stock prices are obtained:

$$p_k = \ln S_0 + \frac{2k - M - 1}{M - 1} I_p, \quad k = 1, 2, \dots, M .$$

Here  $M$  is an odd integer to ensure that  $\ln S_0$  falls in the middle of the nodes, i.e.,  $p_{\frac{M+1}{2}} = \ln S_0$ .

Let  $\mathbf{Y} := \{\mathbf{Y}_t | t \in \mathcal{T}\}$  be a discrete-time, finite-state Markov chain on  $(\Omega, \mathcal{F}, \mathcal{Q}_{\theta^*})$  with the canonical state space representation  $\mathcal{E}_Y := \{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \dots, \boldsymbol{\epsilon}_M\} \subset \mathfrak{R}^M$ , where the  $l^{th}$  component of  $\boldsymbol{\epsilon}_k$  is the Kronecker delta  $\delta_{kl}$  for each  $k, l = 1, 2, \dots, M$ . The chain  $\mathbf{Y}$  is the AMC. Indeed, the Markov chain approximation method is to approximate the time- $t$  logarithmic stock prices as follows:

$$\ln S(t) \approx \langle \mathbf{p}, \mathbf{Y}_t \rangle ,$$

which means that  $p_k$  is used to approximate the logarithmic stock price when the AMC  $\mathbf{Y}$  is in the  $k$ th state, for each  $k = 1, 2, \dots, M$ .

The probability transition matrix is modulated by the Markov chain  $\mathbf{X}$ , where  $\boldsymbol{\Pi}(t) := \sum_{i=1}^N \langle \mathbf{X}(t), \mathbf{e}_i \rangle \boldsymbol{\Pi}^i$ . Here,  $\boldsymbol{\Pi}^i := [\pi_{kl}^i]_{k,l=1,2,\dots,M}$  represents the probability transition matrix of the Markov chain  $\mathbf{Y}$  under  $\mathcal{Q}_{\theta^*}$  when the economic state is  $\mathbf{e}_i$ , where  $\pi_{kl}^i$  is the transition probability of the chain  $\mathbf{Y}$  from state  $\boldsymbol{\epsilon}_k$  to state  $\boldsymbol{\epsilon}_l$  when the economy state is  $\mathbf{e}_i$ . To compute  $\boldsymbol{\Pi}^i$ , for each  $i = 1, 2, \dots, N$ , the following  $M$  cells are defined:

$$C_k = [c_k, c_{k+1}), \quad k = 1, 2, \dots, M ,$$

on  $[0, \infty)$  such that

$$c_k = \begin{cases} -\infty, & k = 1, \\ \frac{p_k + p_{k-1}}{2}, & k = 2, \dots, M, \\ +\infty, & k = M + 1. \end{cases}$$

The probability of the chain  $\mathbf{Y}$  from state  $\epsilon_k$  to state  $\epsilon_l$  in the  $i$ th economic state equals the probability of the logarithmic stock price landing in the cell  $C_l$  given a current price  $p_k$  and an economic state  $\mathbf{e}_i$ . Let  $m$  be the number of steps in the time discretization and  $T$  be the maturity time. If the stock price is in the  $k$ th node at time  $d\Delta$  and changes to the  $l$ th node at time  $(d+1)\Delta$ , the AMC jumps from the  $k$ th state to the  $l$ th state during the time interval  $[d\Delta, (d+1)\Delta]$ , where  $\Delta := \frac{T}{m}$  and  $d = 0, 1, \dots, m$ .  $S_{d\Delta, k}$  represents the stock price of the  $k$ th node at time  $d\Delta$ , for  $d = 0, 1, \dots, m$ . Then under  $\mathcal{Q}_{\theta^*}$ ,  $\pi_{kl}^i$  is calculated as follows:

$$\begin{aligned} \pi_{kl}^i &:= \mathcal{Q}_{\theta^*}[\ln S_0 + Z((d+1)\Delta) \in C_l | \ln S_0 + Z(d\Delta) = p_k, \mathbf{X}(d\Delta) = \mathbf{e}_i] \\ &= \mathcal{Q}_{\theta^*}[c_l \leq \ln S_0 + Z((d+1)\Delta) < c_{l+1} | \ln S_0 + Z(d\Delta) = p_k, \mathbf{X}(d\Delta) = \mathbf{e}_i] \\ &= \Phi\left(\frac{c_{l+1} - p_k - r_i\Delta + \frac{1}{2}\sigma_i^2\Delta}{\sigma_i\sqrt{\Delta}}\right) - \Phi\left(\frac{c_l - p_k - r_i\Delta + \frac{1}{2}\sigma_i^2\Delta}{\sigma_i\sqrt{\Delta}}\right), \end{aligned}$$

where  $\Phi(\cdot)$  is the cumulative distribution function of a standard normal distribution.

The value of the European option at the  $k$ th node at time  $d\Delta$  in the  $i$ th economy state is denoted as  $V_{d\Delta, k, i}$ , while the value of the American option at the  $k$ th node at time  $d\Delta$  in the  $i$ th economic state is denoted as  $V_{d\Delta, k, i}^{Ame}$ . Also, the transition probability of the Markov chain  $\mathbf{X}$  over  $[d\Delta, (d+1)\Delta]$  can be calculated from the transition rate matrix  $\mathbf{A}$  by

$$\mathbf{O} = e^{\mathbf{A}\Delta} = \mathbf{I} + \sum_{i=1}^{\infty} \frac{\mathbf{A}^i(\Delta)^i}{i!},$$

where  $\mathbf{O} := [o_{ij}]_{i, j=1, 2, \dots, N}$  with  $o_{ij}$  representing the transition probability of the MMC  $\mathbf{X}$  from state  $\mathbf{e}_i$  to state  $\mathbf{e}_j$  over the time interval  $[d\Delta, (d+1)\Delta]$ .

Since the Markov chain  $\mathbf{X}$  is independent of the Brownian motion, the probability matrix  $\mathbf{O}$  remains the same after the measure change. Then, the price of a  $T$ -maturity European option can be computed with the following recursion:

$$V_{d\Delta, k, i} = e^{-r_i\Delta} \left[ \sum_{j=1}^N \sum_{l=1}^M o_{ij} \pi_{kl}^i V_{(d+1)\Delta, l, j} \right].$$

For American options, we need to consider their early exercise privileges. Consequently, during the time interval  $[d\Delta, (d+1)\Delta]$ , the time- $d\Delta$  value of the American option equals the larger value of its intrinsic value at time  $t$  and the discounted expected time- $(d+1)\Delta$  value of the American option. Let  $f$  be a random variable having the following binomial distribution:

$$f = \begin{cases} 1, & \text{The option is a call option,} \\ -1, & \text{The option is a put option.} \end{cases}$$

In our analysis, the following recursion is used:

$$V_{d\Delta, k, i}^{Ame} = \max \left( \max(f(S_{d\Delta, k} - K), 0), e^{-r_i\Delta} \left[ \sum_{j=1}^N \sum_{l=1}^M o_{ij} \pi_{kl}^i V_{(d+1)\Delta, l, j} \right] \right).$$

Both iterations start from the maturity time  $T$  with the terminal payoff:

$$V_{T,k,i} = (f(S_{T,k} - K))_+, \quad \text{for all states ,}$$

where  $S_{T,k} = e^{p_k}$ .

To price barrier options under regime-switching models, following Duan et al. [3], we consider an index variable  $b_t$ , which follows a Bernoulli distribution, i.e.,

$$b_t = \begin{cases} 1, & \text{The barrier is triggered before or at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Define a knock-out set  $\mathcal{U}$ , which means that the barrier option will knock out if the state of the logarithmic stock price falls into this set. Let  $h$  and  $H$  be the lower barrier and upper barrier, respectively. Mathematically,  $\mathcal{U}$  is defined as follows:

$$\mathcal{U} = \begin{cases} \{k \in \{1, \dots, M\} : e^{p_k} \leq h\}, & \text{for a down-and-out option,} \\ \{k \in \{1, \dots, M\} : e^{p_k} \geq H\}, & \text{for a up-and-out option,} \\ \{k \in \{1, \dots, M\} : e^{p_k} \leq h \text{ or } e^{p_k} \geq H\}, & \text{for a double-barrier-out option.} \end{cases}$$

Considering the impacts of the barrier, the probability of transiting from  $p_k$  to  $p_l$  given a current price  $p_k$  and an economic state  $\mathbf{e}_i$  can be calculated as:

$$\begin{aligned} \pi_{kl}^{io} &:= \mathcal{Q}_{\theta^*}[\ln S_0 + Z((d+1)\Delta) \in C_l, b_{(d+1)\Delta} = 0 | \ln S_0 + Z(d\Delta) = p_k, \\ &\quad \mathbf{X}(d\Delta) = \mathbf{e}_i, b_{d\Delta} = 0] \\ &= \begin{cases} \pi_{kl}^i, & \text{if } k, l \notin \mathcal{U}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let  $V_{d\Delta,k,i;b_{d\Delta}=0}^{Bar}$  denote the conditional value of the barrier option at the  $k$ th node at time  $d\Delta$  in the  $i$ th economic state given that the barrier is not triggered before or at time  $d\Delta$ . If the barrier is triggered before or at time  $d\Delta$ , the value of the barrier option equals zero. Then, the following recursive pricing formula for the corresponding barrier option is obtained:

$$V_{d\Delta,k,i;b_{d\Delta}=0}^{Bar} = e^{-r_i \Delta} \left[ \sum_{j=1}^N \sum_{l=1}^M o_{ij} \pi_{kl}^{io} V_{(d+1)\Delta,l,j;b_{(d+1)\Delta}=0} \right].$$

**4. Numerical results and analysis.** In this section, we perform a numerical analysis for option valuation based on the Markov chain approximation method presented earlier. For the ease of comparison, we also provide the numerical results of the valuation of options under the regime-switching trinomial tree method given in Yuen and Yang [28]. To simplify our computation, we consider a two-state modulating Markov chain  $\mathbf{X}$ , where State 1 and State 2 of the chain represent a ‘Good’ economy and a ‘Bad’ economy, respectively. We write  $\mathbf{X}(t) = (1, 0)'$  and  $\mathbf{X}(t) = (0, 1)'$  for State 1 and State 2, respectively.

In what follows, configurations of the parameters values are presented. The rate matrix of the chain  $\mathbf{X}$  under  $\mathcal{Q}_{\theta^*}$  is given by

$$\mathbf{A} = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}.$$

The larger  $a$  is, the more volatile the economy is. That is, the probability of the transition of the economy from one state to another increases with  $a$ . Note that when  $a = 0$ , the regime-switching effect is degenerate. Generally speaking, the

main features of the financial market in a ‘Bad’ (‘Good’) economy are low (high) appreciation rate, low (high) interest rate and high (low) volatility.

In the following, we use the valuation of put options as an example, i.e.,  $f = -1$ . The prices of the options in different economic states are computed by Matlab programs. We compute the option prices using the trinomial tree method in Yuen and Yang [28], the Monte Carlo method, the Markov chain approximation with regime-switching and the Markov chain approximation without regime switches in Duan and Simonato [4]. For notational simplicity, we denote the three methods by  $Y\&Y$ ,  $MC$ ,  $RSMCA$  and  $D\&S$ , respectively.

TABLE 1. European put option prices using different methods

European put option I					
$S_0$	State 1				
	$Y\&Y$	$MC$	$RSMCA$	$D\&S$	
94	4.8074	4.8107	4.8149	3.8553	
96	3.8978	3.8870	3.8989	2.9622	
98	3.1308	3.1429	3.1288	2.2301	
100	2.4853	2.4812	2.4895	1.6448	
102	1.9714	1.9420	1.9674	1.1889	
104	1.5520	1.5509	1.5466	0.8421	
State 2					
94	7.8567	7.8001	7.7903	8.5570	
96	6.9006	6.8894	6.8310	7.5888	
98	6.0372	6.0042	5.9641	6.7035	
100	5.2557	5.2089	5.1857	5.8979	
102	4.5679	4.4892	4.4909	5.1689	
104	3.9531	3.8607	3.8741	4.5121	
European put option II					
$S_0$	State 1				
	$Y\&Y$	$MC$	$RSMCA$	$D\&S$	
94	5.3588	5.3359	5.3555	3.8553	
96	4.4362	4.4326	4.4308	2.9622	
98	3.6452	3.6379	3.6370	2.2301	
100	2.9673	2.9530	2.9650	1.6448	
102	2.4135	2.4083	2.4036	1.1889	
104	1.9505	1.9274	1.9395	0.8421	
State 2					
94	7.4447	7.3826	7.3962	8.5570	
96	6.4906	6.4466	6.4392	7.5888	
98	5.6344	5.5823	5.5797	6.7035	
100	4.8642	4.8380	4.8133	5.8979	
102	4.1924	4.1171	4.1346	5.1689	
104	3.5966	3.5197	3.5370	4.5121	

Table 1 shows European put option prices corresponding to different values of the initial stock price  $S_0$ , computed using the above procedure with maturity time  $T = 1$ , strike price  $K = 100$  and the discretization number of the underlying stock price  $M = 501$ . Note that a  $((M - 1)/2)$ -step trinomial tree with recombination



also yields  $M$  different stock prices. Let  $m$  denote the number of time steps. For ease of comparison, we assume  $m = (M - 1)/2$  in Table 1. The function  $\delta(M)$  is set to be  $\ln(\ln(M))$ . The transition rate matrices used in Table 1 are  $\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$  and  $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ .

The prices of the European put option I are calculated under the assumption that the transition rate matrix is  $\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$ , while the prices of the European put option II are calculated under the assumption that the transition rate matrix is  $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ . We consider the following vectors for the values of the risk-free interest rate and the volatility, respectively:

$$\mathbf{r} = (0.06, 0.04)' , \quad \boldsymbol{\sigma} = (0.1, 0.2)' .$$

As indicated in Table 1, the option prices calculated by applying the Markov chain approximation method are close to those obtained from the trinomial tree method in Yuen and Yang [28] and the Monte Carlo method. For the same level of the initial stock price, the option prices in State 1 are systematically lower than those in State 2 using all three methods. This makes intuitive sense. In State 1 ('Good' economy), the interest rate is higher and the volatility is lower, when comparing with State 2 ('Bad' economy). Consequently, it is not unreasonable that the option prices in State 1 are lower than the corresponding prices in State 2 since an additional amount of risk premium is required to compensate for a 'Bad' economic condition. On the other hand, using the Markov chain approximation method, the option prices calculated under the *RSMCA* model are higher than the corresponding prices obtained under the *D&S* model in State 1, but are lower than the prices obtained using the *D&S* model in State 2. Under the *D&S* model, the option prices obtained in State 1 are lower than those in State 2 with the same initial stock price. Consequently, in the *RSMCA* model, when the impact of the transition between the two states is incorporated, the option prices obtained under the *RSMCA* model are higher than those under the *D&S* model in State 1, while are lower in State 2. For the above two hypothetical data sets, the option prices of the European put option I are smaller than those of the European put option II in State 1, while the prices of European put option I are higher than those of the European put option II in State 2. This is consistent with the results in Yuen and Yang [28] and the Monte Carlo method. One possible explanation for the results is that the larger  $a$  is, the more volatile the economy is. In other words, the probability of the transition of the economy from one state to another increases with  $a$ , leading to the results in Table 1.

From Table 2 and Table 3, we see that the convergence rate of the European put options is fast across different initial stock prices and maturities. The convergence analysis of European call options, American put options and American call options can be done similarly. Here, the number of time steps is assumed equal to the number of days to maturity. To investigate the convergence rate of the Markov chain approximation method, we assumed a pair of larger volatilities as  $\boldsymbol{\sigma} = (0.25, 0.35)'$  in Table 2 and Table 3, while the transition rate matrix is  $\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$ . As indicated in Table 2 and Table 3, the convergence rates are fast and the computation times sound reasonable.

TABLE 2. European put option prices with different  $S_0$  and  $M$ 

$S_0$	80	90	100	110	120
State 1					
$M = 501$	18.3466	12.3093	7.9112	4.9131	2.9729
$M = 1001$	18.3490	12.3088	7.9100	4.9135	2.9754
$M = 2001$	18.3517	12.3110	7.9124	4.9160	2.9793
$M = 3001$	18.3531	12.3123	7.9138	4.9177	2.9812
$M = 4001$	18.3539	12.3132	7.9147	4.9187	2.9824
State 2					
$M = 501$	21.0190	15.2373	10.7887	7.4922	5.1180
$M = 1001$	21.0511	15.2715	10.8261	7.5383	5.1642
$M = 2001$	21.0722	15.2947	10.8518	7.5620	5.1954
$M = 3001$	21.0810	15.3046	10.8627	7.5740	5.2086
$M = 4001$	21.0861	15.3103	10.8690	7.5810	5.2162

TABLE 3. European put option prices with different  $T$  and  $M$ 

$T$	10/365	30/365	60/365	90/365	270/365
State 1					
$M = 501$	1.4952	2.5943	3.5909	4.3176	6.9768
$M = 601$	1.4953	2.5945	3.5912	4.3180	6.9765
$M = 701$	1.4954	2.5947	3.5915	4.3183	6.9765
$M = 801$	1.4954	2.5949	3.5918	4.3186	6.9766
$M = 901$	1.4955	2.5950	3.5920	4.3189	6.9769
$M = 1001$	1.4955	2.5951	3.5921	4.3191	6.9772
State 2					
$M = 501$	2.1162	3.6942	5.1303	6.1660	9.6946
$M = 601$	2.1186	3.6998	5.1384	6.1756	9.7064
$M = 701$	2.1204	3.7040	5.1446	6.1828	9.7154
$M = 801$	2.1218	3.7073	5.1494	6.1886	9.7226
$M = 901$	2.1230	3.7100	5.1534	6.1932	9.7285
$M = 1001$	2.1239	3.7123	5.1567	6.1971	9.7334

Table 4 gives the prices of American put options, assuming the transition rate matrix is  $\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$ . When compared with the prices of European put option I in Table 1, the prices of the American put option are higher than the corresponding prices of the European put option. A possible explanation is that there exist possible scenarios leading to an early exercise of the American put option.

Table 5 illustrates the valuation of an up-and-out barrier put option. Here, Diff represents the difference between the option prices computed with various discretization numbers of the underlying stock prices. Seen from Table 1 and Table 5, under the assumption that the barrier level is set as 120, the prices of the up-and-out barrier put option are lower than those of the corresponding European put option in both states when  $S_0 = 100$ . On the other hand, the differences of the up-and-out put option prices between the two states are smaller than those of the European put option prices. These make intuitive sense. When compared with the European

TABLE 4. American put option prices calculated using different methods

$S_0$	State 1			State 2		
	<i>Y&amp;Y</i>	<i>RSMCA</i>	<i>D&amp;S</i>	<i>Y&amp;Y</i>	<i>RSMCA</i>	<i>D&amp;S</i>
94	6.5096	6.5151	6.0105	8.5086	8.5068	9.3308
96	5.2366	5.2439	4.3827	7.3667	7.3647	8.2550
98	4.2070	4.2112	3.1553	6.3519	6.3488	7.2788
100	3.3670	3.3738	2.2403	5.4520	5.4517	6.3970
102	2.6947	2.6972	1.5674	4.6677	4.6648	5.6040
104	2.1517	2.1527	1.0798	3.9818	3.9785	4.8937

put option, the price of a barrier option should be lower due to the presence of the up-and-out barrier. For state 2, although the higher volatility may lead to a potential higher payoff at expiration, the higher volatility may also lead to higher probability to hit the up-and-out barrier. Consequently, the difference between the prices of the up-and-out barrier put option in the two states will be smaller. It is worth noting that the convergence rate is reasonably fast. This may be desirable from the practical perspective.

TABLE 5. Prices of the up-and-out barrier put option

$M$	State 1		State 2	
	Price	Diff	Price	Diff
101	2.4683	-0.0026	4.9941	0.0566
501	2.4657	-0.0014	5.0507	0.0113
1001	2.4643	-0.0009	5.0620	0.0079
2001	2.4634	-0.0004	5.0699	0.0033
3001	2.4630	-0.0002	5.0732	0.0018
4001	2.4628	-0.0002	5.0750	0.0012
5001	2.4626		5.0762	

Finally, we provide an empirical study based on market prices of S&P 500 index options. For simplicity, we use the European-style option prices to illustrate the implementation of the method. The dataset includes European-style option prices written on the S&P 500 index for seven consecutive trading days from 1 October 2012 to 9 October 2012, obtained from the Datastream Database of Reuters. For each trading day, there are 13 strikes ranging from 1300 to 1600. The in-sample dataset consists of the option prices from 1 October 2012 to 5 October 2012 and the out-of-sample dataset consists of the rest option prices on 8 October 2012 to 9 October 2012. To calibrate model parameters, we adopt the method of nonlinear least squares using the in-sample dataset. We denote the regime-switching model and the model without regime-switching as the *RSMCA* model and the *D&S* model, respectively. Then, we illustrate how the *RSMCA* model might improve the performance of the *D&S* model based on the market data. Table 6 reports the the root mean square error (RMSE) for in-sample fitting errors and out-of-sample prediction errors of both the *RSMCA* model and the *D&S* model. As indicated in Table 6, the *RSMCA* model has lower RMSEs for both fitting and prediction errors than the *D&S* model. This provides some evidence that the *RSMCA* model may improve the performance of the *D&S* model.

TABLE 6. In-sample fitting errors and out-of-sample prediction errors

Errors	<i>RSMCA model</i>	<i>D&amp;S model</i>
In-sample	0.3594%	0.3789%
Out-of-sample	0.6194%	0.6685%

**5. Conclusion.** We investigated option valuation under the regime-switching model using the Markov chain approximation method. This method is intuitive and easy to implement. It may be applied to various pricing problems. We considered here two Markov chains, namely the modulating Markov chain and the approximating Markov chain. The states of the former one represent the states of an economy. When the economic condition changes, the state of the modulating Markov chain also changes. Consequently, the model can incorporate the impacts of economic conditions. Using the Markov chain approximation method, we numerically implemented the valuation of European put options, American put options and up-and-out barrier put options. Numerical examples and empirical analysis illustrated the impacts of regime-switching on option prices.

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