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A WEAK CONDITION FOR GLOBAL STABILITY OF DELAYED NEURAL NETWORKS

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Abstract. The classical analysis of asymptotical and exponential stability of neural networks needs assumptions on the existence of a positive Lyapunov function V and on the strict negativity of the function dV/dt, which often come as a result of boundedness or uniformly almost periodicity of the activation functions. In this paper, we investigate the asymptotical stability problem of Hopfield neural networks with time delays under weaker conditions. By constructing a suitable Lyapunov function, sufficient conditions are derived to guarantee global asymptotical stability and exponential stability of the equilibrium of the system. These conditions do not require the strict negativity of dV/dt, nor do they require that the activation functions to be bounded or uniformly almost periodic.

1. **Introduction.**

Hopfield neural networks were introduced in 1984 [11], they have been successfully applied to many disciplines such as combinatorial optimization [1, 24, 26], image processing [21, 22], pattern recognition [23], signal processing [15], and communication [3]. Much research has been attracted to this area in the past decades [18, 28, 29, 9, 33, 20, 32, 4, 30, 5, 6, 7, 8, 10, 13, 14, 17, 19, 25, 27, 34, 12, 31].

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Generally, a Hopfield neural network can be described by the following differential equation

$$\dot{x}(t) = -Dx(t) + Ag(x(t)) + I, \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons. $g(x) = (g_1(x_1), g_2(x_2), \cdots, g_n(x_n))^T$ denotes the neuron activation, in which $g_i(x_i)$ are continuous on $[0, +\infty)$. $D = \text{diag}\{d_1, \cdots, d_n\}$, with $d_i > 0$. The $n \times n$ connection matrix $A = (a_{ij})_{n\times n}$ tells how the neurons are connected in the network. $I = (I_1, \cdots, I_n)^T$ is the vector of constant neuron inputs.

We are concerned with the stability properties of system (1). Usually, the approaches used in the existing investigation for stability of neural networks are those based on the Lyapunov direct method. This method requires constructing a Lyapunov functions $V(t)$, taking the derivative of the function, and keeping negativity of the derivative according to the given conditions. The main difficulty for stability analysis of neural networks comes from the nonlinearity of the activation functions $g_i(i = 1, 2, \cdots, n)$. Almost all stability analysis results are made under some special assumptions on g_i, including differentiability, boundedness, and others

In [33], Zhang obtained a weaker condition for global asymptotical stability and exponential stability of system (1). He assumed that the activation functions g_j satisfy $0 < D^+g_j(s) < D^+g_j(0)$, $\forall s \neq 0, j = 1, \cdots, n$, where D^+ stands for the upper right Dini derivative defined in Section 2.1. These activation functions may be neither bounded nor differentiable, and it is proved that (1) is globally asymptotically stable and exponentially stable under $\dot{V}(t) \leq 0$ rather than $\dot{V}(t) < 0$. In [12], Ignatyev discussed the general form of system (1), that is $\dot{x}(t) = f(t, x)$, under the weaker condition that $\dot{V}(t) \leq 0$, but $f(t, x)$ and $V(t)$ are all assumed to be uniformly almost periodic in t.

In practice, due to finite switching speed of amplifiers and communication speed between the neurons, time delays are inevitably encountered in the electronic implementations of neural networks. Time delays can change the dynamics of a network, such as inducing a network to exhibit oscillations or other unstable behaviours [2]. Since the existence of time delays is often a source of instability of neural networks, considerable efforts have been focused on the stability analysis of neural networks with time delays. A variety of results on global asymptotical stability and global exponential stability have been proposed in [20, 32, 4, 30, 5, 6, 7, 8, 10, 13, 14, 34]. One of the most common models of the neural networks with time delays can be presented below in (2).

Motivated by the above considerations, in this paper, we will further investigate the global asymptotical stability and exponential stability for the following Hopfield neural networks with time delays

$$\dot{x}(t) = -Dx(t) + Ag(x(t)) + Bg(x(t - \tau(t))) + I, \tag{2}$$

under a weak condition, where $B = (b_{ij})_{n\times n}$ with the constants b_{ij} denoting the delayed connection weights, $g(x(t-\tau(t))) = (g_1(x_1(t-\tau_1(t))), \cdots, g_n(x_n(t-\tau_n(t))))$, $\tau_i(t)$ is the discrete time delay of ith neuron at time t, and satisfies $0 \leq \tau_i(t) \leq \tau_i$, $\dot{\tau}_i(t) \leq \tau_i^* \leq 1$, $i = 1, \cdots, n$, and the other parameters are the same as those in (1).

In [16], Lao et al investigated the exponential stability and estimated the exponential convergence rates for (2) with constant or time-varying delays. Their conditions require that the activation functions must be bounded. In this paper, we remove the restriction and do not require that the activation functions to be bounded or uniformly almost periodic.

2. Preliminaries.

2.1. Definitions and assumptions.
We need to make some definitions and assumptions. We define the upper right Dini derivative as follows.

For any continuous function $f : \mathbf{R} \to \mathbf{R}$, the upper right Dini derivative is defined as

$$D^+ f(t) = \lim_{\Delta t \to 0^+} \sup \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

We assume that the continuous activation functions g_i in (2) satisfy the following conditions for all $j = 1, \cdots, n$,

$$0 < \frac{g_i(z_2) - g_i(z_1)}{z_2 - z_1} \leq D^+ g_j(0) \quad (z_1 \neq z_2,\ j = 1, \cdots, n). \tag{3}$$

Thus,

$$0 < D^+ g_j(s) < D^+ g_j(0) \quad (\forall s \neq 0, j = 1, \cdots, n). \tag{4}$$

Since the activation function of (2) satisfies the Lipschitz conditions (3), by using the Brouwer's fixed-point theorem, it can be easily proven that there exists one equilibrium point for (2), this result can be seen in many literature [20, 32, 4, 30, 5, 6]. Assume that $x^* = (x_1^*, \cdots, x_n^*)^T$ is an equilibrium point of (2) and use the transformation $y(t) = x(t) - x^*$, (2) can be rewritten as the following system:

$$\dot{y}(t) = -Dy(t) + Af(y(t)) + Bf(y(t - \tau(t))), \tag{5}$$

where $f(y) = g(y+x^*) - g(x^*)$. Obviously, $f(0) = 0$, and each component f_j of the vector function $f(y) = (f_1(y_1), \cdots, f_n(y_n))^T$ satisfies (3) and (4), that is

$$0 < \frac{f_i(z_2) - f_i(z_1)}{z_2 - z_1} \leq D^+ g_j(0) \quad (j = 1, \cdots, n), \tag{6}$$

$$0 < D^+ f_j(s) < D^+ g_j(0) \quad (j = 1, \cdots, n). \tag{7}$$

2.2. Notations.
For the sake of convenience, we make use of the following notations:

(I) $\eta_1 = y(t),\ \eta_2 = f(y(t)),\ \eta_3 = f(y(t - \tau(t)))$;
(II) $G_i = D^+ g_i(0)$, $G = \mathrm{Diag}\{G_i\}$, $G^* = \max\{G_i\}$, where $i = 1, \cdots, n; \tau^* = \max\{\tau_1^*, \cdots, \tau_n^*\}$;
(III) $\|y(t)\| = \sqrt{y_1^2(t) + y_2^2(t) + \cdots + y_n^2(t)}$;
(IV) The superscript T denotes the matrix transpose and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathrm{Diag}\{\cdots\}$ denotes the block diagonal matrix. E denotes the identity matrix.

By using (I), system (5) can be rewritten as

$$\dot{y}(t) = -D\eta_1 + A\eta_2 + B\eta_3. \tag{8}$$

2.3. Necessary lemmas.
A very useful tool is the so-called Schur complementary lemma as follows.

Lemma 2.1 (Schur complement [34])**.** *Let S be a symmetric matrix given by*

$$S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

where both A, and C are symmetric. Assume that C is positive definite. Then the following properties are equivalent:
(a) *S is positive semi-definite.*

(b) *The Schur complement of C in S, defined as the matrix $A - BC^{-1}B^T$, is positive semi-definite.*

3. **Main results.** In this section, we will establish some sufficient conditions of global asymptotical stability and exponential stability for system (2) under a weak condition.

Theorem 3.1. *Suppose that the activation function $f(y)$ of system (5) satisfies (6). If $2DG^{-1} - A - A^T - E - (1-\tau^*)^{-1}BB^T$ is positive semi-definite, then system (5) is globally asymptotically stable and exponentially stable.*

Proof. We define the following Lyapunov functional candidate for system (5)

$$V(t) = 2\sum_{i=1}^{n}\int_{0}^{y_i(t)} f_i(s)ds + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} f_i^2(y_i(s))ds, \tag{9}$$

Obviously, $V(t)$ is radially unbounded. By (7), we see that $V(t) \geq 0$ for all t, and $V(t) = 0$ for a certain t if and only if $y(t) = 0$.

Using (6), we obtain

$$\begin{aligned}
&0 < \frac{f_i(y_i(t))}{y_i(t)} \leq G_i,\\
\Rightarrow &0 < f_i(y_i(t))f_i(y_i(t)) \leq G_i y_i(t) f_i(y_i(t)),\\
\Rightarrow &f_i(y_i(t))y_i(t) \geq \frac{1}{G_i} f_i(y_i(t))f_i(y_i(t)).
\end{aligned}$$

Noting that $d_i > 0$ $(i = 1, 2, \cdots, n)$, and $G = \mathrm{Diag}\{G_i\}$ $(i = 1, 2, \cdots, n)$, we have

$$\begin{aligned}
-f^T(y(t))Dy(t) &= -\sum_{i=1}^{n} f_i(y_i(t))d_i y_i(t)\\
&\leq -\sum_{i=1}^{n} f_i(y_i(t))\frac{d_i}{G_i} f_i(y_i(t))\\
&= -f^T(y(t))DG^{-1}f(y(t)).
\end{aligned} \tag{10}$$

It follows from (I) that (10) can be rewritten in the form of

$$-\eta_2^T D\eta_1 \leq -\eta_2^T DG^{-1}\eta_2.$$

Then, differentiating both sides of (9), and using notation (I), we can get

$$\begin{aligned}
\dot{V}(t) =& 2f^T(y(t))\dot{y}(t) + f^T(y(t))f(y(t)) - \sum_{i=1}^{n}(1-\dot{\tau}_i(t))f_i^2(y_i(t-\tau_i(t)))\\
\leq& 2\eta_2^T(-D\eta_1 + A\eta_2 + B\eta_3) + \eta_2^T\eta_2 + (\tau^*-1)\eta_3^T\eta_3\\
=& -2\eta_2^T D\eta_1 + 2\eta_2^T A\eta_2 + 2\eta_2^T B\eta_3 + \eta_2^T\eta_2 + (\tau^*-1)\eta_3^T\eta_3\\
\leq& -2\eta_2^T DG^{-1}\eta_2 + 2\eta_2^T A\eta_2 + 2\eta_2^T B\eta_3 + \eta_2^T\eta_2 + (\tau^*-1)\eta_3^T\eta_3\\
=& \eta_2^T(-2DG^{-1})\eta_2 + (\eta_2^T A\eta_2 + \eta_2^T A^T\eta_2)\\
&+ (\eta_2^T B\eta_3 + \eta_3^T B^T\eta_2) + \eta_2^T\eta_2 + (\tau^*-1)\eta_3^T\eta_3\\
=& \eta_2^T(-2DG^{-1} + A + A^T + E)\eta_2 + \eta_2^T B\eta_3 + \eta_3^T B^T\eta_2 + (\tau^*-1)\eta_3^T\eta_3\\
=& \begin{pmatrix}\eta_2^T & \eta_3^T\end{pmatrix}\begin{bmatrix}-2DG^{-1} + A + A^T + E & B\\ B^T & (\tau^*-1)E\end{bmatrix}\begin{pmatrix}\eta_2\\ \eta_3\end{pmatrix} \leq 0.
\end{aligned} \tag{11}$$

The last inequality above is due to the facts that $(\tau^* - 1)E$ is positive definite, $2DG^{-1} - A - A^T - E - (1-\tau^*)^{-1}BB^T$ is positive semidefinite, and according to the Schur complementary lemma (Lemma 2.1), we have

$$\begin{bmatrix} 2DG^{-1} - A - A^T - E & -B \\ -B^T & (1-\tau^*)E \end{bmatrix} \geq 0 \tag{12}$$

that is

$$\begin{bmatrix} -2DG^{-1} + A + A^T + E & B \\ B^T & (\tau^* - 1)E \end{bmatrix} \leq 0.$$

Therefore, $\dot{V}(t) \leq 0$, and $V(t)$ is a decreasing function. In addition, $V(t) \geq 0$, so $V^* = \lim_{t\to 0} V(t)$ exist and $V^* \geq 0$.

Next, we prove that $V^* = 0$ by contradiction. If $V^* \neq 0$, we assume that $V^* > 0$. Then there must be a $t_1 > 0$ such that $V(t) \geq V^*/2$ for all $t \geq t_1$. According to the definition of $V(t)$, there is a constant $\delta_1 > 0$, such that $\|y(t)\| \geq \delta_1$, i.e., $\sqrt{y_1^2(t) + y_2^2(t) + \cdots + y_n^2(t)} \geq \delta_1$, for $t \geq t_1$; Let $N = \delta_1/\sqrt{n}$, then, for each fixed $t \geq t_1$, there must be a $y_q(t) \in \{y_1(t), y_2(t), \cdots, y_n(t)\}$ such that $y_q(t) \geq N$, all this "q" compose a new sequence $\{n_1, n_2, \cdots, n_r\}$. On the other hand, because $\lim_{t\to 0} V(t)$ exists, we get that each of $y_j(t)$ in $\{y_1(t), y_2(t), \cdots, y_n(t)\}$ is bounded, that is

$$|y_j(t)| \leq M, \ \forall t \geq t_1, \ j = 1, \cdots, n, \tag{13}$$

in which the constant $M > \delta_1$. On the other hand, if (13) is not true, without loss of generality, we assume that there exists an increasing sequence $\{\alpha_i\}$, with $\alpha_i \geq t_1$, and $\alpha_i \to +\infty$, such that $y_1(\alpha_i) \to +\infty$. Then, there exists an integer $p > 0$, such that $y_1(\alpha_i) > 0$ for $i > p$. Because $f_1(y_1(\alpha_p)) > 0$, we see that

$$V(y(\alpha_i)) \geq \int_{y_1(\alpha_p)}^{y_1(\alpha_i)} f_1(y_1(\alpha_p)) \geq f_1(y_1(\alpha_p))(y_1(\alpha_i) - y_1(\alpha_p)) \to +\infty, \text{ as } i \to +\infty.$$

Thus, $\lim_{t\to 0} V(t)$ exists, which causes a contradiction. Hence (13) is true.

From (11), for each fixed $t \geq t_1$, we have

$$\begin{aligned}
\dot{V}(t) =& f^T(y(t))D(-2y(t) + 2G^{-1}f(y(t))) \\
& - \begin{pmatrix} \eta_2^T & \eta_3^T \end{pmatrix} \begin{bmatrix} 2DG^{-1} - A - A^T - E & -B \\ -B^T & -(\tau^* - 1)E \end{bmatrix} \begin{pmatrix} \eta_2 \\ \eta_3 \end{pmatrix} \\
=& -2\sum_{i=1}^{n} f_i(y_i(t))d_i \left[1 - \frac{1}{G_i}\frac{f_i(y_i(t))}{y_i(t)}\right] y_i(t) \\
& - \begin{pmatrix} \eta_2^T & \eta_3^T \end{pmatrix} \begin{bmatrix} 2DG^{-1} - A - A^T - E & -B \\ -B^T & -(\tau^* - 1)E \end{bmatrix} \begin{pmatrix} \eta_2 \\ \eta_3 \end{pmatrix} \\
\leq& -2f_q(y_q(t))d_q \left[1 - \frac{1}{G_q}\frac{f_q(y_q(t))}{y_q(t)}\right] y_q(t) \\
\leq& -2d_q \left[1 - \frac{1}{G_q}\frac{f_q(y_q(t))}{y_q(t)}\right] \frac{(f_q(y_q(t)))^2}{G_q} \\
=& -2d_q \left[1 - \frac{1}{G_q}\frac{f_q(y_q(t))}{y_q(t)}\right] \frac{(f_q(y_q(t)))^2}{G_q}
\end{aligned}$$

According to (7) and the conditions of theorem, we note that $f_i(y_i)y_i \geq 0$, $1 - G_j^{-1}f_j(y_i)/y_j \geq 0$, and $2DG^{-1} - A - A^T - E - (1-\tau^*)^{-1}BB^T \geq 0$. Since f_q is an increasing and continuous function and satisfies $f_q(0) = 0$, consider $|y_q(t)| \geq N$, we can see that $|f_q(y_q(t))| \geq \min\{f_q(N), -f_q(-N)\}$. Let $b_q = \min\{f_q(N), -f_q(-N)\}$, and $b = \min\{b_j : j = 1, \cdots, n\}$. Obviously $b_q > 0$ and $b > 0$. Therefore $|f_q(y_q(t))| \geq b$. Using (7), we have

$$B_j = \max\{f_j(s)/s : N \leq |s| \leq M\} < G_j,\ j = 1, \cdots, n.$$

Then consider $|y_j(t)| \leq M,\ \forall t \geq t_1,\ j = 1, \cdots, n$, and $|y_q(t)| \geq N$, for each fixed $t \geq t_1$, we have $f_q(y_q(t))/y_q(t) \leq B_q$. Let $\Lambda_q = 2d_q \left[1 - \frac{B_q}{G_q}\right] \frac{b^2}{G_q}$, obviously $\Lambda_q > 0$, therefore

$$\dot{V}(t) \leq -2d_q \left[1 - \frac{1}{G_q} \frac{f_q(y_q(t))}{y_q(t)}\right] \frac{(f_q(y_q(t)))^2}{G_q} \leq -2d_q \left[1 - \frac{B_q}{G_q}\right] \frac{b^2}{G_q} = -\Lambda_q.$$

Let $\Lambda = \min\{\Lambda_q :\ q = n_1, n_2, \cdots, n_r\}$, then we have

$$\dot{V}(t) \leq -\Lambda. \tag{14}$$

By taking the indefinite integral of both sides of (14) from t_1 to t, we have: $V(t) \leq V(t_1) - \Lambda * (t - t_1)$. When t is sufficiently large, we see that $V(t) < 0$. This is a contradiction to $V(t) > 0$. Therefore, this further implies that $y(t) \to 0$ as $t \to +\infty$. Hence, system (5) is asymptotically stable, and this also implies that the system (2) is asymptotically stable at x^*.

Next, we will show the exponential stability of system (5).

According to the above analysis, (2) is globally asymptotically stable at $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$, we suppose that $x_j^* \neq 0$.

By (7), $0 < f_j(s)/s < G_i$, for $s \neq 0,\ j = 1, \cdots, n$, where $G_i = D^+ g_j(0)$, and continuity of f_j, we see that there is a constant $0 < \varepsilon_j < 1$, such that $f_j(s)/s \leq G_j^* < G_i$, in which $G_j^* = \varepsilon_j G_i$. Let $\xi = \max\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$, we can see that $0 < \xi < 1$, and $G_j^* \leq \xi G_i$. Therefore, $0 \leq |f_j(y_j(t))| \leq \xi G_i |y_j(t)|$, $0 < |f_j(y_j(t)) f_j(y_j(t))| \leq \xi G_i |y_j(t) f_j(y_j(t))|$.

From (7), it is easy to know that the signs of $f_j(y_j(t))$ and $y_j(t)$ are the same (both positive, or both negative). So we have

$$f_j(y_j(t)) d_j y_j(t) \geq \frac{d_j}{\xi G_i} f_j(y_j(t)) f_j(y_j(t))$$

By (12), we see that $\begin{bmatrix} 2DG^{-1} - A - A^T - E & -B \\ -B^T & -(\tau^* - 1)E \end{bmatrix} \geq 0$, therefore

$$\begin{aligned}
\dot{V}(t) =& f^T(y(t)) D(-2y(t) + 2G^{-1} f(y(t))) \\
& - \begin{pmatrix} However, \eta_2^T & \eta_3^T \end{pmatrix} \begin{bmatrix} 2DG^{-1} - A - A^T - E & -B \\ -B^T & -(\tau^* - 1)E \end{bmatrix} \begin{pmatrix} \eta_2 \\ \eta_3 \end{pmatrix} \\
\leq & -2 \sum_{i=1}^{n} d_i f_i(y_i(t)) y_i(t) + 2 f(y(t))^T D G^{-1} f(y(t)) \\
\leq & -2 \sum_{i=1}^{n} \frac{d_i}{\xi G_i} f_i(y_i(t)) f_i(y_i(t)) + 2 f(y(t))^T D G^{-1} f(y(t)) \\
= & -2\xi^{-1} f(y(t))^T D G^{-1} f(y(t)) + 2 f(y(t))^T D G^{-1} f(y(t)) \\
= & -2(\xi^{-1} - 1) f(y(t))^T D G^{-1} f(y(t)) \\
\leq & -2(\xi^{-1} - 1) \min_{j=1,\cdots,n} \{d_j G_j^{-1}\} f(y(t))^T f(y(t)) \\
= & -\alpha f(y(t))^T f(y(t)),
\end{aligned} \tag{15}$$

where $\alpha = 2(\xi^{-1} - 1) \min_{j=1,\cdots,n}\{d_j G_j^{-1}\}$.

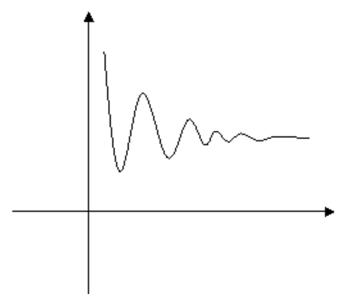

FIGURE 1. Schematic diagram of global asymptotical stability of a one-dimensional system

Because of global asymptotical stability of system (2) (a one-dimensional diagram shown in Figure 1), for a given $\sigma_{1j} > 0$, there is a constant $t_{1j} > 0$, such that $|y_j(t)| \leq \sigma_{1j}$, i.e. $-\sigma_{1j} \leq y_j(t) \leq \sigma_{1j}$, for $t \geq t_{1j}$, where $y_j(t) = x_j(t) - x_j^*$. Since f_j is increasing, $j = 1, \cdots, n$, we can see that $f_j(-\sigma_{1j}) \leq f_j(y_j(t)) \leq f_j(\sigma_{1j})$, for $t \geq t_{1j}$. Let $M_j = \max\{|f_j(-\sigma_{1j})|, f_j(\sigma_{1j})\}$, we have $0 < |f_j(y_j(t))| \leq M_j$ or $0 < (f_j(y_j(t)))^2 \leq M_j^2$. Let $t_1 = \max\{t_{1j}\}$, $M^* = \max\{M_j\}$, $j = 1, \cdots, n$, we have $0 < (f_j(y_j(t)))^2 \leq M^*$, as $t > t_1$.

On the other hand, according to the definition of $V(t)$, there is a $\sigma_{2j} > 0$, corresponds to a $t_{2j} > 0$, such that $y_j(t) \geq \sigma_{2j} > 0$, as $t \geq t_{2j}$. Because f_j is increasing, we have $f_j(y_j(t)) \geq f_j(\sigma_{2j}) > 0$, $(f_j(y_j(t)))^2 \geq (f_j(\sigma_{2j}))^2 > 0$, for all $t \geq t_{2j}$. Let $t_2 = \max\{t_{2j}\}$, $m^* = \min\{(f_j(\sigma_{2j}))^2\}$, $j = 1, \cdots, n$, $t_0 = \max\{t_1, t_2\}$, we see that $0 < m^* \leq (f_j(y_j(t)))^2 \leq M^*$, for all $t > t_0$.

Since $f_i(y_i(t))$ are continuous on closed interval $[t-\tau, t]$, according to the definite integral mean value theorem, there must be a $T \in [t - \tau, t]$, such that

$$\int_{t-\tau(t)}^{t} f_j^2(y_j(s))ds = f_j^2(y_j(T))\tau(t) \leq \frac{M^*}{m^*} f_j^2(y_j(t))\tau(t) \leq \beta f_j^2(y_j(t))\tau,$$

where $\beta = \frac{M^*}{m^*}$, and $t > t_0$. Therefore

$$\begin{aligned} V(t) \leq& 2\sum_{j=1}^{n} f_j(y_j(t))y_j(t) + \sum_{i=1}^{n}\int_{t-\tau(t)}^{t} f_i^2(y_i(s))ds \\ \leq& 2\xi^{-1} f^T(y(t))f(y(t)) + \sum_{i=1}^{n} \beta f_i^2(y_i(t))\tau(t) \\ \leq& 2\xi^{-1} f^T(y(t))f(y(t)) + \beta f^T(y(t))f(y(t))\tau \\ =& (2\xi^{-1} + \beta\tau) f^T(y(t))f(y(t)), \text{ for } \forall t \geq t_0. \end{aligned} \tag{16}$$

Since $2\xi^{-1} + \beta\tau > 0$, by (15) and (16), we see that

$$\begin{aligned} &V(t) \leq -\frac{\alpha}{2\xi^{-1} + \beta\tau} V(t), \\ &\ln V(t) - lnV(t_0) \leq -\lambda(t - t_0), \\ &V(t) \leq e^{-\lambda(t-t_0)} V(t_0) \end{aligned}$$

where $\lambda = \frac{\alpha}{2\xi^{-1}+\beta\tau}$. Applying the definition of $V(t)$ and (15) again, we have

$$V(t) \geq 2\sum_{j=1}^{n}\int_0^{y_i(t))} f_i(s)ds \geq 2\sum_{i=1}^{n}\int_0^{y_i(t)} \xi s ds = \xi\sum_{i=1}^{n}(y_i(t))^2, \ \forall t \geq t_0.$$

Therefore

$$\|y(t)\|^2 \leq (\xi^{-1}V(t_0))e^{-\lambda(t-t_0)}, \ \forall t \geq t_0.$$

This proves that system (5) is exponentially stable. This completes the proof. □

4. **Numerical examples.** In this section, we use an example to test the validity of our results. We consider the following two-dimensional neural network model,

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = -D\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + A\begin{pmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \end{pmatrix} + B\begin{pmatrix} f_1(x_1(t-\tau_1(t))) \\ f_2(x_2(t-\tau_2(t))) \end{pmatrix} \tag{17}$$

where $f_i(x_i) = \ln(1+x_i)(i = 1, 2$, the same as below), and $\tau_i(t) = \dfrac{1}{e^{-t}+1}$. Obviously, f_i is unbounded, and satisfies (3) and (4), furthermore $G_i = 1$, $\tau^* = 0.5$. Let $\Gamma = 2DG^{-1} - A - A^T - E - BB^T$. If we consider $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 0.01 & 0.03 \\ 0.04 & 0.02 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0.3 \\ 0.1 & 0 \end{bmatrix}$, then eig $(\Gamma) = \begin{pmatrix} 0.7710 \\ 0.9690 \end{pmatrix}$, we get $2DG^{-1} - A - A^T - E - BB^T > 0$. According to Theorem 3.1, the trivial solution of system (17) is globally asymptotically stable and exponentially stable. The system state trajectory and phase diagram can be shown in Figure 2.

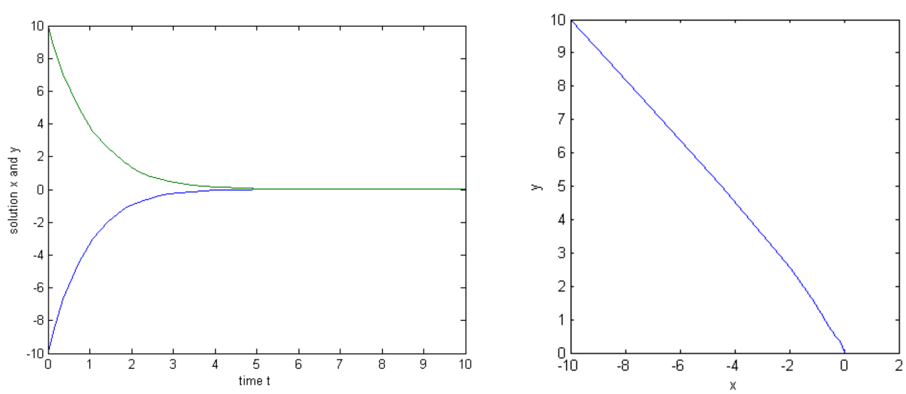

FIGURE 2. System (17) state trajectory and phase diagram ($\Gamma > 0$)

If we consider $D = \begin{bmatrix} 5.5 & 0 \\ 0 & 5.5 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, then $\Gamma = 2DG^{-1} - A - A^T - E - BB^T = 0$, According to the Theorem 3.1, the trivial solution of system (17) is also globally asymptotically stable and exponentially stable. The system state and trajectory phase diagram can be seen in Figure 3.

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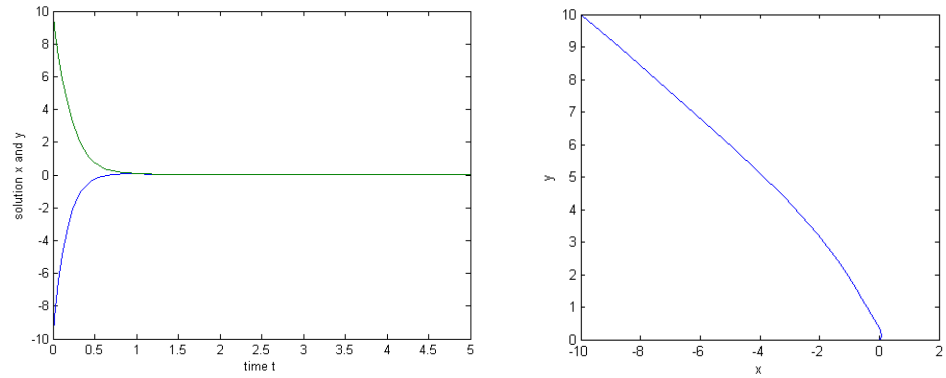

FIGURE 3. System (17) state trajectory and phase diagram ($\Gamma = 0$)

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