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pTH MOMENT ABSOLUTE EXPONENTIAL STABILITY OF STOCHASTIC CONTROL SYSTEM WITH MARKOVIAN SWITCHING

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Abstract. In this paper we discuss the pth moment absolute exponential stability of stochastic control system with Markovian switching. We first give a new concept of pth moment absolute exponential stability, then we establish some theorems under different hypotheses to guarantee the system pth moment absolutely exponentially stable. These sufficient conditions in our theorems are algebraic criteria in terms of matrix inequalities, and we introduce an M-method with MATLAB to compute them. Finally, some examples are given to illustrate our results.

1. **Introduction.** The absolute stability of Lurie's control system is a very important problem in automatic control. It was first proposed by Lurie and Postnikov [11] in 1944. Since then it has been studied by many mathematicians and engineers, and a lot of results have been obtained, for example, see [4, 8, 10, 19, 26] and references therein.

The absolute stability of Itô type stochastic Lurie's control system has attracted many mathematicians and engineers since 1970's. For example, using the frequency-domain method, Mahalanabis and Purkayastha [12] discussed the global asymptotical stability with probility 1. Korenevskii [5] gave the algebraic criteria in the approach of Lyapunov function. The review [18] shows more results. There are some results on the pth moment absolute stability of stochastic Lurie's control system. For example, using functional analysis technique, Brusin and Ugrinovskii [2] investigated the case $p = 2$ and obtained the criteria for global asymptotical stability in the mean square.

However, there are numerous phenomenons whose dynamic behavior required several systems to describe in real world. The mathematical model of these phenomenons is switched system which consists of finite subsystems together with a switching law to determine which subsystem is active at every instant of time ([9, 21]). Switched systems are widely used in various fields, such as automatic control, economics, physics, chemistry, engineering, biology, etc. Some recent research on these systems can be found in [3, 13, 23, 24, 25, 27, 28] and references

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therein. Moreover, these systems are always perturbed by some nonnegligible noise, which are called stochastic switched systems. Stability problem of these systems has attracted much attention during recent decades and many results have been obtained (see [1, 14, 15, 16, 20] and references therein). However, to the best of our knowledge, there are few papers on the pth moment absolute stability of stochastic switched systems because of the theoretical difficulties and complexity. So it is a meaningful and challenging problem, that is our motivation.

In this article, we investigate the pth moment absolute exponential stability of stochastic control system with Markovian switching. At first, the concept of pth moment absolute exponential stability for the stochastic switched systems is proposed. Then using the method of Lyapunov function and stochastic analysis technique, we study the pth moment absolute exponential stability under different hypotheses. We first assume that the diffusion coefficient satisfies an inequality (hypothesis (H2)) following Mahalanabis and Purkayastha [12], and establish Theorem 3.2. Then we extend the results under a weaker assumption (H3) and obtain Theorem 3.3. These results are algebraic criteria in terms of matrix inequalities which are convenient to check, and a concise approach named M-method is introduced to compute these matrix inequalities.

The remainder of this paper is organized as follows. Section 2 introduces some notations, definitions and lemmas. In Section 3, we first give the concept of pth moment absolute exponential stability of stochastic switched systems, then establish some theorems to show the sufficient conditions in terms of matrix inequalities under different assumptions, and introduce an approach named M-method to verify our results using MATLAB. Some examples are given in Section 4 to illustrate our results. At last, conclusions are given in Section 5.

2. Problem statement and preliminaries.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual condition (see [17]), $\mathbb{R}_+$ denote the interval $[0, +\infty)$, I denote the identity matrix. For matrix $A = (a_{ij})_{m\times n}$, the norm of A is defined by

$$|A| := \sqrt{\text{trace}(A^T A)} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}.$$

Consider the stochastic switching system

$$\begin{cases} \mathrm{d}x = \big[A\big(r\big)x + bu\big]\,\mathrm{d}t + D\big(x,t,r\big)\,\mathrm{d}B, \\ y = c^T x, \\ u = -\varphi(y), \\ r : \mathbb{R}_+ \times \Omega \to \mathbb{S} = \{1, 2, \cdots, N\}, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $A(i) \in \mathbb{R}^{n\times n}$ is the plant matrix, $D(x,t,i) \in \mathbb{R}^{n\times m}$ is the noise matrix, $B(t)$ is an m-dimensional Brownian motion, $y \in \mathbb{R}$ is the output, $b, c \in \mathbb{R}^n$, $\varphi \in F_{(0,k)}$ is a controller, where

$$F_{(0,k)} := \big\{\varphi : \varphi(0) = 0,\, 0 < y\varphi(y) < ky^2,\, \forall y \neq 0, \\ \text{and } \varphi \text{ satisfies the local Lipschitz condition}\big\}.$$

$r(t)$ is a Markov process taking values in $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & j \neq i, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & j = i, \end{cases}$$

where $\Delta > 0$, $\gamma_{ij} \geq 0$ is the transition rate from state i to j when $j \neq i$, and

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij} \leq 0.$$

In this paper, we always assume that $r(t)$ is independent of $B(t)$.

If φ is fixed, then system (1) can be rewritten as

$$\mathrm{d}x(t) = \left[A\big(r(t)\big)x(t) - b\varphi\big(c^T x(t)\big)\right]\mathrm{d}t + D\big(x(t), t, r(t)\big)\,\mathrm{d}B(t), \tag{2}$$

In this paper, we will focus on the pth moment absolute stability of system (2) with some of following conditions:

(**H1**) $D(x, t, i)$ satisfies the Lipschitz condition: $\exists K > 0$ such that

$$|D(x_1, t, i) - D(x_2, t, i)| \leq K|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n, \forall (t, i) \in \mathbb{R}_+ \times \mathbb{S}. \tag{3}$$

(**H2**) There exists a constant $\lambda > 0$, such that for any positive-definite matrix $P \in \mathbb{R}^{n\times n}$,

$$\mathrm{trace}\big(D^T(x, t, i)PD(x, t, i)\big) \leq \lambda x^T P x, \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}. \tag{4}$$

(**H3**) $D(0, t, i) = 0$, $\forall (t, i) \in \mathbb{R}_+ \times \mathbb{S}$.

It is easy to know that (H2) implies (H3) and (H1)(H3) imply that $D(x, i)$ satisfies the linear growth condition, which means that there exists a unique solution to equation (2) corresponding to any initial value $x(t_0)$.

We now present some useful lemmas.

Lemma 2.1. *Let $P \in \mathbb{R}^{n\times n}$ be a positive-definite matrix, then for any vector $x \in \mathbb{R}^n$ and matrix $D \in \mathbb{R}^{n\times m}$, we have*

$$|x^T PD|^2 \leq (x^T Px)\mathrm{trace}(D^T PD). \tag{5}$$

Proof. Since P is positive-definite, there exists a reversible real matrix $Q \in \mathbb{R}^{n\times n}$ such that $P = Q^T Q$. Then

$$\begin{aligned} & |x^T PD|^2 = |x^T Q^T QD|^2 \leq |x^T Q^T|^2 |QD|^2 \\ = \; & \big(x^T Q^T Qx\big)\,\mathrm{trace}\big(D^T Q^T QD\big) = (x^T Px)\,\mathrm{trace}(D^T PD). \end{aligned}$$

□

Lemma 2.2. *Let $P \in \mathbb{R}^{n\times n}$ be a positive-definite matrix, $D(x, t, i)$ is a $\mathbb{R}^{n\times m}$-valued function matrix satisfying (H1) and (H3), then we have*

$$\mathrm{trace}\big(D^T(x, t, i)PD(x, t, i)\big) \leq \mathrm{trace}(P)K^2|x|^2, \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}. \tag{6}$$

Proof. P is positive-definite, then there exists a reversible real matrix $Q \in \mathbb{R}^{n\times n}$ such that $P = Q^T Q$. By (H1) and (H3), we have

$$\begin{aligned} & \mathrm{trace}\big(D^T(x, t, i)PD(x, t, i)\big) = \mathrm{trace}\big(D^T(x, t, i)Q^T QD(x, t, i)\big) \\ = \; & |QD(x, t, i)|^2 \leq |Q|^2 |D(x, t, i)|^2 \leq \mathrm{trace}(Q^T Q)K^2|x|^2 = \mathrm{trace}(P)K^2|x|^2. \end{aligned}$$

□

Lemma 2.3. *Suppose $P, H \in \mathbb{R}^{n\times n}$ and P is a positive-definite matrix. If*

$$-PH - H^T P > 0, \tag{7}$$

then $\mathrm{Re}\lambda(H) < 0$, which means that all eigenvalues of H have negative real parts.

Lemma 2.4. *$A_i \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, $\Gamma = (\gamma_{ij})_{N\times N}$, and $k > 0$ are given in the system (1). $\lambda > 0$ is given by hypothesis (H2). $p > 1$ is some constant. If there exist $\alpha_i > 0$, $\beta_i > 1$, and positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{S}$, such that*

$$R_i - \frac{k}{\alpha_i} d_i d_i^T \geq 0, \quad \forall i \in \mathbb{S}, \tag{8}$$

then $\forall i \in \mathbb{S}$, we have

$$\mathrm{Re}\lambda\Big(A_i + \frac{\gamma_{ii}}{p} I\Big) < 0 \quad \textit{and} \quad \mathrm{Re}\lambda\Big(A_i - \frac{k}{2} bc^T + \frac{\gamma_{ii}}{p} I\Big) < 0, \tag{9}$$

where

$$\begin{cases} -R_i = P_i A_i + A_i^T P_i + \big(|p-2|+1\big)\lambda\beta_i P_i + \dfrac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \displaystyle\sum_{j\neq i} \gamma_{ij} P_j + \frac{2}{p}\gamma_{ii} P_i, \\ d_i = P_i b - \dfrac{1}{2}\alpha_i c, \end{cases} \tag{10}$$

$$\lambda_M = \max_{i\in\mathbb{S}} \lambda_{\max}(P_i), \quad \lambda_m = \min_{i\in\mathbb{S}} \lambda_{\min}(P_i).$$

Proof. Substitute (10) into (8), we have

$$\begin{aligned} & R_i - \frac{k}{\alpha_i} d_i d_i^T \\ = & - P_i A_i - A_i^T P_i - \big(|p-2|+1\big)\lambda\beta_i P_i - \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j - \frac{2}{p}\gamma_{ii} P_i - \frac{k}{\alpha_i} d_i d_i^T \\ = & - P_i G_i - G_i^T P_i - \big(|p-2|+1\big)\lambda\beta_i P_i - \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j - \frac{k}{\alpha_i} d_i d_i^T, \end{aligned}$$

and

$$\begin{aligned} & R_i - \frac{k}{\alpha_i} d_i d_i^T \\ = & - P_i A_i - A_i^T P_i - \big(|p-2|+1\big)\lambda\beta_i P_i - \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j - \frac{2}{p}\gamma_{ii} P_i - \frac{k}{\alpha_i} d_i d_i^T \\ = & - P_i A_i - A_i^T P_i - \big(|p-2|+1\big)\lambda\beta_i P_i - \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j - \frac{2}{p}\gamma_{ii} P_i \\ & - \frac{k}{\alpha_i} P_i bb^T P_i + \frac{k}{2} cb^T P_i + \frac{k}{2} P_i bc^T - \frac{k}{4}\alpha_i cc^T \\ = & - P_i H_i - H_i^T P_i - \big(|p-2|+1\big)\lambda\beta_i P_i - \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j \\ & - \frac{k}{\alpha_i} P_i bb^T P_i - \frac{k}{4}\alpha_i cc^T. \end{aligned}$$

where

$$G_i := A_i + \frac{\gamma_{ii}}{p} I \quad \text{and} \quad H_i := A_i - \frac{k}{2} bc^T + \frac{\gamma_{ii}}{p} I \tag{11}$$

Then

$$-P_i G_i - G_i^T P_i = R_i - \frac{k}{\alpha_i} d_i d_i^T + \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j + \frac{k}{\alpha_i} d_i d_i^T + \big(|p-2|+1\big)\lambda\beta_i P_i,$$

and

$$\begin{aligned} &-P_i H_i - H_i^T P_i \\ = \;& R_i - \frac{k}{\alpha_i} d_i d_i^T + \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j + \frac{k}{\alpha_i} P_i bb^T P_i + \frac{k}{4}\alpha_i cc^T + \big(|p-2|+1\big)\lambda\beta_i P_i. \end{aligned}$$

It's easy to know that $d_i d_i^T$, $P_i bb^T P_i$, and cc^T are all positive semi-definite, then the necessary condition of (8) is that

$$-P_i G_i - G_i^T P_i > 0 \quad \text{and} \quad -P_i H_i - H_i^T P_i > 0. \tag{12}$$

Then by Lemma 2.3, we obtain (9). □

Lemma 2.5. *If we replace (10) in Lemma 2.4 by following*

$$\begin{cases} -R_i = P_i A_i + A_i^T P_i + \big(|p-2|+1\big)\beta_i \mathrm{trace}(P_i) K^2 I + \dfrac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \displaystyle\sum_{j\neq i} \gamma_{ij} P_j + \dfrac{2}{p}\gamma_{ii} P_i, \\ d_i = P_i b - \dfrac{1}{2}\alpha_i c, \end{cases} \tag{13}$$

then the result in Lemma 2.4 is also true.

Lemma 2.6. [22] *Suppose $H \in \mathbb{R}^{n\times n}$ and $\mathrm{Re}\lambda(H) < 0$. Then for all $Q \in \mathbb{R}^{n\times n}$, equation*

$$PH + H^T P = -Q$$

has a unique solution

$$P = \int_0^\infty \mathrm{e}^{H^T t} Q \mathrm{e}^{Ht}\, \mathrm{d}t.$$

Moreover, if Q is positive-definite, then so is P.

Lemma 2.7. [17] *Suppose (H1) and (H3) hold. Then for all $\varphi \in F_{(0,k)}$, for all $x_0 \neq 0$ in $\mathbb{R}^n$, for all $r_0 \in \mathbb{S}$, we have*

$$P\{x(t; t_0, x_0, r_0) \neq 0 \text{ on } t \geq t_0\} = 1.$$

For the stochastic switched system

$$\mathrm{d}x(t) = f\big(x(t), t, r(t)\big)\,\mathrm{d}t + g\big(x(t), t, r(t)\big)\,\mathrm{d}B(t)$$

and $V \in C^{2,1}\big(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}\big)$, we introduce an operator $LV : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ given by

$$\begin{aligned} LV(x,t,i) =& V_t(x,t,i) + V_x(x,t,i) f(x,t,i) \\ &+ \frac{1}{2}\mathrm{trace}\big[g^T(x,t,i) V_{xx}(x,t,i) g(x,t,i)\big] + \sum_{j=1}^N \gamma_{ij} V(x,t,j). \end{aligned}$$

3. Main Results.

3.1. pth moment absolute exponential stability. In this section, we give the new concept of pth moment absolute exponential stability for the stochastic switched systems and establish some theorems to guarantee the system pth moment absolutely exponentially stable under different hypotheses.

Definition 3.1. System (2) is said to be pth moment absolutely exponentially stable with respect to (w.r.t.) $F_{(0,k)}$, if $\forall \varphi \in F_{(0,k)}$, the trivial solution of system (2) is pth moment exponentially stable, which means that there exists a constant $\nu > 0$ such that the pth moment Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t} \log\big(E|x(t;t_0,x_0,r_0)|^p\big) < -\nu$$

for all $(t_0, x_0, r_0) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$.

Theorem 3.2. *Assume that the system (2) satisfies hypotheses (H1), (H2) and (9), for some constant $p > 1$, if there exist $\alpha_i > 0$, $\beta_i > 1$, and positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{S}$, such that*

$$R_i - \frac{k}{\alpha_i} d_i d_i^T \geq 0, \quad \forall i \in \mathbb{S},$$

then system (2) is pth moment absolutely exponentially stable w.r.t. $F_{(0,k)}$, where

$$\begin{cases} -R_i = P_iA_i + A_i^TP_i + \big(|p-2|+1\big)\lambda\beta_iP_i + \dfrac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \displaystyle\sum_{j\neq i} \gamma_{ij}P_j + \frac{2}{p}\gamma_{ii}P_i, \\ d_i = P_ib - \dfrac{1}{2}\alpha_i c, \end{cases} \tag{14}$$

and

$$\lambda_M = \max_{i\in\mathbb{S}} \lambda_{\max}(P_i), \quad \lambda_m = \min_{i\in\mathbb{S}} \lambda_{\min}(P_i).$$

Proof. It's obvious that $x(t;t_0,x_0,r_0) \equiv 0$, when $x_0 = 0$, then

$$\limsup_{t\to\infty} \frac{1}{t} \log\big(E|x(t;t_0,0,r_0)|^p\big) = -\infty.$$

So we only need to consider the case $x_0 \neq 0$. By Lemma 2.7, for almost surely,

$$x(t;t_0,x_0,r_0) \neq 0, \quad \text{on } t \geq t_0.$$

Let $V(x,t,i) = (x^TP_ix)^{\frac{p}{2}}$, $i = 1,2,\cdots,N$, then $\forall x \neq 0$, $\forall (t,i) \in \times\mathbb{R}_+ \times \mathbb{S}$,

$$V_t(x,t,i) = 0, \quad V_x(x,t,i) = p(x^TP_ix)^{\frac{p}{2}-1}x^TP_i,$$

$$V_{xx}(x,t,i) = 2p\Big(\frac{p}{2}-1\Big)(x^TP_ix)^{\frac{p}{2}-2}P_ixx^TP_i + p(x^TP_ix)^{\frac{p}{2}-1}P_i.$$

It's easy to know that

$$\lambda_m^{\frac{p}{2}}|x|^p \leq V(x,t,i) \leq \lambda_M^{\frac{p}{2}}|x|^p, \quad \forall (x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}.$$

Let

$$c_1 = \lambda_m^{\frac{p}{2}} \quad \text{and} \quad c_2 = \lambda_M^{\frac{p}{2}},$$

then we have

$$c_1|x|^p \leq V(x,t,i) \leq c_2|x|^p, \quad \forall (x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}.$$

According to the criteria of pth moment exponential stability for the SDE with Markovian switching ([14, Theorem 3.1]), we only need to show that there exists $c_3 > 0$ such that

$$LV(x,t,i) \le -c_3|x|^p, \quad \forall x \neq 0,\, \forall (t,i) \in \mathbb{R}_+ \times \mathbb{S},\, \forall \varphi \in F_{(0,k)}.$$

By the definition of $F_{(0,k)}$, we know that

$$\alpha_i \varphi(y)\Big(y - \frac{1}{k}\varphi(y)\Big) \ge 0, \quad \forall \alpha_i \ge 0.$$

Then by (H2) and Lemma 2.2, $\forall x \neq 0$, $\forall t \ge 0$, $\forall i \in \mathbb{S}$, $\forall \varphi \in F_{(0,k)}$,

$$\begin{aligned}
&LV(x,t,i)\\
=\;& p(x^TP_ix)^{\frac{p}{2}-1}x^TP_i\big[A(i)x - b\varphi(y)\big] + \sum_{j=1}^{N}\gamma_{ij}(x^TP_jx)^{\frac{p}{2}}\\
&+\frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}\operatorname{trace}\big[D^T(x,t,i)P_iD(x,t,i)\big]\\
&+p\Big(\frac{p}{2}-1\Big)(x^TP_ix)^{\frac{p}{2}-2}\operatorname{trace}\big[D^T(x,t,i)P_ixx^TP_iD(x,t,i)\big]\\
\le\;& \frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}x^T\bigg[P_iA_i + A_i^TP_i + \lambda P_i + |p-2|\lambda P_i\\
&\quad+\frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}}\sum_{j\neq i}\gamma_{ij}P_j + \frac{2}{p}\gamma_{ii}P_i\bigg]x\\
&-\frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}2b^TP_ix\varphi(y) + \frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}\alpha_i\varphi(y)\Big(c^Tx - \frac{1}{k}\varphi(y)\Big)\\
&-\frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}\alpha_i\varphi(y)\Big(y - \frac{1}{k}\varphi(y)\Big)\\
\le\;& \frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}x^T\bigg[P_iA_i + A_i^TP_i + \big(|p-2|+1\big)\lambda\beta_iP_i\\
&\quad+\frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}}\sum_{j\neq i}\gamma_{ij}P_j + \frac{2}{p}\gamma_{ii}P_i\bigg]x\\
&-\frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}2\Big(P_ib - \frac{1}{2}\alpha_ic\Big)^Tx\varphi(y) - \frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}\frac{\alpha_i}{k}\varphi^2(y)\\
&+\frac{1}{2}p\big(|p-2|+1\big)\lambda(1-\beta_i)(x^TP_ix)^{\frac{p}{2}}\\
\le\;& -\frac{1}{2}p(x^TP_ix)^{\frac{p}{2}-1}\big(x^T,\varphi(y)\big)\begin{bmatrix} R_i & d_i\\ d_i^T & \alpha_i/k\end{bmatrix}\begin{bmatrix} x\\ \varphi(y)\end{bmatrix}\\
&-\frac{1}{2}p\big(|p-2|+1\big)\lambda(\beta_i-1)\lambda_{\min}^{\frac{p}{2}}(P_i)|x|^p.
\end{aligned}$$

We know that

$$\begin{bmatrix} I & -kd_i/\alpha_i\\ 0 & 1\end{bmatrix}\begin{bmatrix} R_i & d_i\\ d_i^T & \alpha_i/k\end{bmatrix}\begin{bmatrix} I & 0\\ -kd_i^T/\alpha_i & 1\end{bmatrix} = \begin{bmatrix} R_i - kd_id_i^T/\alpha_i & 0\\ 0 & \alpha_i/k\end{bmatrix},$$

so if

$$R_i - \frac{k}{\alpha_i}d_id_i^T \ge 0, \quad \forall i \in \mathbb{S},$$

then

$$M_i := \begin{bmatrix} R_i & d_i \\ d_i^T & \alpha_i/k \end{bmatrix} \geq 0, \quad \forall i \in \mathbb{S}.$$

Let

$$c_3 = \frac{1}{2}p\big(|p-2|+1\big)\lambda \min_{i\in\mathbb{S}} \Big\{(\beta_i - 1)\lambda_{\min}^{\frac{p}{2}}(P_i)\Big\} > 0,$$

then, by $M_i \geq 0$, $\forall i \in \mathbb{S}$, we have

$$LV(x,t,i) \leq -c_3|x|^p, \quad \forall x \neq 0, \forall (t,i) \in \mathbb{R}_+ \times \mathbb{S}, \forall \varphi \in F_{(0,k)}.$$

So we prove the result. □

Noting that $\big(E|x|^p\big)^{1/p} \leq \big(E|x|^{\hat{p}}\big)^{1/\hat{p}}$ for $0 < p \leq \hat{p}$, we see that the $\hat{p}$th moment absolute exponential stability implies the pth moment absolute exponential stability. So we have the following corollary.

Corollary 1. *Assume that the system* (2) *satisfies hypotheses (H1), (H2) and* (9)*, for some constant $\hat{p} > 1$, if there exist $\alpha_i > 0$, $\beta_i > 1$, and positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{S}$, such that*

$$R_i - \frac{k}{\alpha_i} d_i d_i^T \geq 0, \quad \forall i \in \mathbb{S},$$

then $\forall p \in (1, \hat{p}]$, system (2) *is pth moment absolutely exponentially stable w.r.t. $F_{(0,k)}$, where*

$$\begin{cases} -R_i = P_i A_i + A_i^T P_i + \big(|p-2|+1\big)\lambda\beta_i P_i + \dfrac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \displaystyle\sum_{j\neq i} \gamma_{ij} P_j + \frac{2}{p}\gamma_{ii} P_i, \\ d_i = P_i b - \dfrac{1}{2}\alpha_i c. \end{cases} \tag{15}$$

By Theorem 3.2 and Lemma 2.6, it's easy to prove the following result.

Corollary 2. *Assume that the system* (2) *satisfies hypotheses (H1), (H2) and* (9)*. $p \geq 2$ is some integer. If there exist $\alpha_i > 0$, $\beta_i > 1$ and positive-definite martices $Q_i \in \mathbb{R}^{n\times n}$(or $\tilde{Q}_i \in \mathbb{R}^{n\times n}$), $i \in \mathbb{S}$, such that $\forall i \in \mathbb{S}$,*

$$\begin{aligned} E_i :=& Q_i - \big(|p-2|+1\big)\lambda\beta_i \Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big] \\ &- \frac{2\hat{\lambda}_M^{\frac{p}{2}-1}}{p\hat{\lambda}_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} \Big[\int_0^\infty \mathrm{e}^{G_j^T t} Q_j \mathrm{e}^{G_j t}\,\mathrm{d}t\Big] \\ &- \frac{k}{\alpha_i}\bigg(\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big] b - \frac{1}{2}\alpha_i c\bigg)\bigg(\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big] b - \frac{1}{2}\alpha_i c\bigg)^T \\ \geq& 0, \end{aligned}$$

where

$$\hat{\lambda}_M = \max_{i\in\mathbb{S}} \lambda_{\max}\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big],$$

$$\hat{\lambda}_m = \min_{i\in\mathbb{S}} \lambda_{\min}\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big],$$

(or

$$\begin{aligned}\tilde{E}_i :=& \tilde{Q}_i - \big(|p-2|+1\big)\lambda\beta_i\Big[\int_0^\infty \mathrm{e}^{H_i^T t}\tilde{Q}_i\mathrm{e}^{H_i t}\,\mathrm{d}t\Big] \\ &- \frac{2\hat{\lambda}_M^{\frac{p}{2}-1}}{p\hat{\lambda}_m^{\frac{p}{2}-1}}\sum_{j\neq i}\gamma_{ij}\Big[\int_0^\infty \mathrm{e}^{H_j^T t}\tilde{Q}_j\mathrm{e}^{H_j t}\,\mathrm{d}t\Big] \\ &- \frac{k}{\alpha_i}\Big[\int_0^\infty \mathrm{e}^{H_i^T t}\tilde{Q}_i\mathrm{e}^{H_i t}\,\mathrm{d}t\Big]bb^T\Big[\int_0^\infty \mathrm{e}^{H_i^T t}\tilde{Q}_i\mathrm{e}^{H_i t}\,\mathrm{d}t\Big] - \frac{k}{4}\alpha_i cc^T \geq 0,\end{aligned}$$

where

$$\hat{\lambda}_M = \max_{i\in\mathbb{S}}\lambda_{\max}\Big[\int_0^\infty \mathrm{e}^{H_j^T t}\tilde{Q}_j\mathrm{e}^{H_j t}\,\mathrm{d}t\Big],$$

$$\hat{\lambda}_m = \min_{i\in\mathbb{S}}\lambda_{\min}\Big[\int_0^\infty \mathrm{e}^{H_j^T t}\tilde{Q}_j\mathrm{e}^{H_j t}\,\mathrm{d}t\Big],$$

) then system (2) is pth moment absolutely exponentially stable w.r.t. $F_{(0,k)}$.

In Theorem 3.2, we assume that the diffusion coefficient satisfies (H2) following [12]. However, this assumption can be simplified to (H3), which is shown in the following theorem.

Theorem 3.3. *Assume that the system (2) satisfies hypotheses (H1), (H3) and (9), for some constant* $p > 1$, *if there exist* $\alpha_i > 0$, $\beta_i > 1$, *and positive-definite matrices* $P_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{S}$, *such that*

$$R_i - \frac{k}{\alpha_i}d_i d_i^T \geq 0, \quad \forall i \in \mathbb{S},$$

then system (2) is pth moment absolutely exponentially stable w.r.t. $F_{(0,k)}$, *where*

$$\begin{cases} -R_i = P_iA_i + A_i^TP_i + \big(|p-2|+1\big)\beta_i\,\mathrm{trace}(P_i)K^2I \\ \qquad\qquad + \dfrac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}}\displaystyle\sum_{j\neq i}\gamma_{ij}P_j + \frac{2}{p}\gamma_{ii}P_i, \\ d_i = P_ib - \dfrac{1}{2}\alpha_i c. \end{cases} \tag{16}$$

Proof. Let $V(x,t,i) = (x^TP_ix)^{\frac{p}{2}}, i = 1,2,\cdots,N$. Similarly, we only need to show that there exists $c_3 > 0$ such that

$$LV(x,t,i) \leq -c_3|x|^p, \quad \forall x \neq 0,\ \forall (t,i) \in \mathbb{R}_+ \times \mathbb{S},\ \forall \varphi \in F_{(0,k)}.$$

By (H3) and Lemma 2.2, $\forall x \neq 0$, $\forall t \geq 0$, $\forall i \in \mathbb{S}$, $\forall \varphi \in F_{(0,k)}$,

$$\begin{aligned}
& LV(x,t,i) \\
= \;& p(x^T P_i x)^{\frac{p}{2}-1} x^T P_i \big[A(i)x - b\varphi(y)\big] + \sum_{j=1}^{N} \gamma_{ij} (x^T P_j x)^{\frac{p}{2}} \\
& + \frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} \operatorname{trace}\big[D^T(x,t,i) P_i D(x,t,i)\big] \\
& + p\Big(\frac{p}{2} - 1\Big) (x^T P_i x)^{\frac{p}{2}-2} \operatorname{trace}\big[D^T(x,t,i) P_i x x^T P_i D(x,t,i)\big] \\
\leq \;& \frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} x^T \Bigg[P_i A_i + A_i^T P_i + \operatorname{trace}(P_i) K^2 I \\
& \quad + |p-2| \operatorname{trace}(P_i) K^2 I + \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j \neq i} \gamma_{ij} P_j + \frac{2}{p} \gamma_{ii} P_i \Bigg] x \\
& - \frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} 2 b^T P_i x \varphi(y) + \frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} \alpha_i \varphi(y) \Big(c^T x - \frac{1}{k} \varphi(y) \Big) \\
& - \frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} \alpha_i \varphi(y) \Big(y - \frac{1}{k} \varphi(y) \Big),
\end{aligned}$$

then

$$\begin{aligned}
& LV(x,t,i) \\
\leq \;& \frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} x^T \Bigg[P_i A_i + A_i^T P_i \\
& \quad + \big(|p-2| + 1\big) \beta_i \operatorname{trace}(P_i) K^2 I + \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j \neq i} \gamma_{ij} P_j + \frac{2}{p} \gamma_{ii} P_i \Bigg] x \\
& - \frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} \big(2 b^T P_i - c^T\big) x \varphi(y) - \frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} \frac{1}{k} \varphi^2(y) \\
& + \frac{1}{2} p \big(|p-2| + 1\big)(1 - \beta_i)(x^T P_i x)^{\frac{p}{2}-1} \operatorname{trace}(P_i) K^2 x^T x \\
\leq \;& -\frac{1}{2} p (x^T P_i x)^{\frac{p}{2}-1} \big(x^T, \varphi(y)\big) \begin{bmatrix} R_i & d_i \\ d_i^T & \alpha_i / k \end{bmatrix} \begin{bmatrix} x \\ \varphi(y) \end{bmatrix} \\
& - \frac{1}{2} p \big(|p-2| + 1\big) K^2 (\beta_i - 1) \operatorname{trace}(P_i) \lambda_{\min}^{\frac{p}{2}-1}(P_i) |x|^p.
\end{aligned}$$

Similarly, if

$$R_i - \frac{k}{\alpha_i} d_i d_i^T \geq 0, \quad \forall i \in \mathbb{S},$$

then

$$M_i := \begin{bmatrix} R_i & d_i \\ d_i^T & \alpha_i / k \end{bmatrix} \geq 0, \quad \forall i \in \mathbb{S}.$$

Let

$$c_3 = \frac{1}{2} p \big(|p-2| + 1\big) K^2 \min_{i \in \mathbb{S}} \Big\{ (\beta_i - 1) \operatorname{trace}(P_i) \lambda_{\min}^{\frac{p}{2}-1}(P_i) \Big\} > 0,$$

then we have

$$LV(x,t,i) \leq -c_3 |x|^p, \quad \forall x \neq 0, \forall (t,i) \in \times \mathbb{R}_+ \times \mathbb{S}, \forall \varphi \in F_{(0,k)}.$$

The proof is complete. □

Similarly, we can obatin the corollaries below.

Corollary 3. *Assume that the system (2) satisfies hypotheses (H1), (H3) and (9), for some constant $\hat{p} > 1$, if there exist $\alpha_i > 0$, $\beta_i > 1$, and positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{S}$, such that*

$$R_i - \frac{k}{\alpha_i} d_i d_i^T \geq 0, \quad \forall i \in \mathbb{S},$$

then $\forall p : 1 < p \leq \hat{p}$, system (2) is pth moment absolutely exponentially stable w.r.t. $F_{(0,k)}$, where

$$\begin{cases} -R_i = P_i A_i + A_i^T P_i + \big(|\hat{p} - 2| + 1\big)\beta_i \operatorname{trace}(P_i) K^2 I \\ \qquad + \dfrac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \displaystyle\sum_{j\neq i} \gamma_{ij} P_j + \frac{2}{p}\gamma_{ii} P_i, \\ d_i = P_i b - \dfrac{1}{2}\alpha_i c. \end{cases} \tag{17}$$

Corollary 4. *Assume that the system (2) satisfies hypotheses (H1), (H3) and (9). $p \geq 2$ is some integer. If there exist $\alpha_i > 0$, $\beta_i > 1$ and positive-definite martices $Q_i \in \mathbb{R}^{n\times n}$(or $\tilde{Q}_i \in \mathbb{R}^{n\times n}$), $i \in \mathbb{S}$, such that $\forall i \in \mathbb{S}$,*

$$\begin{aligned} E_i :=& Q_i - \big(|p-2|+1\big)\beta_i \operatorname{trace}\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big] K^2 I \\ &- \frac{2\hat{\lambda}_M^{\frac{p}{2}-1}}{p\hat{\lambda}_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} \Big[\int_0^\infty \mathrm{e}^{G_j^T t} Q_j \mathrm{e}^{G_j t}\,\mathrm{d}t\Big] \\ &- \frac{k}{\alpha_i}\bigg(\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big] b - \frac{1}{2}\alpha_i c\bigg)\bigg(\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big] b - \frac{1}{2}\alpha_i c\bigg)^T \\ &\geq 0, \end{aligned}$$

where

$$\hat{\lambda}_M = \max_{i\in\mathbb{S}} \lambda_{\max}\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big],$$
$$\hat{\lambda}_m = \min_{i\in\mathbb{S}} \lambda_{\min}\Big[\int_0^\infty \mathrm{e}^{G_i^T t} Q_i \mathrm{e}^{G_i t}\,\mathrm{d}t\Big],$$

(or

$$\begin{aligned} \tilde{E}_i :=& \tilde{Q}_i - \big(|p-2|+1\big)\beta_i \operatorname{trace}\Big[\int_0^\infty \mathrm{e}^{H_i^T t} \tilde{Q}_i \mathrm{e}^{H_i t}\,\mathrm{d}t\Big] K^2 I \\ &- \frac{2\hat{\lambda}_M^{\frac{p}{2}-1}}{p\hat{\lambda}_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} \Big[\int_0^\infty \mathrm{e}^{H_j^T t} \tilde{Q}_j \mathrm{e}^{H_j t}\,\mathrm{d}t\Big] \\ &- \frac{k}{\alpha_i}\Big[\int_0^\infty \mathrm{e}^{H_i^T t} \tilde{Q}_i \mathrm{e}^{H_i t}\,\mathrm{d}t\Big] bb^T \Big[\int_0^\infty \mathrm{e}^{H_i^T t} \tilde{Q}_i \mathrm{e}^{H_i t}\,\mathrm{d}t\Big] - \frac{k}{4}\alpha_i cc^T \geq 0, \end{aligned}$$

where

$$\hat{\lambda}_M = \max_{i\in\mathbb{S}} \lambda_{\max}\Big[\int_0^\infty \mathrm{e}^{H_j^T t} \tilde{Q}_j \mathrm{e}^{H_j t}\,\mathrm{d}t\Big],$$
$$\hat{\lambda}_m = \min_{i\in\mathbb{S}} \lambda_{\min}\Big[\int_0^\infty \mathrm{e}^{H_j^T t} \tilde{Q}_j \mathrm{e}^{H_j t}\,\mathrm{d}t\Big],$$

) then system (2) is pth moment absolutely exponentially stable w.r.t. $F_{(0,k)}$.

3.2. **M-method with MATLAB.** In this subsection, we introduce a technique called M-method to verify our results. It is a convenient method to compute P_i, $i \in \mathbb{S}$.

We know that in the software MATLB, there is a function "care" to solve the continuous time algebraic Riccati equation

$$A^T X + XA - XBR^{-1}B^T X + Q = 0, \tag{18}$$

where A, B, R, Q are known. In the proof of Lemma 2.4, we obtain that $\forall i \in \mathbb{S}$,

$$\begin{aligned} & R_i - \frac{k}{\alpha_i} d_i d_i^T \\ =& - P_i H_i - H_i^T P_i - \big(|p-2|+1\big)\lambda\beta_i P_i - \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j - \frac{k}{\alpha_i} P_i bb^T P_i - \frac{k}{4}\alpha_i cc^T \\ \geq& 0, \end{aligned}$$

or

$$\begin{aligned} & P_i H_i + H_i^T P_i + \big(|p-2|+1\big)\lambda\beta_i P_i \\ + & \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j + \frac{k}{\alpha_i} P_i bb^T P_i + \frac{k}{4}\alpha_i cc^T \leq 0, \end{aligned} \tag{19}$$

where H_i, $i \in \mathbb{S}$, are given by (11).

Compare (19) and (18), if we choose some constant $M > 0$ and let

$$A = H_i, \quad B = \sqrt{\frac{k}{\alpha_i}} b, \quad R = -I, \quad Q = \frac{k}{4}\alpha_i cc^T + \big[\big(|p-2|+1\big)\lambda\beta_i - \gamma_{ii}\big] M Q_0,$$

in (18), where Q_0 is a given positive-definite matrix and usually it can be given by $Q_0 = I$ or $Q_0 = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, then we can use the function "care" to get $\tilde{P}_i$ such that

$$\tilde{P}_i H_i + H_i^T \tilde{P}_i + \big[\big(|p-2|+1\big)\lambda\beta_i - \gamma_{ii}\big] M Q_0 + \frac{k}{\alpha_i} \tilde{P}_i bb^T \tilde{P}_i + \frac{k}{4}\alpha_i cc^T = 0, \quad \forall i \in \mathbb{S},$$

when α_i have been given. If M is good enough such that

$$MQ_0 - \frac{\tilde{\lambda}_M^{\frac{p}{2}-1}}{\tilde{\lambda}_m^{\frac{p}{2}-1}} \tilde{P}_j \geq 0, \quad \forall i \in \mathbb{S},$$

where

$$\tilde{\lambda}_M = \max_{i\in\mathbb{S}} \lambda_{\max}\big(\tilde{P}\big), \quad \tilde{\lambda}_m = \min_{i\in\mathbb{S}} \lambda_{\min}\big(\tilde{P}\big),$$

then we can choose $P_i = \tilde{P}_i$, $i \in \mathbb{S}$.

If the system (1) satisfies hypotheses (H1) and (H3), (19) should be replaced by

$$\begin{aligned} & P_i H_i + H_i^T P_i + \big(|p-2|+1\big)\beta_i \,\mathrm{trace}(P_i) K^2 I \\ + & \frac{2\lambda_M^{\frac{p}{2}-1}}{p\lambda_m^{\frac{p}{2}-1}} \sum_{j\neq i} \gamma_{ij} P_j + \frac{k}{\alpha_i} P_i bb^T P_i + \frac{k}{4}\alpha_i cc^T \leq 0, \end{aligned} \tag{20}$$

where H_i, $i \in \mathbb{S}$, are given by (11).

In this case, we can let

$$A = H_i, \ \ B = \sqrt{\frac{k}{\alpha_i}} b, \ \ R = -I, \ \ Q = \frac{k}{4}\alpha_i cc^T + \big[\big(|p-2|+1\big)\beta_i n K^2 - \gamma_{ii}\big] M Q_0,$$

in (18), then use the function "care" to find P_i, $i \in \mathbb{S}$.

4. **Examples.** In this section, we give some examples to illustrate our results. Example 4.1 is with hypotheses (H1) and (H2), while Example 4.2 is with (H1) and (H3).

Example 4.1. Let $B(t)$ be a 1-dimensional Brownian motion, $S = \{1, 2, 3\}$, $r(t)$ be a right-continuous Markov chain with generator

$$\Gamma = \begin{bmatrix} -1 & 0.5 & 0.5 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}.$$

Let

$$A_1 = \begin{bmatrix} -4 & 0 & 1 \\ 1 & -2 & 0 \\ -1 & 0 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & 1 & 2 \\ 0 & -5 & 0 \\ -1 & 0 & -6 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -7 & 0 & 3 \\ 0 & -8 & 0 \\ 1 & 0 & -9 \end{bmatrix},$$

where $A_i = A(i)$, $i = 1, 2, 3$, $b = (1, 1, 1)^T$, $c = (1, 1, 1)^T$, $k = 5$. Let

$$D(x, t, i) = \sin(2\pi i t)x, \quad i = 1, 2, 3,$$

then $\lambda = 1$.

It's easy to know that hypothesis (H1)(H2) are satisfied. And we get

$$\max \mathrm{Re}\lambda(G_i) = -2.5 < 0, \quad \max \mathrm{Re}\lambda(H_i) = -3.6595 < 0,$$

then (9) is satisfied, so we can use Theorem 3.2.

Let $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = \beta_3 = 1.5$, $Q_0 = I$, $M = 1$, $p = 2$, using the M-method, we can obtain

$$\tilde{P}_1 = \begin{bmatrix} 0.3013 & 0.0649 & 0.0023 \\ 0.0649 & 0.5054 & 0.0031 \\ 0.0023 & 0.0031 & 0.3690 \end{bmatrix}, \quad \begin{matrix} \lambda_{\min}(\tilde{P}_1) = 0.2824, \\ \lambda_{\max}(\tilde{P}_1) = 0.5243, \end{matrix}$$

$$\tilde{P}_2 = \begin{bmatrix} 0.3451 & 0.0349 & 0.0405 \\ 0.0349 & 0.3058 & 0.0177 \\ 0.0405 & 0.0177 & 0.2719 \end{bmatrix}, \quad \begin{matrix} \lambda_{\min}(\tilde{P}_2) = 0.2538, \\ \lambda_{\max}(\tilde{P}_2) = 0.3829, \end{matrix}$$

$$\tilde{P}_3 = \begin{bmatrix} 0.2786 & 0.0100 & 0.0638 \\ 0.0100 & 0.2505 & 0.0120 \\ 0.0638 & 0.0120 & 0.2405 \end{bmatrix}, \quad \begin{matrix} \lambda_{\min}(\tilde{P}_3) = 0.1928, \\ \lambda_{\max}(\tilde{P}_3) = 0.3291. \end{matrix}$$

And it's obvious that $MQ_0 - \tilde{P}_i > 0$, $\forall i \in \mathbb{S}$, so we can choose $P_i = \tilde{P}_i$, then by Theorem 3.2, this system is 2th moment absolutely exponentially stable w.r.t. $F_{(0,5)}$. Actually, we have

$$R_1 - \frac{k}{\alpha_1} d_1 d_1^T = \begin{bmatrix} 1.7362 & -0.1197 & -0.0555 \\ -0.1197 & 1.4638 & -0.0195 \\ -0.0555 & -0.0195 & 1.6902 \end{bmatrix}, \ \lambda_{\min}\left(R_1 - \frac{k}{\alpha_1} d_1 d_1^T\right) = 1.4134,$$

$$R_2 - \frac{k}{\alpha_2} d_2 d_2^T = \begin{bmatrix} 2.4025 & -0.1272 & -0.1268 \\ -0.1272 & 2.2854 & -0.0417 \\ -0.1268 & -0.0417 & 2.4826 \end{bmatrix}, \ \lambda_{\min}\left(R_2 - \frac{k}{\alpha_2} d_2 d_2^T\right) = 2.1674,$$

$$R_3 - \frac{k}{\alpha_3} d_3 d_3^T = \begin{bmatrix} 3.0906 & -0.1496 & -0.1789 \\ -0.1496 & 3.0073 & -0.0565 \\ -0.1789 & -0.0565 & 3.2264 \end{bmatrix}, \ \lambda_{\min}\left(R_3 - \frac{k}{\alpha_3} d_3 d_3^T\right) = 2.8294.$$

Example 4.2. In this example, we let $B(t)$, $r(t)$, Γ, $A(i) = A_i$, b, c, k be all the same as those in Example 4.1, but we change $D(x,t,i)$ into

$$D(x,t,i) = \frac{i}{30}(x_n, x_1, \cdots, x_{n-1})^T.$$

It's easy to know that $D(x,t,i)$ satisfies hypothesis (H1)(H3) with $K = 0.1$, but doesn't satisfy (H2) because $\forall \lambda > 0$, we can choose $P = \mathrm{diag}(1800\lambda, 1, \cdots, 1)$ and $x = (0, \cdots, 0, x_n)^T \neq 0$, then

$$\mathrm{trace}\big(D^T(x,t,i)PD(x,t,i)\big) = 2i^2\lambda x_n^2 > \lambda x_n^2 = \lambda x^T P x.$$

Besides, (9) is satisfied too, so we can use Theorem 3.3.

Let $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = \beta_3 = 1.5$, $Q_0 = I$, $M = 1$, $p = 2$, using the M-method, we can obtain

$$\tilde{P}_1 = \begin{bmatrix} 0.1559 & 0.0575 & 0.0357 \\ 0.0575 & 0.2363 & 0.0409 \\ 0.0357 & 0.0409 & 0.1963 \end{bmatrix}, \qquad \begin{aligned} \lambda_{\min}(\tilde{P}_1) = 0.1229, \\ \lambda_{\max}(\tilde{P}_1) = 0.2948, \end{aligned}$$

$$\tilde{P}_2 = \begin{bmatrix} 0.2167 & 0.0382 & 0.0420 \\ 0.0382 & 0.1991 & 0.0311 \\ 0.0420 & 0.0311 & 0.1799 \end{bmatrix}, \qquad \begin{aligned} \lambda_{\min}(\tilde{P}_2) = 0.1512, \\ \lambda_{\max}(\tilde{P}_2) = 0.2757, \end{aligned}$$

$$\tilde{P}_3 = \begin{bmatrix} 0.2006 & 0.0186 & 0.0556 \\ 0.0186 & 0.1809 & 0.0202 \\ 0.0556 & 0.0202 & 0.1756 \end{bmatrix}, \qquad \begin{aligned} \lambda_{\min}(\tilde{P}_3) = 0.1307, \\ \lambda_{\max}(\tilde{P}_3) = 0.2551. \end{aligned}$$

And it's obvious that $MQ_0 - \tilde{P}_i > 0$, $\forall i \in \mathbb{S}$, so we can choose $P_i = \tilde{P}_i$, then by Theorem 3.3, this system is 2th moment absolutely exponentially stable w.r.t. $F_{(0,5)}$. Actually, we have

$$R_1 - \frac{k}{\alpha_1}d_1 d_1^T = \begin{bmatrix} 0.8275 & -0.0284 & -0.0488 \\ -0.0284 & 0.8461 & -0.0257 \\ -0.0488 & -0.0257 & 0.8584 \end{bmatrix}, \lambda_{\min}\left(R_1 - \frac{k}{\alpha_1}d_1 d_1^T\right) = 0.7721,$$

$$R_2 - \frac{k}{\alpha_2}d_2 d_2^T = \begin{bmatrix} 1.6796 & -0.0762 & -0.0913 \\ -0.0762 & 1.6188 & -0.0611 \\ -0.0913 & -0.0611 & 1.6641 \end{bmatrix}, \lambda_{\min}\left(R_2 - \frac{k}{\alpha_2}d_2 d_2^T\right) = 1.5007,$$

$$R_3 - \frac{k}{\alpha_3}d_3 d_3^T = \begin{bmatrix} 2.4473 & -0.1340 & -0.1197 \\ -0.1340 & 2.4021 & -0.1031 \\ -0.1197 & -0.1031 & 2.4806 \end{bmatrix}, \lambda_{\min}\left(R_3 - \frac{k}{\alpha_3}d_3 d_3^T\right) = 2.2011.$$

5. **Conclusions.** In this paper, a new definition of pth moment absolute exponential stability for stochastic switching system is presented. And we use the Lypunov functions to obtain the algebraic criteria in terms of matrix inequalities under different assumptions. We also introduce a technique called M-method to verify our results with MATLAB.

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