

GAUSSIAN ESTIMATES FOR A HEAT EQUATION ON A NETWORK

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(Communicated by Benedetto Piccoli)

ABSTRACT. We consider a diffusion problem on a network on whose nodes we impose Dirichlet and generalized, non-local Kirchhoff-type conditions. We prove well-posedness of the associated initial value problem, and we exploit the theory of sub-Markovian and ultracontractive semigroups in order to obtain upper Gaussian estimates for the integral kernel. We conclude that the same diffusion problem is governed by an analytic semigroup acting on all L^p -type spaces as well as on suitable spaces of continuous functions. Stability and spectral issues are also discussed. As an application we discuss a system of semilinear equations on a network related to potential transmission problems arising in neurobiology.

1. Introduction. Evolution equations taking place in networks or, more generally, in ramified structures have been first considered in pioneering articles by K. Ruedenberg and C. Scherr back in the 1950s, cf. [35], and, at a more mathematical level, in a series of papers by R. Mills and E. Montroll and by G. Lumer in the 1970s, cf. [29]–[28] and [24]–[25], respectively. Shortly afterwards, F. Ali Mehmeti, J. von Below, S. Nicaise, and J.P. Roth among others began a systematic study of properties of elliptic operators acting on spaces of functions over networks, cf. e.g. the monographs [33], [2], [23], and references therein. Ever since, such problems have aroused broad interest among mathematicians working on partial differential equations, control, and spectral theory – as well as among theoretical physicists interested in scattering theory of guided waves, photonic crystals, and quantum wires, resulting in a literature so vast that it can by no means be summarized here.

Throughout this paper we consider a finite, unitarily parametrized, connected network whose structure is given by a suitable graph. On it we study a general diffusion equation. Adopting a setting which is standard in literature, the node

2000 *Mathematics Subject Classification.* Primary: 47D06, 47A07; Secondary 90B10.

Key words and phrases. Evolution equations on networks; Ultracontractive semigroups of operators; Gaussian estimates.

I gratefully acknowledge the hospitality of the AGFA - Working group on Functional Analysis at the Eberhard-Karl University of Tübingen, where this paper was begun in February 2005. I would also like to thank Rainer Nagel, Susanna Piazzera, Stefan Teufel and Roderich Tumulka (University of Tübingen), as well as Abdelaziz Rhani (University of Marrakech) and Stefano Cardanobile (University of Ulm) for helpful comments and motivating discussions.

conditions impose continuity and Kirchhoff-type transmission laws in ramification vertices. However, we extend known results by allowing more general, *non-local* node conditions, cf. Section 2 below. Roughly speaking, in each node v_i of the network we allow an absorption that does not only depend on the value of the function in v_i itself, but also on other nodes v_h . Using the arguments presented in [11], one can promptly obtain well-posedness of such a general parabolic network problem; we are thus more interested in qualitative properties. In some sense, our results complement those obtained in [18] and [19], where even more general node conditions are allowed, and where positivity of the semigroups is also discussed, but where the quantum physical viewpoint was motivated to mainly discuss those conditions leading to self-adjointness.

In this paper we pursue an approach to parabolic equations on networks based on the theory of sesquilinear forms and associated sub-Markovian semigroups. Following the approach presented in previous works (see [11], [2, Chapt. 2], and [20]), we draw some conclusions about several issues, including L^∞ -contractivity of the semigroup governing the problem, its $L^2 - L^\infty$ -stability and its analyticity in suitable spaces of continuous as well as of L^1 functions over the graph. The key arguments are upper estimates for the integral kernel of the generated semigroup (which, roughly speaking, yields the Green function of the problem). The theory of such *Gaussian estimates* has become a mature one that proves extremely powerful when applied to diffusion problems on domains – and, as a matter of fact, on networks, too; we refer to [14], [3], and [34] for an introduction to this subject.

While the integral kernel for the diffusion problem on a network has already been explicitly computed in [32] in the special case of constant coefficients for the heat operator, Gaussian estimates are, to our knowledge, new in the case of *variable* coefficients, even in the case of local nodal conditions. Observe also that Gaussian estimates for the semigroup that governs the *discrete* diffusion problem on a graph have recently been proved in [22], cf. also [8].

2. General framework. We consider a finite connected network, represented by a finite graph G with m edges e_1, \dots, e_m and n vertices v_1, \dots, v_n . We normalize and parametrize the edges on the interval $[0, 1]$. The structure of the network is given by the $n \times m$ matrix $\Phi^+ := (\phi_{ij}^+)$ and $\Phi^- := (\phi_{ij}^-)$ defined by

$$\phi_{ij}^+ := \begin{cases} 1, & \text{if } e_j(0) = v_i, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \phi_{ij}^- := \begin{cases} 1, & \text{if } e_j(1) = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

The $n \times m$ matrix $\Phi := (\phi_{ij})$, defined by $\Phi := \Phi^+ - \Phi^-$, is thus the *incidence matrix* of the graph G . Further, let $\Gamma(v_i)$ be the set of all the indices of the edges having an endpoint at v_i , i.e.,

$$\Gamma(v_i) := \{j \in \{1, \dots, m\} : e_j(0) = v_i \text{ or } e_j(1) = v_i\}, \quad 1 \leq i \leq n-1.$$

For the sake of simplicity, we denote the value of the functions $c_j(\cdot)$ and $u_j(t, \cdot)$ at 0 or 1 by $c_j(v_i)$ and $u_j(t, v_i)$, if $e_j(0) = v_i$ or $e_j(1) = v_i$, respectively. With an abuse of notation, we also set $u_j'(t, v_i) = c_j(v_i) := 0$ whenever $j \notin \Gamma(v_i)$.

In the literature networks are usually considered where Dirichlet conditions are imposed on n_0 boundary vertices (i.e., vertices of degree 1). Since no process takes place in such boundary vertices, we may and do identify all of them. Thus, we instead consider an equivalent diffusion equation on a rearranged graph where these n_0 nodes of degree 1 are replaced by *only one node of degree n_0* , on which a Dirichlet condition is imposed. Without loss of generality we assume that this node is v_n .

For $t \geq 0$ we are now in the position to consider the network diffusion problem

$$\begin{cases} \dot{u}_j(t, x) &= (c_j u'_j)'(t, x), & x \in (0, 1), j = 1, \dots, m, \\ u_j(t, \mathbf{v}_i) &= u_\ell(t, \mathbf{v}_i) =: d_i^u(t), & j, \ell \in \Gamma(\mathbf{v}_i), i = 1, \dots, n, \\ \sum_{h=1}^{n-1} b_{ih} d_h^u(t) &= \sum_{j=1}^m \phi_{ij} c_j(\mathbf{v}_i) u'_j(t, \mathbf{v}_i), & i = 1, \dots, n-1, \\ d_n^u(t) &= 0, \end{cases}$$

on the network, with initial conditions

$$u_j(0, x) = u_{0j}(x), \quad x \in (0, 1), j = 1, \dots, m.$$

Throughout this paper we assume that the coefficients c_j satisfy $0 < c_j \in C^1[0, 1]$, $j = 1, \dots, m$, while for the time being the $(n-1) \times (n-1)$ matrix $B := (b_{ih})_{1 \leq i, h \leq n-1}$ is arbitrary, i.e., $B \in M_{n-1}(\mathbb{C})$. The third equation above is a generalized, non-local Kirchhoff-type law on the first $n-1$ nodes, while the fourth one prescribes a Dirichlet condition in \mathbf{v}_n .

We introduce the $n \times m$ matrices $\Phi_w^+ := (\omega_{ij}^+)$ and $\Phi_w^- := (\omega_{ij}^-)$ defined by

$$\omega_{ij}^+ := \begin{cases} c_j(\mathbf{v}_i), & \text{if } \phi_{ij}^+ = 1 \text{ and } i \leq n-1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\omega_{ij}^- := \begin{cases} c_j(\mathbf{v}_i), & \text{if } \phi_{ij}^- = 1 \text{ and } i \leq n-1, \\ 0, & \text{otherwise.} \end{cases}$$

In the following we will repeatedly and without further notice write the functions u_j in vector form, i.e.,

$$u \equiv \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

With these notations, one can directly check that the second, third and fourth equations of our network diffusion problem can be rewritten as

$$\begin{aligned} \exists d^u(t) \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t. } & (\Phi^+)^T d^u(t) = u(t, 0), \quad (\Phi^-)^T d^u(t) = u(t, 1), \\ & \text{and } \Phi_w^- u'(t, 1) - \Phi_w^+ u'(t, 0) = \mathbb{B} d^u(t), \end{aligned} \quad t \geq 0,$$

where we have introduced the $n \times n$ matrices

$$\mathbb{B} := \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark 2.1. Inspired by the non-local boundary conditions introduced above, and motivated by the theory of ephaptic coupling of myelinated nerve fibers (see [36, Chapt. 8]) one may also consider an ever more general parabolic system over the network, where for $t \geq 0$ the first equation of the problem is replaced by

$$\dot{u}_j(t, x) = \sum_{\ell=1}^m (c_{j\ell} u'_\ell)'(t, x), \quad x \in (0, 1), j = 1, \dots, m.$$

Under suitable assumptions on the diffusion coefficients c_{ij} one can study well-posedness and qualitative properties of such a system. However, this introduces new technical difficulties, as for instance one sees that usual Kirchhoff-conditions do not ensure dissipativity of such a system any more, and one has to formulate appropriate, more general conditions in the nodes. We will discuss this kind of problems in the forthcoming paper [13].

It is already well-known that the above network diffusion problem is well-posed: in fact, this has been shown in a Hilbert space setting in [11] for the case of variable coefficients and diagonal B , cf. also the references therein for earlier results on less general cases. We also remark that, at least in the case of $c_1 = \dots = c_m \equiv 1$ and $b_{ih} = 0$, $i, h = 1, \dots, n$. S. Nicaise has derived in [32] an explicit formula for the solution, thus showing the well-posedness of the problem in all L^p -type spaces as well as on suitable spaces of continuous functions (see also [31]). Our first goal is to establish a meaningful L^p -theory for the general case of variable diffusion coefficients and non-local node conditions. To this aim, we need the following.

Definition 2.2. For given functions $f_j : [0, 1] \rightarrow \mathbb{C}$, $j = 1, \dots, n$, we define a mapping $Uf : [0, m] \rightarrow \mathbb{C}$ by

$$Uf(x) := \tilde{f}(x) := f_j(x - j + 1) \quad \text{if } x \in (j - 1, j), \quad j = 1, \dots, m.$$

With this notation one sees that the following holds.

Lemma 2.3. *The mapping U is one-to-one from $X_p := (L^p(0, 1))^m$ onto $L^p(0, m)$ for all $p \in [1, \infty]$, and in fact it is an isometry if we endow $(L^p(0, 1))^m$ with the canonical l^p -norm, i.e.,*

$$\|f\|_{X_p} := \left(\sum_{j=1}^m \|f_j\|_{L^p(0,1)}^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

or

$$\|f\|_{X_\infty} := \max_{1 \leq j \leq m} \|f_j\|_{L^\infty(0,1)}.$$

In the following we will hence regard X_p as an L^p -space over a finite measure space, so that $X_p \hookrightarrow X_q$ for all $1 \leq q \leq p \leq \infty$. Moreover, each X_p is a Banach lattice, and its positive cone can be identified with the positive cone of $L^p(0, m)$.

3. Basic results. We are now in the position to consider an abstract reformulation of our diffusion problem. First we consider the (complex) Hilbert space $X_2 = (L^2(0, 1))^m$ endowed with the natural inner product

$$(f | g)_{X_2} := \sum_{j=1}^m \int_0^1 f_j(x) \overline{g_j(x)} dx, \quad f, g \in X_2.$$

On X_2 we define an operator

$$A := \begin{pmatrix} \frac{d}{dx} \left(c_1 \frac{d}{dx} \right) & & 0 \\ & \ddots & \\ 0 & & \frac{d}{dx} \left(c_m \frac{d}{dx} \right) \end{pmatrix} \quad (3.1)$$

with domain

$$D(A) := \left\{ f \in (H^2(0, 1))^m : \begin{array}{l} \exists df \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^T df = f(0), \quad (\Phi^-)^T df = f(1), \\ \text{and } \Phi_w^- f'(1) - \Phi_w^+ f'(0) = B df \end{array} \right\}. \quad (3.2)$$

We can finally rewrite the concrete network diffusion problem as an abstract Cauchy problem

$$\begin{cases} \dot{u}(t) &= Au(t), & t \geq 0, \\ u(0) &= u_0, \end{cases} \quad (\text{ACP})$$

on X_2 . In order to discuss the semigroup generator property of A it is convenient to use a variational method.

Recall that we are assuming the network to be connected throughout the paper. This is crucial for the proof of the following.

Lemma 3.1. *The linear space*

$$V_0 := \left\{ f \in (H^1(0,1))^m : \begin{array}{l} \exists d^f \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^{\top} d^f = f(0) \text{ and } (\Phi^-)^{\top} d^f = f(1) \end{array} \right\}.$$

is densely and compactly embedded in X_2 . It becomes a Hilbert space when equipped with the inner product

$$(f | g)_{V_0} := \sum_{j=1}^m \int_0^1 f'_j(x) \overline{g'_j(x)} dx, \quad f, g \in V_0. \quad (3.3)$$

Proof. It is well-known that V_0 is densely and compactly imbedded in X_2 : this follows from the inclusions $(C_c^\infty(0,1))^m \subset V_0 \subset (H^1(0,1))^m$ and the Rellich–Khondrakov Theorem.

We are going to show that the inner product defined in (3.3) is equivalent to that induced by the Hilbert space $(H^1(0,1))^m$, i.e., to

$$(f | g) := \sum_{j=1}^m \int_0^1 \left(f'_j(x) \overline{g'_j(x)} + f_j(x) \overline{g_j(x)} \right) dx, \quad f, g \in (H^1(0,1))^m.$$

Let e_j be a general edge. By the connectedness of G we can find a set of edges e_{j_1}, \dots, e_{j_r} linking any $x \in e_j$ with v_n . More precisely, there exist $j_1, \dots, j_r \in \{1, \dots, m\}$ such that

- $\phi_{nj_1}^- = 1$, i.e., v_n is the head of e_{j_1} ;
- for all $h = 1, \dots, r$ there exists a vertex $v_{i_{h+1}}$ such that $\phi_{i_{h+1}j_h}^+ = 1 = \phi_{i_{h+1}j_{h+1}}^-$, i.e., $v_{i_{h+1}}$ is the tail of e_{j_h} and the head of $e_{j_{h+1}}$;
- $e_{j_r} = e_j$.

Then, for every $f \in V$ and at any $x \in (0,1)$ one has $f_{j_r}(x) = d_{i_{r-1}}^f + \int_0^x f'_{j_r}(s) ds$, and therefore

$$\begin{aligned} |f_{j_r}(x)| &\leq |d_{i_{r-1}}^f| + \int_0^1 |f'_{j_r}(s)| ds \\ &\leq |d_{i_{r-2}}^f| + \int_0^1 |f'_{j_{r-1}}(s)| ds + \int_0^1 |f'_{j_r}(s)| ds \\ &\leq \dots \\ &\leq |d_n^f| + \sum_{h=1}^r \int_0^1 |f'_{j_h}(s)| ds \\ &= \sum_{h=1}^r \int_0^1 |f'_{j_h}(s)| ds, \end{aligned}$$

due to the Dirichlet condition satisfied at v_n by all $f \in V$. We conclude that

$$\sum_{j=1}^m \int_0^1 |f_j(x)| dx \leq m \sum_{j=1}^m \int_0^1 |f'_j(x)| dx.$$

Having proved such a Poincaré-type inequality, the claim follows directly. □

The following lemma extends known results (see e.g. [1], [32], and [11]) to the case of non-local node conditions. Thus, for the sake of self-containedness we do not omit the proofs, although basic elements and techniques involved are essentially well-known.

Proposition 3.2. *Consider the sesquilinear form*

$$\mathfrak{a}(f, g) := \sum_{j=1}^m \int_0^1 c_j(x) f'_j(x) \overline{g'_j(x)} dx - \sum_{i,h=1}^{n-1} b_{ih} d_h^f \overline{d_i^g}, \quad f, g \in V_0,$$

on the Hilbert space X_2 . Then \mathfrak{a} enjoys the following properties:

- \mathfrak{a} is X_2 -elliptic, i.e.,: there exist $\omega \in \mathbb{R}$, $\alpha > 0$ such that

$$\operatorname{Re} \mathfrak{a}(f, f) + \omega \|f\|_{X_2}^2 \geq \alpha \|f\|_{V_0}^2 \quad \text{for all } f \in V_0;$$

- \mathfrak{a} is continuous, i.e., there exists $M > 0$ such that

$$|\mathfrak{a}(f, g)| \leq M \|f\|_{V_0} \|g\|_{V_0} \quad \text{for all } f, g \in V_0;$$

- \mathfrak{a} is symmetric, i.e.,

$$\mathfrak{a}(f, g) = \overline{\mathfrak{a}(g, f)} \quad \text{for all } f, g \in V_0,$$

if and only if B is self-adjoint;

- \mathfrak{a} is coercive, i.e., it is X_2 -elliptic with $\omega = 0$, if and only if it is accretive, i.e.,

$$\operatorname{Re} \mathfrak{a}(f, f) \geq 0 \quad \text{for all } f \in V_0,$$

if and only if B is dissipative.

Proof. First of all, we show that \mathfrak{a} is X_2 -elliptic. Taking into account (3.3), it suffices to find $\omega \in \mathbb{R}$ such that for all $f \in V_0$

$$\sum_{j=1}^m \int_0^1 |f'_j(x)|^2 dx + \omega \sum_{j=1}^m \int_0^1 |f_j(x)|^2 dx \geq \operatorname{Re} \sum_{i,h=1}^{n-1} \frac{b_{ih}}{c} d_h^f \overline{d_i^f}$$

where

$$c := \frac{1}{2} \min_{1 \leq j \leq m} \min_{x \in [0,1]} \|c_j(x)\| > 0. \quad (3.4)$$

Observe that

$$\operatorname{Re} \sum_{i,h=1}^{n-1} \frac{b_{ih}}{c} d_h^f \overline{d_i^f} = \frac{1}{c} \operatorname{Re}(\mathbb{B} d^f \mid d^f) \leq \frac{\|\mathbb{B}\|}{c} \|d^f\|^2 \leq M \sum_{j=1}^m (\|f'_j\|_{L^2}^2 + \|f_j\|_{L^2}^2)$$

for some constant M . Here we have used the fact that, due to the continuous imbedding of $H^1(0, 1)$ into $C[0, 1]$, there holds

$$|d_i^f| \leq \max_{1 \leq j \leq m} \max_{x \in [0,1]} |f_j(x)| \leq \max_{1 \leq j \leq m} \|f_j\|_{H^1(0,1)} \leq \sum_{j=1}^m \|f_j\|_{H^1(0,1)}. \quad (3.5)$$

for all $f \in V_0$ and all $i = 1, \dots, n-1$. Now the claim follows, since $\|f_j\|_{L^2} \neq 0$ whenever $f \in V_0$ and $\|d^f\| \neq 0$.

Set now

$$C := \max_{1 \leq j \leq m} \max_{x \in [0,1]} c_j(x) \quad \text{and} \quad b := \max_{1 \leq i,h \leq n-1} |b_{ih}|.$$

Recall that by (3.3) there exists a constant N such that $\sum_{j=1}^m \|f_j\|_{H^1(0,1)} \leq N\|f\|_{V_0}$. Take $f, g \in V_0$ and observe that

$$\begin{aligned} |\mathfrak{a}(f, g)| &\leq C \left| \sum_{j=1}^m \int_0^1 f'_j(x) \overline{g'_j(x)} dx \right| + \sum_{i,h=1}^{n-1} |b_{ih}| |d_i^f| |d_h^g| \\ &\leq C |(f|g)_{V_0}| + b \sum_{i,h=1}^{n-1} N^2 \|f\|_{V_0} \|g\|_{V_0} \\ &\leq M \|f\|_{V_0} \|g\|_{V_0} \end{aligned}$$

by the Cauchy–Schwartz inequality, where $M := C + bN^2(n-1)^2$. This completes the proof of the continuity of \mathfrak{a} .

Since by assumptions the coefficients c_1, \dots, c_m are strictly positive,

$$\mathfrak{a}(f, g) = \overline{\mathfrak{a}(g, f)} \quad \text{if and only if} \quad \sum_{i,h=1}^{n-1} b_{ih} d_h^f \overline{d_i^g} = \sum_{i,h=1}^{n-1} \overline{b_{ih}} d_i^f \overline{d_h^g}, \quad f, g \in V_0,$$

and this is the case if and only if B is self-adjoint.

Let finally \mathfrak{a} be coercive. Then it is clearly accretive. If \mathfrak{a} is accretive, then a direct computation shows that

$$\operatorname{Re} \sum_{i,h=1}^{n-1} b_{ih} d_h^f \overline{d_i^f} \leq 0$$

holds for all $f \in V_0$. Due to the arbitrariness of the nodal values d_2^f, \dots, d_n^f of $f \in V_0$, one sees that this already implies that B is dissipative. Finally, if B is dissipative, then there holds

$$\begin{aligned} \operatorname{Rea}(f, f) &= \sum_{j=1}^m \int_0^1 c_j(x) |f'_j(x)|^2 dx - \operatorname{Re} \sum_{i,h=1}^{n-1} b_{ih} d_h^f \overline{d_i^f} \\ &\geq c \sum_{j=1}^m \int_0^1 |f'_j(x)|^2 dx, \end{aligned}$$

where c is defined as in (3.4). By Lemma 3.1, we have thus obtained that $\operatorname{Rea}(f, f) \geq \alpha \|f\|_{V_0}^2$ for some $\alpha > 0$. This concludes the proof. \square

Corollary 3.3. *The operator associated with \mathfrak{a} is densely defined, sectorial, and resolvent compact, hence it generates a strongly continuous, analytic, compact semigroup $(T_2(t))_{t \geq 0}$ on X_2 . If moreover B is dissipative (resp., self-adjoint), then $(T_2(t))_{t \geq 0}$ is contractive and uniformly exponentially stable (resp., self-adjoint).*

Proof. It follows from Proposition 3.2 and [34, Prop. 1.51, Thm. 1.52] that the operator associated with \mathfrak{a} is densely defined, sectorial, and resolvent compact.

Let B be dissipative and take f in the null space of the operator associated with \mathfrak{a} . Then by the proof of the above lemma there exists $c > 0$ such that

$$0 = \operatorname{Rea}(f, f) \geq c \sum_{j=1}^m \int_0^1 |f'_j(x)|^2 dx.$$

This means that f_j is constant for all $j = 1, \dots, m$, hence f is constant on the whole network, due to the continuity condition in the nodes. In particular, $f \equiv d_n^f = 0$, hence the operator associated with \mathfrak{a} is one-to-one. \square

In the following we are able to identify the operator associated with \mathfrak{a} . The following result is already known in the literature in the special case where the matrix B is diagonal, see e.g. [2, Chapt. 2].

Lemma 3.4. *The operator associated with the form \mathfrak{a} is $(A, D(A))$ defined in (3.1)–(3.2).*

Proof. Denote by $(C, D(C))$ the operator associated with \mathfrak{a} , which by definition is given by

$$\begin{aligned} D(C) &:= \{f \in V_0 : \exists g \in X_2 \text{ s.t. } \mathfrak{a}(f, h) = (g | u)_{X_2} \forall u \in V_0\}, \\ Cf &:= -g. \end{aligned}$$

Let us first show that $A \subset C$. Take $f \in D(A)$. Then for all $u \in V_0$

$$\begin{aligned} \mathfrak{a}(f, u) &= \sum_{j=1}^m \int_0^1 c_j(x) f'_j(x) \overline{u'_j(x)} dx - \sum_{i,h=1}^{n-1} b_{ih} d_h^f \overline{d_i^u} \\ &= \sum_{j=1}^m [c_j f'_j \overline{u_j}]_0^1 - \sum_{j=1}^m \int_0^1 (c_j f'_j)'(x) \overline{u_j(x)} dx - \sum_{i,h=1}^{n-1} b_{ih} d_h^f \overline{d_i^u}. \end{aligned}$$

Using the incidence matrix $\Phi = \Phi^+ - \Phi^-$ and recalling that $d_n^u = 0$ as $u \in V_0$, we can write

$$\sum_{j=1}^m [c_j f'_j \overline{u_j}]_0^1 = \sum_{j=1}^m \sum_{i,h=1}^{n-1} c_j(\mathbf{v}_i) (\phi_{ij}^- - \phi_{ij}^+) f'_j(\mathbf{v}_i) \overline{d_i^u} = \sum_{i,h=1}^{n-1} \overline{d_i^u} \sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) f'_j(\mathbf{v}_i).$$

Using the generalized Kirchhoff condition $\Phi_w^- f'(1) - \Phi_w^+ f'(0) = \mathbb{B}d^f$, which holds for all functions $f \in D(A)$, we obtain that

$$\begin{aligned} \mathfrak{a}(f, u) &= \sum_{i=1}^{n-1} \overline{d_i^u} \underbrace{\sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) f'_j(\mathbf{v}_i)}_{=\sum_{h=1}^{n-1} b_{ih} d_h^f} - \sum_{j=1}^m \int_0^1 (c_j f'_j)'(x) \overline{u_j(x)} dx \\ &\quad - \sum_{i,h=1}^{n-1} b_{ih} d_h^f \overline{d_i^u} \\ &= - \sum_{j=1}^m \int_0^1 (c_j f'_j)'(x) \overline{u_j(x)} dx = -(Af | u)_{X_2}, \end{aligned}$$

which makes sense because $Af \in X_2$. The proof of the inclusion $A \subset C$ is completed.

To check the converse inclusion take $f \in D(C)$. By definition, there exists $g \in X_2$ such that

$$\mathfrak{a}(f, u) = (g | u)_{X_2} = \sum_{j=1}^m \int_0^1 g_j(x) \overline{u_j(x)} dx \quad \text{for all } u \in V_0, \quad (3.6)$$

hence in particular for all functions of the form

$$\begin{pmatrix} 0 \\ \vdots \\ u_j \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}} \text{ row}, \quad u_j \in H_0^1(0, 1). \quad (3.7)$$

From this follows that (3.6) in fact implies

$$\int_0^1 c_j(x) f'_j(x) \overline{u'_j(x)} dx = \int_0^1 g_j(x) \overline{u_j(x)} dx \quad \text{for all } j = 1, \dots, m, \quad u_j \in H_0^1(0, 1).$$

By definition of weak derivative this means that $c_j \cdot f'_j \in H^1(0, 1)$ for all $j = 1, \dots, m$. Since $0 < c_j \in C^1[0, 1]$, there follows that $f'_j \in H^1(0, 1)$ for all $j = 1, \dots, m$. We conclude that $f \in (H^2(0, 1))^m$. Moreover, integrating by parts as in the proof of the first inclusion we see that if (3.6) holds for all $u \in V_0$, then there also holds

$$\sum_{i,h=1}^{n-1} \overline{d_i^u} \sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) f'_j(\mathbf{v}_i) = \sum_{i,h=1}^{n-1} b_{ih} d_h^f \overline{d_i^u}.$$

Since $u \in V_0$ is arbitrary, this means that

$$\sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) f'_j(\mathbf{v}_i) = \sum_{h=1}^{n-1} b_{ih} d_h^f \quad \text{for all } i = 1, \dots, n-1,$$

that is, $\Phi_w^- f'(1) - \Phi_w^+ f'(0) = \mathbb{B} d^f$. Therefore $f \in D(A)$ and

$$-\sum_{j=1}^m \int_0^1 (c_j f'_j)'(x) \overline{u_j(x)} dx = \sum_{j=1}^m \int_0^1 g_j(x) \overline{h_j(x)} dx$$

holds for all $h \in V_0$. This implies that $Af = -g$, and the proof is complete. \square

By the above results, the operator A generates on X_2 a semigroup $(T_2(t))_{t \geq 0}$: thus, the abstract Cauchy problem (ACP) (and hence the concrete diffusion problem on the network) is well-posed in X_2 . We can characterize several features of $(T_2(t))_{t \geq 0}$ by those of $(e^{tB})_{t \geq 0}$, and hence of the scalar matrix B : we are going to show that $(T_2(t))_{t \geq 0}$ is real, positive, and X_∞ -contractive (i.e., the unit ball of X_∞ is invariant under $T_2(t)$ for all $t \geq 0$), respectively, if and only if the semigroup $(e^{tB})_{t \geq 0}$ generated by B is real, positive, and ℓ^∞ -contractive (i.e., once endowed \mathbb{C}^{n-1} with the equivalent ∞ -norm, e^{tB} leaves the unit ball of \mathbb{C}^{n-1} invariant for all $t \geq 0$), respectively.

Theorem 3.5. *The semigroup $(T_2(t))_{t \geq 0}$ on X_2 associated with \mathbf{a} enjoys the following properties:*

- *it is real if and only if the matrix B has real entries;*
- *it is positive if and only if the matrix B has real entries that are positive off-diagonal,*

If moreover B is dissipative, then $(T_2(t))_{t \geq 0}$ is X^∞ -contractive if and only if

$$\operatorname{Re} b_{ii} + \sum_{h \neq i} |b_{ih}| \leq 0, \quad i = 1, \dots, n-1. \quad (3.8)$$

Proof. By Corollary 3.3 \mathbf{a} is densely defined, continuous, and elliptic. Thus, by [34, Prop. 2.5 and Thm. 2.6] and Proposition 3.2, and taking into account the rescaling argument discussed in [5, § 9.2], the semigroup associated to \mathbf{a} is real and positive, respectively, if and only if

- $f \in V_0 \Rightarrow \operatorname{Re} f \in V_0$ and $\mathbf{a}(\operatorname{Re} f, \operatorname{Im} f) \in \mathbb{R}$, and
- $f \in V_0 \Rightarrow (\operatorname{Re} f)^+ \in V_0$, $\mathbf{a}(\operatorname{Re} f, \operatorname{Im} f) \in \mathbb{R}$, and $\mathbf{a}((\operatorname{Re} f)^+, (\operatorname{Re} f)^-) \leq 0$,

respectively.

Furthermore, if B is dissipative, then by Proposition 3.2 \mathfrak{a} is accretive and by [34, Thm. 2.15] the semigroup $(T_2(t))_{t \geq 0}$ is X_∞ -contractive if and only if

$$f \in V_0 \Rightarrow (1 \wedge |f|)\text{sign}f \in V_0 \text{ and } \text{Re}((1 \wedge |f|)\text{sign}f, (|f| - 1)^+\text{sign}f) \geq 0,$$

where sign denotes the generalized (complex-valued) function defined by

$$\text{sign}f(x) := \begin{cases} \frac{f(x)}{|f(x)|}, & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0. \end{cases}$$

If $k \in H^1(0, 1)$, then it is clear that $\text{Re } k \in H^1(0, 1)$ as well as $(\text{Re } k)^+ \in H^1(0, 1)$. Furthermore, $(\text{Re } k)' = \text{Re}(k')$ and $((\text{Re } k)^+)' = \text{Re}(k')\mathbb{1}_{\{k \geq 0\}}$. Moreover the functions defined by

$$((1 \wedge |k|)\text{sign}k)(x) = \begin{cases} k(x), & \text{if } |k(x)| \leq 1, \\ \frac{k(x)}{|k(x)|}, & \text{if } |k(x)| \geq 1, \end{cases}$$

as well as

$$(|k| - 1)^+\text{sign}k(x) = \begin{cases} 0, & \text{if } |k(x)| \leq 1, \\ k(x) - \frac{k(x)}{|k(x)|}, & \text{if } |k(x)| \geq 1, \end{cases}$$

are in $H^1(0, 1)$, with $((1 \wedge |k|)\text{sign}k)' = k'\mathbb{1}_{\{|k| \leq 1\}}$ and $(|k| - 1)^+\text{sign}k)' = k'\mathbb{1}_{\{|k| \geq 1\}}$.

By definition, the subspace V_0 contains exactly those functions on the network that are continuous in the vertices and vanish in the vertex v_n . Then for every $f \in V_0$ we have $\text{Re}(f_j) = (\text{Re}f)_j$, $1 \leq j \leq m$. It follows from the above arguments that $\text{Re}f \in (H^1(0, 1))^m$, and one can see that the continuity of the values attained by f in the vertices is preserved after taking the real part $\text{Re}f$. All in all, $\text{Re}f \in V_0$. Moreover, $\mathfrak{a}(\text{Re}f, \text{Im}f)$ is the sum of m integrals and n numbers. Recall that the weights c_1, \dots, c_m are real-valued, positive functions. Since $d^{\text{Re}f}, d^{\text{Im}f} \in \mathbb{R}^n$, it follows that $\mathfrak{a}(\text{Re}f, \text{Im}f) \in \mathbb{R}$ if and only if

$$\sum_{i,h=1}^{n-1} b_{ih} d_h^{\text{Re}f} d_i^{\text{Im}f} \in \mathbb{R}.$$

This holds for all $f \in V_0$ if and only if $b_{ih} \in \mathbb{R}$.

Moreover, if $f \in V_0$, then $((\text{Re}f)^+)_j = (\text{Re}(f_j))^+$, $1 \leq j \leq m$, and one sees as above that $(\text{Re}f)^+ \in V_0$. In particular, for all $i = 1, \dots, n-1$ there holds

$$d_i^{(\text{Re}f)^+} = \begin{cases} 0, & \text{if } \text{Red}_i^f \leq 0, \\ \text{Red}_i^f, & \text{if } \text{Red}_i^f \geq 0, \end{cases} \quad \text{and} \quad d_i^{(\text{Re}f)^-} = \begin{cases} -\text{Red}_i^f, & \text{if } \text{Red}_i^f \leq 0, \\ 0, & \text{if } \text{Red}_i^f \geq 0. \end{cases}$$

Accordingly,

$$\begin{aligned} \mathfrak{a}((\text{Re}f)^+, (\text{Re}f)^-) &= - \sum_{i,h=1}^{n-1} b_{ih} d_h^{(\text{Re}f)^+} d_i^{(\text{Re}f)^-} \\ &= - \sum_{\substack{1 \leq i,h \leq 2 \\ i \neq h}} b_{ih} (\text{Red}_h^f)^+ (\text{Red}_i^f)^-. \end{aligned}$$

Thus, $\mathfrak{a}((\text{Re}f)^+, (\text{Re}f)^-) \leq 0$ if B has positive off-diagonal entries. Conversely assume that $b_{i_0 h_0} < 0$ for some $i_0 \neq h_0$. Then consider the function f with boundary

values $d_{i_0}^f = -1$, $d_{h_0}^f = 1$, and $d_i^f = 0$ for all other i . Then $\mathfrak{a}((\text{Ref})^+, (\text{Ref})^-) = -b_{i_0 h_0} > 0$.

Finally, we discuss the property of X_∞ -contractivity for the semigroup associated to \mathfrak{a} . Thus, take $f \in V_0$. Then $(1 \wedge |f|)\text{sign}f \in (H^1(0, 1))^m$ and, again, the continuity of f in the vertices imposes the same property to the function $(1 \wedge |f|)\text{sign}f$, and in fact

$$d_i^{(1 \wedge |f|)\text{sign}f} = (1 \wedge |d_i^f|)\text{sign}d_i^f = \begin{cases} d_i^f, & \text{if } |d_i^f| \leq 1, \\ \text{sign}d_i^f, & \text{if } |d_i^f| > 1, \end{cases}$$

as well as

$$d_i^{(|f|-1)^+\text{sign}f} = (|d_i^f| - 1)^+\text{sign}d_i^f = \begin{cases} 0, & \text{if } |d_i^f| \leq 1, \\ d_i^f - \text{sign}d_i^f, & \text{if } |d_i^f| > 1, \end{cases}$$

for all $i = 1, \dots, n-1$. Now a direct computation yields

$$\begin{aligned} \mathfrak{a}((1 \wedge |f|)\text{sign}f, (|f| - 1)^+\text{sign}f) &= - \sum_{i,h=1}^{n-1} b_{ih} (1 \wedge |d_h^f|)\text{sign}d_h^f \overline{(|d_i^f| - 1)^+\text{sign}d_i^f} \\ &= b((1 \wedge |d^f|)\text{sign}d^f, (|d^f| - 1)^+\text{sign}d^f), \end{aligned}$$

where b denotes the sesquilinear, accretive form on \mathbb{C}^{n-1} associated to the matrix B . Since the nodal values d_1^f, \dots, d_{n-1}^f are arbitrary, by [34, Thm. 2.15] (which of course also applies to the accretive form b) one sees that the property of X_∞ -contractivity for $(T_2(t))_{t \geq 0}$ is equivalent to that of ℓ^∞ -contractivity for the semigroup $(e^{tB})_{t \geq 0}$ generated by B on \mathbb{C}^{n-1} . Now the claim follows by Lemma 6.1. \square

We recall that if $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ are semigroups on a Banach lattice X , then $(T(t))_{t \geq 0}$ is said to *dominate* $(S(t))_{t \geq 0}$ in the sense of positive semigroups if

$$|S(t)f| \leq T(t)|f| \quad \text{for all } f \in X, t \geq 0.$$

Proposition 3.6. *Let B, \tilde{B} be $(n-1) \times (n-1)$ dissipative matrices and denote by $(e^{tB})_{t \geq 0}$ and $(e^{t\tilde{B}})_{t \geq 0}$ the semigroups they generate on \mathbb{C}^{n-1} . Assume that B is real and positive off-diagonal, so that $(e^{tB})_{t \geq 0}$ is positive. Denote by $\mathfrak{a}_B, \mathfrak{a}_{\tilde{B}}$ the coercive form \mathfrak{a} with coefficients given by B and \tilde{B} , respectively, and by $(T_B(t))_{t \geq 0}, (T_{\tilde{B}}(t))_{t \geq 0}$ the associated semigroups on X_2 . Then $(T_B(t))_{t \geq 0}$ dominates $(T_{\tilde{B}}(t))_{t \geq 0}$ in the sense of positive semigroups if and only if $(e^{tB})_{t \geq 0}$ dominates $(e^{t\tilde{B}})_{t \geq 0}$ in the sense of positive semigroups.*

Proof. Observe that V_0 , the domain of both \mathfrak{a}_B and $\mathfrak{a}_{\tilde{B}}$ is an ideal of itself by [34, Prop. 2.20]. A direct computation shows that

$$\mathfrak{a}_B(|f|, |g|) \leq \text{Re} \mathfrak{a}_{\tilde{B}}(f, g) \quad \text{for all } f, g \in V_0 \text{ s.t. } f\bar{g} \geq 0$$

if and only if

$$\sum_{i,h=1}^{n-1} b_{ih} |d_h^f| |d_i^g| \geq \text{Re} \sum_{i,h=1}^{n-1} \tilde{b}_{ih} d_h^f \overline{d_i^g} \quad \text{for all } f, g \in V_0 \text{ s.t. } f\bar{g} \geq 0.$$

Due to the arbitrariness of the nodal values of $f, g \in V_0$, such a condition is satisfied if and only if

$$\sum_{i,h=1}^{n-1} b_{ih} |x_h| |y_i| \geq \text{Re} \sum_{i,h=1}^{n-1} \tilde{b}_{ih} x_h \overline{y_i} \quad \text{for all } x, y \in \mathbb{C}^{n-1} \text{ s.t. } x\bar{y} \geq 0.$$

Thus, applying [34, Thm. 2.21] to the forms $\mathbf{a}_B, \mathbf{a}_{\tilde{B}}$ as well as to the forms associated to the matrices B, \tilde{B} yields the claim. \square

It has been shown in [20, § 5] that the positive semigroup governing a diffusion problem on a network *without Dirichlet conditions on any node* is irreducible if and only if G is connected. This is no more true in the setting considered in this paper, unless we replace the notion of connectedness by a stronger one.

Proposition 3.7. *Let B be dissipative with real, positive off-diagonal entries. Then the following hold.*

1. *If the positive semigroup $(T_2(t))_{t \geq 0}$ is not irreducible, then G is the union of two non-trivial subgraphs $G_1, G_2 \subset G$ containing vertices v_1, \dots, v_{n_0} and $v_{n_0+1}, \dots, v_{n-1}$, respectively, and such that $G_1 \cap G_2 \subset \{v_n\}$.*
2. *The converse also holds, if further B is assumed to be block-diagonal, i.e., of the form*

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1n_0} & & & \\ \vdots & \ddots & \vdots & & & 0 \\ b_{n_0 1} & \cdots & b_{n_0 n_0} & & & \\ & & & b_{n_0+1 \ n_0+1} & \cdots & b_{n_0+1 \ n-1} \\ & 0 & & \vdots & \ddots & \vdots \\ & & & b_{n \ n_0+1} & \cdots & b_{n-1 \ n-1} \end{pmatrix}$$

3. *Let B strictly positive off-diagonal. Then $(T(t))_{t \geq 0}$ is irreducible.*

Proof. Recall that by [34, Thm. 2.10] the positive semigroup $(T_2(t))_{t \geq 0}$ is irreducible if and only whenever \tilde{G} is an open subset of G such that

$$f \mathbf{1}_{\tilde{G}} \in V_0 \quad \text{and} \quad \text{Rea}(f \mathbf{1}_{\tilde{G}}, f \mathbf{1}_{G \setminus \tilde{G}}) \geq 0 \quad \text{for all } f \in V_0, \quad (3.9)$$

then either \tilde{G} or its complement has zero measure.

In particular, let $(T_2(t))_{t \geq 0}$ be irreducible and let $f \in V_0$ be a function that agrees with the constant 1 on all edges of \tilde{G} that are *not* incident to v_n , and is smooth elsewhere. Then $f \mathbf{1}_{\tilde{G}}$ is of class H^1 on each edge, continuous on each node of G_1 and vanishes in v_n . Since $f \mathbf{1}_{\tilde{G}}$ also vanishes on all nodes of G_2 , it follows that $f \mathbf{1}_{\tilde{G}} \in V_0$. We deduce that if \tilde{G} contains an interior point x of some edge e_j , $j \notin \Gamma(v_n)$, then the whole e_j belongs to \tilde{G} (otherwise $f \mathbf{1}_{\tilde{G}}$ would be discontinuous at x , and in particular not of class H^1 .)

1) Let $(T(t))_{t \geq 0}$ be not irreducible. We want to show that G is the union of nontrivial subgraphs whose intersection is $\{v_n\}$ or, in other words, that for all pair of points $x, y \in G$ every path $p(x, y)$ connecting them contains v_n . Since the semigroup is not irreducible, there exists $\tilde{G} \subset G$ such that $\mu(\tilde{G}) \neq 0 \neq \mu(G \setminus \tilde{G})$ and such that (3.9) holds. As remarked before, \tilde{G} and $G \setminus \tilde{G}$ can thus only contain whole edges: let thus e_{j_0}, e_{ℓ_0} be edges contained in \tilde{G} and $G \setminus \tilde{G}$, respectively. Assume now that then there exists a path $p \subset G$ that connects some interior point of e_{j_0} to some interior point of e_{ℓ_0} and such that $v_n \notin p$. However, by a continuity argument the function $f \mathbf{1}_{\tilde{G}}$ would be constant 1 along the path p , a contradiction. Here we have denoted by f a smooth function in V_0 that agrees with the constant 1 everywhere beside on the edged incident to v_n .

2) Take $\tilde{G} = G_1$ and let $f \in V_0$. Then $f \mathbf{1}_{G_1}$ is a function that equals f on the edges of G_1 : by definition, $f \mathbf{1}_{G_1}$ is continuous in the vertices of G_1 and it vanishes

in v_n . Furthermore, $f\mathbb{1}_{G_1}$ vanishes on the edges of G_2 and all the vertices adjacent to them, thus in particular it is continuous in the vertices of G_2 , too. Summing up, $f\mathbb{1}_{G_1} \in V_0$ and moreover

$$\operatorname{Rea}(f\mathbb{1}_{G_1}, f\mathbb{1}_{G \setminus G_1}) = -\operatorname{Re} \sum_{i=1}^{n_0} \sum_{h=n_0+1}^{n-1} b_{ih} d_h^f d_i^f = 0.$$

However, G_1 is not a set of measure zero.

3) Let $\tilde{G} \subset G$ with $\mu(\tilde{G}) \neq 0$, and assume that (3.9) holds. We are going to show that $\mu(G \setminus \tilde{G}) = 0$. In fact, let $\mu(G \setminus \tilde{G}) \neq 0$, i.e., let $G \setminus \tilde{G}$ contain (at least) one whole interval. We can thus assume that there exist nodes $v_{i_0} \in \tilde{G}$ and $v_{h_0} \in G \setminus \tilde{G}$. Let us now consider some function $f \in V_0$ such that $d_{i_0}^f = d_{h_0}^f = 1$ and $d_i^f = 0$ for all $i \neq i_0, i \neq h_0$. Then there holds

$$\mathfrak{a}(f\mathbb{1}_{\tilde{G}}, f\mathbb{1}_{G \setminus \tilde{G}}) = -b_{i_0 h_0} d_{h_0}^{f\mathbb{1}_{\tilde{G}}} d_{i_0}^{f\mathbb{1}_{G \setminus \tilde{G}}} = -b_{i_0 h_0} < 0,$$

a contradiction to (3.9). \square

4. Extrapolating semigroups, ultracontractivity, and Gaussian estimates.

Let the matrix B and hence its adjoint B^* be dissipative. One can easily see that the adjoint of the form \mathfrak{a} is given by the form \mathfrak{a}^* defined by

$$\mathfrak{a}^*(f, g) := \sum_{j=1}^m \int_0^1 c_j(x) f'_j(x) \overline{g'_j(x)} dx - \sum_{i,h=1}^{n-1} \overline{b_{hi}} d_h^f d_i^g, \quad f, g \in V_0.$$

By Theorem 3.5 the semigroup associated to \mathfrak{a}^* is X_∞ -contractive if and only if

$$\operatorname{Re} b_{ii} + \sum_{h \neq i} |b_{hi}| \leq 0, \quad i = 1, \dots, n-1. \quad (4.1)$$

By duality, this is the case if and only if $(T_2(t))_{t \geq 0}$ is X_1 -contractive. By interpolation we thus conclude that $(T_2(t))_{t \geq 0}$ is X_p -contractive for all $p \in [1, \infty]$ if and only if both (3.8) and (4.1) are satisfied. We thus obtain the following.

Theorem 4.1. *Let the matrix B be dissipative, and let it satisfy the assumptions (3.8)–(4.1). Then $(T_2(t))_{t \geq 0}$ extrapolates to a family of contractive semigroups $(T_p(t))_{t \geq 0}$ on X_p , $p \in [1, \infty]$, which are strongly continuous for $p \in [1, \infty)$ and analytic for $p \in (1, \infty)$.*

Furthermore, if B has real (resp., real and positive off-diagonal) entries, then $(T_p(t))_{t \geq 0}$ is real for $p \in [1, \infty]$ (resp., positive for $p \in [1, \infty]$).

Finally, for $p \in (1, \infty)$ the spectrum of A_p is independent of p , where A_p denotes the generator of $(T_p(t))_{t \geq 0}$, and $(T_p(t))_{t \geq 0}$ is uniformly exponentially stable with common growth bound $\omega(T_p)$ given by $s(A)$.

Proof. We are in the position to apply the results summarized, e.g., in [4, § 7.2] and deduce the existence of an extrapolation semigroup $(T_p(t))_{t \geq 0}$ on X_p , $1 \leq p \leq \infty$. A list of the all properties of $(T_2(t))_{t \geq 0}$ inherited by $(T_p(t))_{t \geq 0}$ can be found in [4, § 7.2.2]. This yields all the claimed properties up to exponential stability. To check this, we combine the uniform exponential stability of the semigroup associated with \mathfrak{a} , cf. Corollary 3.3, and the p -independence of the spectrum of the analytic semigroups $(T_p(t))_{t \geq 0}$, $p \in (1, \infty)$. Hence, the growth bound of $(T_p(t))_{t \geq 0}$ agrees with the spectral bound $s(A_p) = s(A_2) = s(A) < 0$. \square

Remarks 4.2. 1) Let $(T_2(t))_{t \geq 0}$ be positive and irreducible. If the semigroup extrapolates to X_p , $p \in [1, \infty]$, then it is known that also $(T_p(t))_{t \geq 0}$ are irreducible, $p \in [1, \infty)$, cf. [4, § 7.2].

2) Consider the case of $c_1 = \dots = c_m \equiv 1$ and $b_{ih} = 0$, $i, h = 1, \dots, n$. Then, it follows by [32, Théo. 2.4 and Théo. 3.1] that

$$-\left(\frac{\pi}{m+1}\right)^1 \leq s(A) = \omega(T_p) \leq -\left(\frac{\pi}{2m}\right)^2, \quad p \in (1, \infty).$$

In other words, we can say that *the more edges belong to the network, the slower is the heat dissipation*. A further, more involved upper estimate on $s(A)$ is shown in [32, Théo. 3.2], showing that the inner structure of the graph does influence the asymptotical behavior of the diffusion problem.

We have shown that, if B is dissipative and satisfies (3.8)–(4.1), then the semigroup $(T_p(t))_{t \geq 0}$ is contractive on X_p , $p \in [1, \infty]$. In fact, we can characterize a stronger property.

Lemma 4.3. *Let the matrix B be dissipative.*

1. *If B satisfies (3.8), then the semigroup $(T_2(t))_{t \geq 0}$ on X_2 associated with \mathbf{a} is ultracontractive. In particular, it satisfies the estimate*

$$\|T_2(t)f\|_{X_\infty} \leq Mt^{-\frac{1}{4}}\|f\|_{X_2} \quad \text{for all } t > 0, f \in X_1, \quad (4.2)$$

for some constant M .

2. *If instead B satisfies (4.1), then*

$$\|T_2(t)f\|_{X_2} \leq Mt^{-\frac{1}{4}}\|f\|_{X_1} \quad \text{for all } t > 0, f \in X_1, \quad (4.3)$$

holds.

Proof. By [34, Thm. 6.3] it suffices to show that there holds

$$\|f\|_{X_2} \leq M\mathbf{a}(f, f)^{\frac{1}{6}} \cdot \|f\|_{X_1}^{\frac{2}{3}} \quad \text{for all } f \in V_0,$$

for some constant M . Recall that

$$\begin{aligned} \|k\|_{L^2(0,1)} &\leq M_1 (\|k'\|_{L^2(0,1)} + \|k\|_{L^1(0,1)})^{\frac{1}{3}} \cdot \|k\|_{L^1(0,1)}^{\frac{2}{3}} \\ &\leq M_1 \|k\|_{H^1(0,1)}^{\frac{1}{3}} \cdot \|k\|_{L^1(0,1)}^{\frac{2}{3}}, \end{aligned}$$

is valid for all $k \in H^1(0,1)$ and some constant M_1 , cf. [27, Thm. 1.4.8.1].

Take finally $f \in V_0$ and observe that by the above Nash-type inequality

$$\begin{aligned} \|f\|_{X_2}^2 &= \sum_{j=1}^m \|f_j\|_{L^2(0,1)}^2 \leq M_1^2 \sum_{j=1}^m \|f_j\|_{H^1(0,1)}^{\frac{2}{3}} \cdot \|f_j\|_{L^1(0,1)}^{\frac{4}{3}} \\ &\leq M_2 \left(\sum_{j=1}^m \|f_j\|_{H^1(0,1)} \right)^{\frac{2}{3}} \cdot \left(\sum_{j=1}^m \|f_j\|_{L^1(0,1)} \right)^{\frac{4}{3}} \\ &\leq M_3 \|f\|_{V_0}^{\frac{2}{3}} \cdot \|f\|_{X_1}^{\frac{4}{3}} \\ &\leq M_3 \mathbf{a}(f, f)^{\frac{1}{3}} \cdot \|f\|_{X_1}^{\frac{4}{3}}. \end{aligned}$$

We have so far shown that

$$\|T_2(t)f\|_{X_\infty} \leq Mt^{-\frac{1}{4}}\|f\|_{X_2} \quad \text{for all } t > 0, f \in X_2.$$

If instead (4.1) holds, then the claim follows by duality. \square

Remark 4.4. In the terminology of N.Th. Varopoulos ([37, § 0.1], cf. also [4, § 7.3.2]), Lemma 4.3 says that the dimension of the semigroup $(T_2(t))_{t \geq 0}$ is 1. This is true regardless of the structure of the underlying graph.

Combining the stability and ultracontractivity results we have obtained, we can finally show that $X_2 - X_\infty$ uniform exponential stability holds.

Corollary 4.5. *Let the matrix B be dissipative and satisfy (3.8). Then the semigroup $(T_2(t))_{t \geq 0}$ on X_2 satisfies the estimate*

$$\|T_2(t)f\|_{X_\infty} \leq M \left(\frac{1 - t\omega}{t} \right)^{\frac{1}{4}} e^{t\omega} \|f\|_{X_2} \quad \text{for all } t > 0, f \in X_2,$$

where M is the constant that appears in (4.2) and ω is the strictly negative growth bound of $(T_2(t))_{t \geq 0}$.

Proof. Taking into account Lemma 4.3 and Theorem 4.1, the claim follows directly from [34, Lemma 6.5]. \square

In order to discuss the well-posedness of the problem in an L^p -setting, we want to identify the generators of the $(T_p(t))_{t \geq 0}$.

Proposition 4.6. *Let the matrix B be dissipative, and let it satisfy the assumptions (3.8)–(4.1). Then for all $p \in [1, \infty]$ the generator A_p of the semigroup $(T_p(t))_{t \geq 0}$ is the operator whose action on the domain*

$$D(A_p) := \left\{ f \in (W^{2,p}(0,1))^m : \begin{array}{l} \exists df \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^{\top} df = f(0), (\Phi^-)^{\top} df = f(1), \\ \text{and } \Phi_w^+ f'(0) - \Phi_w^- f'(1) = B df \end{array} \right\}. \quad (4.4)$$

is formally given in (3.1). In particular, A_p has compact resolvent for all $p \in [1, \infty]$.

Proof. The proof goes in two steps. We first consider the case of $p \in (2, \infty]$, then discuss the case $p \in [1, 2)$ by duality.

1) Let $p > 2$. We have already remarked that $X_p \hookrightarrow X_q$ for all $1 \leq q \leq p \leq \infty$. Moreover, for all $p > 2$ the space X_p is invariant under $(T_p(t))_{t \geq 0}$ by the ultracontractivity of $(T_2(t))_{t \geq 0}$, $p > 2$. Thus, by [16, Prop. II.2.3] the generator of $(T_p(t))_{t \geq 0}$ is the part of A in X_p . A direct computations yields the claim.

2) Take now some p with $1 \leq p < 2$, and let q such that $p^{-1} + q^{-1} = 1$. By an argument similar to that of [14, Thm. 1.4.1] one has that the adjoint semigroup $((T_p)^*(t))_{t \geq 0}$ of $(T_p(t))_{t \geq 0}$ on X_p is actually the extrapolation semigroup $((T^*)_q(t))_{t \geq 0}$ of $((T^*)_2(t))_{t \geq 0}$. Since the generator $(A^*)_q$ of $((T^*)_q(t))_{t \geq 0}$ also satisfies the assumption of the theorem, by 1) we deduce that

$$D((A^*)_q) := \left\{ f \in (W^{2,q}(0,1))^m : \begin{array}{l} \exists df \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^{\top} df = f(0), (\Phi^-)^{\top} df = f(1), \\ \text{and } \Phi_w^+ f'(0) - \Phi_w^- f'(1) = B^* df \end{array} \right\}.$$

Set

$$D_p := \left\{ f \in (W^{2,p}(0,1))^m : \begin{array}{l} \exists df \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^{\top} df = f(0), (\Phi^-)^{\top} df = f(1), \\ \text{and } \Phi_w^+ f'(0) - \Phi_w^- f'(1) = B df \end{array} \right\}.$$

Consider the operator A_p whose action on D_p is formally given by (3.1). We are going to show that the adjoint $(A_p)^*$ of (A_p, D_p) actually agrees with $(A^*)_q$. Then, since the generator of the pre-adjoint semigroup $(T_p(t))_{t \geq 0}$ on X_p of $((T^*)_q(t))_{t \geq 0}$

on X_q is the pre-adjoint operator A_p of A_q , we conclude that (A_p, D_p) generates the C_0 -semigroup $(T_p(t))_{t \geq 0}$ on X_p , and the claim follows.

By definition we have that the adjoint of (A_p, D_p) is the operator $((A_p)^*, D_p^*)$ given by

$$\begin{aligned} D_p^* &= \{f \in X_p^* : \exists g \in X_p^* : \langle A_p u, f \rangle = \langle u, g \rangle \quad \forall u \in D_p\} \\ &= \{f \in X_q : \exists g \in X_q : \langle A_p u, f \rangle = \langle u, g \rangle \quad \forall u \in D_p\}, \\ (A_p)^* f &= g. \end{aligned}$$

Let us first show that $D_p^* \subset D((A^*)_q)$. Take $f \in D_p^*$ and observe that the identity $\langle A_p u, f \rangle = \langle u, g \rangle$ holds in particular for all u of the form introduced in (3.7), with $u_j \in C_c^\infty(0, 1)$. Thus, we obtain that for all $j = 1, \dots, m$ there exists $g_j \in L^q(0, 1)$ such that

$$\int_0^1 (c_j u_j')'(x) \overline{f_j(x)} dx = \int_0^1 u_j(x) \overline{g_j(x)} dx.$$

Integrating by parts one thus obtains that $(c_j f_j')' = g_j$ (in the sense of distributions), and since $g_j \in L^q(0, 1)$ it follows from the definition of Sobolev space of order 2 that $f_j \in W^{2,q}(0, 1)$. Thus, we conclude that $f \in (W^{2,q}(0, 1))^m$.

In order to check that the node conditions are also verified, let us perform a computation similar to that in Lemma 3.4. The condition $\langle A_p u, f \rangle = \langle u, g \rangle$ for $u \in D_p$ reads then

$$\begin{aligned} \sum_{j=1}^m \int_0^1 u_j(x) \overline{((c_j f_j')'(x) - g_j(x))} dx = \\ - \sum_{j=1}^m c_j(\mathbf{v}_n) \phi_{nj} u_j'(\mathbf{v}_n) \overline{f_j(\mathbf{v}_n)} \\ + \sum_{i=1}^{n-1} \sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) \left(d_i^u \overline{f_j'(\mathbf{v}_i)} - u_j'(\mathbf{v}_i) \overline{f_j(\mathbf{v}_i)} \right) \end{aligned} \quad (4.5)$$

for some $g \in X_q$. Since $u \in D_p$ is arbitrary (and therefore so are its derivative's nodal values), we deduce that all terms on both the left and the right hand sides must vanish identically. In particular, whenever the edge e_j is incident to \mathbf{v}_n the number $c_j(\mathbf{v}_n) \phi_{nj} u_j'(\mathbf{v}_n)$ is arbitrary (recall that $c_j(x) \geq c > 0$ for all $x \in [0, 1]$ and $j = 1, \dots, m$), so that necessarily $f_j(\mathbf{v}_n) = 0$ for all $j \in \Gamma(\mathbf{v}_n)$. Now we invoke again the arbitrariness of $u \in D_p$ (and hence of its and its derivative's nodal values) and, considering functions u s.t. $d_i^u = 0$ for $i = 1, \dots, n$, we conclude that there holds

$$\sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) u_j'(\mathbf{v}_i) \overline{f_j(\mathbf{v}_i)} = 0 \quad \text{for all } i = 1, \dots, n-1. \quad (4.6)$$

Moreover, observe that the generalized Kirchhoff law for $u \in D_p \cap (H_0^1(0, 1))^m$ becomes

$$\sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) u_j'(\mathbf{v}_i) = 0 \quad \text{for all } i = 1, \dots, n-1. \quad (4.7)$$

Let us reduce (4.6) and (4.7) to pairwise relations. More precisely, pick u in such a way that only exactly two values $(\omega_{i\ell}^- - \omega_{i\ell}^+) u_\ell'(\mathbf{v}_i)$, $(\omega_{ik}^- - \omega_{ik}^+) u_k'(\mathbf{v}_i)$ are non-zero, $1 \leq \ell, k \leq m$. Then, we obtain from (4.6)–(4.7) that for given i the vector

$$\begin{pmatrix} (\omega_{i\ell}^- - \omega_{i\ell}^+) u_\ell'(\mathbf{v}_i) \\ (\omega_{ik}^- - \omega_{ik}^+) u_k'(\mathbf{v}_i) \end{pmatrix} \text{ is orthogonal to } \begin{pmatrix} \overline{f_\ell(\mathbf{v}_i)} \\ \overline{f_k(\mathbf{v}_i)} \end{pmatrix} \text{ as well as to } \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This promptly yields that $f_\ell(\mathbf{v}_i) = f_k(\mathbf{v}_i)$. Repeating the argument $m-1$ times we conclude that the nodal values $f_j(\mathbf{v}_i) := d_i^f$ do not depend on $j \in \Gamma(\mathbf{v}_i)$.

Thus, the second term on the right hand side of (4.5) can be written as

$$\sum_{i=1}^{n-1} \left(d_i^u \sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) \overline{f_j'(\mathbf{v}_i)} - \overline{d_i^f} \sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) u_j'(\mathbf{v}_i) \right)$$

or rather, taking into account the generalized Kirchhoff condition satisfied by u , as

$$\sum_{i=1}^{n-1} \left(d_i^u \sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) \overline{f_j'(\mathbf{v}_i)} - \overline{d_i^f} \sum_{h=1}^{n-1} b_{ih} d_h^u \right). \quad (4.8)$$

Since this expression vanishes identically and because of the arbitrariness of the nodal values of u and u' , we conclude that

$$\sum_{j=1}^m (\omega_{ij}^- - \omega_{ij}^+) f_j'(\mathbf{v}_i) = \sum_{h=1}^{n-1} \overline{b_{hi}} d_h^f, \quad i = 1, \dots, n-1.$$

Summing up, f satisfies the Dirichlet condition at \mathbf{v}_n as well as the generalized Kirchhoff law at $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ for coefficients given by B^* , thus $f \in D((A^*)_q)$.

Let us now check that $D((A^*)_q) \subset D_p^*$. Take $f \in (W^{2,q}(0,1))^m$ satisfying the continuity condition on the nodal values as well as the Dirichlet condition on \mathbf{v}_n and the generalized Kirchhoff law on $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ for coefficients given by B^* . Set $g_j = (c_j f_j')'$, so that $g \in X_q$. We only have to prove that for all $u \in D_p$ there holds $\langle A_p u, f \rangle = \langle u, g \rangle$, i.e.,

$$\sum_{j=1}^m \int_0^1 (c_j u_j')'(x) \overline{f_j(x)} dx = \sum_{j=1}^m \int_0^1 u_j(x) \overline{(c_j f_j')'(x)} dx.$$

Integrating by parts as in the proof of the converse inclusion and recalling that the nodal values of both f and u do not depend on j we see that this is the case if and only if the expression in (4.8) vanishes identically. \square

As a direct consequence of the ultracontractivity of $(T_2(t))_{t \geq 0}$ and the Dunford–Petty Theorem, the semigroup has an integral kernel for all $t > 0$, cf. [4, § 7.3]. More precisely, denote by $(\tilde{T}_p(t))_{t \geq 0}$ the semigroup on $L^p(0, m)$ that is similar to $(T_p(t))_{t \geq 0}$ on X_p with a similarity transformation given by the isometry U introduced in Definition 2.2. Then for all $p \in [1, \infty]$ the action of $(\tilde{T}_p(t))_{t \geq 0}$ is given by

$$\tilde{T}_p(t)g(\cdot) = \int_0^m \tilde{K}_t(\cdot, y)g(y)dy, \quad t > 0, g \in L^p(0, m),$$

for a suitable kernel $\tilde{K}_t \in L^\infty((0, m) \times (0, m))$.

The existence of integral kernels is a typical feature of diffusion equations. Also in view of its consequences in the theory of evolution equations (see e.g. [3]), it is of great interest to estimate such kernels and compare them with the standard Gaussian one, which is associated to the heat equation on the whole space. This is usually done by the so-called Davies' trick, that amounts to prove uniform L^∞ -(quasi)contractivity estimates for a class of perturbed semigroups.

More precisely, introduce a class W of functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ constructed in the following way: first, we consider smooth functions $\phi \in (C_b^\infty(0, 1))^m$ such that

$$\exists d^\phi \in \mathbb{C}^n \text{ s.t. } (\Phi^+)^T d^\phi = \phi(0), (\Phi^-)^T d^\phi = \phi(1), \|\phi'\|_\infty \leq 1 \text{ and } \|\phi''\|_\infty \leq 1.$$

We then stretch via the isometry U such functions over the network to functions $\tilde{\phi} : (0, m) \rightarrow \mathbb{R}$. We finally call W the class of all smooth, bounded continuous

extensions φ of $\tilde{\phi}$ to the whole line such that also $\|\varphi'\|_\infty \leq 1$ and $\|\varphi''\|_\infty \leq 1$. Finally, for fixed $\varphi \in W$ we define an operator L_ρ on $L^2(\mathbb{R})$ by $L_\rho f := e^{-\rho\varphi} f$ and perturbed semigroups $(\tilde{T}_2^\rho(t))_{t \geq 0}$, where $\tilde{T}_2^\rho(t) := L_\rho \tilde{T}_2(t) L_\rho^{-1}$, $\rho \in \mathbb{R}$. Observe that, by construction, $U^{-1} L_\rho U f \in V_0$ for all $f \in V_0$, $\rho \in \mathbb{R}$.

In the remainder of this section we consider X_p as real spaces, $1 \leq p \leq \infty$. Then, by [7, Thm. 3.3] Gaussian estimates for $(\tilde{T}_2(t))_{t \geq 0}$ are equivalent to ultracontractivity estimates for $(\tilde{T}_2^\rho(t))_{t \geq 0}$, uniformly in $\rho \in \mathbb{R}$ and $\varphi \in W$. This can be done by applying the above presented theory to the similar semigroups $(T_2^\rho(t))_{t \geq 0}$ on X_2 , which by a direct computation are associated to the bilinear forms \mathfrak{a}^ρ defined by

$$\begin{aligned} \mathfrak{a}^\rho(f, g) &:= \sum_{j=1}^m \int_0^1 c_j(x) f'_j(x) g'_j(x) dx \\ &\quad + \rho \sum_{j=1}^m \int_0^1 c_j(x) \phi'_j(x) (f_j(x) g'_j(x) - f'_j(x) g_j(x)) dx \\ &\quad - \rho^2 \sum_{j=1}^m \int_0^1 c_j(x) \phi'_j(x)^2 f_j(x) g_j(x) dx \\ &\quad - \sum_{i,h=1}^{n-1} b_{ih} e^{\rho(d_h^\phi - d_i^\phi)} d_h^f d_i^g, \end{aligned}$$

for all $f, g \in V_0$. In the following we restrict ourselves to the local case, i.e., to the case where B is a diagonal matrix. In fact, by considering the above form \mathfrak{a}^ρ with coefficients $c \equiv 1$ and $B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ on the domain $H^1(0, 1)$ (i.e., for the sake of simplicity, on a graph with a single edge without Dirichlet boundary conditions), one sees that it is not possible to find $\omega \in \mathbb{R}$ such that the shifted form $\mathfrak{a}^\rho + \omega(1 + \rho^2)$ is accretive uniformly in $\rho \in \mathbb{R}$ and $\phi \in W$, i.e., such that

$$\begin{aligned} 0 \leq \mathfrak{a}^\rho(f, f) + \omega(1 + \rho^2) \|f\|_{X_2}^2 &= \int_0^1 f'(x)^2 + (\rho^2 \omega + \omega - \rho^2 \phi'(x)^2) f(x)^2 dx \\ &\quad - (e^{\rho(\phi(1) - \phi(0))} + e^{\rho(\phi(0) - \phi(1))}) f(1) f(0) \\ &\quad + f(0)^2 + f(1)^2 \end{aligned}$$

for all $\rho \in \mathbb{R}$, $\phi \in W$, and $f \in H^1(0, 1)$. This can be checked by taking f constant, $\rho = 1$, $\phi(x) := (1 + \omega)x$. The uniform accretivity of the forms \mathfrak{a}^ρ seems to be an essential ingredient of the method of proof explained in [7]: we thus derive in the following Gaussian estimates only in the local case of B diagonal.

Theorem 4.7. *Let the matrix B be diagonal with negative entries. Then the semigroup $(T_2(t))_{t \geq 0}$ has Gaussian estimates. More precisely, there exist constants $b, c > 0$ such that \tilde{K}_t satisfies*

$$0 \leq \tilde{K}_t(x, y) \leq ct^{-\frac{1}{2}} e^{-\frac{b|x-y|^2}{t} + t}, \quad x \in (0, m), \quad (4.9)$$

uniformly in $t > 0$.

Proof. Under our assumptions it is already known that $(T_2(t))_{t \geq 0}$ is positive and contractive with respect to the X_1 and X_∞ norm. Moreover, a direct computation shows that the shifted form $\mathfrak{a}^\rho + (1 + \rho^2)$ is accretive.

Observe now that V_0 is not an ideal of $(H^1(0,1))^m$, but indeed of

$$V := \left\{ f \in (H^1(0,1))^m : \begin{array}{l} \exists df \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^{\top} df = f(0) \text{ and } (\Phi^-)^{\top} df = f(1) \end{array} \right\}.$$

The proof can now be concluded by mimicking [7, Thm. 4.4]. \square

We are finally able to obtain an optimal result on the analyticity of the semigroup generated by A . We stress that we are *not* imposing any assumption on B .

Theorem 4.8. *Consider the operator A_p whose action on the domain defined in (4.4) is formally given by 3.1. Then A_p generates on X_p a strongly continuous, analytic semigroup $(T_p(t))_{t \geq 0}$ of angle $\frac{\pi}{2}$, $p \in [1, \infty)$.*

Proof. The proof goes in three steps.

1) Let us first consider the case of $B = 0$, $p = 2$. Then, it follows by Proposition 3.2 that the form a is symmetric and coercive, hence in particular the associated operator A is self-adjoint and dissipative, and the statement follows by the spectral theorem.

2) If $B = 0$, then for general $p \in [1, \infty]$ the semigroup $(T_2(t))_{t \geq 0}$ extrapolates to a semigroup $(T_p(t))_{t \geq 0}$ on X_p that is analytic of angle $\frac{\pi}{2}$, by Theorem 4.7 and [34, Thm. 6.16].

3) Finally, consider the case of general B . Then we can apply the theory developed in [17, § 2], after setting $X := X^p$, $\partial X := \mathbb{C}^{n-1}$, and

$$Y := \left\{ f \in (W^{2,p}(0,1))^m : \begin{array}{l} \exists df \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^{\top} df = f(0), (\Phi^-)^{\top} df = f(1) \end{array} \right\},$$

as well as

$$Lu := \begin{pmatrix} \sum_{j=1}^m \phi_{1j} c_j(v_1) u'_j(v_1) \\ \vdots \\ \sum_{j=1}^m \phi_{n-1j} c_j(v_{n-1}) u'_j(v_{n-1}) \end{pmatrix}, \quad u \in D(A),$$

and

$$\Phi u := \mathbb{B} d^u, \quad u \in V_0.$$

We consider the operator A with maximal domain Y and observe that the restriction of A to $\ker(L)$ is the operator considered in 2), hence the generator of an analytic semigroup of angle $\frac{\pi}{2}$. Since the boundary perturbation operator $\Phi : V_0 \rightarrow \partial X$ is compact, the claim follows by [17, Thm. 2.6] (observe that in that theorem is also proved, although not explicitly stated, that the angle of analyticity remains invariant under admissible boundary perturbations). \square

Remark 4.9. Gaussian estimates like (4.9) are a key argument for discussing a number of different issues that go far beyond the scope of this paper. Without going into details, we recall that Theorem 4.7 implies at once, among other, the property of maximal regularity for $(T_p(t))_{t \geq 0}$ for $p \in (1, \infty)$, upper estimates for the time derivative of the kernel K_t , L^p -estimates for Schrödinger and wave equations, and the fact that A_p has bounded H^∞ -calculus on each sector (and therefore that it has bounded imaginary powers) for $p \in (1, \infty)$, cf. [34, § 6.5 and Chapt. 7], [4, § 7.4] and references therein. Observe that, even if Gaussian estimates can only be obtained for local nodal conditions, most of the above mentioned consequences also hold in the general case by perturbation methods.

Theorem 4.10. *The first order network diffusion problem introduced in Section 2 is well-posed on X_p , $p \in [1, \infty)$, i.e., for all initial data $u_0 \in X_p$ it admits a unique mild solution that continuously depends on the initial data. Such a solution is of class C^∞ in both variables x, t and its ∞ -norm tends to 0 in time. If furthermore $c_j \in C^\infty[0, 1]$, $j = 1, \dots, m$, then the solution $u(t, \cdot)$ is of class C^∞ with respect to the space variable.*

Proof. The well-posedness and boundedness results follow from the fact that the semigroup $(T_2(t))_{t \geq 0}$ is ultracontractive and extends to a family of semigroups $(T_2(p))_{t \geq 0}$ that, by Proposition 4.6, actually govern the network diffusion problem. The decay of the solution is ensured by the uniform exponential stability of all semigroups.

Finally, if $c_j \in C^\infty[0, 1]$, $j = 1, \dots, m$, then one sees that $D(A_p^\infty) \subset (C^\infty[0, 1])^m$ for all $p \in (1, \infty)$. Since the semigroup $(T_p(t))_{t \geq 0}$ is analytic, it maps X_p into $D(A_p^\infty)$, and the claim follows. \square

Observe that if we replace the Dirichlet condition in v_n by continuity of the values of $u_j(t, v_n)$, $t \geq 0$, $j \in \Gamma(v_n)$, plus a Kirchhoff-type condition analogous to that imposed on the other nodes, we obtain the system

$$\begin{cases} \dot{u}_j(t, x) &= (c_j u_j')'(t, x), & x \in (0, 1), j = 1, \dots, m, \\ u_j(t, v_i) &= u_\ell(t, v_i) =: d_i^u, & j, \ell \in \Gamma(v_i), i = 1, \dots, n, \\ \sum_{h=1}^n b_{ih} d_h^u &= \sum_{j=1}^m \phi_{ij} c_j(v_i) u_j'(t, v_i), & i = 1, \dots, n, \end{cases} \quad (4.10)$$

for $t \geq 0$, with initial conditions

$$u_j(0, x) = u_{0j}(x), \quad x \in (0, 1), j = 1, \dots, m.$$

Here b_{1h}, b_{i1} , are arbitrary numbers, $1 \leq i, h \leq n-1$. Such an initial-value problem has been proved to be well-posed in [20] (in the special case of $B = 0$): we can compare its solution and that to the original network diffusion problem and obtain the following. For the sake of simplicity, in the following we restrict to the case of purely Kirchhoff nodal conditions. However, one can see that a similar proof also works in the general case.

Proposition 4.11. *Let the coefficients $b_{ih} = 0$, $1 \leq i, h \leq n$. Then the semigroup $(T_2(t))_{t \geq 0}$ governing the original network diffusion problem (as in § 2) is dominated by the semigroup $(\tilde{T}_2(t))_{t \geq 0}$ governing (4.10) in the sense of positive semigroups.*

Proof. As shown in [20], $(\tilde{T}_2(t))_{t \geq 0}$ is a sub-Markovian semigroup that comes from a form with domain

$$V = \left\{ f \in (H^1(0, 1))^m : \exists d^f \in \mathbb{C}^n \text{ s.t. } (\tilde{\Phi}^+)^T d^f = f(0) \text{ and } (\tilde{\Phi}^-)^T d^f = f(1) \right\},$$

where $\tilde{\Phi}^+ = (\tilde{\phi}_{ij}^+)$ and $\tilde{\Phi}^- = (\tilde{\phi}_{ij}^-)$ represent the incidence matrices defined by

$$\tilde{\phi}_{ij}^+ := \begin{cases} 1, & \text{if } e_j(0) = v_i, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{\phi}_{ij}^- := \begin{cases} 1, & \text{if } e_j(1) = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since a Dirichlet condition in the node v_n implies in particular continuity on a function in that vertex, one sees that $V_0 \subset V$. Accordingly, by [34, Cor. 2.22] it suffices to prove that V_0 is an ideal of V , i.e., that the following conditions are satisfied:

- $f \in V_0 \Rightarrow |f| \in V$,
- $f \in V_0$, $g \in V$, and $|g| \leq |f| \Rightarrow g \cdot \text{sign} f \in V_0$.

To check the first condition, observe that $H_0^1(0, 1)$ is an ideal of $H^1(0, 1)$, and that the continuity of the values of $f \in (H^1(0, 1))^m$ in the nodes is not affected by taking the absolute value of f . Let now $f \in V_0$ and $g \in V$. If $|g| \leq |f|$, then in particular $|g_j(v_n)| \leq |f_j(v_n)| = 0$ for all $j \in \Gamma(v_n)$, i.e., $g \in V_0$. Finally, since A generates a positive semigroup, V_0 is an ideal of itself and this yields that $g \cdot \text{sign} f \in V_0$. \square

5. The heat equation on spaces of continuous functions. Also in view of applications, we are now interested to extend the previous L^p -type well-posedness results to a setting where continuous functions are considered instead.

Consider the part \tilde{A}_∞ of A in the Banach space $\tilde{X}_\infty := (C[0, 1])^m$, whose domain is given by

$$D(\tilde{A}_\infty) = \left\{ f \in (C^2(0, 1) \cap C^1[0, 1])^m : \begin{array}{l} \exists d^f \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^T d^f = f(0), (\Phi^-)^T d^f = f(1), \\ \text{and } \Phi_w^+ f'(0) - \Phi_w^- f'(1) = B d^f \end{array} \right\}.$$

Lemma 5.1. *The operator \tilde{A}_∞ is sectorial on \tilde{X}_∞ , and it generates a compact analytic semigroup of angle $\frac{\pi}{2}$. If the matrix B is a diagonal with negative entries, then such a semigroup is also positive and contractive.*

Proof. 1) Let us first consider the case $B = 0$. Then by Theorem 4.1 and Theorem 4.8 we deduce that all the operators A_p , $p \in [1, \infty]$, are dissipative and sectorial of angle $\frac{\pi}{2}$. In particular, for each $\epsilon \in (0, \frac{\pi}{2})$ there exists $M_\epsilon \geq 1$ such that the estimate

$$\|\lambda R(\lambda, A_\infty)\|_{\mathcal{L}(X_\infty)} \leq M_\epsilon \quad (5.1)$$

holds for all $\lambda \in \{\mu \in \mathbb{C} : |\arg \mu| < \pi - \epsilon\}$.

Now observe that if $f \in \tilde{X}_\infty$, then $R(\lambda, A_\infty)f = R(\lambda, A_2)f \in D(A_2)$. But one has $D(A_2) \subset (H^2(0, 1))^m \hookrightarrow (C[0, 1])^m$, with compact embedding. Therefore, we see that $R(\lambda, A_\infty)f$ is a continuous function for all $\lambda \in \{\mu \in \mathbb{C} : |\arg \mu| < \pi - \epsilon\}$. It follows that the analogous of (5.1) also holds with respect to the norm of \tilde{X}_∞ , hence \tilde{A}_∞ is sectorial and it generates an analytic semigroup of angle $\frac{\pi}{2}$.

2) The case of general B can be treated as in the proof of Theorem 4.8, by means of the theory of boundary perturbation for sectorial operators discussed in [17]. \square

The main motivation for considering semigroups on $(C[0, 1])^m$ comes from applications involving semilinear equations, since we can then effectively apply the theory developed, e.g., in [26] in order to discuss well-posedness and stability. As an elementary, yet motivating example we mention the following system, related to a Hodgkin–Huxley-model describing the transmission of potential along neurons (see [36] and references therein).

Proposition 5.2. *Let $\psi_j \in C^2(\mathbb{R})$, $j = 1, \dots, m$. Consider for $t > 0$ the semilinear parabolic network problem given by*

$$\begin{cases} \dot{u}_j(t, x) &= (c_j u_j')'(t, x) + (\psi_j(u_j(t, x)))', & x \in (0, 1), j = 1, \dots, m, \\ u_j(t, v_i) &= u_\ell(t, v_i) =: d_i^u, & j, \ell \in \Gamma(v_i), i = 1, \dots, n, \\ \sum_{h=1}^{n-1} b_{ih} d_h^u &= \sum_{j=1}^m \phi_{ij} c_j(v_i) u_j'(t, v_i), & i = 1, \dots, n-1, \\ d_n^u(t) &= 0. \end{cases}$$

Then for all $u_0 \in (C[0, 1])^m$ the Cauchy problem associated to such a system admits a unique (global) mild solution u that depends continuously (with respect to the sup-norm) on the initial data. In fact, u satisfies the problem pointwise for $t > 0$.

Proof. Rewrite the above system as a semilinear abstract Cauchy problem

$$\begin{cases} \dot{u}(t) &= \tilde{A}_\infty u(t) + \Psi(u(t)), \quad t > 0, \\ u(0) &= u_0, \end{cases}$$

on \tilde{X}_∞ . Here Ψ is the Nemitsky operator defined by

$$\Psi(u)(\cdot) := \begin{pmatrix} \frac{d}{dx}(\psi_1(u_1(\cdot))) \\ \vdots \\ \frac{d}{dx}(\psi_m(u_m(\cdot))) \end{pmatrix}.$$

Taking into account Lemma 5.1, we are in a setting that is analogous to that of [26, § 7.3.3]. Now, mimicking the proof of [26, Prop. 7.3.6] the claim follows. \square

A thorough treatment of well-posedness and stability of semilinear diffusion problems over networks goes beyond the scope of this paper. We will deal with such an issue in the forthcoming paper [12].

Even in the linear case (i.e., $\psi_1 \equiv 0$, $j = 1, \dots, m$), the problem considered in Proposition (5.2) is not well-posed in a classical sense. In fact, albeit sectorial (hence the generator of an analytic semigroup), the operator \tilde{A}_∞ is not densely defined in \tilde{X}_∞ , thus the generated semigroup is not strongly continuous.

Let us define

$$C_0(G) := \left\{ f \in (C[0, 1])^m : \begin{array}{l} \exists df \in \mathbb{C}^{n-1} \times \{0\} \text{ s.t.} \\ (\Phi^+)^T df = f(0) \text{ and } (\Phi^-)^T df = f(1) \end{array} \right\},$$

which by the theorem of Stone–Weierstrass is the closure of $D(\tilde{A}_\infty)$. (Observe that $C_0(G)$ can be looked at as the space of all continuous functions on the graph G that vanish in v_n).

Theorem 5.3. *The part \mathbf{A} of \tilde{A}_∞ in $C_0(G)$ generates a compact, strongly continuous semigroup which is analytic of angle $\frac{\pi}{2}$. If further B is diagonal with negative entries, then such a semigroup is contractive, real, positive, and uniformly exponentially stable.*

Proof. Reasoning as in the proof of Proposition 5.2, we deduce from Theorem 4.1 that \mathbf{A} is a resolvent positive operator on $C_0(G)$. Since \mathbf{A} is also densely defined, by [6, Thm. 3.11.9] it generates a positive strongly continuous semigroup $(\mathbf{T}(t))_{t \geq 0}$.

By Lemma 5.1, we see that \mathbf{A} is sectorial and dissipative: this yields the analyticity (with angle $\frac{\pi}{2}$) and the contractivity of $(\mathbf{T}(t))_{t \geq 0}$. Observe further that the p -independence of the spectrum of A_p (by Theorem 4.1) yields the invertibility of \mathbf{A} , hence the uniform exponential stability of $(\mathbf{T}(t))_{t \geq 0}$.

Finally, in order to show that the semigroup is compact, observe that due to its analyticity $T_2(t)$ maps X_2 into $D(A_2^\infty) \subset (C^\infty[0, 1])^m \cap C_0(G) \subset D(\mathbf{A})$ for all $t > 0$. Thus, denoting by $X_{\mathbf{A}}$ the Banach space obtained by endowing $D(\mathbf{A})$ with the graph norm, we have

$$\mathbf{T}(t) = i_{X_{\mathbf{A}}, C_0(G)} \circ T_2(t) \circ i_{C_0(G), X_2}, \quad t > 0.$$

Here $i_{C_0(G), X_2}$ and $i_{X_{\mathbf{A}}, C_0(G)}$ denote the canonical imbeddings of $C_0(G)$ into X_2 and of $X_{\mathbf{A}}$ into $C_0(G)$, respectively. It follows from the theorem of Ascoli–Arzelà that the latter imbedding is compact, so that also $\mathbf{T}(t)$ is compact for $t > 0$, and the claim follows. \square

We can finally draw a conclusion that is similar to Theorem 4.10, and can be proved likewise.

Theorem 5.4. *The network diffusion problem is well-posed on $C_0(G)$, i.e., for all initial data $u_0 \in C_0(G)$ it admits a unique classical solution that continuously depends on the initial data. The sup-norm of the solution tends to 0 in time.*

6. A technical lemma. The following result seems to be of independent interest. Its proof is due to Wolfgang Arendt, whom we warmly thank.

Lemma 6.1. *Let $A = (a_{ih})$ be a $n \times n$ matrix with complex-valued coefficients. Then the semigroup $(e^{tA})_{t \geq 0}$ generated by A is ℓ^∞ -contractive, i.e.,*

$$\|e^{tA}x\|_\infty \leq \|x\|_\infty, \quad t \geq 0, x \in \mathbb{C}^n,$$

if and only if

$$\operatorname{Re} a_{ii} + \sum_{h \neq i} |a_{ih}| \leq 0 \quad \text{for all } i = 1, \dots, n. \quad (6.1)$$

Proof. The proof goes in two steps.

1) Let us first assume the semigroup $(e^{tA})_{t \geq 0}$ generated by the matrix $A = (a_{ih})$ to be positive, i.e., to have real entries that are positive off-diagonal. Then it is known that $(e^{tA})_{t \geq 0}$ is ℓ^∞ -contractive if and only if $A\mathbf{1} \leq 0$, i.e., if and only if

$$a_{ii} + \sum_{h \neq i} a_{ih} \leq 0 \quad \text{for all } i = 1, \dots, n. \quad (6.2)$$

2) Let us now consider the case of a general matrix A and define a new matrix $A^\sharp = (a_{ih}^\sharp)$ by

$$a_{ih}^\sharp := \begin{cases} \operatorname{Re} a_{ii} & \text{if } h = i, \\ |a_{ih}| & \text{if } i \neq h. \end{cases}$$

It is known (see [15]) that A^\sharp generates the modulus semigroup of $(e^{tA})_{t \geq 0}$, i.e., the (unique) semigroup $(e^{tA^\sharp})_{t \geq 0}$ that dominates $(e^{tA})_{t \geq 0}$ in the sense of positive semigroups, and is dominated by any other semigroup also dominating $(e^{tA})_{t \geq 0}$.

Let us first assume that (6.1) holds. Since $(e^{tA^\sharp})_{t \geq 0}$ is positive, by part 1) it is also ℓ^∞ -contractive. Now, since $(e^{tA^\sharp})_{t \geq 0}$ dominates $(e^{tA})_{t \geq 0}$, it follows that $(e^{tA})_{t \geq 0}$ is ℓ^∞ -contractive as well.

Conversely, let $(e^{tA})_{t \geq 0}$ be ℓ^∞ -contractive. In order to show that (6.1) holds, consider the modulus semigroup $(e^{tA^\sharp})_{t \geq 0}$, which is positive and, by [34, Prop. 2.26], also ℓ^∞ -contractive. One can check directly that the adjoint $(e^{tA^{\sharp*}})_{t \geq 0}$ of the modulus semigroup also dominates the adjoint $(e^{tA^*})_{t \geq 0}$, which by duality is ℓ^1 -contractive. Now it follows from the proof of [9, Prop. 2.5] that the semigroup $(e^{tA^{\sharp*}})_{t \geq 0}$ is also ℓ^1 -contractive, and by duality the positive semigroup $(e^{tA^\sharp})_{t \geq 0}$ is ℓ^∞ -contractive. Thus, by part 1) the entries of A^\sharp satisfy the condition (6.2), i.e., (6.1) holds. This concludes the proof. \square

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Received February 2006; revised December 2006.

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