NETWORKS AND HETEROGENEOUS MEDIA
Website: http://aimSciences.org

Volume **2**, Number **1**, March **2007**
pp. **37–54**

A MULTI LAYER METHOD APPLIED TO A MODEL OF PHYTOPLANKTON

Khalid Boushaba

Department of Mathematics
Iowa State University
Ames, Iowa, 50011, USA

(Communicated by Thomas Hillen)

Abstract. In this paper, we develop a multi layer method to solve a generalized case of a phytoplankton model introduced in [7]. It is treated by means of a sequence of approximations: the mixed layer is subdivided into a finite number of thin layers within each of which horizontal velocity can be considered constant with respect to depth. Existence, uniqueness and non negativity of solutions are investigated.

1. Introduction.

In the marine life cycle, phytoplankton is the first level of living species, known as primary production: using photosynthesis and organic as well as mineral nutrients to grow and proliferate (see [4]). Zooplankton is the secondary production: it eats the phytoplankton, and larger animals eat the zooplankton and other animals. This is to say how essential the dynamics of phytoplankton is in the sustainability and development of marine biotope. Models of phytoplankton dynamics are many and correspond to a variety of motivations ([27, 19, 20, 17] etc). As an example, there is the phenomenon known as phytoplankton blooms which occurs at certain periods, characterized by rapid and, most of the time, short lasting enormous proliferation of phytoplankton cells. Modeling the occurrence of such bursts is a challenging issue, with the appeal exerted on dynamicists by the prospect of complex behavior (see [24]). The purpose of this work is not, however, to investigate such properties. What we aim at here is to look at a model rich enough to encompass both physical and biological features, and that can yet be solved. The model considered here stresses three main factors: 1) Transport entailed by the currents: the currents are computed using Navier-Stokes equations and are introduced in the equations of the phytoplankton as time-dependent coefficients; 2) Vertical diffusion induced by vertical mixing in the upper part of the water column; 3) Production of new phytoplankton as a result of photosynthesis which depends on the quantity of light that a cell receives.

Regarding the movement in the sea, we treat phytoplankton cells as passive particles, displaced by the currents and the vertical diffusion entailed by eddies. Horizontal diffusion has been neglected here: a justification is that although it may be several orders of magnitude larger than vertical diffusion, it acts on scales (several tens of kilometers) which are even bigger than the scale of vertical movement (a few

2000 *Mathematics Subject Classification.* Primary: 47D06, 65M20,65M25, 74G55, 34K18.
Key words and phrases. Phytoplankton dynamics, multi layer method, semigroup, method of lines, vertical diffusion, sea current.

hundreds of meters). Reducing both scales to similar orders of magnitude entails dividing the horizontal diffusion coefficients by a number which is about 10^4 times as big as the one dividing the vertical diffusion.

The part of the equation which has to do with migration is linear. Nonlinear effects are to be found in biological interactions, namely, density-dependence comes into effect during periods of high activity, in the form of superficial layers of phytoplankton preventing the lower layers from getting as much light as could possibly reach these layers. This is known in the literature as the shading effect ([10]). From the above considerations, the model under study in this work reads as

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + div\left[\kappa V\left(t,P\right)\varphi\right] = \dfrac{\partial}{\partial z}\left(h\left(t,P\right)\dfrac{\partial \varphi}{\partial z}\right) - \mu\left(t,P\right)\varphi + rJ\left(t,P,\varphi\right)\varphi \\ \varphi\left(0,P\right) = \varphi_0\left(P\right) \\ h(t,x,y,0)\dfrac{\partial \varphi}{\partial z}\left(t,x,y,0\right) - V_3\left(t,x,y,0\right)\varphi\left(t,x,y,0\right) = 0 \\ \dfrac{\partial \varphi}{\partial z}\left(t,x,y,z^*\right) = 0 \end{cases}$$

The parameter κ multiplying V in equation (1) is a number between 0 and 1 which indicates how fast the current "entrains" the phytoplankton. It is a simplified model of the effect of viscosity. We assume it to be constant although it would be natural to assume κ depends on the phytoplankton density. h is the mixing coefficient. μ is the mortality rate. At first sight, this model is the same as the one treated in [7] with the exception of the boundary condition at $z = z^*$ the distance from the surface to a region below the thermocline. Here we are assuming a no-flux condition while the condition assumed in [7] is the homogeneous Dirichlet condition. In fact, in [7], it was considered that the vertical mixing coefficient $h(t,P) = 0$ on the thermocline, while the vertical component of the velocity was not vanishing. In fact, one does not have $h(t,P) = 0$, at $z = z^*$, h is just very small but $V_3(t,P)$ may be even smaller: in this case, the above condition is feasible (see [3]). More fundamentally, the hypotheses considered here differ from those made in [7] in such respects that have rendered necessary the development of another method for solving the equation, what we call the multilayer approach. Before we present this approach, we point out that, in both cases, the equation is considered without its nonlinear growth term, first: this term is added later on as a nonlinear perturbation. Comparing migration and mortality terms considered in [7] to the same expressions here, there is no restriction here on the physical parameters, the fluid velocity and the vertical diffusion are functions of all the variables, and the same is true for the mortality, while in [7] it was supposed that the mixing coefficient and the mortality depend on z only, and the horizontal velocities depend on time and horizontal components only. As a consequence of incompressibility, the vertical component of the velocity was supposed to be a function of z only. So the partial differential equation governing the evolution of phytoplankton could be divided into a horizontal first order hyperbolic equation and a vertical second order parabolic equation. It was then possible to decompose the study into two steps. First, the problem was solved (in the horizontal component) along the characteristic lines of the horizontal field. On these lines, it reduced to a one dimensional parabolic equation with respect to the vertical component, which was solved by means of semigroup theory. One advantage of the method we used there is that it allowed us to obtain explicit formulae for the solutions.

The main emphasis here is put on the currents and the diffusion. Based on data provided to us by P. Lazure and A.-M. Jegou ([21]), the domain of study has notably been restricted to the upper layer, the so-called mixed layer. As far as the results we are looking for here, we have in mind analytical results. As far as possible, we look for explicit formulae or formulae which can be expressed in terms of simpler ones. We focus mainly on the coupling of the horizontal transport and the vertical diffusion. In this paper we deal with the general case where the horizontal velocity changes with depth. This case is treated by means of a sequence of approximations: the mixed layer is subdivided into a finite number of thin layers within each of which horizontal velocity can be considered constant with respect to depth.

The paper is organized as follows. Section 2 is devoted to a detailed presentation of the model and the main assumptions. Section 3 is devoted to proving existence and positivity of solutions of the equation under consideration: subsection 3.1 addresses specifically the approximate model, and subsection 3.2 is devoted convergence of this approximation to the exact solution of the model. Section 4 deals with the non linear case: the non linearity is incurred by the competition for the photoenergy. A short review of other models is made in a final discussion.

2. **The multilayer model.**

The domain under consideration is approximately

$$\Omega = D \times]0, z^*[\,,$$

where D is an open subset of the surface, that is D is a portion of a plane and z^* is the distance from the surface to a region below the thermocline. In this region the density of phytoplankton is approximately constant (see [20]).

The phytoplankton is characterized by its density, that is to say, at each time $t \in [0, T]$ where T is the maximal time of observation, $\varphi(t, P)$ can be thought of as the phytoplankton biomass per unit of volume evaluated at the point P at that time

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + div\left[\kappa V(t,P)\,\varphi\right] = \dfrac{\partial}{\partial z}\left(h(t,P)\dfrac{\partial \varphi}{\partial z}\right) - \mu(t,P)\,\varphi + rJ(t,P,\varphi)\,\varphi \\ \varphi(0,P) = \varphi_0(P) \qquad P \in \Omega \\ h(t,x,y,0)\dfrac{\partial \varphi}{\partial z}(t,x,y,0) - V_3(t,x,y,0)\,\varphi(t,x,y,0) = 0 \\ \dfrac{\partial \varphi}{\partial z}(t,x,y,z^*) = 0 \end{cases} \tag{1}$$

- $V(t, P)$ is the velocity of the sea currents and is given by $V = (V_1, V_2, V_3)$. Since the sea water is incompressible, we conclude that

$$div\ V(t,P) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 0 \tag{2}$$

  (see [5]). The parameter κ multiplying V in equation (1) is a number between 0 and 1 which indicates how fast the current "entrains" the phytoplankton. Which is a simplified model of the effect of viscosity. We assume it to be constant although it would be natural to assume κ depends on the phytoplankton density.
- h is the mixing coefficient which reflect the molecular diffusion.
- μ is the mortality rate.

- $J(t,P,\varphi)$ is the irradiance intensity defined by the following equation 3:

$$J(t,P,\varphi) = J_0(t)\exp\left[-k_0 z - k_1\int_0^z \int_{D(x,y,(z-\tau)tan(\alpha))} \varphi(t,\eta,\xi,\tau)\, d\eta d\xi d\tau\right] \quad (3)$$

$J_0(t)$ is the irradiance intensity hitting the sea surface at time t.

k_0 is the diffuse attenuation coefficient in the water due to water alone.

k_1 is the diffuse attenuation coefficient due to the phytoplankton alone.

The set of all particles in competition with a given one is a cone with vertex at the particle, the axis is the vertical and the angle is the maximum angle $0 < \alpha < \pi$ for which the irradiance received by the particle can be perceived.

$D(x,y,\delta)$ is the disk of center (x,y) and radius $\delta > 0$

- r, the concentration of nutrient transformed by the phytoplankton, is assumed to be constant, ([10] and [23])
- φ_0, the initial conditions is a functions defined on $\Omega = D \times]0,z^*[$, with horizontal projection of the support inside a compact subset of the interior of D. Some natural properties of φ_0 are that it is non negative and lives in $L^1(\Omega)$. We define the maximal time of observation as

$$T = T_{\varphi_0} = \sup\{t > 0 : \ \Phi(t,x,y,z) \in D, \ \forall (x,y,z) \in \text{support} \quad \varphi_0\}.$$

From the assumption made about the sea surface, we can conclude that $V_3(t,x,y,0) = 0$, and since h is roughly deceasing with depth so $h(t,x,y,0) > 0$, thus the condition at $z = 0$ is just

$$\frac{\partial \varphi}{\partial z}(t,x,y,0) = 0$$

For a detailed discussion of the parameters and functions of the model we refer to [7].

We divide the water column into $n+1$ layers

$$\Omega = \bigcup_{i=1}^{n+1} D \times]z_{i-1}, z_i[\,.$$

For $1 \le i \le n+1$ we denote respectively by

$$\begin{aligned}
V_k^i(t,x,y) &= \frac{1}{z_i - z_{i-1}}\int_{z_{i-1}}^{z_i} V_k(t,P)\,dz \quad k = 1,2 \\
V_3^i(t,x,y,z) &= \frac{V_3(t,x,y,z_i) - V_3(t,x,y,z_{i-1})}{z_i - z_{i-1}}(z - z_{i-1}) + V_3(t,x,y,z_{i-1}) \\
\alpha_i(t,x,y) &= \frac{1}{z_i - z_{i-1}}\int_{z_{i-1}}^{z_i} h(t,P)\,dz \\
\mu^i(t,x,y) &= \frac{1}{z_i - z_{i-1}}\int_{z_{i-1}}^{z_i} \mu(t,P)\,dz
\end{aligned}$$

the averaged horizontal velocity of the sea currents, the affine approximation of the vertical velocity of the sea currents, the averaged mixing function and mortality rate respectively in the layer $D \times]z_{i-1}, z_i[$. Note that V^i gives an approximation of the velocity of the sea currents and satisfies the incompressibility property in the layer $D \times]z_{i-1}, z_i[$; we have

$$\frac{\partial V_1^i}{\partial x} + \frac{\partial V_2^i}{\partial y} + \frac{\partial V_3^i}{\partial z} = \frac{1}{z_i - z_{i-1}}[\int_{z_{i-1}}^{z_i} \frac{\partial V_1}{\partial x}dz + \frac{\partial V_2}{\partial y}dz + V_3(t,x,y,z_i) - V_3(t,x,y,z_{i-1})]$$

from the incompressibility property (2) we conclude that

$$\frac{\partial V_1^i}{\partial x}+\frac{\partial V_2^i}{\partial y}+\frac{\partial V_3^i}{\partial z}=0.$$

We denote by $\varphi^i=\varphi|_{[z_{i-1},z_i]}$. the restriction of φ to $[z_{i-1},z_i]$.

In each layer; for $P\in D\times\,]z_{i-1},z_i[\,,\ 1\leq i\leq n+1$ equation (1) can be approximated as follows

$$\frac{\partial\varphi^i}{\partial t}+div\left[\kappa V^i\varphi^i\right]=\alpha_i\left(t,x,y\right)\frac{\partial^2\varphi^i}{\partial z^2}-\mu^i\left(t,x,y\right)\varphi^i+rJ\left(t,P,\varphi\right)\varphi^i$$

with the initial data

$$\begin{aligned}\varphi^i\left(0,P\right) &= \varphi_0|_{[z_{i-1},z_i]}\left(P\right)\\ &= \varphi_0^i\left(P\right),\end{aligned}$$

the boundary conditions

$$\begin{aligned}\frac{\partial\varphi^1}{\partial z}\left(t,x,y,0\right) &= 0\\ \frac{\partial\varphi^{n+1}}{\partial z}\left(t,x,y,z^*\right) &= 0.\end{aligned}$$

The flux at the interface of each layer and the continuity of the solution give rise to the following supplementary boundary conditions

$$\alpha_i\left(t,x,y\right)\frac{\partial\varphi^i}{\partial z}\left(t,x,y,z_i\right)=\alpha_{i+1}\left(t,x,y\right)\frac{\partial\varphi^{i+1}}{\partial z}\left(t,x,y,z_i\right)$$

and

$$\varphi^i\left(t,x,y,z_i\right)=\varphi^{i+1}\left(t,x,y,z_i\right).$$

for $1\leq i\leq n$. What we have in mind is that φ^i should be the restriction of φ to the layer represented by the interval $[z_{i-1},z_i]$. In fact, it will be shown to be an approximation of this restriction.

Remark 1. In the equation (2) φ is used inside $J(t,P,\varpi)$. This makes the model coupled through boundary conditions and J.

Using the incompressibility hypothesis , the full multilayer model can be written in the form

$$\left\{\begin{array}{l}\frac{\partial\varphi^i}{\partial t}+\kappa V_1^i\left(t,x,y\right)\frac{\partial\varphi^i}{\partial x}+\kappa V_2^i\left(t,x,y\right)\frac{\partial\varphi^i}{\partial y}=\alpha_i\left(t,x,y\right)\frac{\partial^2\varphi^i}{\partial z^2}-\kappa V_3^i\left(t,x,y,z\right)\frac{\partial\varphi^i}{\partial z}\\ \qquad\qquad\qquad -\mu^i\left(t,x,y\right)\varphi^i+rJ\left(t,P,\varphi\right)\varphi^i \qquad 1\leq i\leq n+1\\ \varphi^i\left(0,P\right)=\varphi_0^i\left(P\right)\\ \frac{\partial\varphi^1}{\partial z}\left(t,x,y,0\right)=0\\ \alpha_i\left(t,x,y\right)\frac{\partial\varphi^i}{\partial z}\left(t,x,y,z_i\right)=\alpha_{i+1}\left(t,x,y\right)\frac{\partial\varphi^{i+1}}{\partial z}\left(t,x,y,z_i\right)\\ \varphi^i\left(t,x,y,z_i\right)=\varphi^{i+1}\left(t,x,y,z_i\right)\\ \frac{\partial\varphi^{n+1}}{\partial z}\left(t,x,y,z^*\right)=0\end{array}\right. \tag{4}$$

It is the equation we are going to study now.

3. **The linear equation.** In this section, we deal with the linear equation associated with equation (4), that is, we drop the quantity $rJ(t,P,\varphi)\varphi$ or we assume $r=0$. It will be further designated as equation $(4)_0$. The aim of this section is to show that the multilayer model $(4)_0$ possesses a non negative solution. and its solution along characteristic lines defined by the horizontal flow in each layer, tends to the desired solution of equation $(1)_0$ where $(1)_0$ denotes equation (1) with r=0, as the number of layers $n\to+\infty$, in $L^2\left([0,T],W^{1,2}(0,z^*)\right)$. We now state the assumptions of this section:

(H_1): V_1^i V_2^i, V_3^i are Lipschitz continuous in (x,y), $V_3(t,x,y,z^*)=0$ and $V_3(t,x,y,0)=0$.

(H_2): $\mu\in L^\infty((0,T)\times\Omega)$, $\kappa V_3'-\mu\le 0$.

(H_3): $h\in L^\infty\left((0,T),W^{2,2}(0,z^*)\right)$, $h>0$.

3.1. **Existence, uniqueness and positivity of multilayer model.** We use an approach by the method of characteristics to build a one-dimensional time dependent parabolic equation whose resolution will yield solutions of equation $(4)_0$. More precisely, z being fixed, equation $(4)_0$ reduces to a first order hyperbolic equation in (x,y) which can be solved by integration along the characteristic lines, that is to say, the curves with parametric representation of the form: $\left(\overline{t}(s),\overline{x}^i(s),\overline{y}^i(s)\right)$, solutions of the following system of ordinary differential equations:.

$$\begin{cases}\frac{d\overline{t}}{ds}(s)=1\\ \frac{d\overline{x}^i}{ds}(s)=\kappa V_1^i\left(\overline{t}(s),\overline{x}^i(s),\overline{y}^i(s)\right)\\ \frac{d\overline{y}^i}{ds}(s)=\kappa V_2^i\left(\overline{t}(s),\overline{x}^i(s),\overline{y}^i(s)\right)\end{cases}\tag{5}$$

and the initial value

$$\left(\overline{t}(0),\overline{x}^i(0),\overline{y}^i(0)\right)=(0,x_0,y_0).$$

In fact $\overline{t},\overline{x}^i,\overline{y}^i$ are also functions of the initial values x_0, y_0 and should be written as

$$\overline{t}=\overline{t}(s,x_0,y_0),\overline{x}^i=\overline{x}(s,x_0,y_0),\overline{y}^i=\overline{y}^i(s,x_0,y_0),$$

if we denote Φ^i the map defined by

$$\Phi^i(s,x_0,y_0)=\left(\overline{x}^i(s),\overline{y}^i(s)\right),$$

we have :

$$(x_0,y_0)=\Phi^i\left(-s,\overline{x}^i(s),\overline{y}^i(s)\right).$$

We denote $\overline{\varphi}^i(s,z)$, or $\overline{\varphi}^i(s,z,x_0,y_0)$ the restriction of the solution along the characteristic line emanating from the point $(0,x_0,y_0)$,

$$\overline{\varphi}^i(s,x_0,y_0,z)=\varphi^i\left(\overline{t}(s),\overline{x}^i(s),\overline{y}^i(s),z\right),$$

In terms of $\overline{\varphi}$, equation $(4)_0$ reads

$$\frac{\partial\overline{\varphi}^i}{\partial s}(s,z)=\overline{\alpha}_i(s)\frac{\partial^2\overline{\varphi}^i}{\partial z^2}-\kappa\overline{V}_3^i(s,z)\frac{\partial\overline{\varphi}^i}{\partial z}-\overline{\mu}^i(s)\overline{\varphi}^i$$

The condition

$$\frac{\partial\varphi^1}{\partial z}(t,x,y,0)=0$$
$$\frac{\partial\varphi^{n+1}}{\partial z}(t,x,y,z^*)=0$$

yields

$$\frac{\partial\overline{\varphi}^1}{\partial z}(s,0)=0$$
$$\frac{\partial\overline{\varphi}^{n+1}}{\partial z}(s,z^*)=0,$$

and the condition

$$\begin{aligned}\alpha_i(t,x,y)\frac{\partial\varphi^i}{\partial z}(t,x,y,z_i) &= \alpha_{i+1}(t,x,y)\frac{\partial\varphi^{i+1}}{\partial z}(t,x,y,z_i)\\ \varphi^i(t,x,y,z_i) &= \varphi^{i+1}(t,x,y,z_i) \quad 1\le i\le n\end{aligned}$$

yields

$$\begin{aligned}\overline{\alpha}_i(s)\frac{\partial\overline{\varphi}^i}{\partial z}\left(s,\Phi^i\left(-s,\Phi^{i+1}(s,x_0,y_0)\right),z_i\right) &= \overline{\alpha}_{i+1}(s)\frac{\partial\overline{\varphi}^{i+1}}{\partial z}(t,x_0,y_0,z_i)\\ \overline{\varphi}^i\left(s,\Phi^i\left(-s,\Phi^{i+1}(s,x_0,y_0)\right),z_i\right) &= \overline{\varphi}^{i+1}(t,x_0,y_0,z_i) \quad 1\le i\le n\end{aligned}$$

So, to each (x_0,y_0), we have associated the following system of equations:

$$\begin{cases}\frac{\partial u^i}{\partial s}(s,z)=\overline{\alpha}_i(s)\frac{\partial^2u^i}{\partial z^2}-\kappa\overline{V}_3^i\frac{\partial u^i}{\partial z}-\overline{\mu}^i(s)\,u^i, \quad 1\le i\le n+1\\ u^i(0,x_0,y_0,z)=\varphi_0^i(x_0,y_0,z)\\ \frac{\partial u^1}{\partial z}(s,0)=0\\ \overline{\alpha}_i(s)\frac{\partial u^i}{\partial z}\left(s,\Phi^i\left(-s,\Phi^{i+1}(s,x_0,y_0)\right),z_i\right)=\overline{\alpha}_{i+1}(s)\frac{\partial u^{i+1}}{\partial z}(s,x_0,y_0,z_i)\\ u^i\left(s,\Phi^i\left(-s,\Phi^{i+1}(s,x_0,y_0)\right),z_i\right)=u^{i+1}(s,x_0,y_0,z_i), \quad 1\le i\le n\\ \frac{\partial u^{n+1}}{\partial z}(s,x_0,y_0,z^*)=0\end{cases} \tag{6}$$

Conversely, once problem (6) is solved, we have:

$$\overline{\varphi}^i(s,x_0,y_0,z)=u^i(s,x_0,y_0,z)$$
$$\varphi^i\left(\overline{t}(s),\overline{x}(s),\overline{y}(s),z\right)=\overline{\varphi}^i(s,x_0,y_0,z)$$

So, the solution of equation $(4)_0$ can be determined in terms of the solutions of problem (6).

Let us consider for a moment problem (6). The unknown functions u^i can be treated as functions on the product $R^+\times[0,z^*]$, parameterized by (x_0,y_0). Each of the functions of the parameterized family satisfies a partial differential equation in the interior of the layers. The initial value also is a parameterized family of functions with the horizontal coordinates as parameters. In fact each solution and its initial value correspond to the same parameter. This does not hold for the boundary conditions which, indeed, involve functions associated with several values of the parameters, more precisely, the boundary conditions are expressed as relationships between the solution corresponding to the value (x_0,y_0) of the parameters, on the one hand, and the solution corresponding to $\Phi^i\left(-s,\Phi^{i+1}(s,x_0,y_0)\right)$, on the other hand. This fact makes the study of equation (6) difficult. On the other hand, one may observe that, as n becomes large

$$\Phi^i\left(-s,\Phi^{i+1}(s,x_0,y_0)\right)\to(x_0,y_0).$$

A natural simplification of equation (6) which, asymptotically (as n tends to ∞) will lead to the same result is to express the transmission conditions at the same (x_0,y_0).

So from now on, we will consider as an approximation the following system of equations:

$$\begin{cases}\frac{\partial u^i}{\partial s}(s,z)=\overline{\alpha}_i(s)\frac{\partial^2u^i}{\partial z^2}-\kappa\overline{V}_3^i(s,z)\frac{\partial u^i}{\partial z}-\overline{\mu}^i(s)u^i, \quad 1\le i\le n+1\\ u^i(0,z,x_0,y_0)=\varphi_0^i(x_0,y_0,z)\\ \frac{\partial u^1}{\partial z}(s,0)=0\\ \overline{\alpha}_i(s)\frac{\partial u^i}{\partial z}(s,x_0,y_0,z_i)=\overline{\alpha}_{i+1}(s)\frac{\partial u^{i+1}}{\partial z}(t,x_0,y_0,z_i)\\ u^i(s,x_0,y_0,z_i)=u^{i+1}(s,x_0,y_0,z_i), \quad 1\le i\le n\\ \frac{\partial u^{n+1}}{\partial z}(s,x_0,y_0,z^*)=0\end{cases} \tag{7}$$

We denote $\mathcal{A}^n(s)$ the operator defined by the right hand side of problem (7). We are now ready to define the operator suited to the problem at hand. We still call it $\mathcal{A}^n(s)$.

$\mathcal{A}^n(s)$ is given by

$$\mathcal{A}^n(s)\, f|_{(z_{i-1},\, z_i)} = \frac{\partial}{\partial z}\left(\overline{\alpha}(s)\frac{\partial f_i}{\partial z}\right) - \overline{\beta}(s,z)\frac{\partial f_i}{\partial z} - \gamma(s,z)\, f_i$$

with $\overline{\alpha}(s,z)|_{(z_{i-1},z_i)} = \overline{\alpha}_i(s)$, $\overline{\beta}(s,z)\big|_{(z_{i-1},z_i)} = \kappa \overline{V}_3^i(s,z)$ and $\gamma(s,z)|_{(z_{i-1},z_i)} = \overline{\mu}^i(s)$. $f_i = f|_{(z_{i-1},z_i)}$ is the restriction of the function f to the interval $[z_{i-1}, z_i]$, for $1 \leq i \leq n+1$ and

$$\mathcal{D}(\mathcal{A}^n(s)) = \Big\{ f \in W^{1,2}(0,z^*) \cap W^{2,2}(z_{i-1},z_i) : \frac{\partial f}{\partial z}(0) = \frac{\partial f}{\partial z}(z^*) = 0,$$
$$\overline{\alpha}_i(s)\frac{\partial f}{\partial z}\left(z_i^-\right) = \overline{\alpha}_{i+1}(s)\frac{\partial f}{\partial z}\left(z_i^+\right)\ 1 \leq i \leq n. \Big\}$$

Proposition 1. *Assume (H_2) and (H_3) hold. Let $v_0 \in \mathcal{D}(\mathcal{A}^n(0))$, $v_0 \geq 0$. Then equation (7) has a unique non negative solution $v^n \in L^2\left([0,T], W^{1,2}(0,z^*)\right) \cap \mathcal{C}^0\left([0,T], L^2(0,z^*)\right)$.*

Proof. To show this result we use a variational formulation method, we consider the bilinear form $a(s,u,v) = \int_0^{z^*} \overline{\alpha}(s)\frac{\partial u}{\partial z}(z)\frac{\partial v}{\partial z}(z)\,dz + \int_0^{z^*}\overline{\beta}(s,z)\frac{\partial u}{\partial z}(z)\,v(z)\,dz$ $+ \int_0^{z^*}\gamma(s,z)\,u(z)v(z)dz$, on $W^{1,2}(0,z^*)$. We show that conditions below hold.

(i) The map $s \mapsto a(s;u,v)$ is measurable for any $u,\ v \in W^{1,2}(0,z^*)$;

(ii) There exists $M \in (0,+\infty)$ such that for almost any $s \in (0,T)$ and for any $u,\ v \in W^{1,2}(0,z^*)$ we have

$$|a(s;u,v)| \leq M\|u\|_{W^{1,2}(0,z^*)} \cdot \|v\|_{W^{1,2}(0,z^*)};$$

(iii) There exist constants $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$a(s;v,v) \geq \alpha\|v\|^2_{W^{1,2}(0,z^*)} - \beta\|v\|^2_{L^2(0,z^*)}$$

for almost every $s \in (0,T)$ and for every $v \in W^{1,2}(0,z^*)$; we have that $\overline{\alpha}$, $\overline{\beta}$ and γ are measurable consequently the map $s \mapsto a(s;u,v)$ is measurable. So condition (i) is satisfied. For any $q,\ w \in W^{1,2}(0,z^*)$ and for almost any $s \in (0,T)$ we get

$$|a(s;q,w)| \leq C_1\int_0^{z^*}|\frac{\partial q}{\partial z}|\cdot|w|dz + C_2\int_0^{z^*}|\frac{\partial q}{\partial z}|\cdot|\frac{\partial w}{\partial z}|dz + C_3\int_0^{z^*}|q|\cdot|w|dz$$

(where $C_1 = \max_{(s,z)\in[0,T]\times[0,z^*]}\left|\overline{\beta}(s,z)\right|$, $C_2 = \max_{(s,z)\in[0,T]\times[0,z^*]}\overline{\alpha}(s,z)$,
$C_3 = ess\ sup_{(s,z)\in(0,T)\times[0,z^*]}\gamma(s,z)$)

$$\begin{aligned} &\leq C_1\|q\|_{W^{1,2}}\cdot\|w\|_{L^2} + C_2\|q\|_{W^{1,2}}\cdot\|w\|_{W^{1,2}} + C_3\|q\|_{L^2}\cdot\|w\|_{L^2} \\ &\leq M\|q\|_{W^{1,2}}\cdot\|w\|_{W^{1,2}}, \end{aligned}$$

where $M \in (0,+\infty)$ is a non negative constant independent of q and w. So, (ii) is fulfilled.

Let us now prove that $a(s;w,w) \geq \alpha\|w\|^2_{W^{1,2}} - \beta\|w\|^2_{L^2}$ for almost every $s \in (0,T)$ and for any $w \in W^{1,2}(0,z^*)$, where $\alpha > 0$ and $\beta \in \mathbb{R}$ are constants, we have

$$\begin{aligned} a(s;w,w) &= \int_0^{z^*} \overline{\alpha}(s)\left(\frac{\partial w}{\partial z}(z)\right)^2 dz + \int_0^{z^*} \overline{\beta}(s,z)\frac{\partial w}{\partial z}(z)\,w(z)\,dz \\ &\quad + \int_0^{z^*} \gamma(s,z)\,w^2(z)dz \\ &\geq \int_0^{z^*} \overline{\alpha}(s)\left(\frac{\partial w}{\partial z}(z)\right)^2 dz + \overline{\beta}(s,z)\frac{\partial w}{\partial z}(z)\,w(z)\,dz. \end{aligned}$$

Since $\overline{\alpha} > 0,\ \overline{\beta} \in L^\infty((0,T)\times(0,z^*))$, then there exist non negative constants ζ and δ such that

$$a(s;w,w) \geq -\zeta \int_\Omega |\frac{\partial w}{\partial z}|\cdot|w|dz + \delta\int_\Omega |\frac{\partial w}{\partial z}|^2 dz.$$

On the other hand we have

$$\int_\Omega |\frac{\partial w}{\partial z}|\cdot|w|dz \leq \int_\Omega (\rho|\frac{\partial w}{\partial z}|^2 + \rho^{-1}|w|^2),\ \forall\rho > 0$$

and this implies

$$\begin{aligned} a(s;w,w) &\geq -\zeta\int_\Omega (\rho|\frac{\partial w}{\partial z}|^2 + \rho^{-1}|w|^2) + \delta\int_\Omega |\frac{\partial w}{\partial z}|^2 dP \\ &\geq (\delta - \zeta\rho)\int_\Omega |\frac{\partial w}{\partial z}|^2 dz - \zeta\rho^{-1}\int_\Omega |w|^2 dz. \end{aligned}$$

Choosing now $\rho = \dfrac{\delta}{2\zeta}$ we get

$$\begin{aligned} a(s;w,w) &\geq \frac{\delta}{2}\int_\Omega |\frac{\partial w}{\partial z}|^2 dz - \frac{2\zeta^2}{\delta}\int_\Omega |w|^2 dz \\ &\geq \frac{\delta}{2}\|w\|^2_{W^{1,2}} - \frac{2\zeta^2}{\delta}\|w\|^2_{L^2}. \end{aligned}$$

We conclude that (iii) is verified. Condition (i), (ii) and (iii) allow us to conclude via a well known result of J.L. Lions (see [8], p. 218) that problem

$$\begin{cases} \int_0^T \left[a(s,u(s,.),v)\,\phi(s) - \int_0^{z^*} u(s,z)v(z)\,\phi'(s)\,dz\right] ds = \left(\int_0^{z^*} u^0(z)v(z)\,dz\right)\phi(0) \\ \text{for all } \phi \in \mathcal{D}(0,T) \text{ and for all } v \in W^{1,2}(0,z^*). \end{cases} \tag{8}$$

has a unique *weak solution* u, i.e.,

$$u \in L^2(0,T;W^{1,2}(0,z^*)) \cap \mathcal{C}^0(0,T;L^2(0,z^*)),\ u(0,z) = u^0(z)$$

We choose $v \in \mathcal{D}((z_{i-1},z_i))$, $i = 1,..n+1$ in (8), we deduce that

$$\frac{\partial u^i}{\partial s} = \overline{\alpha}_i(s)\frac{\partial^2 u^i}{\partial z^2} - \kappa\overline{V}^i_3(s,z)\frac{\partial u^i}{\partial z} - \overline{\mu}^i(s)u^i \text{ in } \mathcal{D}'((0,T)\times(z_{i-1},z_i))$$

with u^i is the restriction of the function u to the interval $[z_{i-1},z_i]$, for $1 \leq i \leq n+1$. On the other hand integration by parts in (8), we can see the boundary condition in 0 and in z^*

$$\frac{\partial u}{\partial z}(s,0)=\frac{\partial u}{\partial z}(s,z^*)=0.$$

It remains to give interpretation of transmission conditions. First, we know that $u\in\mathcal{C}^0(0,T;L^2(0,z^*))$ so we have continuity in z_i

$$u^i(s,z_i)=u^{i+1}(s,z_i)\text{ for }i=1,..n.$$

now, we show that

$$\overline{\alpha}_i(s)\frac{\partial u^i}{\partial z}(s,z_i)=\overline{\alpha}_{i+1}(s)\frac{\partial u^{i+1}}{\partial z}(s,z_i)\text{ for }i=1,..n. \tag{9}$$

The Green formula in $(0,z_i)$ yields

$$\begin{aligned}&\int_0^T\overline{\alpha}_i(s)\tfrac{\partial u^i}{\partial z}(s,z_i)vz_ids-\left(\int_0^{z_i}u^0(z)v(z)dz\right)\phi(0)dz\\&=\int_0^T[\int_0^{z_i}\overline{\alpha}_i(s)\tfrac{\partial^2u^i}{\partial z^2}+\kappa\overline{V}_3^i(s,z)\tfrac{\partial u^i}{\partial z}v+\overline{\mu}^i(s)u^iv\phi(s)-\int_0^{z_i}u(s,z)v(z)\phi'(s)dz]ds\end{aligned} \tag{10}$$

and in (z_i,z^*)

$$\begin{aligned}&-\int_0^T\overline{\alpha}_{i+1}(s)\tfrac{\partial u^{i+1}}{\partial z}(s,z_i)v(z_i)-\left(\int_{z_i}^{z^*}u^0(z)v(z)dz\right)\phi(0)\\&=\int_0^T\left[\int_{z_i}^{z^*}\overline{\alpha}_i(s)\tfrac{\partial^2u^i}{\partial z^2}+\kappa\overline{V}_3^i(s,z)\tfrac{\partial u^i}{\partial z}v+\overline{\mu}^i(s)u^iv\phi(s)-\int_{z_i}^{z^*}u(s,z)v(z)\phi'(s)dz\right]\end{aligned} \tag{11}$$

for all $v\in\mathcal{D}([0,z^*])$. Since u is solution of (8), according to equalities (10) and (11) we can see that

$$\int_0^T\left(\overline{\alpha}_i(s)\frac{\partial u^i}{\partial z}(s,z_i)-\overline{\alpha}_{i+1}(s)\frac{\partial u^{i+1}}{\partial z}(s,z_i)\right)v(z_i)=0\text{ for all }v\in\mathcal{D}([0,z^*])$$

then we conclude the transmission conditions (9).

Non negativity can be proved by standard arguments. For $f\in\mathcal{D}(\mathcal{A}^n(0))$, if u^n denotes the solution of equation (7), such that

$$u^n(0)=f,$$

then, u^n is a strong solution, we can multiply both sides of equation (7) by $u^{n-}=\max(0,-u^n)$ and integrate on $[0,z^*]$. Standard arguments can be used to arrive at the following

$$\frac{1}{2}\frac{d}{ds}\left|u^{n-}\right|^2_{L^2(0,z^*)}=-\sum_{i=1}^{n+1}\int_{z_{i-1}}^{z_i}\alpha_i\left(\frac{du^{n-}}{dz}\right)^2-\sum_{i=1}^{n+1}\int_{z_{i-1}}^{z_i}\left(\kappa\frac{\partial}{\partial z}V_3^i-\mu^i\right)\left(u^{n-}\right)^2$$

Then, the above equality leads to the following differential inequality

$$\frac{d}{dt}\left|u^{n-}\right|^2_{L^2}\le0,$$

This implies that:

$$\left|u^{n-}(s)\right|^2_{L^2}\le\left|u^{n-}(0)\right|^2_{L^2}.$$

So, if we assume that $f\ge0$, that is, $u^{n-}(0)=0$, we obtain that :

$$u^{n-}(s)=0,\forall t\ge0$$

so

$$u(s)\ge0,\forall s\ge0.$$

According to proposition 1 the solution of the linear problem (7) is non negative and given by

$$\varphi^n(t,x,y,z)=u^n\left(t,z,\Phi^i(-t,x,y)\right)\quad\text{for }z\in[z_{i-1},z_i]$$

□

3.2. **The exact solution.** In this subsection we show that solution along characteristic lines defined by the horizontal flow in each layer, tends to the desired unique solution of equation $(1)_0$, in $L^2\left([0,T], W^{1,2}(0,z^*)\right)$ as the number of layers $n \to +\infty$.

Theorem 1. *Let $u_0 \in L^2(0,z^*)$ then there exists a sequence $(u_0^n)_{n\geq 1}$ $u_0^n \in \mathcal{D}(\mathcal{A}^n(0))$ for all $n \geq 1$, such that*

$$u_0^n \to u_0$$

in $L^2(0,z^*)$ and there exist $u \in L^2\left([0,T], W^{1,2}(0,z^*)\right)$ such that the solution u^n of equation (7)

$$u^n \rightharpoonup u$$

in $L^2\left([0,T], W^{1,2}(0,z^*)\right)$ with respect to the weak norm. Finally u satisfies the equation

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial z}\left(\overline{h}(s,z)\frac{\partial u}{\partial z}\right) - \kappa \overline{V}_3(s,z)\frac{\partial u}{\partial z} - \overline{\mu}(s,z)\,u$$

with initial value

$$u(0,z) = u_0(z)$$

Proof. Let $u_0 \in L^2(0,z^*)$, there exists a sequence $(v_n)_{n\geq 1}$ in

$$\left\{u \in W^{2,2}(0,z^*),\ u'(z^*) = u'(0) = 0\right\}$$

such that

$$v_n \overset{L^2(0,z^*)}{\to} u_0.$$

We determine a new initial value u_0^n such that $u_0^n \in \mathcal{D}(\mathcal{A}^n(0))$ and

$$u_0^n \overset{L^2(0,z^*)}{\to} u_0$$

we choose u_0^n as a polynomial perturbation of u_0,

$$u_0^n = v_n + \sum_{i=2}^{n+1} Q_i \chi_{(z_{i-1},z_i)}$$

where Q_i is a second degree polynomial. So

$$Q_i(z_{i-1}) = Q_i(z_i) = 0$$

and

$$\overline{\alpha}_1(0)\,v_n'(z_1) = \overline{\alpha}_2(0)\left(v_n'(z_1) + Q_2'(z_1)\right) \tag{12}$$

$$\overline{\alpha}_i(0)\left(v_n'(z_i) + Q_i'(z_i)\right) = \overline{\alpha}_{i+1}(0)\left(v_n'(z_i) + Q_{i+1}'(z_i)\right) \tag{13}$$

for $2 \leq i \leq n$, we denote by

$$\varepsilon_{i,n}^- = Q_i'(z_i),\ \varepsilon_{i,n}^+ = Q_i'(z_{i-1})$$

the polynomial Q_i is given by

$$Q_i(z) = \frac{\varepsilon_{i,n}^+}{(z_i - z_{i-1})}(z - z_{i-1})(z_i - z)$$

with

$$\varepsilon_{i,n}^+ = -\varepsilon_{i,n}^-.$$

Using the equalities (12) and (13)we show that

$$\begin{aligned} \varepsilon_{2,n}^{+} &= \frac{(\overline{\alpha}_1(0) - \overline{\alpha}_2(0))}{\overline{\alpha}_2(0)} v_n'(z_1) \\ \overline{\alpha}_{i+1}(0)\varepsilon_{i+1,n}^{+} + \overline{\alpha}_i(0)\varepsilon_{i,n}^{+} &= (\overline{\alpha}_{i+1}(0) - \overline{\alpha}_i(0)) v_n'(z_i) \text{ for } i = 3, ..n, \end{aligned}$$

$\varepsilon_{2,n}^{+} \to 0$ as n tend to $+\infty$ and by induction we can see that for $\forall i$, $\varepsilon_{i,n}^{+} \to 0$ as $n \to +\infty$. We know that

$$\max_{z_{i-1} \leq z \leq z_i} Q_i(z) = \frac{\varepsilon_{i,n}^{+}(z_i - z_{i-1})}{4},$$

then

$$u_0^n \xrightarrow{L^2} u_0.$$

From proposition 1 there exists $u^n \in L^2\left(0, T, W^{1,2}(0, z^*)\right) \cap \mathcal{C}^0\left([0,T], L^2(0, z^*)\right)$ solution of problem (7). Multiplying the first equation of system $(7)_0$ by the test function $\theta \in \mathcal{C}^0\left([0,T], W_0^{2,2}(0, z^*)\right)$

$$\begin{aligned} \frac{d}{dt}\langle u^n, \theta\rangle &= \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \alpha_i(\triangle u^n)(t,z)\,\theta(t,z)\,dz - \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \kappa V_3^i \frac{du^n}{dz}\theta(t,z) \\ &\quad - \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \mu_i u^n \theta(t,z) \\ &= \sum_{i=1}^{n+1} (\alpha_i - \alpha_{i-1})\, u^n(s, z_i) \frac{\partial \theta}{\partial z}(t, z_i) + \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \alpha_i u^n \frac{\partial^2 \theta}{\partial^2 z}(t,z)\,dz \\ &\quad + \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \kappa V_3^i u^n \frac{d\theta}{dz}(t,z) + \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \kappa \frac{dV}{dz}{}_3^i u^n \theta(t,z) \\ &\quad - \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \mu_i u^n \theta(t,z)\,dz \end{aligned}$$

We integrate the first equality from 0 to t

$$\begin{aligned} \langle u^n, \theta\rangle(t) - \langle u^n, \theta\rangle(0) &= \int_0^t \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \overline{\alpha}_i u^n \frac{\partial^2 \theta}{\partial z^2}(s,z)\,dz \\ &\quad + \int_0^t \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \kappa \overline{V}_3^i u^n \frac{d\theta}{dz}(s,z)\,ds \\ &\quad + \int_0^t \sum_{i=1}^{n+1} (\overline{\alpha}_i - \overline{\alpha}_{i-1})\, u^n(s, z_i) \frac{\partial \theta}{\partial z}(s, z_i)\,ds \\ &\quad + \int_0^t \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \frac{\partial \overline{V}_3^i}{\partial z} u^n(s, z_i)\,\theta(s, z_i)\,ds \\ &\quad - \int_0^t \sum_{i=1}^{n+1} \int_{z_{i-1}}^{z_i} \overline{\mu}^i u^n \theta(s,z)\,dzds \end{aligned}$$

if we take $\theta = u^n$ we find that

$$\|u^n(t)\|_{L^2} + (\min_{1\le i\le n+1}\overline{\alpha}_i)\int_0^T \|u^n(s)\|_{W^{1,2}}\,ds \quad + \quad (\min_{1\le i\le n+1}\overline{\mu}^i)\int_0^T \|u^n(s)\|_{W^{1,2}}\,ds \le \|u_0^n\|_{L^2} \tag{14}$$

where $\min_{1\le i\le n+1}\overline{\alpha}_i > 0$, then u^n is bounded in $L^2\left([0,T], W^{1,2}(0,z^*)\right)$ so there exists a subsequence of u^n such that $u^n \rightharpoonup v$ on $L^2\left([0,T], W^{1,2}(0,z^*)\right)$ (see [8]).
It is convenient for further computations to express the integral in z on (z_{i-1}, z_i) in each integral quantity in the form of

$$\int_0^{z^*} \chi_{(z_{i-1},z_i)}(z)\,dz$$

where χ_J denotes the indicator function of set J, for instance we have

$$\int_0^t \sum_{i=1}^{n+1}\int_{z_{i-1}}^{z_i} \overline{\alpha}_i u^n \frac{\partial^2\theta}{\partial^2 z}(s,z)\,dzds = \int_0^t\int_0^{z^*}\sum_{i=1}^{n+1}\overline{\alpha}_i\chi_{(z_{i-1},z_i)}u^n\frac{\partial^2\theta}{\partial^2 z}(s,z)\,dzds$$

according to the definition of $\overline{\alpha}_i$, $\overline{\mu}^i$ and $\overline{V}_3^i$ we have the convergence of

$$\int_0^t \sum_{i=1}^{n+1}\int_{z_{i-1}}^{z_i}\overline{\alpha}_i u^n\frac{\partial^2\theta}{\partial^2 z}dz \overset{n\to+\infty}{\to} \int_0^t\int_0^{z^*}\overline{h}v\frac{\partial^2\theta}{\partial^2 z}dzds$$

$$\int_0^t \sum_{i=1}^{n+1}\int_{z_{i-1}}^{z_i}\kappa V_3^i u^n\frac{d\theta}{dz}(s,z)\,dzds \overset{n\to+\infty}{\to} \int_0^t\int_0^{z^*}\kappa\overline{V}_3 v\frac{d\theta}{dz}dzds$$

$$\int_0^t \sum_{i=1}^{n+1}\int_{z_{i-1}}^{z_i}\overline{\mu}^i(s)\,u^n(s,z)\,\theta(s,z)\,dzds \overset{n\to+\infty}{\to} \int_0^t\int_0^{z^*}\overline{\mu}v\theta(s,z)\,dzds$$

We have u^n bounded in $L^2\left([0,T], W^{1,2}(0,z^*)\right)$ and $\theta\in\mathcal{C}^0\left([0,T], W_0^{2,2}(0,z^*)\right)$.
Then, one can see that

$$\begin{aligned}&\textstyle\int_0^t\sum_{i=1}^{n+1}(\overline{\alpha}_i-\overline{\alpha}_{i-1})\,u^n(s,z_i)\,\frac{\partial\theta}{\partial z}(s,z_i)\\ &=\textstyle\int_0^t\sum_{i=1}^{n+1}(\overline{\alpha}_i-\overline{\alpha}_{i-1})\int_0^{z_i}\frac{\partial}{\partial z}\left(u^n(s,z)\frac{\partial\theta}{\partial z}(s,z)\right)+u^n(s,0)\frac{\partial\theta}{\partial z}(s,0)\\ &=\textstyle\int_0^t\sum_{i=1}^{n+1}(\overline{\alpha}_i-\overline{\alpha}_{i-1})\int_0^{z_i}\frac{\partial}{\partial z}\left(u^n(s,z)\frac{\partial\theta}{\partial z}(s,z)\right)\end{aligned}$$

and

$$\begin{aligned}&\textstyle\int_0^z\frac{\partial}{\partial z}\left(u^n(s,z)\frac{\partial\theta}{\partial z}(s,z)\right)-\int_0^{z_i}\frac{\partial}{\partial z}\left(u^n(s,z)\frac{\partial\theta}{\partial z}(s,z)\right)\\ &\le\sqrt{|z-z_i|}\left(\int_0^{z^*}\left(u^n(s,z)\frac{\partial\theta}{\partial z}(s,z)\right)^2\right)^{1/2}\end{aligned}$$

So for all $s\in(0,T)$

$$\sum_{i=1}^{n+1}(\overline{\alpha}_i-\overline{\alpha}_{i-1})\,u^n(s,z_i)\frac{\partial\theta}{\partial z}(s,z_i)\overset{n\to\infty}{\to}\int_0^{z^*}\frac{\partial\overline{h}}{\partial z}\int_0^z\frac{\partial}{\partial z}\left(u(s,z)\frac{\partial\theta}{\partial z}(s,z)\right)$$

using the Lebesgue dominated convergence, we can see that

$$\int_0^t\sum_{i=1}^{n+1}(\overline{\alpha}_i-\overline{\alpha}_{i-1})\,u^n(s,z_i)\frac{\partial\theta}{\partial z}(s,z_i)\overset{n\to+\infty}{\to}\int_0^t\int_0^{z^*}\frac{\partial\overline{h}}{\partial z}v\frac{\partial\theta}{\partial z}$$

and

$$\int_0^t\sum_{i=1}^{n+1}\int_{z_{i-1}}^{z_i}\frac{\partial\overline{V}_3^i}{\partial z}u^n(s,z_i)\,\theta(s,z_i)\overset{n\to+\infty}{\to}\int_0^t\int_0^{z^*}\frac{\partial\overline{V}_3^i}{\partial z}v\theta$$

We integrate the equation (7) from 0 to t with a test function

$$\theta \in L^2\left([0,T], W_0^{2,2}[0,z^*]\right)$$

and let n tend to $+\infty$ (a subsequence) we find

$$\begin{aligned}
&\langle v,\theta\rangle(t) - \langle u_0,\theta\rangle(0)\\
&= \int_0^t \int_0^{z^*} \frac{\partial \overline{h}}{\partial z} v \frac{\partial \theta}{\partial z} + \int_0^t \int_0^{z^*} \overline{h} v \frac{\partial^2 \theta}{\partial^2 z} + \int_0^t \int_0^{z^*} \kappa \overline{V}_3 v \frac{d\theta}{dz} dz ds\\
&+ \int_0^t \int_0^{z^*} \kappa \frac{\partial \overline{V}_3}{\partial z} v\theta - \int_0^t \int_0^{z^*} \mu v\theta\\
&= \int_0^t \int_0^{z^*} \frac{\partial}{\partial z}\left(\overline{h}\frac{\partial v}{\partial z}\right)\theta(s,z)\, dz ds - \int_0^t \int_0^{z^*} \kappa \overline{V}_3 v \frac{dv}{dz} \theta dz ds - \int_0^t \int_0^{z^*} \overline{\mu} v\theta
\end{aligned}$$

Thus v is the weak solution of equation (7), and uniqueness of the solution of equation (7) implies that $u = v$. □

Now we consider the characteristic lines $\left(\overline{t}(s), \overline{x}(s), \overline{y}(s)\right)$, solutions of the following system of ordinary differential equations:.

$$\begin{cases}
\frac{d\overline{t}}{ds}(s) = 1\\
\frac{d\overline{x}}{ds}(s) = \kappa V_1\left(\overline{t}(s), \overline{x}(s), \overline{y}(s), z\right)\\
\frac{d\overline{y}}{ds}(s) = \kappa V_2\left(\overline{t}(s), \overline{x}(s), \overline{y}(s), z\right)
\end{cases}$$

and the initial value

$$\left(\overline{t}(0), \overline{x}(0), \overline{y}(0)\right) = (0, x_0, y_0).$$

if we denote Φ the map defined by

$$\Phi(s, x_0, y_0, z) = (\overline{x}(s,z), \overline{y}(s,z)),$$

we have:

$$(x_0, y_0) = \Phi(-s, \overline{x}(s,z), \overline{y}(s,z), z).$$

The restriction of the solution of the linear equation $(1)_0$ along the characteristic line emanating from the point $(0, x_0, y_0)$,

$$\overline{\varphi}(s, z, x_0, y_0) = \varphi\left(\overline{t}(s), \overline{x}(s), \overline{y}(s), z\right),$$

Theorem 2. *Under the assumptions (H_1), (H_2), and (H_3), we have that for $\varphi_0 \in W^{2,2}(\Omega)$ there exist a unique nonnegative solution of the linear model $(1)_0$ for $t \in [0,T]$.*

Proof. For $z \in [z_{i-1}, z_i]$, using the definition of V_1^i and V_2^i, we have

$$\frac{\partial}{\partial s}\Phi^i(s, x_0, y_0) \to \frac{\partial}{\partial s}\Phi(s, x_0, y_0, z)$$

as $n \to +\infty$ and

$$\Phi^i(0, x_0, y_0) = \Phi(0, x_0, y_0, z)$$

so

$$\Phi^i(s, x_0, y_0) \overset{n \to +\infty}{\to} \Phi(s, x_0, y_0, z).$$

we know that the solution of the linear problem (7) is given by

$$\varphi^n(t, x, y, z) = u^n\left(t, z, \Phi^i(-t, x, y)\right)$$

and by applying theorem (1)

$$\varphi^n(t, x, y, z) \to u(t, z, \Phi(-t, x, y, z))$$

the solution of linear problem $(1)_0$ is non negative and given by

$$\varphi(t, x, y, z) = u(t, z, \Phi(-t, x, y, z)).$$

□

4. **Nonlinear equation.** As a preliminary step in the proof of existence in the nonlinear case, we derive a priori estimates for the solutions of equation (1). We have to assume that
$H_4 : J_0 \in L^{\infty}_{+}(0,T)$ is satisfied.
The notation $\|.\|$ corresponds to the usual norm in $L^2(\Omega)$.

Proposition 2. *If equation (1) has a non negative solution φ, such that its initial value $\varphi_0 \in L^1(\Omega)$ and if it has a compact support, then there exists a constant c such as*

$$\|\varphi(t)\| \le \|\varphi_0\| \exp(ct)$$

Proof. By integration of the equation (1) and multiplying by a test function φ, we found that

$$\begin{aligned}\int_\Omega \frac{\partial \varphi}{\partial t}\varphi + \int_\Omega div\left[\kappa V(t,P)\varphi\right]\varphi &= \int_\Omega \frac{\partial}{\partial z}\left(h(z)\frac{\partial \varphi}{\partial z}\right)\varphi - \int_\Omega \mu(z)\varphi^2 \\ &\quad + \int_\Omega rJ(t,P,\varphi)\varphi^2 \\ &= \int_D \left[h\frac{d\varphi}{dz}\varphi\right]_0^{z^*} - \int_\Omega h\left(\frac{df}{dz}\right)^2 dz - \int_\Omega \mu(z)\varphi^2 \\ &\quad + \int_\Omega rJ(t,P,\varphi)\varphi^2 \qquad (15)\end{aligned}$$

we have that

$$\begin{aligned}\int_\Omega div\left[\kappa V(t,P)\varphi\right]\varphi &= \int_{\partial\Omega} \varphi^2 \kappa V(t,P).\eta(P) - \int_\Omega \varphi\kappa V(t,P).\nabla\varphi \\ &= \int_{D\times\{z^*\}} \varphi^2\kappa V(t,P)\eta(P) + \int_{D\times\{0\}} \varphi^2\kappa V(t,P)\eta(P) \\ &\quad + \int_{\partial D\times[0,z^*]} \varphi^2\kappa V(t,P)\eta(P) - \int_\Omega \varphi\kappa V(t,P).\nabla\varphi\end{aligned}$$

where $\eta(P)$ is the outward normal vector at point P and $\nabla\varphi$ is the gradient of φ. The first term

$$\int_{D\times\{z^*\}} \varphi^2\kappa V(t,P)\eta(P) = 0,$$

since, for $P \in D\times\{z^*\}$

$$\begin{aligned}V(t,P).\eta(P) &= V_3|_{z=z^*} \\ &= 0.\end{aligned}$$

Also, for $P \in D\times\{0\}$

$$\begin{aligned}V(t,P).\eta(P) &= V_3|_{z=0} \\ &= 0.\end{aligned}$$

We integrate the equation within a time interval during which the horizontal projection of the support is contained in the interior of D, as long as $t \in [0, T_{\varphi_0}]$. So for $P \in \partial D\times[0,z^*]$

$$V(t,P)\eta(P)\varphi = 0.$$

Thus

$$\begin{aligned}\int_\Omega \varphi\kappa V(t,P).\nabla\varphi &= \frac{1}{2}\int_\Omega \kappa V(t,P)\nabla\varphi^2 \\ &= \frac{1}{2}\int_{\partial\Omega}\varphi^2\kappa V(t,P).\eta(P) - \frac{1}{2}\int_\Omega \varphi^2 div(\kappa V(t,P)) \\ &= 0.\end{aligned}$$

We know that

$$\int_\Omega \frac{\partial\varphi}{\partial t}\varphi = \frac{1}{2}\frac{d}{dt}\|\varphi(t)\|^2,$$

so

$$\begin{aligned}\frac{d}{dt}\|\varphi(t)\|^2 &\le 2r\|J_0\|_{L^\infty(0,T)}\|\varphi(t)\|^2 \\ &\le c\|\varphi(t)\|^2,\end{aligned}$$

which immediately yields

$$\|\varphi(t)\|^2 \le \|\varphi_0\|^2\exp(ct).$$

□

Theorem 3. *Under assumptions (H_1), (H_2), (H_3) and (H_4), we have that for each $\varphi_0 \in L^2_+(\Omega)$ with a compact support, equation (1) has one and only one solution, defined on the maximal time interval on which it remains with a compact support. Moreover, the solution is non negative on its domain.*

Proof. Denoting $u(t) = \varphi(t,.)$, where φ is a possible mild solution, we have the following

$$u(t) = T_0(t,0)u(0) + r\int_0^t T_0(t,s)(J(s,.,u(s))u(s))ds \tag{16}$$

$\varphi \to T_0(t,s)\varphi$ is a solution of the linear problem (7). Uniqueness that it entails is an evolution operator on $L^2(\Omega)$ (see [8]). It is also non negative. From (3), it is obvious that the function J takes only non negative values. So, the right hand side of formula (16) is a non negative operator. Given any $\zeta \in [0,T]$, let us consider the convex cone

$$\Gamma = \left\{u, u : [0,\zeta] \to L^2_+(\Omega), \text{ continuous and } \|u(0)\| \le R\right\},$$

Γ is in fact a closed subset of the space of continuous functions from $[0,\zeta]$ into $L^2(\Omega)$, endowed with the usual norm which we will denote $\|.\|$. Given u_0 in $L^2_+(\Omega)$, we consider the map, denoted $\mathcal{G}$, defined by the right hand side of (16) on the set Γ. Clearly, for each $v \in \Gamma$, we have $\mathcal{G}(\nu) \in \Gamma$. On the other hand, we can see that, for any pair ν_1, υ_2 of elements in Γ, we have

$$\|\mathcal{G}(\nu_2)(t) - \mathcal{G}(\nu_1)(t)\| \le C\exp(\alpha t)\sup_{0\le s\le t}\|\nu_2(s) - \nu_1(s)\|,$$

from which we deduce by induction

$$\left\|\mathcal{G}^k(\nu_2)(t) - \mathcal{G}^k(\nu_1)(t)\right\| \le C^k\frac{t^{k-1}}{(k-1)!}\exp(\alpha t)\sup_{0\le s\le t}\|\nu_2(s) - \nu_1(s)\|.$$

This inequality shows that there exists k^* such that, for each $k \ge k^*$, $\mathcal{G}^k$ is a strict contraction from Γ into itself. From this, we conclude that $\mathcal{G}$ has a unique fixed point. This being true for $\zeta \in [0,T]$, we obtain existence and uniqueness of a non negative solution of (16), defined for all $t \in [0, T]$. So, problem (1) has, at least, a mild solution. □

5. **Conclusion.** The phytoplankton has for long generated a lot of interest from various perspectives. With no attempt to being exhaustive, let us mention the following works: on the mathematical side, a typical work is the one by S. Ruan who investigates qualitative features- stability, bifurcations- of delay differential equations, with possibly infinite delay and/or partial differential equations. In principle, the delay is related to nutrient recycling and delayed growth response Beretta et al. [6], S. Ruan [22] (see also J.Wu [30] for a comprehensive presentation of reaction diffusion equations with delay). Restrictions in such models (constant coefficients, no transport) make them far from being adapted to the oceans. There is a category of models which deal only with some physical or chemical aspects of the phytoplankton growth, see for example C. Zonneveld [31, 32] and references therein: these are generally systems of ordinary differential equations from which it is possible to estimate some parameters. Most spatial models are treated by simulations: this in principle allows the consideration of very complicated models. Examples of such studies can be found in articles of P. Franks and his co-workers [19, 20, 17].

In this paper we have considered a three dimensional model for phytoplankton a model where the water column is divided into small layers, inside each of which it is reasonable to assume that the current velocity depends only on the horizontal variables. Firstly we showed that the multilayer model has a unique non negative solution, secondly we investigated the convergence of this approximation to the exact solution of the model.

The model neglects the effect of nutrient concentration gradients. It would be interesting to include these in a model, as they have a substantial effect on the phytoplankton distributions.

REFERENCES

[1] O. Arino, K. Boushaba and A. Boussouar, *A mathematical model of the dynamics of the phytoplankton-nutrient system* , Journal of Non Linear Analysis and Application B, **1** (2000), 69-87.

[2] O. Arino, K. Boushaba and A. Boussouar, *Modelization of the role of currents and turbulence on the growth and dispersion of the marine phytoplankton*, Compte. Rendu de l'Académie des Sciences, **323** (2000), 113-118.

[3] O. Arino, A. Boussouar and P. Prouzet, *Modeling of the larval stage of the anchovy of the Bay of Biscay. Estimation of the rate of recruitment in the juvenile stage* , Final Report, (1999) Project 96/048 DG XIV EC.

[4] R. S. K. Barnes and R. N. Hughes, "An Introduction to Marine Ecology," Blackwell Scientific Publication, 1988.

[5] G. K. Batchelor, "An Introduction to Fluid Dynamics," Cambridge Univ. Press. London, New York, 1967.

[6] E. Beretta, G. I. Bischi and F. Solimano, *Stability in chemostat equations with delayed nutrient recycling*, J. Math. Biol., **28** (1990), 99-111.

[7] K. Boushaba, O. Arino and A.Boussouar, *A Mathematical Model of Phytoplankton*, Mathematical Models and Methods in Applied Sciences, **12** (2002), 871-901.

[8] H. Brezis, "Analyse Fonctionnelles et Application," Masson, 1984.

[9] N. F. Britton, "Reaction-diffusion Equations and Their Application to Biology," London, Academic Press, 1986.

[10] N.F. Britton, U. Timm, *Effects of competition and shading in planktonic communities*, J. Math. Biol., **31** (1993), 655-673.

[11] N.F. Britton, *Spatial structures and periodic traveling waves in an integro-differential reaction-diffusion population model,* SIAM J. Appl. Math., **50** (1990), 1663-1688.

[12] R. Dautray and J. L. Lions, "Analyse Mathématique et Calcul Numérique," Masson, 1985.
[13] K. Deimling, "Nonlinear Functinal Analysis," Springer-Verlag, 1985.
[14] E. DiBenedetto, "Partial Differential Equations," Birkhäuser Boston, Basel, Berlin, 1995.
[15] M. D. Frague, *Reducibilité des systèmes héréditaire à des systèmes dynamiques*, C. Acad. Sci. Paris. Sér. B, **277** (1973), 471-473.
[16] P. S. Franks, *Models of harmful lagal blooms*, Limnol. Oceanogr., **42** (1997), 1273-1282.
[17] P. J. S. Franks, Leonardo J Walstad, *Phytoplankton pathes at fronts : A model of formulation and response to wind events*, J. Mar. Res., **55** (1997), 1-29.
[18] P. J. S. Franks, Sink or swim, *Accumulation of biomass at fronts*, Mar. Ecol. Prog. Ser., **82** (1992), 1-12.
[19] P. J. S. Franks and C. Chen, *Plankton in tidal fronts : A model of Georges Bank in summer*, J. Mar. Res., **54** (1996), 631-651.
[20] P. J. S. Franks, J. Marra, *A simple new formulation for Phytoplankton photoresponse and an application in wind-driven mixed-Layer model*, Mar. Ecol. Prog. Ser., **11** (1994), 143-153.
[21] P. Lazure and A.M.Jegou, *3D-modeling of seasonal evolution of Loire and Gironde plumes on Biscay bay continental shelf*, to appear in Oceanologia Acta.
[22] S. Ruan, *Global Stability in chemostat-type competition models with nutrient recycling*, SIAM J. Appl. Math., **58**(1998), 170-192.
[23] N. Shgesada and A.Okubo, *Analysis of self-shading effect on algal vertical distribution in the natural waters*, J. Math. Biol, **12** (1981), 311-326.
[24] S. A. Levin and Pacala, S.W., *Theories of simplification and scaling in ecological systems*, in Spatial Ecology: " The Role of Space in Population of Dynamics and Interspecific Interactions" (eds. D. Tilman and P. Kareiva), Princeton University Press, Princeton, NJ, 1997.
[25] J. D. Murray, "Mathematical Biology," Springer-Verlag, Berlin, 1989.
[26] Nagel R. et al., "One-parameter Semigroups of Positve Operators," Springer-Verlag, 1986.
[27] A. Okubo, "Diffusion and Ecological Problems: Mathematical Models," Springer-Verlag, Berlin, 1980.
[28] A.Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, 1983.
[29] J. Pedlosky, "Geophysical Fluid Dynamics," Springer-Verlag, 1979.
[30] J. Wu, "Theory and Applications of Partial Functional Differential Equations," Springer-Verlag, New York, 1996.
[31] Zonneveld C., *A cell-based model for chlorophyll a to carbon ratio in phytoplankton*, Ecol. Modelling, 113(1-3) (1998), 55-70.
[32] Zonneveld C., *Modeling Effects of Photadaptation on the Photosynthesis-Irradiance Curve* , J. theor. Biol., **186** (1997), 381-388.

Received July 2006; revised December 2006.

E-mail address: boushaba@iastate.edu