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MULTISCALE STOCHASTIC HOMOGENIZATION OF MONOTONE OPERATORS

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Abstract. Multiscale stochastic homogenization is studied for divergence structure parabolic problems. More specifically we consider the asymptotic behaviour of a sequence of realizations of the form

$$\frac{\partial u_\varepsilon^\omega}{\partial t} - \operatorname{div}\left(a\left(T_1(\frac{x}{\varepsilon_1})\omega_1, T_2(\frac{x}{\varepsilon_2})\omega_2, t, Du_\varepsilon^\omega\right)\right) = f.$$

It is shown, under certain structure assumptions on the random map $a(\omega_1, \omega_2, t, \xi)$, that the sequence $\{u_\varepsilon^\omega\}$ of solutions converges weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ to the solution u of the homogenized problem $\frac{\partial u}{\partial t} - \operatorname{div}(b(t, Du)) = f$.

1. **Introduction.** In this paper we consider the homogenization problem for the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u_\varepsilon^\omega}{\partial t} - \operatorname{div}\left(a\left(T_1(\frac{x}{\varepsilon_1})\omega_1, T_2(\frac{x}{\varepsilon_2})\omega_2, t, Du_\varepsilon^\omega\right)\right) = f \text{ in } \Omega\times(0,T), \\ u_\varepsilon^\omega(x,0) = u_0(x) \text{ in } \Omega, \\ u_\varepsilon^\omega(x,t) = 0 \text{ in } \partial\Omega\times(0,T), \end{cases} \tag{1}$$

where Ω is an open bounded set in $\mathbb{R}^n$ and T is a positive real number. We associate two probability spaces $(X_k, \mathcal{F}_k, \mu_k)$, $k = 1, 2$. Each $\mathcal{F}_k$ is a complete σ-algebra and each μ_k is the associated probability measure. We assume that for each $x \in \mathbb{R}^n$, X_k is acted on by the dynamical system

$$T_k(x) : X_k \to X_k$$

We also assume that ε_1 and ε_2 are two well separated functions (scales) of $\varepsilon > 0$ which converge to zero as ε tends to zero. Well separatedness means

$$\lim_{\varepsilon\to 0} \frac{\varepsilon_2}{\varepsilon_1} = 0.$$

Further we assume that the map $a = a(\omega_1, \omega_2, t, \xi)$ is monotone and continuous and satisfies certain coercivity and growth conditions in ξ and is measurable in (ω_1, ω_2, t). With these structure conditions (to be precisely defined below) it is well-known that for given data $f \in L^q(0,T;W^{-1,q}(\Omega))$ and $u_0 \in L^2(\Omega)$ there exists a unique solution $u_\varepsilon^\omega \in L^p(0,T;W_0^{1,p}(\Omega))$ to (1) with time derivative $\frac{\partial u_\varepsilon^\omega}{\partial t} \in L^q(0,T;W^{-1,q}(\Omega))$ for

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every fixed $\varepsilon > 0$ and almost all $(\omega_1, \omega_2) \in X_1 \times X_2$, where p and q are the dual exponents.

The multiscale stochastic homogenization problem for (1) consists in studying the asymptotic behavior of the solutions u_ε^ω as ε tends to zero.

Homogenization problems with more than one oscillating scale is in the periodic setting was first introduced in [2] for linear elliptic problems. Recently the monotone elliptic case has been studied in [7].

In the present report we prove a stochastic homogenization theorem (Theorem 7) for the corresponding monotone parameter dependent elliptic problem to (1) and use this result and a comparison result to prove the homogenization of (1).

We will introduce the reader to the general framework of G-convergence, which can be thought of as a non-periodic "homogenization" or stabilization of sequences of operator equations. Here we show that the general theory also applies to the situation of multiple scales and multiscale stochastic homogenization. The result of Theorem 8 is that the sequence of solutions $\{u_\varepsilon\}$ to (1) converges weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ to the solution u in $L^p(0,T;W_0^{1,p}(\Omega))$ to a homogenized problem of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\ (b\,(t, Du)) = f \ \text{ in } \Omega \times (0,T), \\ u(x,0) = u_0(x) \ \text{ in } \Omega, \\ u(x,t) = 0 \ \text{ in } \partial\Omega \times (0,T), \end{cases} \tag{2}$$

where b depends on t but is no longer oscillating in space with ε. A motivation for the present work is that it is open to homogenize structures which have periodic oscillations in some scales and random oscillations in other. The statement and proof of Theorem 7 below is very explicit and is meant to be a basic tool result for homogenization of multiscale structures. It is also seen from the framework that the result easily extends to any number of well separated scales. A typical situation where periodic and random scales occur is the modeling of porous media. A meso-scale can be modeled as a periodic distribution of solid parts whereas a sub-scale on a finer level can be modeled by a certain random distribution. The homogenization problem for random fields in the linear elliptic case is studied in [6] The extension to monotone operators in the random setting is studied in [8] and has been further studied in a series of papers by Efendiev and Pankov, see [5] and the references therein. They consider single spatial and temporal scales. The paper is organized as follows: In Section 2 we recall the basic terminology of G-convergence of parabolic operators, in Section 3 we introduce some basic facts about monotone operators in reflexive Banach Spaces. Sections 4, 5 and 6 review some basic results for elliptic and parabolic G-convergence that turn out to be very useful in the proof of Theorem 7 and 8. Section 7 is a preparation on multiscale stochastic operators where the framework is based on a dynamical systems setup. The main results Theorems 7 and 8 are stated and proved in Section 8.

2. **General setting - G-convergence.** We study the asymptotic behaviour (as $h \to +\infty$) for a sequence of initial-boundary value problems of the form

$$\begin{cases} u_h' - \text{div}(a_h(x,t,Du_h)) = f \ \text{ in } \ \Omega \times (0,T), \\ u_h(0) = u_0 \ \text{ in } \Omega, \\ u_h \in \mathrm{L}^p(0,\mathrm{T};\mathrm{W}_0^{1,p}(\Omega)), \end{cases} \tag{3}$$

where Ω is an open bounded set in $\mathbb{R}^n$, T is a positive real number and $2 \le p < \infty$. The maps a_h are assumed to be monotone and to satisfy certain boundedness and coerciveness assumptions uniformly with respect to h. We will see that there exists a subsequence still denoted by $\{a_h\}$ and a map a with the same qualitative properties as the maps $\{a_h\}$ such that, as $h \to \infty$,

$$u_h \to u \text{ weakly in } \mathrm{L}^p(0,\mathrm{T};\mathrm{W}_0^{1,p}(\Omega))$$
and
$$a_h(x,t,Du_h) \to a(x,t,Du) \text{ weakly in } \mathrm{L}^q(0,\mathrm{T};\mathrm{L}^q(\Omega;\mathbb{R}^n)),$$

respectively, where $1/p + 1/q = 1$ and where u is the solution of the following initial-boundary value problem:

$$\begin{cases} u' - \operatorname{div}(a(x,t,Du)) = f \text{ in } \Omega \times (0,T) \\ u(0) = u_0 \text{ in } \Omega, \\ u \in \mathrm{L}^p(0,\mathrm{T};\mathrm{W}_0^{1,p}(\Omega)), \end{cases} \tag{4}$$

where the map a only depends on the subsequence $\{a_h\}$. This yields G-convergence of quasilinear parabolic operators. For a complete treatment of G-convergence of monotone parabolic operators we refer to [10] or [11].

3. **Some notations.** Let us introduce some function spaces related to the differential equations studied in this paper. For a nice introduction to partial differential operators in Banach spaces. we refer to the monograph [1] by Barbu. Let V be a reflexive real Banach space, with dual V′ and let H be a real Hilbert space. We introduce the evolution triple.

$$\mathrm{V} \subseteq \mathrm{H} \subseteq \mathrm{V}',$$

with dense embeddings. Further, for positive real-valued T and for $2 \le p < \infty$ let us introduce the spaces $\mathcal{V} = \mathrm{L}^p(0,\mathrm{T};\mathrm{V})$, $\mathcal{H} = \mathrm{L}^2(0,\mathrm{T};\mathrm{H})$ and $\mathcal{V}' = \mathrm{L}^q(0,\mathrm{T};\mathrm{V}')$, where $1/p + 1/q = 1$. Then we can consider the corresponding evolution triple

$$\mathcal{V} \subseteq \mathcal{H} \subseteq \mathcal{V}'$$

also with dense embeddings where the duality pairing $\langle \cdot,\cdot \rangle_{\mathcal{V}}$ between $\mathcal{V}$ and $\mathcal{V}'$ is given by

$$\langle f,u \rangle_{\mathcal{V}} = \int_0^T \langle f(t),u(t) \rangle_V \, dt, \text{ for } u \in \mathcal{V},\ f \in \mathcal{V}'.$$

We define the spaces $\mathcal{W}$ and $\mathcal{W}_0$ as

$$\mathcal{W} = \{v \in \mathcal{V} : v' \in \mathcal{V}'\} \text{ and } \mathcal{W}_0 = \{v \in \mathcal{W} : v(0) = 0\}.$$

Here v' denotes the time derivative of v, where this derivative is taken in distributional sense. Equipped with the graph norm

$$\|v\|_{\mathcal{W}_0} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}'}$$

$\mathcal{W}_0$ becomes a real reflexive Banach space. Moreover, since the embedding $\mathcal{W}_0 \to C(0,\mathrm{T};\mathrm{H})$ is continuous, every function in $\mathcal{W}_0$, with possible modification on a set of measure zero, can be considered as a continuous function with values in H. Let us define the operator $\frac{d}{dt} : \mathcal{V} \to \mathcal{V}'$ given by

$$\frac{d}{dt}u = u' \text{ for } u \in D(\frac{d}{dt}) = \mathcal{W}_0.$$

We will denote by Ω a bounded open set in $\mathbb{R}^n$ and, if nothing else is said, we denote by $\mathrm{V} = \mathrm{W}_0^{1,p}(\Omega)$ with norm $\|u\|_V^p = \int_\Omega |Du|^p \, dx$, $\mathrm{H} = \mathrm{L}^2(\Omega)$ and $\mathrm{V}' = \mathrm{W}^{-1,q}(\Omega)$. Then the evolution triples considered above are well-defined with dense embeddings. We define the spaces

$$\mathrm{U} = \mathrm{L}^p(\Omega;\mathbb{R}^n) \text{ and } \mathrm{U}' = \mathrm{L}^q(\Omega;\mathbb{R}^n)$$

and the spaces

$$\mathcal{U} = \mathrm{L}^p(0, \mathrm{T}; \mathrm{U}) \text{ and } \mathcal{U}' = \mathrm{L}^q(0, \mathrm{T}; \mathrm{U}')$$

Further, we define the pairing $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ between $\mathcal{U}'$ and $\mathcal{U}$ as

$$\langle u, v \rangle_{\mathcal{U}} = \int_0^T \int_\Omega (u, v)\, dxdt, \quad \text{for } u \in \mathcal{U}' \text{ and } v \in \mathcal{U},$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^n$. By $|\cdot|$ we understand the usual Euclidean norm in $\mathbb{R}^n$. Moreover, by $\{h\}$ we understand a sequence in $\mathbb{N}$ tending to $+\infty$.

4. **Parabolic G-convergence.**

Definition 1. Given $0 < \alpha \leq 1$, $2 \leq p < \infty$ and three positive real constants c_0, c_1 and c_2, we define the class $S = S(c_0, c_1, c_2, \alpha)$ of maps

$$a : \Omega \times]0, \mathrm{T}[\times \mathbb{R}^n \to \mathbb{R}^n,$$

satisfying

(i) $|a(x, t, 0)| \leq c_0$ a.e in $\Omega \times (0, T)$.

(ii) $a(\cdot, \cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^n$.

(iii) $|(a(x, t, \xi_1) - a(x, t, \xi_2)| \leq c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha$, a.e. in $\Omega \times (0, T)$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$.

(iv) $(a(x, t, \xi_1) - a(x, t, \xi_2), \xi_1 - \xi_2) \geq c_2 |\xi_1 - \xi_2|^p$, a.e. in $\Omega \times (0, T)$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$.

Let us define the operator $A : \mathcal{V} \to \mathcal{U}'$ as

$$Au = (a(x, t, Du(x, t))). \tag{5}$$

Then, by considering a sequence $\{A_h\}$ of operators of this form, the sequence of problems (3) can be written in the following equivalent form

$$\begin{cases} u_h' - \operatorname{div}(A_h u_h) = f \\ u_h \in \mathcal{W}_0. \end{cases} \tag{6}$$

Further by defining $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ via

$$\mathcal{A}u = -\operatorname{div}(Au), \tag{7}$$

One notices that, by considering a sequence $\{\mathcal{A}_h\}$ of operators of this form, the sequence of parabolic initial-boundary value problems (3) can also be written as a sequence of abstract evolution equations in the form

$$\begin{cases} u_h' + \mathcal{A}_h u_h = f \\ u_h \in \mathcal{W}_0. \end{cases} \tag{8}$$

We define "parabolic" G-convergence in the following way:

Definition 2. The sequence $\{a_h\}$ of maps , corresponding to the sequence of problems (3) is said to G-converge to a map a if, for every $f \in \mathcal{V}'$, the solutions u_h of (3) satisfy

$$\begin{aligned} &u_h \to u \text{ weakly in } \mathcal{V} \\ &\text{and} \\ &a_h(x, t, Du_h) \to a(x, t, Du) \text{ weakly in } \mathcal{U}', \end{aligned}$$

respectively, where u is the unique solution of the problem

$$\begin{cases} u' + \mathcal{A}u = f \\ u \in \mathcal{W}_0 \end{cases} \tag{9}$$

Proposition 1. *Assume that the maps* $a_h \in S(c_0, c_1, c_2, \alpha)$. *Then the sequence of problems* (3) *admits unique solutions* $u_h \in \mathcal{W}_0$ *for every* $f \in \mathcal{V}'$ *and for every* $h \in \mathbb{N}$.

The following G-compactness result is proved in [10]. See also [11].

Theorem 1. *Consider the sequence* $\{a_h\}$ *of maps corresponding to the sequence of problems* (3). *Assume that* $\{a_h\} \subset S(c_0, c_1, c_2, \alpha)$. *Then, there exists a subsequence still denoted by* $\{a_h\}$ *and a map* a *such that* $\{a_h\}$ *G-converges to the map* $a \in S(\tilde{c_0}, \tilde{c_1}, c_2, \gamma)$, *where* $\tilde{c_0}$ *and* $\tilde{c_1}$ *are positive constants depending only on the constants* p, c_0, c_1, c_2, α, *which appear in the structure conditions and where* $\gamma = \alpha/(p - \alpha)$.

5. **Elliptic G-convergence.** For a complete treatment of a large class of (possibly multivalued) elliptic operators we refer to [3] and [4]. We consider the following sequence of Dirichlet boundary value problems

$$\begin{cases} -\mathrm{div}(a_h(x, Du_h)) = f_h, \text{ in } \Omega \\ u_h \in V. \end{cases} \tag{10}$$

Definition 3. The sequence $\{a_h\}$ of maps is said to *G-converge*, in the elliptic sense, to a map a if, for every $f \in \mathcal{V}'$, the sequence of solutions u_h of (10) satisfy

$$\begin{aligned} &u_h \to u \text{ weakly in V} \\ &\text{and} \\ &a_h(x, Du_h) \to a(x, Du) \text{ weakly in U}', \end{aligned}$$

respectively, where u is the unique solution to the problem

$$\begin{cases} -\mathrm{div}(a(x, Du)) = f \\ u \in \mathrm{V}. \end{cases} \tag{11}$$

The following elliptic G-compactness result holds true:

Theorem 2. *Suppose that the sequence* $\{a_h\}$ *of maps belongs to* S^E. *Then there exists a subsequence, still denoted by* $\{h\}$, *such that, for every* $f \in \mathrm{V}'$, *the sequence* $\{a_h\}$ *G-converges to a map* $a \in S^E$.

Proof. See [4]. □

Here S^E denotes the subclass of the class S of maps which do not depend on t.

6. **Parameter-dependent elliptic G-convergence.** We begin by stating a compactness result with respect to elliptic G-convergence for parameter dependent elliptic problems:

Theorem 3. *Suppose that the sequence* $\{a_h\}$ *belongs to* S. *Suppose, in addition, that*

$$|a_h(x, t, \xi) - a_h(x, s, \xi)| \leq B(t - s)(1 + |\xi|^{p-1})$$

for every $\xi \in \mathbb{R}^n$ *and for every* $t \in (0, T)$ *a.e. in* Ω, *where* $B : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing function which is continuous vanishes at the origin. Then, there exists a subsequence still denoted by* $\{h\}$, *such that* $\{a_h(\cdot, t, \cdot)\}$ *G-converges in the elliptic sense to a map* $a(\cdot, t, \cdot)$ *for every* $t \in (0, T)$.

Proof. See [10] or [11]. □

We recall an important comparison result which will be addressed in the proof of Theorem 8.

Theorem 4. *Suppose that*

$$|a_h(x,t,\xi) - a_h(x,s,\xi)| \le B(t-s)(1+|\xi|^{p-1})$$

for all $\xi \in \mathbb{R}^n$ *a.e. in* Ω*, where* $B:\mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing function which is continuous and zero at* 0*. Suppose that* $\{a_h\}$ *G-converges to a in the parabolic sense and that* $\{a_h(\cdot,t,\cdot)\}$ *G-converges to* $b(\cdot,t,\cdot)$ *in the elliptic sense, for every* $t \in (0,T)$*, then* $a = b$*.*

Proof. See [10] or [11]. □

7. **A dynamical systems approach to stochastic multiscale analysis.** For a nice exposition of the framework below we refer to the monograph [6]. Let $(X, \mathcal{F}, \mu)$ denote a probability space, where $\mathcal{F}$ is a complete σ-algebra and μ is a probability measure. We assume that for each $x \in \mathbb{R}^n$, X is acted on by the dynamical system

$$T(x): X \to X$$

where both $T(x)$ and $T(x)^{-1}$ are assumed to be measurable. Moreover we assume that the following (measure preserving) properties are satisfied:

- $T(0)\omega = \omega$ for each $\omega \in X$.
- $T(x+y) = T(x)T(y)$ for $x, y \in \mathbb{R}^n$.
- $\mu(T(x)^{-1}F) = \mu(F)$, for each $x \in \mathbb{R}^n$ and $F \in \mathcal{F}$.
- The set $\{(x,\omega) \in \mathbb{R}^n \times X : T(x)\omega \in F\}$ is a $dx \times d\mu(\omega)$ measurable subset of $\mathbb{R}^n \times X$ for each $F \in \mathcal{F}$ where dx denotes the Lebesgue measure.
- For any measurable function $f(\omega)$ defined on X, the function $f(T(x)\omega)$ defined on $\mathbb{R}^n \times X$ is also measurable where $\mathbb{R}^n$ is endowed with the Lebesgue measure.

The dynamical system is said to be *ergodic* if every invariant function f, (i.e functions f which satisfies $f(T(x)\omega) = f(\omega)$) is constant almost everywhere in X.

Definition 4. We say that a vector field $f \in [L^p_{loc}(\mathbb{R}^n)]^n$ is a potential field if there exists a function $g \in W^{1,p}_0(\mathbb{R}^n)$ such that $f = Dg$.

Definition 5. We say that a vector field $f \in [L^p(X)]^n$ is a potential field if almost all its realizations are potential fields. We denote this field by L^p_{pot}.

Definition 6. We futher define the space of vector fields with mean value zero.

$$V_{pot}(X) = \{f \in [L^p(X)]^n : \int_X f(\omega)\, d\mu(\omega) = 0\}.$$

We observe that by the Fubini Theorem it follows that if $f \in L^p(X)$ then almost all realizations $f(T(x)\omega) \in L^p_{loc}(\mathbb{R}^n)$.

Definition 7. Let $f \in L^1_{loc}(\mathbb{R}^n)$. The number $M(f)$ is called the mean value of f if

$$\lim_{\epsilon \to 0} \int_K f(x/\epsilon)\, dx = |K| M(f)$$

for any Lebesgue measurable bounded set $K \in \mathbb{R}^n$. Alternatively the mean can be expressed in terms of weak convergence. If the family $\{f(\cdot/\epsilon)\}$ is in $L^p(X)$, $p \geq 1$ then $M(f)$ is called the mean value of f if

$$\{f(\cdot/\epsilon)\} \rightharpoonup M(f) \ \ in \ \ L^p(X).$$

We can now formulate the following important:

Theorem 5. *(Birkhoff Ergodic Theorem) Let $f \in L^p(X)$, $p \geq 1$. Then for almost all $\omega \in X$ the realization $f(T(x)\omega)$ possesses a mean value $M(f(T(x)\omega))$. Moreover, as a function of $\omega \in X$, this mean value $M(f(T(x)\omega))$ is invariant and*

$$\int_X f(\omega)\, d\mu(\omega) = \int_X M(f(T(x)\omega))\, d\mu(\omega).$$

If the system $T(x)$ is ergodic then

$$\int_X f(\omega)\, d\mu(\omega) = M(f(T(x)\omega)).$$

Now let $\{(X_k, \mathcal{F}_k, \mu_k)\}_{k=1}^M$ denote a family of probability spaces, where each $\mathcal{F}_k$ is a complete σ-algebra and each μ_k is the associated probability measure. We assume that for each $x \in \mathbb{R}^n$, X_k is acted on by the dynamical system

$$T_k(x) : X_k \to X_k$$

We also formulate the following multiscale extension of Theorem 5:

Theorem 6. *Let $f \in L^p(X_1\times \ldots \times X_M)$, $p \geq 1$. Then for almost all $\omega_k \in X_k$ the realization $f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)$ possesses a mean value $M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))$. Moreover, as a function of $\omega_k \in \Omega_k$, this mean value $M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))$ is invariant and*

$$\langle f \rangle = \int_{X_1} \cdots \int_{X_M} f(\omega_1, \ldots, \omega_M)\, d\mu_1(\omega_1) \ldots d\mu_M(\omega_M) =$$

$$\int_{X_1} \cdots \int_{X_M} M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))\, d\mu_1(\omega_1) \ldots d\mu_M(\omega_M).$$

If the systems $T_k(x)$ are ergodic then

$$\langle f \rangle = M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)).$$

Remark 1. Efendiev and Pankov, see [5] p. 238-239, use another approach where they define an $(n+1)$-parameter space-time action by a dynamical system T_{n+1} acting on a probability space Ω. Their approach will not be scutinized here since in the present work an important aspect is the possibility to allow homogenization where some processes may be periodic and other may be random.

8. **An application to homogenization.** We are interested in the asymptotic behaviour (as $\varepsilon_i \to 0$, $i = 1, 2$) of the following sequence of initial-boundary value problems

$$\begin{cases} (u_\varepsilon^\omega)' - \operatorname{div}(a(T_1(\frac{x}{\varepsilon_1})\omega_1, T_2(\frac{x}{\varepsilon_2})\omega_2, t, Du_\varepsilon^\omega)) = f_\varepsilon, \ \text{ in } \ \Omega \times (0, T), \\ u_\varepsilon^\omega(0) = u_0^\omega, \ \text{ in } \ \Omega, \\ u_\varepsilon^\omega \in \mathrm{L}^p(0, \mathrm{T}; \mathrm{W}_0^{1,p}(\Omega)), \end{cases} \tag{12}$$

where $\varepsilon \in \mathbf{E}$ where $\mathbf{E}$ is a decreasing sequence of positive real numbers tending to zero. Ω is an open bounded set in $\mathbb{R}^n$, T is a positive real number, $2 \leq p < \infty$.

The idea now is to first study the homogenization problem for the corresponding elliptic problem and then use the comparison results from above. We begin by setting the appropriate structure conditions:

Definition 8. Let $(X_k, \mathcal{F}_k, \mu_k)$, $k = 1, 2$, be two probability spaces. Given $0 < \alpha \leq 1$, $2 \leq p < \infty$ and three positive real constants c_0, c_1 and c_2, we define the class $S^\omega = S^\omega(c_0, c_1, c_2, \alpha)$ of maps

$$a : X_1 \times X_2 \times]0, \mathrm{T}[\times \mathbb{R}^n \to \mathbb{R}^n,$$

satisfying

(i) $|a(\omega_1, \omega_2, t, 0)| \leq c_0$ a.e in $X_1 \times X_2 \times (0, T)$.

(ii) $a(\cdot, \cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^n$.

(iii) $|(a(\omega_1, \omega_2, t, \xi_1) - a(\omega_1, \omega_2, t, \xi_2)| \leq c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha}|\xi_1 - \xi_2|^\alpha$, a.e. in $X_1 \times X_2 \times (0, T)$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$.

(iv) $(a(\omega_1, \omega_2, t, \xi_1) - a(\omega_1, \omega_2, t, \xi_2), \xi_1 - \xi_2) \geq c_2|\xi_1 - \xi_2|^p$, a.e. in $X_1 \times X_2 \times (0, T)$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$.

Let us define the operator $A^\omega_\epsilon : \mathcal{V} \to \mathcal{U}'$ as

$$A^\omega_\epsilon(x, t, \xi) = a(T_1(\frac{x}{\varepsilon_1})\omega_1, T_2(\frac{x}{\varepsilon_2})\omega_2, t, \xi). \tag{13}$$

With some abuse of notation we will say that A^ω_ϵ belongs to S^ω if the corresponding map a does. Then (12) can be written in the equivalent form

$$\begin{cases} (u^\omega_\epsilon)' - \operatorname{div}(A^\omega_\epsilon(x, t, Du^\omega_\epsilon)) = f \\ u^\omega_\epsilon \in \mathcal{W}_0. \end{cases} \tag{14}$$

It is a standard result in the theory of monotone operators that (14) possesses a unique weak solution for a.e. $(\omega_1, \omega_2) \in X_1 \times X_2$.

Theorem 7. *Let us consider the sequence of parameter dependent elliptic boundary value problems*

$$\begin{cases} -\operatorname{div}(A^\omega_\epsilon(x, t, Du^\omega_\epsilon)) = f \ \ in \ \ \Omega \\ u^\omega_\epsilon(\cdot, t) \in W^{1,p}_0(\Omega), \ \ t \in [0, T]. \end{cases} \tag{15}$$

Assume that $A^\omega_\epsilon \in S^\omega$ *and that*

$$|A^\omega_\epsilon(x, t, \xi) - A^\omega_\epsilon(x, s, \xi)| \leq B(t - s)(1 + |\xi|^{p-1})$$

where B *is the modulus of continuity function. Also assume that the underlying dynamical systems* $T_1(x)$ *and* $T_2(x)$ *are ergodic and that the realizations* $T_1(x)\omega_1$ *and* $T_2(x)\omega_2$ *are measurable. Then*

$$u^\omega_\epsilon(\cdot, t) \rightharpoonup u \ \ in \ \ W^{1,p}_0(\Omega)$$

and

$$A^\omega_\epsilon(\cdot, t, Du^\omega_\epsilon) \rightharpoonup b(t, Du) \ \ in \ \ [L^q(\Omega)]^n$$

where u *is the solution to the homogenized problem*

$$\begin{cases} -\operatorname{div}(b(t, Du)) = f \ \text{ in } \ \Omega, \\ u(\cdot, t) \in W^{1,p}_0(\Omega), \ \ t \in [0, T]. \end{cases} \tag{16}$$

The operator b *is defined as*

$$b(t, \xi) = \int_{X_1} b_1(\omega_1, t, \xi + z^\xi_1(\omega_1, t))\, d\mu_1(\omega_1)$$

where $z_1^\xi(\omega_1, t) \in V_{pot}(X_1)$ *is the solution to the* ϵ_1*-scale local problem*

$$\langle b_1(\omega_1, t, \xi + z_1^\xi(\omega_1, t), \Phi_1(\omega_1)\rangle = 0$$

for all $\Phi_1(\omega_1) \in V_{pot}(X_1)$*,* $t \in [0, T]$*. The operator* b_1 *is defined as*

$$b_1(\omega_1, t, \xi) = \int_{X_2} a(\omega_1, \omega_2, t, \xi + z_2^\xi(\omega_1, \omega_2, t))\, d\mu_2(\omega_2)$$

where $z_2^\xi(\omega_1, \omega_2, t) \in V_{pot}(X_2)$ *is the solution to the* ϵ_2*-scale local problem*

$$\langle a(\omega_1, \omega_2, t, \xi + z_2^\xi(\omega_1, \omega_2, t), \Phi_2(\omega_2)\rangle = 0$$

for all $\Phi_2(\omega_2) \in V_{pot}(X_2)$ *a.e.* $\omega_1 \in X_1$*,* $t \in [0, T]$*.*

Proof. By the structure conditions (Definition 8) it follows that for every (ϵ_1, ϵ_2) there exists a unique solution

$$u_\epsilon^\omega(\cdot, t) \in W_0^{1,p}(\Omega), \quad t \in [0, T]$$

for a.e. $(\omega_1, \omega_2) \in X_1 \times X_2$.

By the structure conditions and by the reflexivity of $W_0^{1,p}(\Omega))$ and $[L^p(\Omega)]^n$ it follows that (up to subsequences)

$$u_\epsilon^\omega \rightharpoonup u^* \ \ in \ \ \mathcal{W}_0$$

and

$$A_\epsilon^\omega(x, t, Du_\epsilon^\omega) \rightharpoonup \xi^* \ \ in \ \ \mathcal{U}'.$$

The rest of the proof aims at verifying that $\xi^* = b(t, Du^*)$ and at the same time characterizing b explicitly. We start out by focusing on the "fastest" ϵ_2-process. Let us fix $\xi \in \mathbb{R}^n$. For a.e $\omega_1 \in X_1$ we let $z_2^\xi(\omega_1, \omega_2, t) \in V_{pot}(X_2)$ be the solution to

$$\langle a(\omega_1, \omega_2, t, \xi + z_2^\xi(\omega_1, \omega_2, t), \Phi_2(\omega_2)\rangle = 0$$

for all $\Phi_2(\omega_2) \in V_{pot}(X_2)$, $t \in [0, T]$. The existence and uniqueness of solution $z_2^\xi(\omega_1, \omega_2, t) \in V_{pot}(X_2)$ follow by a Weyl decomposition type argument, see [6] p. 228-229. Consider the realization

$$v_2^{\xi,\omega_2}(\omega_1, x, t) = z_2^\xi(\omega_1, T_2(x)\omega_2, t).$$

By the definition of $V_{pot}(X_2)$ there exists a function $q_2^{\xi,\omega_2} \in W_{loc}^{1,p}(\mathbb{R}^n)$ such that $v_2^{\xi,\omega_2} = Dq_2^{\xi,\omega_2}$ for a.e $\omega_2 \in X_2$, $t \in [0, T]$. Let us now define

$$w_{\epsilon_2}^{\xi,\omega_2} = \langle \xi, x\rangle + \epsilon_2\, q_2^{\xi,\omega_2}(\omega_1, \frac{x}{\epsilon_2}, t).$$

By construction

$$Dw_{\epsilon_2}^{\xi,\omega_2} = \xi + Dq_2^{\xi,\omega_2}(\omega_1, \frac{x}{\epsilon_2}, t).$$

By the Birkhoff ergodic theorem and by the properties of $V_{pot}(X_2)$ keeping in mind that T_2 is ergodic we have

$$\int_{X_2} z_2^\xi\, d\mu_2(\omega_2) = 0$$

and hence, as $\epsilon \to 0$,

$$Dw_{\epsilon_2}^{\xi,\omega_2} \rightharpoonup \xi \ \ in \ \ [L^p(\Omega)]^n.$$

Next we consider the realizations

$$a(\omega_1, T_2(x)\omega_2, t, \xi + z_2^\xi(\omega_1, T_2(x)\omega_2, t)).$$

By the structure conditions (Definition 8)

$$a(\omega_1,\omega_2,t,\xi+z_2^\xi(\omega_1,\omega_2,t))\in L^p_{loc}(X_1\times X_2)$$

for every $t\in[0,T]$, and an application of Birkhoff ergodic theorem yields

$$a(\omega_1,T_2(\frac{x}{\epsilon_2})\omega_2,t,\xi+z_2^\xi(\omega_1,T_2(\frac{x}{\epsilon_2})\omega_2,t))\rightharpoonup b_1(\omega_1,t,\xi)=$$

$$\int_{X_2} a(\omega_1,\omega_2,t,\xi+z_2^\xi(\omega_1,\omega_2,t))\,d\mu_2(\omega_2)\ \ in\ \ [L^q(\Omega)]^n.$$

We proceed by solving for the ϵ_1-process. Let us again fix $\xi\in\mathbb{R}^n$. Let $z_1^\xi(\omega_1,t)\in V_{pot}(X_1)$ be the solution to

$$\langle b_1(\omega_1,t,\xi+z_1^\xi(\omega_1,t),\Phi_1(\omega_1)\rangle=0$$

for all $\Phi_1(\omega_1)\in V_{pot}(X_1)$ for a.e $\omega_1\in X_1$, $t\in[0,T]$. The existence and uniqueness of solution $z_1^\xi(\omega_1,,t)\in V_{pot}(X_1)$ follow by the same arguments as for z_2 above. Consider now the realization

$$v_1^{\xi,\omega_1}(x,t)=z_1^\xi(T_1(x)\omega_1,t).$$

By the definition of $V_{pot}(X_1)$ there exists a function $q_1^{\xi,\omega_1}\in W^{1,p}_{loc}(\mathbb{R}^n)$ such that $v_1^{\xi,\omega_1}=Dq_1^{\xi,\omega_1}$ for a.e $\omega_1\in X_1$, $t\in[0,T]$. Let us now define

$$w_{\epsilon_1}^{\xi,\omega_1}=\langle\xi,x\rangle+\epsilon_1 q_1^{\xi,\omega_1}(\frac{x}{\epsilon_1},t).$$

By construction

$$Dw_{\epsilon_1}^{\xi,\omega_1}=\xi+Dq_1^{\xi,\omega_1}(\frac{x}{\epsilon_1},t).$$

By the Birkhoff ergodic theorem and by the properties of $V_{pot}(X_1)$ keeping in mind that T_1 is ergodic we obtain

$$\int_{X_1} z_1^\xi\,d\mu_1(\omega_1)=0$$

and hence, as $\epsilon\to0$,

$$Dw_{\epsilon_1}^{\xi,\omega_1}\rightharpoonup\xi\ \ in\ \ [L^p(\Omega)]^n.$$

Next we consider the realizations

$$b_1(T_1(x)\omega_1,t,\xi+z_1^\xi(T_1(x)\omega_1,t)).$$

By the structure conditions

$$b_1(\omega_1,t,\xi+z_1^\xi(\omega_1,t))\in L^p_{loc}(X_1)$$

for every $t\in[0,T]$. and an application of Birkhoff ergodic theorem yields

$$b_1(T_1(\frac{x}{\epsilon_1})\omega_1,t,\xi+z_1^\xi(T_1(\frac{x}{\epsilon_1})\omega_1,t))\rightharpoonup b(t,\xi)=$$

$$\int_{X_1} b_1(\omega_1,t,\xi+z_1^\xi(\omega_1,t))\,d\mu_1(\omega_1)\ \ in\ \ [L^q(\Omega)]^n.$$

Let us now combine the two steps and define the perturbed test function

$$w_\epsilon^\omega=\langle\xi,x\rangle+\epsilon_1\,q_1^\xi(\frac{x}{\epsilon_1},t)+\epsilon_2\,q_2^{\xi,\omega_1}(\omega_1,\frac{x}{\epsilon_2},t).$$

By construction

$$Dw_\epsilon^\omega=\xi+Dq_1^\xi(\frac{x}{\epsilon_1},t)+Dq_2^{\xi,\omega_1}(\frac{x}{\epsilon_2},t).$$

By the Birkhoff ergodic theorem

$$Dw_\epsilon^\omega \rightharpoonup \xi \ \ in \ \ [L^p(\Omega)]^n.$$

and by the multiscale version of Birkhoff's ergodic theorem

$$a(T_1(\frac{x}{\epsilon_1})\omega_1, T_2(\frac{x}{\epsilon_2})\omega_2, t, Dw_\epsilon^\omega) \rightharpoonup b(t,\xi) \ \ in \ \ [L^q(\Omega)]^n$$

for every $t \in [0,T]$. Let us now finally show that $\xi^* = b(t, Du^*)$. The uniqueness of the solution to the homogenized problem (18) will then imply that the whole sequences converge. First we recall that by the general G-convergence results for monotone operators in [10] or [11] b enjoys all the nice qualitative properties and has the structure conditions of class S from Definition 1. Further, by the monotonicity we have

$$\int_\Omega \langle A_\epsilon^\omega(x,t,Du_\epsilon^\omega) - A_\epsilon^\omega(x,t,Dw_\epsilon^\omega), Du_\epsilon^\omega - Dw_\epsilon^\omega \rangle \, \Phi(x)\, dx \geq 0$$

for every $\Phi \in C_0^\infty(\Omega)$, $\Phi \geq 0$. By using the standard compensated compactness argument, the monotonicity and the multiscale Birkhoff Theorem we obtain after a limit passage ($\epsilon \to 0$)

$$\int_\Omega \langle \xi^* - b(t,\xi), Du^* - \xi \rangle \, \Phi(x)\, dx \geq 0.$$

This implies that

$$\langle \xi^* - b(t,\xi), Du^* - \xi \rangle \geq 0$$

for a.e. $x \in \Omega$. Finally, by the continuity and the maximal monotonicity of b, the Minty trick yields

$$\xi^* = b(t, Du^*)$$

and by uniqueness of solution to the homogenized problem the whole sequences converge and we can put $u = u^*$. □

By combining the above homogenization result for monotone elliptic problems and the comparison result Theorem 4 from the previous section we are able to prove that the sequence $\{u_\varepsilon\}$ of solutions to (12) converges to the unique solution u to the following homogenized problem

$$\begin{cases} u' - \mathrm{div}(b(t,Du)) = f \ \text{ in } \ \Omega \times (0,T), \\ u(0) = u_0 \ \text{ in } \ \Omega, \\ u \in \mathcal{W}_0 \end{cases} \tag{17}$$

We state the following reiterated homogenization result:

Theorem 8. *Consider the sequence of problems* (12).

$$\begin{cases} (u_\varepsilon^\omega)' - \mathrm{div}(a(T_1(\frac{x}{\varepsilon_1})\omega_1, T_2(\frac{x}{\varepsilon_2})\omega_2, t, Du_\varepsilon^\omega)) = f_\varepsilon, \ \text{ in } \ \Omega \times (0,T), \\ u_\varepsilon^\omega(0) = u_0^\omega, \ \text{ in } \ \Omega, \\ u_\varepsilon^\omega \in \mathcal{W}_0, \end{cases}$$

Under the same assumptions as in Theorem 7 it holds true that as $\epsilon \to 0$ the sequence of solutions

$$u_\varepsilon^\omega \rightharpoonup u \ \ in \ \ \mathcal{W}_0$$

and

$$A_\epsilon^\omega(x,t,Du_\epsilon^\omega) \rightharpoonup \ b(t,Du) \ \ in \ \ \mathcal{U}'.$$

where u is the unique solution to the homogenized problem

$$\begin{cases} u' - \operatorname{div}(b(t, Du)) = f \text{ in } \Omega \times (0, T), \\ u(0) = u_0 \text{ in } \Omega, \\ u \in \mathcal{W}_0, \end{cases}$$

Proof. The proof follows immediately by combining the Theorems 1, 2, 3 and 7 in this paper. □

Remark 2. Theorem 8 remains valid also for random stationary oscillatory forcing. See [11] for details in the general G-convergence setting.

Remark 3. Theorem 8 remains valid also for non-homogeneous boundary data. This means that we can impose random oscillatory boundary data. See [11] for the general G-convergence setting.

Remark 4. The result of Theorem 7 and 8 can easily be extended to any number of well separated scales.

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