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CONSERVATION LAWS WITH DISCONTINUOUS FLUX

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Abstract. We consider a hyperbolic conservation law with discontinuous flux. Such a partial differential equation arises in different applications, in particular we are motivated by a model of traffic flow. We provide a new formulation in terms of Riemann Solvers. Moreover, we determine the class of Riemann Solvers which provide existence and uniqueness of the corresponding weak entropic solutions.

1. **Introduction.** There are different models that lead us to consider hyperbolic conservation laws with flux function discontinuous in the state space. Therefore, it is of great interest to provide a complete theory for the problem

$$\begin{cases} u_t + h(x,u)_x = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $h(x,u)$ is discontinuous in a finite number of points x. We restrict ourselves to the case $h(x,u) = H(x)f(u) + (1-H(x))g(u)$, where H is the Heaviside function, thus there is a single point of discontinuity at $x = 0$. This restriction, however, is interesting for our model, and, on the other hand, the general situation can be deduced by this analysis.

A natural condition to impose at the point of discontinuity of the flux is the equality $f(u(t,0+)) = g(u(t,0-))$ for almost every t, which plays the role of the

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Rankine-Hugoniot condition at $x = 0$ and it provides the conservation of the quantity u through the discontinuity. This condition is not sufficient in general to ensure uniqueness of solution to the Cauchy problem (1); hence some other conditions should be added. Our approach consists of providing a good formulation for an appropriate Riemann Solver at $x = 0$. A Riemann Solver is a function $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $R(u_l, u_r) = (R_1(u_l, u_r), R_2(u_l, u_r)) = (u^-, u^+)$, which provides the left and right traces at the boundary $x = 0$ of a solution to the Cauchy problem (1) with initial datum

$$u_0(x) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases}$$

In this paper, we prove that there exists a class of Riemann Solvers, which determines a unique solution to the Cauchy problem (1) depending in a Lipschitz fashion on the initial data.

Let us first illustrate the main results available in the literature. A seminal work about this problem is the paper by Gimse and Risebro [7]. The authors considered a model for two phase flux through a one-dimensional porous medium. Using Darcy's law and conservation of mass, they studied the equation

$$u_t + \{f(u)(v - k(x)g(u))\}_x = 0,$$

where $k(x)$ represents the absolute permeability of the rock type; thus the function k can be discontinuous. Moreover the authors proved existence of weak solutions, rewriting the equation in a triangular 2×2 non-strictly hyperbolic system and using the wave-front tracking method.

In [5] and [6] Diehl studied the case $h(x, u) = H(x)f(u) + (1 - H(x))g(u)$ and he introduced a condition, called Γ-condition, which guarantees uniqueness of solution. This condition corresponds to a particular choice of the Riemann Solver at the discontinuity point.

The special case $h(x, u) = k(x)f(u)$, where f is a strictly concave function, was considered in the papers by Seguin and Vovelle [21] and by Towers [22, 23]. In particular, Towers in [22] proved existence of solutions in the following way: he approximated the solution by using the schemes of Godunov and Engquist-Osher, then he introduced a map $\psi(u, k)$, called a singular map, in order to have BV estimates for the approximations $\psi(u_n, k)$ and to pass to the limit by Helly's theorem. He also proved that, if k is a piecewise C^1 function with a finite number of discontinuity, then the limit function satisfies the entropy formulation

$$\int_0^T \int_{\mathbb{R}} \Big[|u - c|\phi_t + k\,\mathrm{sgn}(u - c)(f(u) - f(c))\phi_x + f(c)|k'(x)|\phi\Big]\,dxdt + \tag{2}$$
$$+f(c)\int_0^T \sum_{i=0}^{n} |k(\xi_i^+) - k(\xi_i^-)|\,dt \geqslant 0.$$

for every $c \in \mathbb{R}$ and $\phi \in C_0^1(\mathbb{R}\times(0, T); [0, +\infty[)$, where ξ_i are the discontinuity points of k. From this characterization he deduced the contractivity of the semigroup for piecewise regular solutions.

Karlsen, Risebro and Towers in [14] studied the Cauchy problem (1) with a hyperbolic relaxation approach. They proved existence of a weak solution using the method of the compensated compactness.

The general case for system (1) was considered in [4, 15, 16, 17]. In particular in [17] the authors considered a flux of the type $h(k(x, t), u)$ and, under appropriate conditions on h and k, proved that a Lax-Friedrichs scheme converges to a weak solution to the Cauchy problem (1). Moreover, assuming k piecewise constant and discontinuous along a finite number of Lipschitz curves $(\gamma_i(t), t)$, they proved that the solutions satisfy the entropy formulation

$$\int_0^T \int_{\mathbb{R}} \Big[|u-c|\phi_t + \operatorname{sgn}(u-c)(h(k,u)-h(k,c))\phi_x\Big] dxdt + \tag{3}$$
$$+ \int_{\mathbb{R}} |u_0 - c|\phi\, dx + \int_0^T \sum_{i=0}^{n} |f(k(\gamma_i(t),t)+,c)) - f(k(\gamma_i(t),t)-,c))|\phi\, dt \geqslant 0,$$

for every $c \in \mathbb{R}$ and $\phi \in C_0^1(\mathbb{R} \times [0,T))$, $\phi \geqslant 0$, which is a generalization of (2).

A different concept of an entropy solution was given in [1] and [2]. The case $h(x,u) = H(x)f(u) + (1-H(x))g(u)$, where the functions f and g have a unique minimum or maximum point, was considered. The motivations for entropy solutions were deduced by a model of two-phase flows in a porous medium. In this model, undercompressive waves are not allowed; thus shocks can not enter simultaneously from both sides of the discontinuity $x = 0$. This leads to entropy solutions, different from those considered in [17]. In [1], using a Godunov-type approximation, the authors proved convergence of the approximations to an entropy solution u in the domains $(0,+\infty) \times (0,T)$ and $(-\infty,0) \times (0,T)$. In [2], Adimurthi, Mishra and Gowda described all the classes of entropy solutions for the Cauchy problem (1), which provide contractivity in L^1. Moreover, they proved that a sequence of approximated solutions, constructed with a Godunov scheme, admits an a.e. converging subsequence. The limit function provides a solution to the Cauchy problem (1) if its discontinuities form a discrete set of Lipschitz curves.

It is interesting to note that various models need different concept of solution for the same problem (1). This depends on the different physical models of the underlying applications. In this paper, we are motivated by a model of traffic flow for which there is no a priori preferable physical solution. Thus we are interested in giving a unifying point of view for all the possible entropy formulations.

In this paper, by using the wave-front tracking method, we prove that a solution to the Cauchy problem (1) exists for every initial datum in $L^1(\mathbb{R})$, which can be approximated by piecewise constant functions with a finite number of discontinuities. Our idea consists in describing all the possible Riemann Solvers at $x = 0$. This permits to completely characterize solutions to the Cauchy Problem (1).

We study the case $h(x,u) = H(x)f(u) + (1-H(x))g(u)$ and we assume that the fluxes $f : [u_a^o, u_b^o] \to \mathbb{R}$, $g : [u_a^i, u_b^i] \to \mathbb{R}$ satisfy the following properties:

1. f and g are strictly concave functions;
2. there exists $\sigma_g \in]u_a^i, u_b^i[$ such that $g(\sigma_g) \geqslant g(u)$ for every $u \in [u_a^i, u_b^i]$;
3. there exists $\sigma_f \in]u_a^o, u_b^o[$ such that $f(\sigma_f) \geqslant f(u)$ for every $u \in [u_a^o, u_b^o]$.

In Section 2 we give a fluidodynamic description for the problem of traffic flow on a simple road network, composed by two roads connected together by a junction (see [10, 13] for a more general network). We use the scalar model, based on the conservation of the number of cars and introduced by Lighthill, Whitham and Richards; see [20, 26]. Then, problem (1) comes naturally by considering different flux functions for the two roads.

In Section 3 we analyze possible solutions to the Riemann problem at the junction, which satisfies the Rankine-Hugoniot and other conditions. Then, for each

Riemann Solver at the junction, we give a consistent definition of admissible solutions. More precisely:

Definition 1. Fix a Riemann Solver R and an initial condition $u_0 \in BV(\mathbb{R})$. We say that u is an entropy solution to problem (1), related to the Riemann Solver R, if and only if

1. u, restricted to $(0,T) \times (-\infty, 0)$, is an entropy solution to
$$u_t + g(u)_x = 0;$$
2. u, restricted to $(0,T) \times (0, +\infty)$, is an entropy solution to
$$u_t + f(u)_x = 0;$$
3. $f(u(t,0+)) = g(u(t,0-))$ for almost every $t \in (0,T)$;
4. $R(u(t,0-), u(t,0+)) = (u(t,0-), u(t,0+))$ for almost every $t \in (0,T)$.

The concepts of solutions given in [1] and in [17] correspond to our entropy solution related to particular choices of the Riemann Solver.

In Section 4 we obtain existence of a solution according to Definition 1, using the method of the wave-front tracking. The key point is to obtain a uniform BV estimate for the sequence of fluxes $h(\cdot, u_n(t,\cdot))$, where u_n are wave-front tracking approximations. In the general case, it is not possible to obtain BV estimates directly for u_n, since interactions of waves with the junction can increase the total variation of the conserved quantity.

In Section 5 we deal with uniqueness, determining the class of Riemann Solvers ensuring this property. The tool to prove uniqueness is the doubling method by Kruzkov.

2. **Description of the problem.** Let us consider a road network composed by two roads I_1 and I_2 connected together by a junction J. I_1 is the incoming road, modeled by the interval $]-\infty, 0]$, while I_2 is the outgoing one, modeled by the interval $[0, +\infty[$. In this case the junction J is at the point $x = 0$.

In the incoming road I_1, the evolution of the traffic is described by the conservation law
$$\begin{cases} u_t(t,x) + g(u(t,x))_x = 0, & \text{if } (t,x) \in (0,T) \times (-\infty, 0), \\ u(0,x) = u_0, & \text{if } x \in (-\infty, 0), \end{cases} \tag{4}$$
where $u(t,x) \in [u_a^i, u_b^i]$ denotes the density of cars at time $t \geqslant 0$ and at the point $x \in I_1$, g is the flux depending on the density u and u_0 represents the initial density.

In the outgoing road I_2, the evolution of traffic is described by the conservation law
$$\begin{cases} u_t(t,x) + f(u(t,x))_x = 0, & \text{if } (t,x) \in (0,T) \times (0, \infty), \\ u(0,x) = u_0, & \text{if } x \in (0, \infty), \end{cases} \tag{5}$$
where $u \in [u_a^o, u_b^o]$ is the density and f is the flux.

Assume that the fluxes $f : [u_a^o, u_b^o] \to \mathbb{R}$ and $g : [u_a^i, u_b^i] \to \mathbb{R}$ satisfy the following properties:

1. f and g are strictly concave functions;
2. there exists $\sigma_g \in]u_a^i, u_b^i[$ such that $g(\sigma_g) \geqslant g(u)$ for every $u \in [u_a^i, u_b^i]$;
3. there exists $\sigma_f \in]u_a^o, u_b^o[$ such that $f(\sigma_f) \geqslant f(u)$ for every $u \in [u_a^o, u_b^o]$.

Define $\gamma_a^i := g(u_a^i)$, $\gamma_b^i := g(u_b^i)$, $\gamma_a^o := f(u_a^o)$, $\gamma_b^o := f(u_b^o)$; see Figure 1. For (4) and (5) we consider weak entropic solutions; see [8].

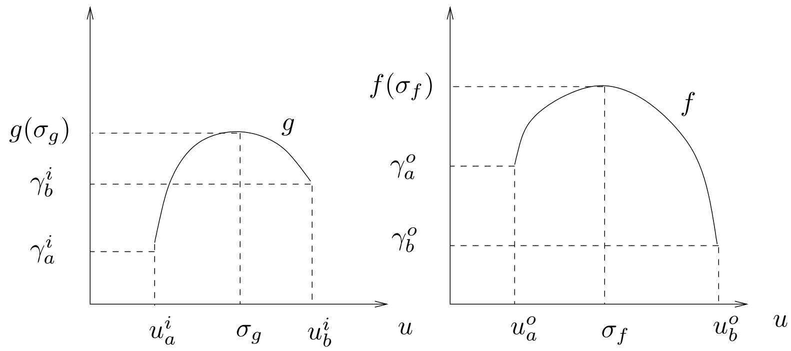

FIGURE 1. Graphs of the fluxes f and g.

Definition 2. A function $u : [0, +\infty[\times] - \infty, 0] \to \mathbb{R}$ is called a weak entropic solution to (4) if

1. for every function $\varphi : [0, +\infty[\times I_1 \to \mathbb{R}$ smooth with compact support on $]0, +\infty[\times] - \infty, 0[$
$$\int_0^{+\infty} \int_{I_1} \left[u(t,x) \frac{\partial}{\partial t} \varphi(t,x) + g(u(t,x)) \frac{\partial}{\partial x} \varphi(t,x) \right] dxdt = 0;$$
2. for every $k \in [u_a^i, u_b^i]$ and for every function $\varphi : [0, +\infty[\times I_1 \to \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times] - \infty, 0[$
$$\int_0^{+\infty} \int_{I_1} |u(t,x) - k| \frac{\partial}{\partial t} \varphi(t,x) dxdt$$
$$+ \int_0^{+\infty} \int_{I_1} \text{sgn}(u(t,x) - k)(g(u(t,x)) - g(k)) \frac{\partial}{\partial x} \varphi(t,x) dxdt \geqslant 0.$$

The definition of weak entropic solution to (5) is analogous.

The previous definition of weak entropic solutions is due to Volpert [25] and it is a generalization of the classical entropy condition in the case of a scalar equation.

Consider the Riemann problem at J
$$\begin{cases} u_t + g(u)_x = 0, & \text{if } x < 0,\, t > 0, \\ u_t + f(u)_x = 0, & \text{if } x > 0,\, t > 0, \\ u(0,x) = u_l, & \text{if } x < 0, \\ u(0,x) = u_r, & \text{if } x > 0, \end{cases} \tag{6}$$
where $u_l \in [u_a^i, u_b^i]$ and $u_r \in [u_a^o, u_b^o]$.

Definition 3. We say that $u^- \in [u_a^i, u_b^i]$ and $u^+ \in [u_a^o, u_b^o]$ determine a weak solution to the Riemann problem (6) at J if

(R-1) the wave (u_l, u^-) on I_1 has negative speed;
(R-2) the wave (u^+, u_r) on I_2 has positive speed;
(R-3) $g(u^-) = f(u^+)$.

The weak solution to the Riemann problem (6) at J is given by the waves (u_l, u^-) and (u^+, u_r) respectively on I_1 and I_2.

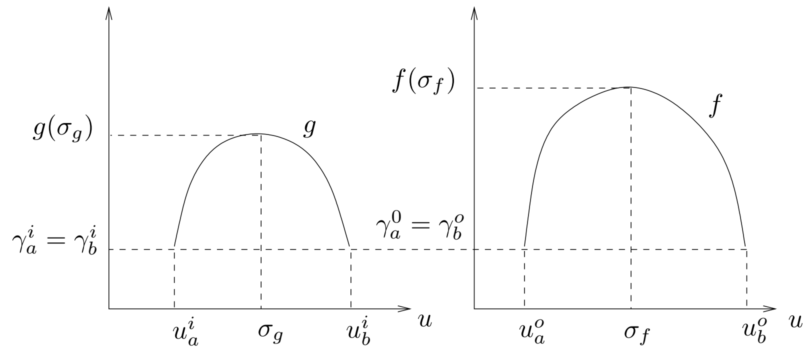

FIGURE 2. The fluxes f and g in the case of Lemma 1.

We look for conditions on γ_a^i, γ_b^i, γ_a^o, γ_b^o in order that, for every $u_l \in [u_a^i, u_b^i]$ and for every $u_r \in [u_a^o, u_b^o]$, the Riemann problem (6) admits at least a weak solution satisfying (R-1), (R-2) and (R-3). The following lemmas hold.

Lemma 1. *Assume $\gamma_a^i \leqslant \gamma_b^i$, $\gamma_a^o \geqslant \gamma_b^o$. The Riemann problem (6) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition if and only if $\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o$.*

Proof. If $\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o$, then

$$u^- = \begin{cases} u_b^i, & \text{if } u_l \neq u_a^i, \\ u_a^i, & \text{if } u_l = u_a^i, \end{cases}$$

and

$$u^+ = \begin{cases} u_a^o, & \text{if } u_r \neq u_b^o, \\ u_b^o, & \text{if } u_r = u_b^o, \end{cases}$$

provide a weak solution to the Riemann problem satisfying (R-1), (R-2) and (R-3).

Suppose now that the Riemann problem (6) admits a least one weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition.

Assume by contradiction that $\gamma_a^i < \gamma_b^i$. Fix u_l such that $g(u_l) < \gamma_b^i$. By (R-1), $u^- = u_l$ and so, by (R-3), $f(u^+) = g(u_l)$. This implies that $\gamma_a^o \leqslant g(u_l)$ and $\gamma_b^o \geqslant g(u_l)$, otherwise, if $\gamma_a^o > g(u_l)$, then the Riemann problem with initial condition $(u_l, u_r) = (u_l, u_a^o)$ does not admit weak solutions, while, if $\gamma_b^o < g(u_l)$, then the Riemann problem with initial condition $(u_l, u_r) = (u_l, u_b^o)$ does not admit weak solutions. Thus we have $g(u_l) = \gamma_a^0 = \gamma_b^o$, that is a contradiction since the arbitrariness of u_l. Therefore $\gamma_a^i = \gamma_b^i$.

By contradiction assume that $\gamma_a^o > \gamma_b^o$. Fixing u_r such that $f(u_r) < \gamma_a^o$ as in the previous case, we conclude that $f(u_r) = \gamma_a^i = \gamma_b^i$, that is a contradiction. So $\gamma_a^o = \gamma_b^o$.

Taking now $(u_l, u_r) = (u_a^i, u_b^o)$, we conclude that $\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o$; see Figure 2. □

Lemma 2. *Assume $\gamma_a^i > \gamma_b^i$, $\gamma_a^o \geqslant \gamma_b^o$. The Riemann problem (6) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition if and only if $\gamma_b^i \leqslant \gamma_b^o \leqslant \gamma_a^o \leqslant \gamma_a^i$.*

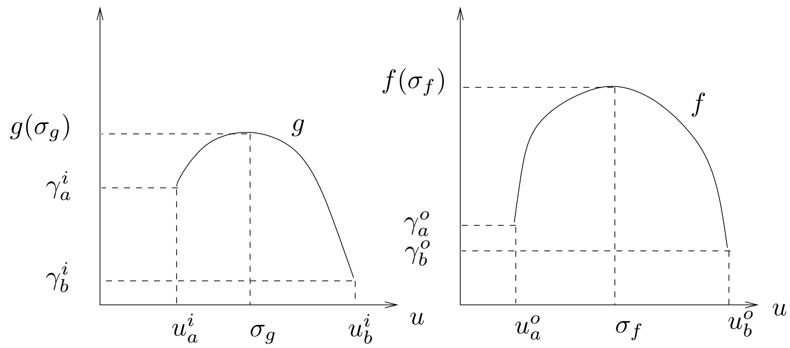

FIGURE 3. The fluxes f and g in the case of Lemma 2.

Proof. Consider first the case $\gamma_b^i \leqslant \gamma_b^o \leqslant \gamma_a^o \leqslant \gamma_a^i$. For every u_l define the set

$$A^-(u_l) := \left\{\tilde{u} \in [u_a^i, u_b^i] : \text{ the wave } (u_l, \tilde{u}) \text{ has negative speed}\right\}.$$

We have that

$$[\gamma_b^i, \gamma_a^i] \subseteq g(A^-(u_l))$$

for every u_l. If u_r satisfies $f(u_r) < \gamma_a^o$, then $u^+ = u_l$ and there exists an element in $A^-(u_l)$ satisfying (R-3). If instead u_r satisfies $f(u_r) \geqslant \gamma_a^o$, then there exists a weak solution such that $u^+ = u_a^o$. Thus the sufficient condition is proved.

Assume now that the Riemann problem (6) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition.

Suppose first by contradiction that $\gamma_b^i > \gamma_b^o$. Consider $u_r = u_b^o$. Then, by (R-2), $u^+ = u_l$ and so it is not possible to satisfy (R-3). Therefore $\gamma_b^i \leqslant \gamma_b^o$.

Suppose now that $\gamma_a^o > \gamma_a^i$. Consider $(u_l, u_r) = (u_a^i, u_a^o)$. By (R-1), u^- satisfies $g(u^-) \leqslant \gamma_a^i$. By (R-2), u^+ satisfies $f(u^+) \geqslant \gamma_a^o$. Then (R-3) is not satisfied and so we get $\gamma_a^o \leqslant \gamma_a^i$; see Figure 3.

This concludes the lemma. □

Lemma 3. *Assume $\gamma_a^i \leqslant \gamma_b^i$, $\gamma_a^o < \gamma_b^o$. The Riemann problem (6) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition if and only if $\gamma_a^o \leqslant \gamma_a^i \leqslant \gamma_b^i \leqslant \gamma_b^o$.*

Proof. The proof is given in the same way as in Lemma 2, since the situation is completely symmetric. □

Lemma 4. *Assume $\gamma_a^i > \gamma_b^i$, $\gamma_a^o < \gamma_b^o$. The Riemann problem (6) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition if and only if $\gamma_a^o \leqslant \gamma_a^i$ and $\gamma_b^i \leqslant \gamma_b^o$.*

Proof. Assume first that $\gamma_a^o \leqslant \gamma_a^i$ and $\gamma_b^i \leqslant \gamma_b^o$. For every u_l and u_r define the sets

$$A^-(u_l) := \left\{\tilde{u} \in [u_a^i, u_b^i] : \text{ the wave } (u_l, \tilde{u}) \text{ has negative speed}\right\}$$

and

$$A^+(u_r) := \{\tilde{u} \in [u_a^o, u_b^o] : \text{ the wave } (\tilde{u}, u_r) \text{ has positive speed}\}.$$

We have that

$$[\gamma_b^i, \gamma_a^i] \subseteq g(A^-(u_l)), \quad [\gamma_a^o, \gamma_b^o] \subseteq f(A^+(u_r)),$$

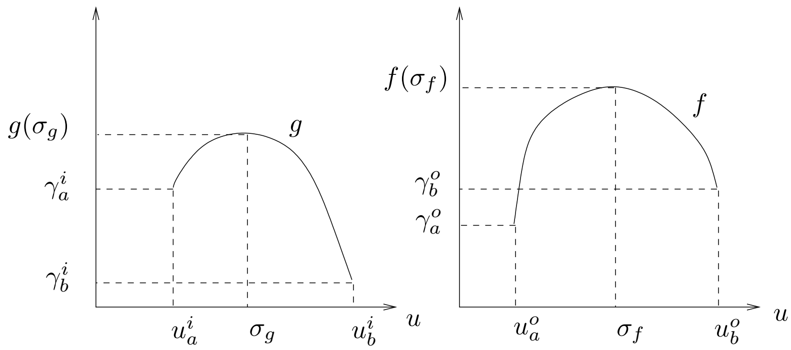

FIGURE 4. The fluxes f and g in the case of Lemma 4.

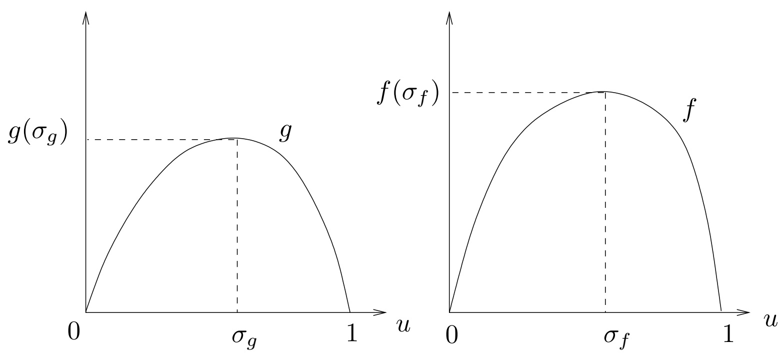

FIGURE 5. The fluxes f and g considered in Section 3.

for every u_l and u_r. By assumption

$$[\gamma_b^i, \gamma_a^i] \cap [\gamma_a^o, \gamma_b^o] \neq \emptyset$$

and so it is possible to find $u^- \in A^-(u_l)$ and $u^+ \in A^+(u_r)$ such that $f(u^+) = g(u^-)$. Hence the sufficient condition is proved.

Assume now that the Riemann problem (6) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition.

Suppose by contradiction that $\gamma_a^i < \gamma_a^o$. If $u_l = u_a^i$, then u^- by (R-1) satisfies $g(u^-) \leqslant \gamma_a^i$ and so (R-3) can not be satisfied. Therefore $\gamma_a^i \geqslant \gamma_a^o$.

Suppose now that $\gamma_b^i > \gamma_b^o$. If $u_r = u_b^o$, then u^+ satisfies $f(u^+) \leqslant \gamma_b^o$ and so (R-3) can not be satisfied. Thus $\gamma_b^i \leqslant \gamma_b^o$ (see Figure 4) and the proof is finished. □

3. **Construction of Riemann solvers.** Consider the Riemann problem at J (6). For simplicity, let us assume that $u_a^i = u_a^o = 0$, $u_b^i = u_b^o = 1$ and $\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o = 0$; see Figure 5.

Definition 4. A Riemann solver for the Riemann problem (6) is a function $R : [0,1] \times [0,1] \to [0,1] \times [0,1]$, $R(u_l, u_r) = (R_1(u_l, u_r), R_2(u_l, u_r)) = (u^-, u^+)$, such that

(H1). $g(u^-) = f(u^+)$;

(H2). the wave (u_l, u^-) has negative speed, while the wave (u^+, u_r) has positive speed;
(H3). the function $(u_l, u_r) \mapsto (g(u^-), f(u^+))$ is continuous;
(H4). $R(R(u_l, u_r)) = R(u_l, u_r)$ for every $u_l \in [0,1]$ and $u_r \in [0,1]$;
(H5). for every $(u_l, u_r) = R(u_l, u_r)$ and $\tilde{u}$ such that the wave $(\tilde{u}, R_1(u_l, u_r))$ has positive speed the following holds:

$$g(R_1(\tilde{u}, u_r)) \in [\min\{g(u_l), g(\tilde{u})\}, \max\{g(u_l), g(\tilde{u})\}]; \tag{7}$$

(H6). for every $(u_l, u_r) = R(u_l, u_r)$ and $\tilde{u}$ such that the wave $(R_2(u_l, u_r), \tilde{u})$ has negative speed the following holds:

$$f(R_2(u_l, \tilde{u})) \in [\min\{f(u_r), f(\tilde{u})\}, \max\{f(u_r), f(\tilde{u})\}]. \tag{8}$$

Definition 5. A couple (u_l, u_r) is said an equilibrium if $R(u_l, u_r) = (u_l, u_r)$.

Remark 1. Observe that conditions (H1) and (H2) are motivated physically by the conservation of mass at the junction and by the fact waves originated at $x = 0$ in I_1 (resp. in I_2) must travel with negative (resp. positive) speed, since I_1 (resp. I_2) is modeled by the interval $(-\infty, 0)$ (resp. $(0, +\infty)$).

Condition (H3) is a regularity property for the Riemann solver, while (H4) is a stability condition, in the sense that the image of R is a fixed point of the same function.

Finally conditions (H5) and (H6) are the key assumptions for some important estimates for the existence of solutions to Cauchy problems, as we see in Section 4.

Example 1. *Assume $f = g$ and $\sigma_g = \sigma_f = \frac{1}{2}$. Let $\Omega = [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ and let $S : \Omega \to [0, \eta]$ ($0 < \eta < g(\frac{1}{2})$) be a continuous function such that*

1. *$S(x, y) = 0$ for every $(x, y) \in \partial\Omega$;*
2. *$S(x, y) = f(x)$ for every $(x, y) \in \Omega$, $f(y) = f(x) \leqslant \eta$.*

Consider the following Riemann Solver R. If $(u_l, u_r) \in ([0,1] \times [0,1]) \setminus \Omega$, then $R(u_l, u_r) = (1, 0)$. If $(u_l, u_r) \in \Omega$, then $R(u_l, u_r) = (u^-, u^+) \in \Omega$, where $g(u^-) = f(u^+) = S(u_l, u_r)$. It is clear that R satisfies (H1)–(H4), but not (H5) and (H6).

The aim of this section is to describe all the possible Riemann solvers for (6). We treat only the case $f(\sigma_f) \geqslant g(\sigma_g)$, the other one similar. We have some different possibilities:

1. $u_l \in [\sigma_g, 1]$ and $u_r \in [0, \sigma_f]$. Since the waves produced must have negative speed in I_1 and positive speed in I_2, then $u^- \in [\sigma_g, 1]$ and $u^+ \in [0, \sigma_f]$. By hypothesis (H3), there exists a continuous function

$$\Gamma : [0, g(\sigma_g)] \times [0, f(\sigma_f)] \to [0, g(\sigma_g)] \tag{9}$$

such that

$$g(u^-) = f(u^+) = \Gamma(g(u_l), f(u_r)).$$

By (H4) we deduce that, if $a \in \operatorname{Im}\Gamma$, then $\Gamma(a, a) = a$ and so, every element of the image of Γ is the flux of an equilibrium for the Riemann problem. Conversely, if (u_l, u_r) is an equilibrium for the Riemann problem, then

$$\Gamma(g(u_l), f(u_r)) = f(u_r) = g(u_l),$$

and so the image of Γ coincides with the set X defined by

$$X := \{s \in [0, g(\sigma_g)] : (u_l, u_r) \in [\sigma_g, 1] \times [0, \sigma_f] \text{ equilibrium}, \ g(u_l) = f(u_r) = s\}. \tag{10}$$

We have the following characterization of the set X.

Lemma 5. *X is a closed, non empty and connected set. Thus $X = [\bar{\gamma}_1, \bar{\gamma}_2]$, with $0 \leqslant \bar{\gamma}_1 \leqslant \bar{\gamma}_2 \leqslant g(\sigma_g)$.*

Proof. X is a connected set since it is the image of a connected set through a continuous function. Moreover X is clearly non empty. Finally we take $x \in \bar{X}$ and a sequence $a_n \to x$ such that $a_n \in X$ for every $n \in \mathbb{N}$. We have:

$$\Gamma(x, x) = \lim_{n \to +\infty} \Gamma(a_n, a_n) = \lim_{n \to +\infty} a_n = x$$

and so $x \in X$. □

From now on with $\bar{\gamma}_1$ and $\bar{\gamma}_2$ we denote respectively the minimum and maximum of the set X.

2. $u_l \in [0, \sigma_g[$ and $u_r \in [0, \sigma_f]$. Some different cases are possible.

a. $g(u_l) \leqslant \bar{\gamma}_1$. By (H2), u^- either is u_l or belongs to $[\sigma_g, 1]$ with $g(u^-) < g(u_l)$. The second possibility can not happen since otherwise $g(u^-) < \bar{\gamma}_1$, a contradiction with (H4); so the solution is given by (u_l, u^+) with $u^+ \in [0, \sigma_f[$, $f(u^+) = g(u_l)$.

b. $g(u_l) > \bar{\gamma}_2$. We claim that $u^- \in [\sigma_g, 1]$ and $g(u^-) \in X$. Indeed, consider the function

$$\begin{array}{rcl} h_{u_r} : [0, \sigma_g] & \to & [0, g(\sigma_g)] \\ u_l & \mapsto & g(u^-) \end{array}$$

giving the flux in I_1 of the solution to the Riemann problem with (u_l, u_r) initial states. It is continuous by (H3). Therefore

$$\lim_{r \to \sigma_g^-} h_{u_r}(r) = h_{u_r}(\sigma_g) \leqslant \bar{\gamma}_2,$$

by the analysis of possibility 1. Moreover there exists a left neighborhood V of σ_g such that $h_{u_r}(s) \leqslant \bar{\gamma}_2$ for every $s \in V$, otherwise, by (H2) on the speed of waves, there exists a sequence $s_n \to \sigma_g^-$ so that $g(s_1) > \bar{\gamma}_2$ and $h_{u_r}(s_n) = g(s_n) \geqslant g(s_1) > \bar{\gamma}_2$ contradicting the continuity of h_{u_r}. Consider the set

$$Y := \{r \in [0, \sigma_g[: g(r) > \bar{\gamma}_2, h_{u_r}(r) > \bar{\gamma}_2\}$$

and suppose by contradiction that $Y \neq \emptyset$. Define $\eta := \sup Y$. The previous analysis implies that

$$0 < \eta < \sigma_g, \qquad \bar{\gamma}_2 < g(\eta)$$

and by continuity of h_{u_r}

$$h_{u_r}(\eta) \geqslant g(\eta) > \bar{\gamma}_2.$$

Moreover

$$\lim_{r \to \eta^+} h_{u_r}(r) \leqslant \bar{\gamma}_2,$$

a contradiction. Thus $Y = \emptyset$ and the claim is proved.

c. $\bar{\gamma}_1 < g(u_l) \leqslant \bar{\gamma}_2$. In this case $h_{u_r}(u_l) \in [\bar{\gamma}_1, \bar{\gamma}_2]$. If $h_{u_r}(u_l) = g(u_l)$, then the solution is given by (u_l, u^+), where $u^+ \in [0, \sigma_f[$ with $f(u^+) = g(u_l)$. Otherwise, if $h_{u_r}(u_l) < g(u_l)$, then $u^- \in [\sigma_g, 1]$.

Remark 2. If $\tilde{\gamma} \in]\bar{\gamma}_1, \bar{\gamma}_2[$ satisfies $h_{u_r}(u_l) = \tilde{\gamma}$ for $u_l \in [0, \sigma_g[$ and $u_r \in [0, \sigma_f]$ with $g(u_l) = f(u_r) = \tilde{\gamma}$, then conditions (H2) and (H5) imply that

$$h_{u_r}(r) = g(r)$$

for every $r \in [0, \sigma_g[$ such that $\bar{\gamma}_1 \leqslant g(r) \leqslant \tilde{\gamma}$.

3. $u_l \in [\sigma_g, 1]$ and $u_r \in]\sigma_f, 1]$. This case is completely symmetric with respect to the previous one.

4. $u_l \in [0, \sigma_g[$ and $u_r \in]\sigma_f, 1]$. We have some different cases.

a. $\min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}_1$. Without loss of generalities we suppose that $g(u_l) \leqslant f(u_r)$. By (H2), u^- either is u_l or $u^- \in]\sigma_g, 1]$ with $g(u^-) < g(u_l)$. Analogously u^+ either is u_r or $u^+ \in [0, \sigma_f[$ with $f(u^+) < f(u_r)$. If $u^- \in]\sigma_g, 1]$, then, by (H1), $u^+ \in [0, \sigma_f[$, but this is not an equilibrium. Thus $u^- = u_l$. If $f(u_r) = g(u_l)$, then $u^+ = u_r$ and the solution is (u_l, u_r). Otherwise if $f(u_r) > g(u_l)$, then $u^+ \in [0, \sigma_f[$, $f(u^+) = g(u_l)$ and the solution is (u_l, u^+).

b. $\bar{\gamma}_1 < \min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}_2$. Without loss of generalities we suppose $g(u_l) \leqslant f(u_r)$. If $g(u_l) < f(u_r)$, then $u^+ \in [0, \sigma_f[$ and the case is completely identical to 2.c.

If $g(u_l) = f(u_r)$, then, by the continuity assumption (H3), the solution is uniquely determined as a limiting procedure by the case $g(u_l) < f(u_r)$.

c. $\min\{g(u_l), f(u_r)\} > \bar{\gamma}_2$. Without loss of generalities we suppose that $g(u_l) \leqslant f(u_r)$. If $g(u_l) < f(u_r)$, then $u^+ \in [0, \sigma_f[$ by (H2) and also $u^- \in]\sigma_g, 1]$ by 2.b.

If $g(u_l) = f(u_r)$, then, by the continuity assumption (H3), the solution is uniquely determined as a limiting procedure by the case $g(u_l) < f(u_r)$.

Given a Riemann solver at the junction J, it is possible to define an admissible weak solution to (4) and (5).

Definition 6. Fix a Riemann solver R. A function $u \in L^\infty((0,T) \times \mathbb{R})$ is an admissible weak solution to (4) and (5) if

1. u is a weak entropic solution to (4) in $(0,T) \times (-\infty, 0)$;
2. u is a weak entropic solution to (5) in $(0,T) \times (0, +\infty)$;
3. for almost every $t \in (0,T)$, the couple $(u(t, 0-), u(t, 0+))$ is an equilibrium for the Riemann solver R.

Observe that the previous definition is well posed, since Vasseur [24] proved existence of the trace for entropy solutions of conservation laws.

3.1. **Case of X singleton.** In this subsection let us consider the special case $X = \{\bar{\gamma}\}$. The Riemann solver is completely described by the following possibilities.

1. $u_l \in [\sigma_g, 1]$ and $u_r \in [0, \sigma_f]$. In this case the solution to the Riemann problem satisfies $u^- \in [\sigma_g, 1]$, $u^+ \in [0, \sigma_f]$ and $g(u^-) = f(u^+) = \bar{\gamma}$.
2. $u_l \in [0, \sigma_g[$ and $u_r \in [0, \sigma_f]$. If $g(u_l) > \bar{\gamma}$, then the solution to the Riemann problem satisfies $u^- \in [\sigma_g, 1]$, $u^+ \in [0, \sigma_f]$ and $g(u^-) = f(u^+) = \bar{\gamma}$.

   If $g(u_l) \leqslant \bar{\gamma}$, then the solution to the Riemann problem satisfies $u^- = u_l$, $u^+ \in [0, \sigma_f]$ and $g(u^-) = f(u^+)$.
3. $u_l \in [\sigma_g, 1]$ and $u_r \in]\sigma_f, 1]$. The situation is completely symmetric to the previous case.
4. $u_l \in [0, \sigma_g[$ and $u_r \in]\sigma_f, 1]$. If $\min\{g(u_l), f(u_r)\} > \bar{\gamma}$, then the solution to the Riemann problem satisfies $u^- \in [\sigma_g, 1]$, $u^+ \in [0, \sigma_f]$ and $g(u^-) = f(u^+) = \bar{\gamma}$.

   If $\min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}$ and $g(u_l) = f(u_r)$, then the solution to the Riemann problem is (u_l, u_r).

   If $\min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}$ and $g(u_l) < f(u_r)$, then the solution to the Riemann problem satisfies $u^- = u_l$, $u^+ \in [\sigma_f, 1]$ and $g(u_l) = f(u^+)$.

   If $\min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}$ and $g(u_l) > f(u_r)$, then the solution to the Riemann problem satisfies $u^- \in [\sigma_g, 1]$, $u^+ = u_r$ and $g(u^-) = f(u_r)$.

Remark 3. If $f(\sigma_f) = g(\sigma_g)$ and $X = \{g(\sigma_g)\}$, then the Riemann solver is completely identical to that used in [1] and [10].

Remark 4. If there exists a unique $u^* \in]0,1[$ such that $f(u^*) = g(u^*)$ and if $X = \{f(u^*)\}$, then the Riemann solver is identical to that used in [5, 6, 17].

4. **Existence of solutions.** In this Section we consider a Riemann Solver R such that the related set X defined by (10) is a singleton. We denote with X_R the set X related to the Riemann Solver R. Using a wave-front tracking method, we prove the existence of an admissible solution of problem (4) and (5) for any fixed Riemann solver R of this kind. We denote with R_γ the Riemann solver such that $X_R = \{\gamma\}$. Observe that for every $\gamma \in (0, g(\sigma_g)]$ we can define R_γ.

The wave front tracking algorithm is very useful for treating systems of conservation laws. In our case, the situation is simpler since we consider a scalar conservation law: this allows us to overcome difficulties due to the discontinuity of the flux. For a detailed description of the algorithm, we refer the reader to [8].

Let us summarize the main points of this approach for our specific case. Fix a sequence of piecewise constant approximations $u_{0,\nu}$ of the initial datum u_0, such that Tot.Var.$u_{0,\nu} \leqslant$ Tot.Var.u_0. We solve the Riemann problems at any discontinuity point and in particular at $x = 0$, where we use the fixed Riemann Solver R_γ. We split rarefaction waves into rarefaction fans formed by rarefaction shocks (i.e. non entropic shocks.) When two waves interact or a wave interact with the junction, we solve a new Riemann problem. Notice that the number of waves can increase only for interactions with the junction. However, in this case at most two waves are produced and any such wave can interact with the junction again only after canceling one wave inside the roads (see also [13].)

Finally, there is a finite number of waves and we can define, for every ν, a function u_ν for every time, which provides a wave front tracking approximate solution (in fact it is a weak solution violating the entropy condition by a quantity going to zero with $\nu \to \infty$; see [8].)

Theorem 1. *Given $\gamma \in (0, \sigma_g]$ and $u_0 \in BV(\mathbb{R})$, there exists an admissible solution u to problem (4) and (5) in the sense of Definition 6 with the Riemann solver R_γ. Moreover such a solution is obtained as an almost everywhere limit of approximate wave front tracking solutions.*

We divide the proof of the previous theorem in some lemmas. First of all we prove an equivalent formulation of an admissible solution valid for the case of singleton.

Lemma 6. *Let $\gamma \in (0, \sigma_g]$. A function $u \in L^\infty((0,T)\times\mathbb{R})$ is an admissible solution to (4) and (5) in the sense of Definition 6 for the Riemann solver R_γ if and only if*

1. *u is a weak entropic solution to (4) in $(0,T) \times (-\infty, 0)$;*
2. *u is a weak entropic solution to (5) in $(0,T) \times (0, +\infty)$;*
3. *for almost every $t \geqslant 0$ the couple $(u(t,0-), u(t,0+))$ satisfies the following conditions*
   (a) *$g(u(t,0-)) = f(u(t,0+)) \leqslant \gamma$;*
   (b) *if $(u(t,0-), u(t,0+)) \in [\sigma_g, 1) \times (0, \sigma_f]$, then*
   $$(u(t,0-), u(t,0+)) = (a_\gamma, b_\gamma)$$
   where a_γ (resp. b_γ) is the unique value in $[\sigma_g, 1)$ (resp. $(0, \sigma_f]$) such that $g(a_\gamma) = \gamma$ (resp. $f(b_\gamma) = \gamma$).

Proof. This result is an immediate consequence of the analysis done in Section 3.1. □

The following lemma shows that the total variation of the flux of a wave front tracking approximate solution does not change when a wave interacts with J. This is due in particular to the properties (H5) and (H6) of Definition 4.

Lemma 7. *Fix an approximate wave front tracking solution $\bar{u}$. If a wave interacts with J at time $\bar{t}$, then*

$$\textit{Tot.Var.}\left[f(\bar{u}(\bar{t}+,\cdot)) + g(\bar{u}(\bar{t}+,\cdot))\right] = \textit{Tot.Var.}\left[f(\bar{u}(\bar{t}-,\cdot)) + g(\bar{u}(\bar{t}-,\cdot))\right].$$

Proof. Fix an equilibrium (u_l, u_r). First suppose that a wave $(\tilde{u}, u_l)$ with positive speed interacts with J from the incoming road I_1. We denote with (u^-, u^+) the solution to the Riemann problem at J with the initial datum $(\tilde{u}, u_r)$. We have

$$\begin{aligned} \text{Tot.Var.}\left[f(\bar{u}(\bar{t}+,\cdot)) + g(\bar{u}(\bar{t}+,\cdot))\right] &= \left|g(\tilde{u}) - g(u^-)\right| + \left|f(u^+) - f(u_r)\right| \\ &= \left|g(\tilde{u}) - g(u^-)\right| + \left|g(u^-) - g(u_l)\right| \\ &= \left|g(\tilde{u}) - g(u_l)\right| \\ &= \text{Tot.Var.}\left[f(\bar{u}(\bar{t}-,\cdot)) + g(\bar{u}(\bar{t}-,\cdot))\right], \end{aligned}$$

where we used (H1) and (H5).

Suppose now that a wave $(u_r, \tilde{u})$ with negative speed interacts with J from the outgoing road I_2. We denote with (u^-, u^+) the solution to the Riemann problem at J with the initial datum $(u_l, \tilde{u})$. We have

$$\begin{aligned} \text{Tot.Var.}\left[g(\bar{u}(\bar{t}+,\cdot)) + f(\bar{u}(\bar{t}+,\cdot))\right] &= \left|g(u_l) - g(u^-)\right| + \left|f(u^+) - f(\tilde{u})\right| \\ &= \left|f(u_r) - f(u^+)\right| + \left|f(u^-) - f(\tilde{u})\right| \\ &= \left|f(\tilde{u}) - f(u_r)\right| \\ &= \text{Tot.Var.}\left[f(\bar{u}(\bar{t}-,\cdot)) + g(\bar{u}(\bar{t}-,\cdot))\right], \end{aligned}$$

where we used (H1) and (H6).

This completes the proof. □

Lemma 8. *Fix an approximate wave front tracking solution $\bar{u}$. For every $t \geqslant 0$, it holds*

$$\textit{Tot.Var.}(h(\cdot, \bar{u}(t,\cdot))) \leqslant \textit{Tot.Var.}(h(\cdot, \bar{u}(0+,\cdot))). \tag{11}$$

Proof. By Lemma 7, we know that the total variation of the flux does not change when a wave approaches the junction J.

If, instead, two waves interact in a road, then the total variation of the flux either remains constant or strictly decreases. □

In order to pass to the limit in the sequence of approximate wave front tracking solutions u_ν we study in depth interactions of waves at the junction. Given an approximate wave front tracking solution $\bar{u}$, we denote with $u^-(t)$ and $u^+(t)$ respectively the values $\bar{u}(t, 0-)$ and $\bar{u}(t, 0+)$. Sometimes to simplify the notation we shall write only u^- and u^+.

From now on, we fix a Riemann Solver R_γ. Given a generic equilibrium (u^-, u^+) for an approximate wave front tracking solution, we classify it in four classes. More precisely we say that u^- is "good" if $u^- \in [a_\gamma, 1]$ instead we say that u^+ is "good" if $u^+ \in [0, b_\gamma]$. If u^- or u^+ is not good, then we say that they are "bad". Using this property we introduce four classes of equilibrium,

I. u^- and u^+ are "good": we denote it by $G|G$. In this case u^- and u^+ are equal to a_γ and b_γ;

II. u^- is "good" and u^+ is "bad": we denote it by $G|B$;

III. u^- is "bad" and u^+ is "good": we denote it by $B|G$;

IV. u^- and u^+ are "bad": we denote it by $B|B$.

We analyze the interaction of a wave $(\tilde{u}, u^-)$ that reaches the junction from the left at time t^* in the four cases. Since the wave reaches the boundary we have that $\tilde{u}$ is "bad".

In the case $G|G$ necessarily $g(\tilde{u}) < g(u^-) = g(a_\gamma) = \gamma$. In this situation for the results proved in Section 3.1 we have that after time t^* the new equilibrium is in the class $B|G$ given by $(\tilde{u}, f_1^{-1}(g(\tilde{u})))$. Where f_1^{-1} is the inverse of the function f restricted in the interval $[0, \sigma_f]$.

In the case $G|B$ we have that $g(\tilde{u}) < g(u^-)$. After time t^* we are in the class $B|G$ and the new equilibrium is $(\tilde{u}, f_1^{-1}(g(\tilde{u})))$.

Suppose now to stay in the case $B|G$; we have to distinguish two cases. If $g(\tilde{u}) \leqslant \gamma$, then the new equilibrium after time t^* is given by $(\tilde{u}, f_1^{-1}(g(\tilde{u})))$ and we remain in the same class $B|G$. Otherwise if $g(\tilde{u}) > \gamma$, then the new equilibrium is given by (a_γ, b_γ) and we are in the class $G|G$.

In the last case $B|B$ there are the same two possibilities of case III, then after the interaction we arrive to the situation $B|G$ or $G|G$.

Symmetric situations happen if we consider a wave $(u^+, \tilde{u})$ that reaches the junction from the right. In view of these results we give the following remarks.

Remark 5. Equilibrium IV is extremely unstable. In fact it can happen only at initial time. If a wave reaches the junction, then this kind of equilibrium is lost and it is impossible to obtain it again.

Remark 6. Suppose that until time T waves reach the junction only on one side, for example from the left. Then $u^+(\cdot)$ can change only one time from "bad" to "good" and it remains "good" at least until time T. It changes type at the moment in which a wave reaches the junction from the right.

Now we are able to prove the following result.

Lemma 9. *For every initial datum u_0 with finite total variation, there exists an entropy solution $u(t,x)$ that satisfies points 1 and 2 of Definition 6.*

Proof. Fix a sequence of initial data $u_{0,\nu}$ such that

$$\text{Tot.Var.}(u_{0,\nu}) \leqslant \text{Tot.Var.}(u_0)$$

for every $\nu \in \mathbb{N}$ and

$$u_{0,\nu} \to u_0$$

in L^1_{loc} as $\nu \to +\infty$. For each $u_{0,\nu}$ we consider a wave-front tracking approximate solution u_ν such that $u_\nu(0,x) = u_{0,\nu}(x)$ and rarefactions are split in rarefaction shocks of size $1/\nu$. If we are able to prove that there exists a subsequence of $\{u_\nu\}$ converging in L^1 to a function u, then, following [8], we conclude that u satisfies conditions 1 and 2 of Definition 6.

By Lemma 8 we deduce, passing to a subsequence, that $f(u_n)$ converges in L^1 to a function $\overline{f}$. We follow the procedure of [10, Definition 5.7 and Theorem 5.10] to conclude the proof.

For every ν we consider the curves Y_-^ν and Y_+^ν such that $Y_-^\nu(0) = Y_+^\nu(0) = 0$ and these follow the generalized characteristics (see [12]) defined for the approximate front tracking solution u_ν letting $Y_-^\nu(t) = 0$ (resp. $Y_+^\nu(t) = 0$) if $Y_-^\nu(t)$ (resp. $Y_-^\nu(t)$) reaches the boundary and $g'(u(t,0-)) \geqslant 0$ (resp. $f'(u(t,0+)) \leqslant 0$). We define the sets

$$D_1^\nu = \left\{(t,x) \in (0,T) \times \mathbb{R} \,:\, Y_-^\nu(t) \leqslant x \leqslant Y_+^\nu(t)\right\},$$

and $D_2^\nu = ((0,T) \times \mathbb{R}) \setminus D_1^\nu$. By definition we see that the set D_2^ν is not influenced by the junction; this gives a priori estimate for the total variation of $u_\nu(\cdot,t)$ in the intervals $(-\infty, Y_-^\nu(t))$ and $(Y_+^\nu(t), +\infty)$ that depends only on the total variation of u_0. Using Remark 6 we can observe that for every t in the intervals $(Y_-^\nu(t), 0]$ there is at most one point $\tilde{x}$ such that $\operatorname{sgn}(u^\nu(t,\tilde{x}-) - a_\gamma)\operatorname{sgn}(u^\nu(t,\tilde{x}+) - a_\gamma) \leqslant 0$. An analogous result is true in the interval $[0, Y_+^\nu(t))$. In particular, for $\gamma < \sigma_g$ inverting g and f we deduce for every t a priori estimate of the total variation of $u_\nu(t,\cdot)$ that depends only on initial data and the constants $\frac{1}{g'(a_\gamma)}$ and $\frac{1}{f'(b_\gamma)}$. When $\gamma = \sigma_g$ a priori estimate for the total variation of u_ν is not true in general. In this case we can divide for every t the intervals $(Y_-^\nu(t), Y_+^\nu(t))$ in a finite number of intervals in which f and g are invertible. This assures that we can extract a subsequence that we call again $\{u_\nu\}$ converging in L^1_{loc} to a function u. This concludes the proof. □

Remark 7. Lemma 9 can be generalized to the case of a general roads network, where junctions could have either one incoming and one outgoing road or one incoming and two outgoing roads or two incoming and one outgoing roads or finally two incoming and two outgoing roads.

We conclude the proof of Theorem 1 if we prove that function u obtained by Lemma 9 verifies condition 3 of Lemma 6. For this aim it is necessary to obtain a priori estimate for the total variation of the flux of a generic approximate solution along the junction. More precisely we have the following lemma.

Lemma 10. *Let $\{u_\nu\}$ be the approximate wave front tracking sequence given in Lemma 9. Then for every ν we have*

$$\begin{aligned} \textit{Tot.Var.}(g(u_\nu^-(\cdot),\,(0,T)) &= \textit{Tot.Var.}(f(u_\nu^+(\cdot),(0,T)) \\ &\leqslant 2\ \textit{Tot.Var.}(h(\cdot,u_0(\cdot)),\mathbb{R}). \end{aligned} \tag{12}$$

Proof. Let us simplify the notations writing v and v_0 instead of the generic function u_ν and of the initial datum $u_{0,\nu}$. Moreover we introduce the following functions

$$M(t) = \lim_{\varepsilon\to 0^+} \text{Tot.Var.}(g(v^-(\cdot)),\,(0,t+\varepsilon));$$
$$F(t,[x_1,x_2]) = \lim_{\varepsilon\to 0^+} \text{Tot.Var.}(h(\cdot,v(t,\cdot)),\,[x_1+\operatorname{sgn}(x_1)\varepsilon, x_2+\operatorname{sgn}(x_2)\varepsilon]);$$
$$S([x_1,x_2]) = F(0,[x_1,x_2]).$$

More precisely M denotes the total variation of $g(v^-)$ in the time interval $(0,t]$ and F is the total variation of $h(\cdot,v(t,\cdot))$ in a spatial interval at a fixed time $t \geqslant 0$.

If the initial datum is chosen in equilibrium (u^-,u^+) near the junction $x=0$, then $M(\cdot)$ is zero until a wave reaches J. It is not restrictive to assume that the first wave comes from the left. Let τ_1 be the time in which a wave reaches the boundary and $Y_{-1}(\cdot)$ the backward minimal characteristic starting from $(0,\tau_1)$. We denote with x_{-1} the point $Y_{-1}(0)$ and with u_{-1} the value $v_0(x_{-1}-)$. Analogously we define t_1 the first time in which a wave reaches the junction from the right; if it does not exist we put $t_1 = +\infty$. When t_1 exists finite we consider the maximal backward characteristic $Y_1(\cdot)$ starting from $(0,t_1)$. Moreover we introduce $x_1 := Y_1(0)$ and

$u_1 := v_0(x_1+)$. Let us introduce the quantities

$$\bar{\tau}_1 := \max\{t \in [\tau_1, t_1) : \exists \textit{ a wave reaching } J \textit{ from the left at time } t\},$$

$\bar{Y}_{-1}(\cdot)$ is the minimal backward characteristic that starts from point $(0, \bar{\tau}_1)$, $\bar{x}_{-1} = \bar{Y}_{-1}(0)$ and $\bar{u}_{-1} = v_0(\bar{x}_{-1}-)$. Obviously $\bar{\tau}_1$ can coincide with τ_1. Suppose that after time t_1 there exists a wave that reaches J from the left; we denote with τ_2 the corresponding interaction time. As before we define the values x_{-2} and u_{-2}. Moreover we introduce the time

$$\bar{t}_1 := \max\{t \in [t_1, \tau_2) : \exists \textit{ a wave reaching } J \textit{ from the right at time } t\},$$

and the corresponding quantities $\bar{x}_1$ and $\bar{u}_1$.

With this procedure we can define four sequences of times, that eventually can become constantly equal to $+\infty$, $\{\tau_n\}$, $\{\bar{\tau}_n\}$, $\{t_n\}$ and $\{\bar{t}_n\}$, such that $\tau_n \leqslant \bar{\tau}_n < t_n \leqslant \bar{t}_n < \tau_{n+1}$, if they are finite.

Moreover we can define four sequences on the x axis, that eventually can become constant, $\{x_{-n}\}$, $\{\bar{x}_{-n}\}$, $\{x_n\}$ and $\{\bar{x}_n\}$, such that

$$0 = x_0 > x_{-1} \geqslant \bar{x}_{-1} > \cdots > x_{-n} \geqslant \bar{x}_{-n} > x_{-n-1} > \cdots$$

and

$$0 = x_0 < x_1 \leqslant \bar{x}_1 < \cdots < x_n \leqslant \bar{x}_n < x_{n+1} < \cdots .$$

Finally we have the sequences $\{u_{-n}\}$, $\{\bar{u}_{-n}\}$, $\{u_n\}$ and $\{\bar{u}_n\}$, where we set $u_0 = u^-$.

Let us prove the following estimates

$$M(\tau_n) \leqslant S([\bar{x}_{-n+1}, \bar{x}_{n-1}]) + \sum_{i=1}^{n-1} (f(\bar{u}_i) - f(u_i)) + \sum_{i=1}^{n-1} (g(\bar{u}_{-i}) - g(u_{-i-1})), \quad (13)$$

for every $n \geqslant 2$ and

$$\begin{aligned} M(t_n) \leqslant S([\bar{x}_{-n}, \bar{x}_{n-1}]) &+ \sum_{i=1}^{n-1} (f(\bar{u}_i) - f(u_i)) \qquad (14) \\ &+ \sum_{i=1}^{n-1} (g(\bar{u}_{-i}) - g(u_{-i-1})) + g(\bar{u}_{-n}) - f(u_n). \end{aligned}$$

for every $n \geqslant 1$.

Let us prove (13) and (14) by induction. Consider the case $n = 1$. At time τ_1 the value u_{-1} reaches the boundary; so we use the analysis made before. Therefore necessarily u_{-1} is a “bad” value and if $g(u_{-1}) \leqslant \gamma$, then the new equilibrium is given by $(u_{-1}, f_1^{-1}(g(u_{-1})))$. Otherwise if $g(u_{-1}) > \gamma$, the new equilibrium is (a_γ, b_γ). In any case since $f(u^+) = g(u^-) \leqslant \gamma$, we have that $M(\tau_1) \leqslant |g(u_{-1}) - g(u^-)| = S([x_{-1}, x_0])$. In the interval $(\tau_1, \bar{\tau}_1]$ waves can arrive only from the interval $[\bar{x}_{-1}, x_{-1})$; this implies that the total variation of the flux at the junction depends only on the total variation of the flux at initial time. More precisely we obtain the estimate $M(\bar{\tau}_1) \leqslant S([\bar{x}_{-1}, 0])$. Using the considerations made before for interactions of waves with J, we know that after time τ_1 and at least until time t_1 the equilibrium is of the type $G|G$ or $B|G$. This means that the value u_1, which reaches J at time t_1, is necessarily “bad” and $f(u_1) < g(\bar{u}_{-1})$. In particular the new equilibrium is $(g_2^{-1}(f(u_1)), u_1)$, where we denote with g_2^{-1} the inverse of the function g restricted to the interval $(\sigma_g, 1)$. From this observation we have that

$$M(t_1) = M(\bar{\tau}_1) + g(\bar{u}_{-1}) - f(u_1) \leqslant S([\bar{x}_{-1}, 0]) + g(\bar{u}_{-1}) - f(u_1)$$

that corresponds to (14) for $n = 1$ and with the choice $\bar{x}_0 = 0$. Reasoning as before we see that $M(\bar{t}_1) \leqslant S([\bar{x}_{-1}, \bar{x}_1]) + g(\bar{u}_{-1}) - f(u_1)$. Moreover in the interval (t_1, τ_2) the equilibrium can only be of the type $G|G$ or $G|B$; thus necessarily u_{-2} is a "bad" value, $g(u_{-2}) < f(\bar{u}_1)$ and the new equilibrium has u_{-2} as left value. This implies that

$$\begin{aligned} M(\tau_2) &= M(\bar{t}_1) + f(\bar{u}_1) - g(u_{-2}) \\ &\leqslant S([\bar{x}_{-1}, \bar{x}_1]) + g(\bar{u}_{-1}) - g(u_{-2}) + f(\bar{u}_1) - f(u_1) \end{aligned}$$

that give (13) for $n = 2$.

Let us assume that (13) is true for a generic n. We prove as before that

$$M(\bar{\tau}_n) \leqslant S([\bar{x}_{-n+1}, \bar{x}_{n-1}]) + \sum_{i=1}^{n-1} (f(\bar{u}_i) - f(u_i)) + \sum_{i=1}^{n-1} (g(\bar{u}_{-i}) - g(u_{-i-1})). \quad (15)$$

In the interval $(\bar{\tau}_n, t_n)$ the equilibrium is of the type $B|G$ or $G|G$. Again this implies that u_n is a "bad" value and $g(\bar{u}_{-n}) > f(u_n)$. Therefore $M(t_n) = M(\bar{\tau}_n) + g(\bar{u}_{-n}) - f(u_n)$. Using (15) and rearranging the terms we obtain (14). Repeating the same arguments we prove that

$$\begin{aligned} M(\bar{t}_n) \leqslant S([\bar{x}_{-n}, \bar{x}_n]) &+ \sum_{i=1}^{n-1} (f(\bar{u}_i) - f(u_i)) \\ &+ \sum_{i=1}^{n-1} (g(\bar{u}_{-i}) - g(u_{-i-1})) + g(\bar{u}_{-n}) - f(u_n). \end{aligned}$$

Finally observing that in the in interval $[\bar{t}_n, \tau_{n+1})$ the equilibrium is of the type $G|B$ or $G|G$, we deduce that u_{-n-1} is a "bad" value and $f(\bar{u}_n) > g(u_{-n-1})$; so $M(\tau_{n+1}) = M(\bar{t}_n) + f(\bar{u}_n) - g(u_{-n-1})$, which gives (13) for $n+1$. This permits us to conclude by an induction argument.

The proof of the lemma is an immediate consequence of (13) and (14). □

By Lemma 10, we deduce that $f(u_\nu^+(\cdot)) = g(u_\nu^-(\cdot))$ is bounded in BV and converges in L^1 to a BV function. This permits us to prove the following result, which concludes the proof of Theorem 1.

Lemma 11. *The function u obtained in Lemma 9 satisfies condition 3 of Lemma 6.*

Proof. By Lemma 6 every equilibria of a generic wave front tracking approximate solution verify condition 3. In particular, letting ν to $+\infty$, the limit u obtained in Lemma 1 satisfies $f(u^+(\cdot)) = g(u^-(\cdot)) \leqslant \gamma$ almost everywhere, which gives the first part of condition 3. We finish the proof if we show that, for almost every t such that $(u^-(t), u^+(t)) \in [\sigma_g, 1) \times (0, \sigma_f]$, we have $(u^-(t), u^+(t)) = (a_\gamma, b_\gamma)$. Let us prove this fact by contradiction. We know that $f(u^+(\cdot))$ and $g(u^-(\cdot))$ are BV functions. Let $\bar{t}$ be a point of continuity for $g(u^-(\cdot))$ such that $(u^-(\bar{t}), u^+(\bar{t})) \in [\sigma_g, 1) \times (0, \sigma_f]$ and $g(u^-(\bar{t})) < \gamma$. Then there exists a neighborhood of $(\bar{t}, 0)$ where $g(u) < \gamma$. This means that in such set, the equilibria are of type $G|B$ or $B|G$ for every ν. Thus it is not restrictive to assume that there exists a subsequence, which we call again $\{u_\nu\}$, such that u_ν^- is of "bad" type. Moreover, since near the point $(\bar{t}, 0)$ the characteristics point outside the domain $(0, T) \times \mathbb{R}^-$, we find a two dimensional neighborhood C of $(\bar{t}, 0)$ such that u_ν takes values in $[0, g_1^{-1}(\gamma))$, where g_1^{-1} denotes the inverse of g restricted to $(0, \sigma_g)$. By Lemma 9, u_ν converges to u in L^1_{loc} and so we conclude that u^- is of the "bad" type almost everywhere in $C \cap \{x = 0\}$. This is a contradiction. □

5. **Uniqueness.** In this section we investigate the problem of uniqueness for admissible solutions to problem (4)–(5). In particular we prove that there is uniqueness if and only if X_R, the set defined in (10), is a singleton.

Suppose that the set X_R is not a singleton. Let $\gamma_1 < \gamma_2 \in X_R$ and take $u_1^-, u_2^- \in [\sigma_g, 1]$, $u_1^+, u_2^+ \in [0, \sigma_f]$ such that

$$g(u_1^-) = f(u_1^+) = \gamma_1, \qquad g(u_2^-) = f(u_2^+) = \gamma_2.$$

Clearly (u_1^-, u_1^+) and (u_2^-, u_2^+) are equilibria. Then for every initial data

$$u_0(x) = \begin{cases} \underline{u}, & \text{if } x \leqslant 0, \\ \overline{u}, & \text{if } x > 0, \end{cases}$$

where $\underline{u} \in [\sigma_g, 1]$ and $\overline{u} \in [0, \sigma_f]$, we can find two admissible solutions u_1 and u_2 to problem (4)–(5) such that the equilibria are respectively (u_1^-, u_1^+) and (u_2^-, u_2^+). This shows that for general initial data there is not uniqueness.

In the following we assume that $X_R = \{\gamma\}$ and denote by R_γ the corresponding Riemann Solver. Let us prove the following result that is crucial to prove uniqueness.

Proposition 1. *Consider a Riemann Solver R_γ. For every two equilibria (u_1^-, u_1^+) and (u_2^-, u_2^+) of R_γ it holds*

$$\operatorname{sgn}(u_1^- - u_2^-)\left[g(u_1^-) - g(u_2^-)\right] \geqslant \operatorname{sgn}(u_1^+ - u_2^+)\left[f(u_1^+) - f(u_2^+)\right]. \qquad (16)$$

Proof. Using that $f(u_1^+) = g(u_1^-)$ and $f(u_2^+) = g(u_2^-)$, inequality (16) is equivalent to

$$[\operatorname{sgn}(u_1^- - u_2^-) - \operatorname{sgn}(u_1^+ - u_2^+)][g(u_1^-) - g(u_2^-)] \geqslant 0. \qquad (17)$$

Since inequality (17) is symmetric in u_1 and u_2, it is not restrictive to assume $u_1^- \leqslant u_2^-$. If $u_1^+ \leqslant u_2^+$, then the conclusion is obvious. Let us consider all the remaining cases assuming that $u_1^+ > u_2^+$.

1. u_1^-, $u_2^- \in [0, \sigma_g]$. Then $g(u_1^-) \leqslant g(u_2^-)$ and this implies (17).
2. $u_1^- \in [0, \sigma_g]$, $u_2^- \in [\sigma_g, 1]$, $u_2^+ \in [0, \sigma_f]$. Using Lemma 6 we deduce that $f(u_2^+) = g(u_2^-) = \gamma$ and $g(u_1^-) \leqslant \gamma$. This gives the result.
3. $u_1^- \in [0, \sigma_g]$, $u_2^- \in [\sigma_g, 1]$, $u_2^+ \in [\sigma_f, 1]$. Then $u_1^+ \in [\sigma_f, 1]$ and $g(u_1^-) = f(u_1^+) < f(u_2^+) = g(u_2^-)$; so we obtain again the inequality.
4. u_1^-, $u_2^- \in [\sigma_g, 1]$, $u_2^+ \in [0, \sigma_f]$. Then we obtain the same conclusion of case 2.
5. u_1^-, $u_2^- \in [\sigma_g, 1]$, $u_2^+ \in [\sigma_f, 1]$. Then we proceed as in the case 3.

The proof is finished. □

Now we are able to prove uniqueness and continuous dependence in L^1 respect to initial data for admissible solutions.

Theorem 2. *Fix $\gamma \in (0, g(\sigma_g)]$ and u_0, $v_0 \in BV(\mathbb{R})$. Let u and v be admissible solutions to problem (4) and (5) in the sense of Definition 6, for the Riemann Solver R_γ and initial data respectively u_0 and v_0. Then for every $C > 0$ and for almost every $t \in (0, T)$*

$$\int_{-C}^{C} |u(t,x) - v(t,x)|\, dx \leqslant \int_{-C-Mt}^{C+Mt} |u_0(x) - v_0(x)|\, dx, \qquad (18)$$

where

$$M = \max\left\{\max_{u\in[-a,a]}|f'(u)|, \max_{u\in[-a,a]}|g'(u)|\right\}$$

and $a = \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$.

Proof. Let u and v be entropy solutions in $\Pi_T^- := (0,T)\times(-\infty,0)$. Using the doubling method by Kruzkov, we obtain

$$\int\int_{\Pi_T^-} |u-v|\phi_t + \operatorname{sgn}(u-v)(g(u)-g(v))\phi_x dxdt \geqslant 0 \tag{19}$$

for any $\phi \in C_0^1(\Pi_T^-)$, $\phi \geqslant 0$; see also [23].

We now choose a particular set of test functions. Consider ϵ, $\theta \in \mathbb{R}^+$ and t', $t'' \in \mathbb{R}$ such that $0 < t' < t'' < T$, $t''+\theta < T$, $t'-\theta > 0$. Define ξ_θ (resp. ξ_ϵ) the corresponding cut-off function, i.e. a smooth function, which approximates the characteristic function of the interval $[-\theta,\theta]$ (resp. $[-\epsilon,\epsilon]$). Set:

$$Y_\theta(x) := \int_{-\infty}^x \xi_\theta(y)\,dy.$$

Letting $I_{(((t-t'')M-C,-2\epsilon))}$ be the characteristic function of the interval $((t-t'')M-C,-2\epsilon)$, we can finally define the following test function:

$$\phi(x,t) = (Y_\theta(t-t') - Y_\theta(t-t''))(I_{(((t-t'')M-C,-2\epsilon))} * \xi_\epsilon)(x).$$

It is easily seen that $\phi \geqslant 0$, $\phi \in C_0^\infty(\Pi_T^-)$. Putting ϕ in the inequality (19) and using the definition of constant M we obtain

$$\int\int_{\Pi_T^-} |u-v|(\xi_\theta(t-t') - \xi_\theta(t-t''))I_{(((t-t'')M-C,-2\epsilon))} * \xi_\epsilon)(x)\,dxdt \tag{20}$$
$$\geqslant \int\int_{\Pi_T^-} H^+(u-v)(g(u)-g(v))\xi_\epsilon(x+2\epsilon)\,dxdt.$$

Passing to limit as $\epsilon \to 0^+$ and using existence of the trace we obtain

$$\int_0^T \int_{(t-t'')M-C}^0 |u-v|(\xi_\theta(t-t') - \xi_\theta(t-t''))\,dxdt \geqslant$$
$$\int_0^T \operatorname{sgn}(u^-(t,0)-v^-(t,0))(g(u^-)-g(v^-))(Y_\theta(t-t') - Y_\theta(t-t''))\,dt.$$

Suppose that t'', t' are Lebesgue points for the function

$$s(t) = \int_{-C-TM}^0 |u(t,x)-v(t,x)|\,dx,$$

for the arbitrariness of ξ_θ and letting t' to 0^+, we obtain

$$\int_{-C}^0 |u(t'',x)-v(t'',x)|\,dx \leqslant \int_{-t''M-C}^0 |u_0(x)-v_0(x)|\,dx$$
$$-\int_0^{t''} \operatorname{sgn}(u^-(t,0)-v^-(t,0))(g(u^-(t,0))-g(v^-(t,0)))\,dt.$$

Proceeding in the same way in the domain $(0,T)\times(0,\infty)$, for almost every t it holds

$$\int_{-C}^{C}|u(t,x)-v(t,x)|\,dx \leqslant \int_{-tM-C}^{tM+C}|u_0(x)-v_0(x)|\,dx$$
$$+\int_0^t \operatorname{sgn}(u^+(s,0)-v^+(s,0))(f(u^+(s,0))-f(v^+(s,0)))$$
$$-\operatorname{sgn}(u^-(s,0)-v^-(s,0))(g(u^-(s,0))-g(v^-(s,0)))\,ds.$$

Since, by Definition 6, $(u^-(0,\cdot),u^+(0,\cdot))$ and $(v^-(0,\cdot),v^+(0,\cdot))$ are almost everywhere equilibria for the Riemann solver R_γ, the conclusion follows from Proposition 1. □

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