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EXISTENCE AND STABILITY OF ENTROPY SOLUTIONS FOR A CONSERVATION LAW WITH DISCONTINUOUS NON-CONVEX FLUXES

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Abstract. We consider a scalar conservation law with a discontinuous flux function. The fluxes are non-convex, have multiple points of extrema and can have arbitrary intersections. We propose an entropy formulation based on interface connections and associated jump conditions at the interface. We show that the entropy solutions with respect to each choice of interface connection exist and form a contractive semi-group in L^1. Existence is shown by proving convergence of a Godunov type scheme by a suitable modification of the singular mapping approach. This extends the results of [3] to the general case of non-convex flux geometries.

1. **Introduction.** We are interested in the following single conservation law in one space dimension,

$$\begin{aligned} u_t + (f(k(x), u))_x &= 0 \\ u(0,x) &= u_0(x) \end{aligned} \tag{1}$$

where the flux f depends on the space variable through a coefficient k which may be discontinuous. The simplest case of (1) is the so called "two flux" case given by

$$\begin{aligned} u_t + (H(x)f(u) + (1-H(x))g(u))_x &= 0 \\ u(0,x) &= u_0(x) \end{aligned} \tag{2}$$

where f and g are Lipschitz continuous functions and H is the Heaviside function. The analysis of (2) serves as a building block in the analysis of (1). See [5] for details. For the rest of this paper, we consider (2).

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The conservation law (1) occurs in several models in Physics and Engineering. In particular, it arises in two phase flow in a heterogeneous porous medium used in petroleum reservoir simulation (See [13]), in the modeling of the ideal Clarifier thickener unit used in waste water treatment plants and in the paper industry (See [7, 6]), in modeling traffic flow on highways with changing surface conditions (See [22]) and in ion etching used in the Semiconductor industry (see [23]). For detailed account of various applications of (1), see [25].

As is standard for conservation laws, we have to look for a suitable form of weak solutions and augment them with extra admissibility criteria or entropy conditions for uniqueness and stability. The development of a proper entropy framework for equations of the type (1) is a major challenge. The equations of the type (1) have been studied extensively over the last decade from both the analytical as well numerical point of view.

Different entropy theories have been proposed by Gimse and Risebro in [9, 10], Diehl in [7, 8], Karlsen, Risebro and Towers in [19] and Adimurthi and Gowda in [1] among others. Concurrently, several existence results for the entropy solutions have been obtained in a series of papers. They use regularization of coefficients as in [17], by explicit formulas for the corresponding Hamilton-Jacobi equations in [1], some are based on front tracking as in [10, 15, 16] while others used numerical schemes of the Godunov or Enquist-Osher type as in [2, 27, 28, 18, 6, 24] and of the Lax-Friedrichs type as in [20].

More recently, the authors have embarked on a systematic study of (1), (2) in a series of papers namely [3, 4, 5]. In these papers, a new entropy framework for (1) is developed. This framework is based on a two-step approach. In the first step, an interface connection is defined and is used to characterize infinite classes of entropy solutions. Each of these classes of solutions is shown to form a stable semi-group in L^1. The existence of solutions is shown by designing Godunov type finite volume schemes based on exact Riemann solvers and showing that they converge to the entropy solution. In the second step, an optimization problem is defined on the set of connections and the optimizer is defined as the optimal entropy solution.

It is now widely accepted that there is more than one valid concept of entropy solutions for (1) depending on the Physics of the problem. These entropy solutions correspond to different semigroups that can be characterized by different connections. The optimal entropy solutions of [3] correspond to the physically meaningful solutions for two-phase flows in heterogeneous porous media whereas a different semigroup (see [6]) is valid for the clarifier-thickener unit. The physical relevance of other semigroups is not encountered so far. Thus, the above solution concept provides flexibility in terms of incorporating different semigroups of solutions for different physical models. The choice of the optimal entropy solution should be based on the physics of the problem modeling higher order small scale effects.

The analysis of [3, 4] was restricted to the case where both the fluxes have at most one extremum in the domain of definition. It is natural to ask whether the same program as above of showing the existence and uniqueness of infinitely many classes of stable solutions can be carried out for more general fluxes with finitely many points of extrema in the domain of definition. Since the analysis in [3] and [4] relied heavily on the flux geometry, it is not a priori clear whether this can be done. The aim of this paper is to address the above question. In this paper, we deal with fluxes f and g where both of them can have finitely many extrema in the domain of the definition (see section 2 for detailed hypothesis). The above hypothesis considered is very general as regards the shape of the fluxes and their intersection but we will concentrate on a slightly simpler class of fluxes satisfying the hypothesis that both of them can have at most two points of extrema in the domain of the definition. This case of fluxes is termed as the *infinity flux case* (see detailed definition in section 2) and is representative of the more general flux geometries. The reason for concentrating on this case are twofold. First, this case is a prototype of the difficulties that have to be faced while dealing with the case of general fluxes while at the same time, the proofs are simpler and the notation is much neater while dealing with this case as compared to the general case. Second, in most practical models where these equations arise, the fluxes have at most two extrema. For example, in the water flooding model from petroleum reservoir simulation, the fluxes f and g both have at most one maximum and in the clarifier thickener model, the flux g can have at most one maximum and no minima whereas f can have at most one maximum and one minimum. The only exception to our notice is while considering ion etching (see [23]) where the fluxes can have many extrema. Due to the above reasons, we will describe the analysis for the infinity flux case in detail and only present the results for the general case.

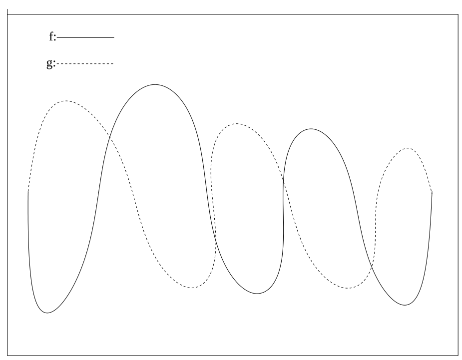

FIGURE 1. possible flux shapes with arbitrary intersections

In this paper, we extend the results of [3] to the more general case and show the wellposedness of infinite classes of entropy solutions. We show uniqueness of the entropy solutions under general assumptions on the fluxes including fluxes violating the “crossing condition” of [19] (see section 2). Another important feature of this paper is the development of Godunov type schemes for the general case based on exact solutions of the Riemann problem. We provide a convergence proof based on a intricate modification of the singular mapping technique (it is well known that the singular mapping approach is hard to implement on non-convex fluxes). We organize this paper as follows - In section 2, we describe the entropy framework for the infinity flux case and show uniqueness of entropy solutions. In section 3, we give the Godunov type scheme for the infinity flux case. The convergence proof for the scheme is described in Section 4. In section 5, we present the corresponding results for a model general case.

2. **Entropy framework.** We start with the precise assumptions on the fluxes considered in this paper.

Definition 1. Admissible Class of fluxes: Let $I = [s, S]$ where $-\infty < s < S < \infty$ and denote the admissible class of fluxes as
$\mathbb{F}(I) = \{h \in \mathrm{Lip}(I) : h$ has finitely many points of extrema $\}$.

For $h \in \mathbb{F}(I)$, let $s = \theta_{0h} < \theta_{1h} < \ldots < \theta_{kh} = S$ be the points of extrema of h. Define the following,

$$E(h) = I \setminus \{\theta_{0h}, \theta_{1h}, \ldots, \theta_{kh}\}$$
$$E_+(h) = \{ \text{ closure of components of } E(h) \text{ on which } h \text{ is increasing } \}$$
$$E_-(h) = \{ \text{ closure of components of } E(h) \text{ on which } h \text{ is decreasing } \}$$
$$\mathrm{N(h)} = \text{number of components of } E(h)$$
$$N_+(h) = \text{number of components of } E_+(h)$$
$$N_-(h) = \text{number of components of } E_-(h)$$

Clearly $N(h) = N_+(h) + N_-(h)$. For $k \geq 0$, let

$$\mathbb{F}_k(I) = \{h \in \mathbb{F}(I) : N(h) = k\}$$

In [2],[3],[4], we have dealt with the case of fluxes being in $\mathbb{F}_2(I)$. In this paper, we will consider fluxes with more number of extrema.

Define the following,

Definition 2. Infinity Flux pair: For $f, g \in \mathbb{F}(I)$, the pair of fluxes (f, g) forms an infinity flux pair if they satisfy the following hypothesis,

(H_1): $f(s) = g(s), \quad f(S) = g(S)$.
(H_2): $f, g \in \mathbb{F}_3(I)$

We call the above class of fluxes as the "infinity"-fluxes because in general, the two fluxes can intersect in the interior of the domain and the graph of both fluxes superposed on each other (see figures 2 and 3), resembles the figure of infinity. We subdivide the infinity flux class of fluxes into the following cases,
1. under-compressive Class: In this case, (f, g) satisfies the relation the

$$N_+(f) + N_-(g) = 2$$

A prototypical example of this case is shown in figure 2. In this case, it is clear that g is decreasing and f is increasing at the point of intersection in the interior. As a result, the characteristics at the interface divege on both sides of the line $x = 0$ and hence the name undercompressive is used for this class of fluxes. *2. over-compressive Class:* In this case, (f, g) satisfies the relation that

$$N_+(f) + N_-(g) \geq 3$$

One example of these class of fluxes is shown in figure 3. In this case, the flux f is decreasing and the flux g is increasing at the interior point of intersection and the characteristics flow into the line $x = 0$ on both sides, hence the name over-compressive for this class of fluxes.

The shape of fluxes which constitute infinity flux pairs is shown in figures 2 and 3. We define the weak solution of the (2) as

Definition 3. Weak Solution: - $u \in L^\infty_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$ is said to be a weak solution of (2) if for all $\varphi \in C_c^\infty(\mathbb{R} \times \overline{R}_+)$, the following integral identity is satisfied,

$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(u \frac{\partial \varphi}{\partial t} + (H(x)f(u) + (1 - H(x))g(u)) \frac{\partial \varphi}{\partial x} \right) dxdt + \int_{-\infty}^{\infty} u_0(x)\varphi(x, 0)dx = 0 \tag{3}$$

It is easy to see that u satisfies (3) if and only if in the weak sense u satisfies

$$\begin{aligned} u_t + g(u)_x &= 0 \quad x < 0, \quad t > 0 \\ u_t + f(u)_x &= 0 \quad x > 0, \quad t > 0 \end{aligned} \tag{4}$$

and at $x = 0, u$ satisfies Rankine-. (RH) condition, namely for almost all t

$$f(u^+(t)) = g(u^-(t)) \tag{5}$$

where $u^+(t) = \lim_{x\to 0+} u(x,t), u^-(t) = \lim_{x\to 0-} u(x,t)$.

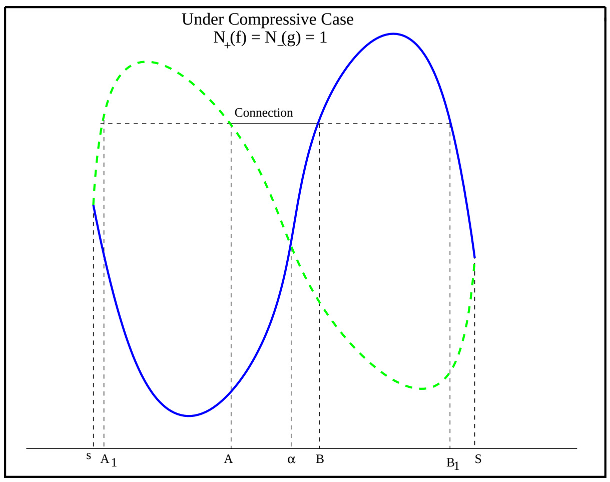

FIGURE 2. Shape of fluxes in the Infinity Flux (Under-Compressive) case.

Definition 4. Entropy-Entropy flux pair: For $i = 1, 2, (\varphi_i, \psi_i)$ are said to be entropy pairs if φ_i is a convex function on $[s, S]$ and $(\psi_1'(\theta), \psi_2'(\theta)) = (\varphi_1'(\theta)f'(\theta), \varphi_2'(\theta)g'(\theta))$ for $\theta \in [s, S]$.

Definition 5. Interior Entropy Condition: A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to satisfy the interior entropy condition if it satisfies in the sense of distributions,

$$\begin{aligned} \frac{\partial \varphi_1(u)}{\partial t} + \frac{\partial \psi_1(u)}{\partial x} &\leq 0 \text{ in } x > 0, \quad t > 0, \\ \frac{\partial \varphi_2(u)}{\partial t} + \frac{\partial \psi_2(u)}{\partial x} &\leq 0 \text{ in } x < 0, \quad t > 0. \end{aligned} \tag{6}$$

As in [3], we need to define a suitable concept of interface connection. This is done below,

Definition 6. Interface connection: Let (f, g) satisfies (H_1) and (H_2). A pair of vectors (A_0, B_0) with $A_0 = (A_1, A_2) \in I \times I$ and $B_0 = (B_1, B_2) \in I \times I$ is called an interface connection vector if they satisfy the following,

1. $A_1, A_2 \in E_-(g), \quad B_1, B_2 \in E_+(f)$ and if $N_+(f) = 2$, then B_1, B_2 lie in disjoint components of $E_+(f)$ and if $N_-(g) = 2$, then A_1, A_2 lie in disjoint components of $E_-(g)$.

2. Let $N_+(f) + N_-(g) = 2$, then $A_1 = A_2, B_1 = B_2$ and $f(B_1) = g(A_1)$. Denote $(A_1, B_1) = (A, B)$.

3. Let $N_+(f) + N_-(g) \geq 3$, then we have to consider several sub cases,

i. $N_+(f) = N_-(g) = 2$.

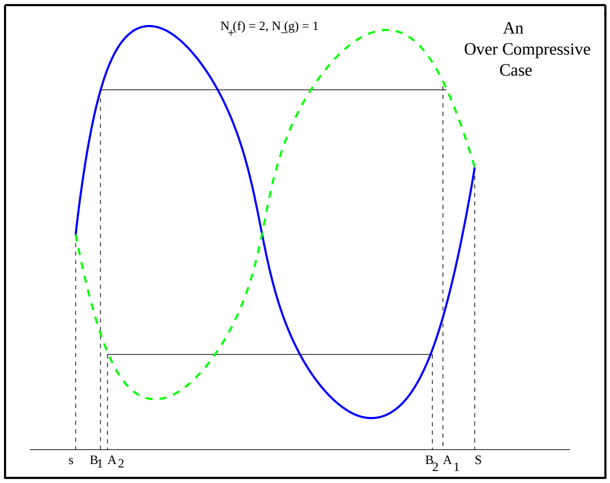

FIGURE 3. Shape of fluxes in the Infinity Flux (Over-Compressive) case.

In this case, we assume that $A_1 \geq \theta_{2g}, A_2 \leq \theta_{1g}$ and $B_1 \leq \theta_{1f}, B_2 \geq \theta_{2f}$ and they satisfy

$$|f(B_1) - g(A_1)| = \min\{|f(\theta) - g(\psi)|; \theta \leq \theta_{1f}, \psi \geq \theta_{2g}\}$$
$$|f(B_2) - g(A_2)| = \min\{|f(\theta) - g(\psi)|; \theta \geq \theta_{2f}, \psi \leq \theta_{1g}\}$$

ii. $N_+(f) = 1, N_-(g) = 2$.

In this case, we assume that $A_1 \geq \theta_{2g}, A_2 \leq \theta_{1g}$ and $B_1, B_2 \in [\theta_{1f}, \theta_{2f}]$ with $B_2 \leq B_1$ such that the following holds,

$$|f(B_1) - g(A_1)| = \min\{|f(\theta) - g(\psi)|; B_2 \leq \theta \leq \theta_{2f}, \psi \geq \theta_{2g}\}$$
$$|f(B_2) - g(A_2)| = \min\{|f(\theta) - g(\psi)|; \theta \in [\theta_{1f}, \theta_{2f}], \geq, \psi \leq \theta_{2g}\}$$

iii. $N_+(f) = 2, N_-(g) = 1$

In this case, we assume that $B_1 \leq \theta_{1f}, B_2 \geq \theta_{2f}$ and $A_1, A_2 \in [\theta_{1g}, \theta_{2g}]$ with $A_1 \leq A_2$ such that the following holds,

$$|f(B_1) - g(A_1)| = \min\{|f(\theta) - g(\psi)|; \theta \leq \theta_{1f}, \psi \in [\theta_{2g}, A_2]\}$$
$$|f(B_2) - g(A_2)| = \min\{|f(\theta) - g(\psi)|; \theta \geq \theta_{2f}, \psi \in [\theta_{1g}, \theta_{2g}]\}$$

For the sake of simplicity, we make the following hypothesis on the connection (A_0, B_0)

(H_3): Whenever $N_+(f) + N_-(g) \geq 3$, then for $i = 1, 2$

$$g(A_i) = f(B_i)$$

The hypothesis H_3 is always satisfied if $f([s, \theta_{1f}]) \cap g([\theta_{2g}, S]) \neq \Phi$ and $g([s, \theta_{1g}]) \cap f([\theta_{2f}, S]) \neq \Phi$. It may not be satisfied in some cases where this may not hold.

This concept of connection appears too complicated at first sight but is better illustrated in the figures 2 and 3 where a connection in each case is given. The interface connection is now a vector pair as compared to a scalar pair that was used in [3]. This is on account of the number of points of extrema in the flux geometry. The under-compressive case is special as the vector pair is reduced to a scalar pair. Next, in order to define the interface entropy condition, we need to define the following, Let $f, g \in Lip(I)$ and let $\alpha, \beta, A, B \in I$, then define

$$I(\alpha, \beta, A, B) = sign(\alpha - A)(g(\alpha) - g(A)) - sign(\beta - B)(f(\beta) - f(B))$$

Definition 7. Interface Entropy condition: Let (f, g) satisfy H_1 and H_2. Let $(A_0, B_0) : A_0 = (A_1, A_2), B_0 = (B_1, B_2)$ be an interface connection vector. Let $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ be such that the traces $u^+(t) = u(0+, t)$ and $u^-(t) = u(0-, t)$ exist for a.e $t > 0$, then u is said to satisfy the interface entropy condition if for a.e $t > 0$, $i = 1, 2$, the following holds,

$$I(u^-(t), u^+(t), A_i, B_i) \geq 0 \tag{7}$$

We define the framework of entropy solutions with respect to each fixed interface connection as follows,

Definition 8. A_0B_0 entropy solution: For a given interface connection vector (A_0, B_0) defined as before, a function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is defined to be an A_0B_0 entropy solution if the following holds,
1. u is a weak solution of (2) i.e u satisfies (3).
2. u satisfies the interior entropy solution (6) .
3. u satisfies the interface entropy condition (7) relative to the connection (A_0, B_0).

Hence a A_0B_0 entropy solution is defined for each choice of connection (A_0, B_0) and we show that each of these classes of solutions are well posed. Hence, we show the well-posedness of infinitely many stable A_0B_0 entropy solutions of (2) in the infinity-flux case. We start with the stability result,

Theorem 1. *For a given interface connection vector (A_0, B_0), Let $u, v \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ be two A_0B_0 entropy solutions for (2) with initial data u_0, v_0 respectively, then for any $\overline{M} \geq \underline{M} = \max\{Lip(f), Lip(g)\}$, $a < 0, b > 0$, $b - a \geq 2\overline{M}t$ the function,*

$$t \mapsto \int\limits_{a+\overline{M}t}^{b-\overline{M}t} |u(x,t) - v(x,t)| dx$$

is non increasing and if $u_0 = v_0$ a.e ,then it follows that $u = v$ a.e

Proof. The proof of this stability theorem follows like the proof of theorem (2.1) of [3]. We need some lemmas below in the proof. The first step is to characterize the interface entropy condition (7) which is done in the following lemma. □

Lemma 1. *Let (f, g) satisfy H_1, H_2 and the connection (A_0, B_0) be such that H_3 is satisfied. Let $u^\pm$ be such that $f(u^+) = g(u^-)$ and $I(u^-, u^+, A_i, B_i) \geq 0$ for $i = 1, 2$. Then (u^-, u^+) satisfy the following conditions,*
1. Let $N_+(f) = N_-(g) = 1$.
Let $\overline{A}_1 \leq \theta_{1g}, \overline{A}_2 \geq \theta_{2g}$ and $\overline{B}_1 \leq \theta_{1f}, \overline{B}_2 \geq \theta_{2f}$ be such that

$$\begin{array}{rcll} g(\overline{A}_1) & = & g(A) & \\ g(\overline{A}_2) & = & \min(g(S), g(A)) & \text{if} \quad g(s) \leq g(A) \\ f(\overline{B}_1) & = & f(B) & \\ f(\overline{B}_2) & = & \max(f(B), f(S)) & \text{if} \quad g(s) \geq g(A) \end{array}$$

let $g(s) \leq g(A)$, then the following holds,
(i). if $u^- \leq A$, then either $u^- = A$ or $u^- = \overline{A}_1$
(ii). if $u^- > A$ then either $g(u^-) \geq g(A)$ or $u^+ \geq \theta_{2f}$
let $g(s) \geq g(A)$, then

(iii). if $u^+ \leq B$, *then either* $u^+ = B$ *or* $u^+ < \overline{B}_1$
(iv). if $u^+ > B$, *then either* $f(u^+) \leq f(B)$ *or* $u^- \geq \theta_{2g}$.

2. Let $N_+(f) = N_-(g) = 2$
For $i = 1, 2$, *let* $\overline{A}_i, s_g, S_g \in [\theta_{1g}, \theta_{2g}], \overline{B}_i \in [\theta_{1f}, \theta_{2f}]$ *be such that* $f(\overline{B}_i) = f(B_i)$, $g(\overline{A}_i) = g(A_i), g(s) = g(s_g), g(S) = g(S_g)$, *then we have that*
(i). If $u^+ \in [s, \overline{B}_1]$ *either* $f(u^+) = f(B_1)$ *or* $u^+ < B_1$ *and* $u^- \in [s_g, \overline{A}_1]$
(ii). If $u^+ \in [\overline{B}_2, S]$, *then either* $f(u^+) = f(B_2)$ *or* $u^+ > B_2$ *and* $u^- \in [\overline{A}_2, S_g]$.

3. Let $N_+(f) = 1, N_-(g) = 2$.
Let $\overline{A}_1, \overline{A}_2 \in [\theta_{1g}, \theta_{2g}], \overline{B} \geq \theta_{2f}, \overline{B}_2 \leq \theta_{1f}$ *be such that for* $i = 1, 2$, *we have that* $g(A_i) = g(\overline{A}_i)$, $f(\overline{B}_i) = f(B_i)$ *then the following holds,*
(i). If $u^+ \geq B_1$, *then either* $f(u^+) = f(B_1)$ *or* $u^+ \geq \overline{B}_1$. *Furthermore, if* $u^+ \geq \overline{B}_1$, *then* $u^- \geq A_2$.
(ii). Let $u^+ \in (B_2, B_1)$ *then* $u^- \in (\overline{A}_2, \overline{A}_1)$
(iii). Let $u^+ \geq B_2$, *then either* $f(u^+) = f(B_2)$ *or* $u^+ \leq \overline{B}_2$. *Furthermore if* $u^+ \leq \overline{B}_2$ *and* $f(u^+) \leq f(B_1)$ *then* $u^- \leq A_1$.

4.Let $N_+(f) = 2, N_-(g) = 1$.
Let $\overline{B}_1, \overline{B}_2 \in [\theta_{1f}, \theta_{2f}], \overline{A_1} \leq \theta_{1g}, \overline{A}_2 \geq \theta_{2g}$ *be such that for* $i = 1, 2$, *we have that* $g(A_i) = g(\overline{A}_i)$, $f(\overline{B}_i) = f(B_i)$ *then the following holds,*
(i). If $u^- \leq A_1$, *then either* $g(u^-) = g(A_1)$ *or* $u^- \leq \overline{A}_1$. *Furthermore, if* $u^- \leq \overline{A}_1$, *then* $u^+ \leq B_2$.
(ii). Let $u^- \in (A_1, A_2)$ *then* $u^+ \in (\overline{B}_1, \overline{B}_2)$
(iii). Let $u^- \geq A_2$, *then either* $g(u^-) = g(A_2)$ *or* $u^- \geq \overline{A}_2$. *Furthermore if* $u^- \geq \overline{A}_2$ *and* $g(u^-) \leq g(A_1)$ *then* $u^+ \geq B_1$

Proof. For $i = 1, 2$, let $I_i = I(u^-, u^+, A_i, B_i)$ then from the Rankine-Hugoniot condition (5) and the interface entropy condition (7), we have that

$$\begin{aligned} 0 \leq I_i &= (g(u^-) - g(A_i))(sign(u^- - A_i) - sign(u^+ - B_i)) \\ &= (f(u^+) - f(B_i))(sign(u^- - A_i) - sign(u^+ - B_i)) \end{aligned}$$

We have to consider the following cases,
1. $N_+(f) = N_-(g) = 1$
Let $g(s) \leq g(A)$ and $u^- < A$. If $u^- \geq \overline{A}_1$, then $g(u^-) > g(A)$ and hence $I_1 < 0$ unless $u^+ < B$. If $u^+ < B$, then $f(u^+) \leq f(B) = g(A) < g(u^-) = f(u^+)$ which is a contradiction. Hence $u^- \leq \overline{A}_1$.
Let $u^- > A$, if $g(u^-) < g(A)$, then $u^+ > B$. By the Rankine-Hugoniot condition (5), we have that $u^+ \geq \theta_{2f}$. Similarly, if $g(s) \geq g(A)$. This proves (*1*).
2. Let $N_+(f) = N_-(g) = 2$
Let $u^+ \leq \overline{B}_1$. Then if $f(u^+) > f(B_1)$, we have that $u^+ > B_1$ and hence $u^- \geq A_1$. Therefore, $f(B_1) < f(u^+) = g(u^-) \leq g(A_1) = f(B_1)$ which is a contradiction. Hence $f(u^+) \leq f(B)$ and this in turn implies that $u^+ \leq B_1$. Let $u^+ > \overline{B}_2$, then if $f(u^+) < f(B_2) = g(A_2) \leq g(u^-) = f(u^+)$ which is a contradiction. Hence $u^+ \geq B_2$ and $u^- \geq A_2$. Again from the Rankine-Hugoniot condition (5), we have that $u^- \in [\overline{A}_2, S_g]$ and this proves (*2*).
3. Let $N_+(f) = 1, N_-(g) = 2$
Let $u^+ > B_1$. Suppose $f(u^+) > f(B_1)$, then from $I_1 > 0$, we have that $u^- > A_1$. This implies that $g(u^-) < g(A_1) = f(B_1) < f(u^+) = g(u^-)$ which is a

contradiction. Hence $u^+ \geq \overline{B}_1$. Suppose that $f(u^+) \geq f(B_2)$, then from $I_2 \geq 0$, we get that $u^- \geq A_2$. If $f(u^+) < f(B_2)$, then by the Rankine-Hugoniot condition (5) we get that $u^- \geq A_2$. This proves (i) of (*3*).

Let $u^+ \in (B_2, B_1)$. Then $I_1 \geq 0$ implies that $u^- < A_1$ and $I_2 \geq 0$ implies that $u^- \geq A_2$. Now from the Rankine-Hugoniot condition, we get that $u^- \in [\overline{A}_2, \overline{A}_1]$. This proves (ii) of (*3.*).

Let $u^+ < B_2$. Suppose $f(u^+) < f(B_2)$, then from $I_2 \geq 0$, we have that $u^- < A_2$. Hence $g(u^-) > g(A_2) = f(B_2) > f(u^+) = g(u^-)$ which is a contradiction. Hence $u^+ \leq \overline{B}_2$. Suppose $f(u^+) \leq f(B_1)$. Then from $I_1 \geq 0$, we have $u^- \geq A_2$. This proves (iii) thus completing the proof of *3*. The proof of *4* follows in a similar way thus completing the proof of the lemma. □

The above lemma helped us to characterize the interface entropy condition. Next we prove the crucial comparison lemma below,

Lemma 2. *Let (f, g), (A_0, B_0) as in lemma (1). Let $u^\pm, v^\pm \in I$ satisfy the Rankine Hugoniot condition $(g(u^-), g(v^-)) = (f(u^+), f(v^+))$ and the interface entropy condition (7) relative to the connection (A_0, B_0). Then*

$$I(u^-, u^+, v^-, v^+) \geq 0$$

Proof. From the Rankine-Hugoniot condition, we have

$$\begin{aligned} I_0 &= (g(u^-) - g(v^-))(sign(u^- - v^-) - sign(u^+ - v^+)) \\ &= (f(u^+) - f(v^+))(sign(u^- - v^-) - sign(u^+ - v^+)) \end{aligned}$$

From the symmetry of $u^\pm, v^\pm$ in I_0, without loss of generality we can assume that $g(u^-) > g(v^-)$ and hence $f(u^+) > f(v^+)$. This implies that $I_0 \geq 0$ whenever $u^- \geq v^-$ or $u^+ \leq v^+$. Therefore we assume that $u^- < v^-$ and $u^+ > v^+$ and show that this case is never possible. We have to consider different cases given below,

1. Let $N_+(f) = N_-(g) = 1$

Let $g(s) \leq g(A)$. If $u^- < A$, then by lemma (1), we get that $u^- \leq \overline{A}_1$. If $v^- \leq A$, then $g(u^-) \leq g(v^-) < g(u^-)$ which is a contradiction. Since $g(v^-) < g(u^-) \leq g(A)$ and hence if $v^- \geq A$, then $v^+ \geq \theta_{2f}$. Since $v^+ < u^+$, therefore by the monotonicity of f in $[\theta_{2f}, S]$, $g(v^-) = f(v^+) > f(u^+) = g(u^-)$ which is a contradiction.

Let $u^- \geq A$, hence we have that $v^- > A$. By lemma (1), if $g(v^-) \geq g(A)$, then $g(u^-) > g(v^-) \geq g(A)$ and hence $u^- > \theta_{2g}$. Since $u^- < v^-$ and by monotonicity of g in $[\theta_{2g}, S]$, we get that $g(u^-) < g(v^-)$ which is a contradiction. If $g(v^-) < g(A)$, then $v^+ \geq \theta_{2f}$ and hence $u^+ \geq v^+ \geq \theta_{2f}$. This implies that $f(u^+) \leq f(v^+) < f(u^+)$ which is a contradiction. Similar argument follows if we assume that $g(s) \geq g(A)$. This proves that $I_0 \geq 0$ in this case.

2. Let $N_+(f) = N_-(g) = 2$

From Rankine-Hugoniot condition, $f(u^+) = g(u^-) > g(v^-) = f(v^+)$. Let $u^+ \in [s, \overline{B}_1]$. Since $v^+ < u^+$ and hence from (i of **2**) of lemma (1), we have that $v^+ \leq B_1$ and $u^-, v^- \in [s_g, \overline{A}_1]$. By monotonicity of g in $[s_g, \overline{A}_1]$, it follows that $g(u^-) \leq g(v^-) < g(u^-)$ which is a contradiction.

Let $u^+ \in [\overline{B}_1, B_2)$, then $u^+ \in [\overline{B}_1, \overline{B}_2]$. By the monotonicity of f in $[\overline{B}_1, \overline{B}_2]$, we get that $f(v^+) > f(u^+) > f(v^+)$ which is a contradiction.

If $v^+ < B_1$, then $v^- \in [s_g, A_1]$ and $u^- < v^-$. Hence $g(u^-) < g(v^-) < g(u^-)$ which is a contradiction.

Let $u^+ \geq B_2$, then $u^- \in [\overline{A}_2, S_g]$. Hence $g(v^-) \geq g(u^-) > g(v^-)$ which is a contradiction. This proves that $I_0 \geq 0$ in this case.

3. Let $N_+(f) = 1, N_-(g) = 2$

Let $u^+ \in [s, B_2]$, then $v^+ < B_2$ and hence $f(v^+) > f(u^+) > f(v^+)$ which is a contradiction. Let $u^+ \in (B_2, B_1]$, suppose that $v^+ \leq B_2$, then $f(v^+) \leq f(u^+) \leq f(B_1)$ and hence from lemma (1) we get that $v^- < \overline{A}_1, u^- \in (\overline{A}_2, \overline{A}_2)$. If $v^+ > B_2$, then $v^+ \in (B_1, B_2)$ and hence $v^- \in (\overline{A}_2, \overline{A}_2)$. Therefore by the monotonicity of g in this region, we get that $g(u^-) < g(v^-) < g(u^-)$ which is a contradiction.

Let $u^+ > B_1$, then from lemma (1), we have that $u^+ \geq \overline{B}_1$ and $A_2 \leq u^- \leq \overline{A}_1$. Since $v^+ < u^+$, $f(v^+) < f(u^+)$ and hence from lemma (1), we have that $A_2 \leq u^- < v^- < \overline{A}_1$. Since $f(u^-) > f(v^-)$ and hence $A_2 \leq u^- < v^- < \overline{A}_2$. Therefore from the Rankine-Hugoniot condition, we get that $v^+ \in (B_2, \overline{B}_2)$. Hence $I(v^-, v^+, A_2, B_2) < 0$ which is a contradiction. This proves $I_0 \geq 0$. A similar proof works in the case where $N_+(f) = 2, N_-(g) = 1$. This completes the proof of the lemma (2). □

The proof of theorem (1) follows exactly in the same way as the proof of theorem (2.1) of [3] and is based on the comparison lemma (2). We omit the details of the the proof. Thus, we have shown that for each choice of connection (A_0, B_0), the corresponding A_0B_0-entropy solutions are stable and and are in fact L^1 contractive.

Remark 1. (1). Let $N_+(f) = 2, N_-(g) = 1$, then the compatibility condition $A_1 \leq A_2$ follows from the fact that (A_1, B_1) and (A_2, B_2) are A_0B_0-entropy solutions.
(2) Similarly if $N_+(f) = 1, N_-(g) = 2$, then $B_1 \leq B_2$

Proof: As (A_1, B_1) and (A_2, B_2) are entropy solutions, this implies that $0 \leq I(A_1, B_1, A_2, B_2) = (sign(A_1 - A_2) - sign(B_1 - B_2))(g(A_1) - g(A_2)$. Suppose that $A_2 < A_1$, then by the flux geometry in this case, we have that $I(A_1, B_1, A_2, B_2) < 0$ which is a contradiction.

In this section, we have formulated the concept of interface connection and the corresponding entropy solutions and showed that they are stable and hence unique. Next, we will show in the remaining part of this paper that the entropy solutions exist.

3. Godunov type numerical scheme.

For a fixed interface connection vector (A_0, B_0), we have shown the uniqueness and stability of the A_0B_0-entropy solutions. Next, we will show that the solutions exist. We propose a Godunov type finite volume scheme based on exact solutions of the Riemann problem associated with (2) and show that the approximate solutions computed by the scheme converge to an A_0B_0-entropy solution.

Definition 9. Godunov Numerical flux: Let $h \in Lip_{loc}(I)$, then the Godunov numerical flux (see [11]) denoted by $H(a, b)$ is given by

$$H(a,b) = \begin{cases} \min\limits_{\theta \in [a,b]} h(\theta) & \text{if} \quad a \leq b \\ \max\limits_{\theta \in [b,a]} h(\theta) & \text{if} \quad a \geq b \end{cases} \tag{8}$$

As in [2],[3], we define an interface flux based on exact solutions of the Riemann problem for (2). The detailed solution of the Riemann problem is similar to those constructed in [9, 7] and is modified to take into account the new interface entropy condition. The detailed solutions of the Riemann problem are provided in [25]. We use them to give explicit formulas for the interface flux below,
i. Let $N_+(f) + N_-(g) = 1$

In this case we have that $A_1 = A_2 = A$ and $B_1 = B_2 = B$. the interface flux is given by,

$$\begin{aligned} F_{A_0B_0}(\alpha,\beta) &= \min(G(\alpha,A),F(S,\beta)) \quad \text{if} \quad g(s)\le g(A) \\ &= \max(G(\alpha,S),F(B,\beta)) \quad \text{if} \quad g(s)\ge g(A) \end{aligned} \tag{9}$$

ii. $N_+(f) = N_-(g) = 2$

Let $\overline{A}_i, \overline{B}_i, s_g$ be as defined before. Further define,

$$\begin{aligned} \tilde{f}(\theta) &= f(B_1) \quad \text{if} \quad \theta \in [B_1, \overline{B}_1] \\ &= f(B_2) \quad \text{if} \quad \theta \in [\overline{B}_2, B_2] \\ &= f(\theta) \quad\quad \text{otherwise} \\ \tilde{g}(\theta) &= g(A_1) \quad \text{if} \quad \theta \in [\overline{A}_1, A_1] \\ &= g(A_2) \quad \text{if} \quad \theta \in [A_2, \overline{A}_2] \\ &= g(\theta) \quad\quad \text{otherwise} \end{aligned}$$

then the interface flux is given by,

$$\begin{aligned} F_{A_0B_0}(\alpha,\beta) &= \max(\tilde{F}(s,\beta),\tilde{G}(\alpha,S)) \quad \text{if} \quad \alpha\le s_g \\ &= \min(\tilde{F}(S,\beta),\tilde{G}(\alpha,s)) \quad \text{if} \quad \alpha\ge s_g \end{aligned} \tag{10}$$

where $\tilde{G}, \tilde{F}$ are the standard Godunov fluxes corresponding to the functions $\tilde{g}$ and $\tilde{f}$ respectively.

iii. $N_+(f) = 1, N_-(g) = 2$

Let $\overline{A}_1, \overline{A}_2 \in [\theta_{1g}, \theta_{2g}]$ be such that $g(A_i) = g(\overline{A}_i)$ for $i = 1, 2$ and define,

$$\begin{aligned} \tilde{g}(\theta) &= g(\theta) \quad\quad \text{if} \quad \theta \in [\overline{A}_2, \overline{A}_1] \\ &= g(A_1) \quad \text{if} \quad \theta \ge \overline{A}_1 \\ &= g(A_2) \quad \text{if} \quad \theta \le \overline{A}_2 \end{aligned}$$

then the interface flux is given by,

$$F_{A_0B_0}(\alpha,\beta) = \max(\min(F(B_1,\beta),\tilde{G}(\alpha,A_1)),F(B_2,\beta)) \tag{11}$$

where $\tilde{G}$ is the standard Godunov flux corresponding to the function $\tilde{g}$.

iv. $N_+(f) = 2, N_-(g) = 1$

Let $\overline{B}_1, \overline{B}_2 \in [\theta_{1f}, \theta_{2f}]$ be such that $f(B_i) = f(\overline{B}_i)$ for $i = 1, 2$ and define,

$$\begin{aligned} \tilde{f}(\theta) &= f(B_1) \quad \text{if} \quad \theta \in [B_1, \overline{B}_1] \\ &= f(B_2) \quad \text{if} \quad \theta \in [\overline{B}_2, B_2] \\ &= f(\theta) \quad\quad \text{otherwise} \end{aligned}$$

then the interface flux is given by,

$$F_{A_0B_0}(\alpha,\beta) = \min(\max(\tilde{F}(B_1,\beta),G(\alpha,A_1)),G(\alpha,A_2)) \tag{12}$$

where $\tilde{F}$ is the standard Godunov flux corresponding to the function $\tilde{f}$.

Next, we describe the discretization in space and time and of the initial data as follows,

Let $h > 0$ and define the space grid points x_j as follows.

$$x_j = \left(\frac{2j-1}{2}\right)h \quad \text{for } j \ge 1, \quad x_j = \left(\frac{2j+1}{2}\right)h \quad \text{for } j \le -1.$$

For time discretization, the time step $\Delta t > 0$ and let $t_n = n\Delta t$. We also introduce $\lambda = \frac{\Delta t}{h}$.

For a function $u_0 \in L^\infty(\mathbb{R})$ we define

$$
\begin{aligned}
u^0_{j+1} &= \frac{1}{h}\int_{x_{j+1/2}}^{x_{j+3/2}} u_0(x)dx \quad \text{if } j \geq 0, u^0_{j-1} = \frac{1}{h}\int_{x_{j-3/2}}^{x_{j-1/2}} u_0(x)dx \quad \text{if } j \leq 0 \\
N^h_{A_0B_0}(f,g,\{u^0_i\}) &= \sum_{i \leq -2} |G(u^0_i, u^0_{i+1}) - G(u^0_{i-1}, u^0_i)| \\
&\quad + \sum_{i \geq 1} |F(u^0_i, u^0_{i+1}) - F(u^0_{i-1}, u^0_i)| \\
&\quad + |F_{A_0B_0}(u^0_{-1}, u^0_1) - G(u^0_{-2}, u^0_{-1})| \\
&\quad + |F(u^0_1, u^0_2) - F_{A_0B_0}(u^0_{-1}, u^0_1)|
\end{aligned}
$$

$$
N_{A_0B_0}(f,g,u_0) = \sup_{h>0} N^h_{A_0B_0}(f,g,u^0_i) \tag{13}
$$

It is easy to see that if $u_0 \in BV(\mathbb{R})$, then $N_{A_0B_0}(f,g,u_0) \leq C\|u_0\|_{BV(\mathbb{R})}$ where $C > 0$ is a constant depending only on the Lipschitz constant of f and g. Now we can define the Godunov type finite difference (finite volume) scheme as,

$$
\begin{aligned}
u^{n+1}_i &= u^n_i - \lambda(F(u^n_i, u^n_{i+1}) - F(u^n_{i-1}, u^n_i)) && \text{if } i \geq 2 \\
u^{n+1}_1 &= u^n_1 - \lambda(F(u^n_1, u^n_2) - F_{A_0B_0}(u^n_{-1}, u^n_1)) \\
u^{n+1}_{-1} &= u^n_{-1} - \lambda(F_{A_0B_0}(u^n_{-1}, u^n_1) - G(u^n_{-2}, u^n_{-1})), \\
u^{n+1}_i &= u^n_i - \lambda(G(u^n_i, u^n_{i+1}) - G(u^n_{i-1}, u^n_i)) && \text{if } i \leq -2
\end{aligned} \tag{14}
$$

Observe that this is a Godunov scheme for $i \neq \pm 1$ and that for $i = \pm 1$, the scheme is not consistent, that is in general $F_{A_0B_0}(u,u)$ need not be equal to $f(u)$ or $g(u)$. Define the following approximating functions,

$$
u^h(x,t) = u^n_i \text{ for } (x,t) \in [x_{i-/2}, x_{i+1/2}) \times [n\Delta t, (n+1)\Delta t), \quad i \neq 0 \tag{15}
$$

Our aim will be to show that for each choice of interface connection vector (A,B), the approximations u^h converge to weak solution of (2) and satisfy the interior entropy condition (6). Furthermore, for each choice of the connection, we will show that the limit will satisfy the corresponding interface entropy condition (7).

4. **Convergence analysis.** The convergence analysis for the scheme (14) is very similar to the convergence analysis of [2, 3]. We start with the following easy to check proposition (without proof).

Proposition 4.1. Let $F_{A_0B_0}$ be as given above and $a, b \in [s,S]$, then the following holds.

(a): $F_{A_0B_0}$ is Lipschitz in each variable with a Lipschitz constant M which is given by $M = \max(\text{ lip } f, \text{ lip } g)$.

(b) : $F_{A_0B_0}$ is non decreasing in a and non increasing in b.

(c): $F_{A_0B_0}(s,s) = f(s) = g(s), F_{A_0B_0}(S,S) = f(S) = g(S)$.

(d): $F_{A_0B_0}$ is not consistent, i.e $F_{A_0B_0}(a,a)$ is not necessarily equal to either $f(a)$ or $g(a)$

Next, we have the following lemma

Lemma 3. *Let $2\lambda M \leq 1$ and $a \in [s,S]$, we have:*

1. The scheme (14) is monotone.

2. The scheme is discrete L^1 contractive, i.e it satisfies,

$$
\sum_{j \neq 0} |u^{n+1}_j - u^n_j| \leq \sum_{j \neq 0} |u^n_j - u^{n-1}_j| \tag{16}
$$

3. Let $u_0 \in L^\infty(\mathbb{R}, [s,S])$ be the initial data and $\{u^n_j\}$ be solutions, calculated by the scheme (14) then

$$
s \leq u^n_j \leq S \quad \forall\, j \text{ and} \forall n \tag{17}
$$

Proof. Follows exactly as in the proof of Lemma (4.3) of [2] by using the CFL condition and the hypothesis that the fluxes intersect at end-points.

The next step as in [2] and [3] is to obtain TV bounds in terms of a singular mapping. It is well acknowledged in literature that the singular mapping is difficult to implement with sign changing coefficients and non-convex fluxes. Some results in this direction were obtained by Towers in [28] for a staggered version of the Enquist-Osher scheme by the use of discrete entropy inequalities. In [24], the problem of sign-changing coefficients was tackled by using a combination of singular mappings instead of one singular mapping. The key point of this section is a suitable modification of the singular mapping approach to handle the non convex fluxes considered in this paper and for the Godunov type scheme (14). In order to obtain bounds in terms of the flux variation, we cannot directly work with the singular mapping defined in terms of the fluxes f and g (see [27],[2], [3]) but we need to split the fluxes into their concave-type and convex-type parts. Then, we show that the total variation of these modified singular mappings can be bounded by the discrete L^1 contractivity estimates. But as we are constrained to have monotone singular mappings, we need to consider separately a residual or defect term whose variation is then shown to be bounded by a "chain" estimate. We start by defining the splitting of the fluxes. Note that we present the analysis in the case where the fluxes are of the under-compressive type (see section 2 and figure 2 for definitions and notation). We fix the interface connection vector (A_0, B_0) for the subsequent presentation. We start with the splitting of the fluxes,

Definition 10. For $h \in \mathbb{F}_3(I)$, let $\theta_{1h} < \theta_{2h}$ be the interior points of extrema. Then define,

$$\begin{aligned} \mathbb{F}_3^+(I) &= \{h \in \mathbb{F}_3(I) : \theta_{1h} \text{ is a local maximum}\} \\ \mathbb{F}_3^-(I) &= \{h \in \mathbb{F}_3(I) : \theta_{1h} \text{ is a local minimum}\} \end{aligned}$$

Let $h \in \mathbb{F}_3$ and H be the standard Godunov flux corresponding to h. Split h into three parts (h_1, h_2, h_3) as follows,

Case 1: Let $h \in \mathbb{F}_3^+(I)$.

Then define $\overline{s}, \overline{S} \in [\theta_{1h}, \theta_{2h}]$ such that

$$\begin{aligned} h(\overline{s}) &= \max(h(s), h(\theta_{2h})) \\ h(\overline{S}) &= \min(h(S), h(\theta_{1h})) \end{aligned}$$

Then define,

$$\begin{aligned} h_1(\theta) &= \min(H(\theta, s), H(S, \theta)) \\ h_2(\theta) &= \max(H(s, \theta), H(\theta, S)) \end{aligned}$$

And h_3 is given by,

(a.) Let $h(s) \geq h(S)$

$$h_3(\theta) = \frac{h(s) + h(S)}{2} \quad \forall \theta \in I$$

(b.) Let $h(s) \leq h(S)$

$$\begin{aligned} h_3(\theta) &= h(S) \quad \text{if} \quad \theta \leq \overline{S} \\ &= h(\theta) \quad \text{if} \quad s \in [\overline{S}, \overline{s}] \\ &= h(s) \quad \text{if} \quad \theta \geq \overline{s} \end{aligned}$$

Case 2: Let $h \in \mathbb{F}_3^-(I)$.

Then define $\overline{s}, \overline{S} \in [\theta_{1h}, \theta_{2h}]$ such that

$$\begin{aligned} h(\overline{s}) &= \min(h(s), h(\theta_{2h})) \\ h(\overline{S}) &= \max(h(S), h(\theta_{1h})) \end{aligned}$$

Then define,

$$\begin{aligned} h_1(\theta) &= \max(H(s,\theta), H(\theta, S)) \\ h_2(\theta) &= \max(H(\theta, s), H(S,\theta)) \end{aligned}$$

And h_3 is given by,
(a.) Let $h(s) \leq h(S)$

$$h_3(\theta) = \frac{h(s)+h(S)}{2} \quad \forall \theta \in I$$

(b.) Let $h(s) \geq h(S)$

$$\begin{aligned} h_3(\theta) &= h(S) \quad \text{if} \quad \theta \leq \overline{S} \\ &= h(\theta) \quad \text{if} \quad s \in [\overline{S}, \overline{s}] \\ &= h(s) \quad \text{if} \quad \theta \geq \overline{s} \end{aligned}$$

Observe that the split fluxes g_1, f_1 are of the concave type (i.e have at most one maximum and no minima) and g_2 and f_2 are of the convex type (i.e have at most one minimum and no maxima). In this case where $g(s) < g(S)$, we end up with a non-constant defect term g_3 in g.

TV bounds: For $a, b \in \mathbb{R}$ and $k \in Lip(I)$ be such that k has at most one strict interior minimum or maximum θ_k. Then define $a_+ = \max(a, 0), a_- = \min(a, 0), a = a_+ + a_-, |a| = a_+ - a_-$ and

$$\chi(a,b) = \begin{cases} 1 & \text{if} a \leq b, \\ 0 & \text{if} a > b, \end{cases} \tag{18}$$

(i). Let θ_k be the strict local maximum, then

$$\chi_+(k'(\theta)) = \begin{cases} 1 & \text{if}\theta < \theta_k \\ 0 & \text{if}\theta \geq \theta_k \end{cases} \qquad \chi_-(k'(\theta)) = \begin{cases} 0 & \text{if}\theta < \theta_k \\ 1 & \text{if}\theta \geq \theta_k \end{cases} \tag{19}$$

(ii.) Let θ_k be the strict local minimum, then

$$\chi_+(k'(\theta)) = \begin{cases} 1 & \text{if}\theta > \theta_k \\ 0 & \text{if}\theta \leq \theta_k \end{cases} \qquad \chi_-(k'(\theta)) = \begin{cases} 1 & \text{if}\theta < \theta_k \\ 0 & \text{if}\theta \geq \theta_k \end{cases} \tag{20}$$

We also recall the following lemma from ([3], lemma (5.4)) below,

Lemma 4. *Let $\{u_1, u_2, u_3\} \in I$, $k \in Lip(I)$ having at most one interior extremum θ_k. Let K be the Godunov flux corresponding to k. Then we have that*
Case 1. Let θ_k be a maximum. then,

$$\chi(u_2,u_1)\int_{u_2}^{u_1} k'_+(\theta)d\theta \leq \chi(u_2,u_1)(\chi_+(k'(u_2))(K(u_1,u_2)-k(u_2)) \tag{21}$$

and if $u_2 \leq \theta_k$, then $k(u_2) \geq K(u_2, u_3)$.

$$-\chi(u_2,u_1)\int_{u_2}^{u_1} k'_-(\theta)d\theta \leq \chi(u_2,u_1)(\chi_-(k'(u_2))(K(u_1,u_2)-k(u_1)) \tag{22}$$

and if $u_1 \geq \theta_k$, then $k(u_1) \geq K(u, u_3)$ for all $u \in I$.
Case 2. Let θ_k be a point of minimum, then

$$\chi(u_2,u_3)\int_{u_2}^{u_3} k'_+(\theta)d\theta \leq \chi(u_2,u_3)(\chi_+(k'(u_3))(k(u_3)-K(u_2,u_3)) \tag{23}$$

and if $u_3 \geq \theta_k$*, then* $k(u_3) \leq K(u_3, u) \quad \forall \quad u \in I$.

$$-\chi(u_2, u_3)\int_{u_2}^{u_3} k'_-(\theta)d\theta \leq \chi(u_2, u_3)(\chi_-(k'(u_2))(k(u_2) - K(u_2, u_3)) \tag{24}$$

and if $u_2 \leq \theta_k$*, then* $k(u_1) \leq K(u_1, u_2)$.

Next, we define the singular mappings in this case. Let $h \in \mathbb{F}_3$ and (h_1, h_2, h_3) be the above defined splitting of the fluxes. For $\alpha \in I$ fixed, $k = 1, 2, 3$, define the singular mappings ψ_k by

$$\psi_k(h, \alpha, u) = \int_{\alpha}^{u} |h'_k(\theta)|d\theta$$

Let $\{v_i\}_{i\in\mathbb{Z}} \subset I$ be a sequence and define the new sequences $\{z_{k,i}\}_{i\in\mathbb{Z}}$ with $k = 1, 2, 3$ as

$$z_{k,i} = \psi_k(h, \alpha, v_i)$$

then we have the following,

Lemma 5. *Let* $h \in \mathbb{F}_3(I)$*,* $\{v_i\} \subset I$ *be a sequence and* $\{z_{k,i}\}$ *be defined as above, then*
Case 1. If $h \in \mathbb{F}_3^+(I)$*, then the following holds*

$$\begin{aligned}
\frac{1}{2}\sum_{-\infty}^{-2}|z_{1,i} - z_{1,i+1}| &\leq \frac{1}{2}(z_{1,-1} - \liminf_{l\to-\infty} z_{1,l}) + \sum_{-\infty}^{-2}|H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| \\
&+ \chi(v_{-1}, v_{-2})\chi_+(h'_1(v_{-1}))(H_1(v_{-2}, v_{-1}) - h_1(v_{-1})) \qquad (25) \\
\frac{1}{2}\sum_{1}^{\infty}|z_{1,i} - z_{1,i+1}| &\leq \frac{1}{2}(\limsup_{l\to\infty} z_{1,l} - z_{1,1}) + \sum_{2}^{\infty}|H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| \\
&+ \chi(v_2, v_1)\chi_-(h'_1(v_1))(H_1(v_1, v_2) - h_1(v_1)) \qquad (26) \\
\frac{1}{2}\sum_{-\infty}^{-2}|z_{2,i} - z_{2,i+1}| &\leq \frac{1}{2}(\limsup_{l\to-\infty} z_{2,l} - z_{2,-1}) + \sum_{-\infty}^{-2}|H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| \\
&+ \chi(v_{-2}, v_{-1})\chi_+(h'_2(v_{-1}))(h_2(v_{-1}) - H_2(v_{-2}, v_{-1})) \qquad (27) \\
\frac{1}{2}\sum_{1}^{\infty}|z_{2,i} - z_{2,i+1}| &\leq \frac{1}{2}(z_{2,1} - \liminf_{l\to\infty} z_{2,l}) + \sum_{2}^{\infty}|H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| \\
&+ \chi(v_{-2}, v_{-1})\chi_+(h'_2(v_1))(h_2(v_1) - H_2(v_1, v_2)) \qquad (28)
\end{aligned}$$

Case 2. Let $h \in \mathbb{F}_3^-(I)$, then the following holds

$$\begin{aligned}
\frac{1}{2}\sum_{-\infty}^{-2}|z_{1,i}-z_{1,i+1}| &\leq \frac{1}{2}(\limsup_{l\to-\infty} z_{1,l}-z_{1,-1})+\sum_{-\infty}^{-2}|H(v_i,v_{i+1})-H(v_{i-1},v_i)| \\
&+ \chi(v_{-2},v_{-1})\chi_+(h_1'(v_{-1}))(h_1(v_{-1})-H_1(v_{-2},v_{-1})) \quad (29)\\
\frac{1}{2}\sum_{1}^{\infty}|z_{1,i}-z_{1,i+1}| &\leq \frac{1}{2}(z_{1,1}-\liminf_{l\to\infty} z_{1,l})+\sum_{2}^{\infty}|H(v_i,v_{i+1})-H(v_{i-1},v_i)| \\
&+ \chi(v_1,v_2)\chi_-(h_1'(v_1))(h_1(v_1)-H_1(v_2,v_2)) \quad (30)\\
\frac{1}{2}\sum_{-\infty}^{-2}|z_{2,i}-z_{2,i+1}| &\leq \frac{1}{2}(z_{2,-1}-\liminf_{l\to-\infty} z_{2,l})+\sum_{-\infty}^{-2}|H(v_i,v_{i+1})-H(v_{i-1},v_i)| \\
&+ \chi(v_{-1},v_{-2})\chi_+(h_2'(v_{-1}))(H_2(v_{-2},v_{-1})-h_2(v_{-1})) \quad (31)\\
\frac{1}{2}\sum_{1}^{\infty}|z_{2,i}-z_{2,i+1}| &\leq \frac{1}{2}(\limsup_{l\to\infty} z_{2,l}-z_{2,1}))+\sum_{-2}^{\infty}|H(v_i,v_{i+1})-H(v_{i-1},v_i)| \\
&+ \chi(v_{-2},v_{-1})\chi_+(h_2'(v_1))(H_2(v_1,v_2)-h_2(v_1)) \quad (32)
\end{aligned}$$

Proof. We will prove (25) and (28) and the rest of the estimates can be proved in a similar way. Let $h \in \mathbb{F}_3^+(I)$, then

$$\begin{aligned}
\sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1})_+ + \sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1})_- &= \sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1}) \\
&= z_{1,l}-z_{1,-1}
\end{aligned}$$

hence,

$$-\sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1})_- = z_{1,-1}-z_{1,l}+\sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1})_+$$

Therefore,

$$\begin{aligned}
\sum_{i=l}^{-2}|z_{1,i}-z_{1,i+1}| &= \sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1})_+ - \sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1})_- \\
&= z_{1,-1}-z_{1,l}+2\sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1})_+
\end{aligned}$$

Hence

$$\sum_{i=-\infty}^{-2}|z_{1,i}-z_{1,i+1}| \leq z_{1,-1}-\liminf_{l\to-\infty} z_{1,l}+2\sum_{i=l}^{-2}(z_{1,i}-z_{1,i+1})_+$$

Since ψ's are non-decreasing functions and hence $(z_{1,i}-z_{1,i+1})_+ \neq 0$ implies that $v_{i+1} < v_i$ and from (21) and (22), we get that

$$\begin{aligned}
(z_{1,i}-z_{1,i+1})_+ &= \chi(v_{i+1},v_i)(z_{1,i}-z_{1,i+1}) \\
&= \chi(v_{i+1},v_i)\int_{v_{i+1}}^{v_i}|h_1'(\theta)|d\theta \\
&= \chi(v_{i+1},v_i)\int_{v_{i+1}}^{v_i}h_{1,+}'(\theta)d\theta - \int_{v_{i+1}}^{v_i}h_{1,-}'(\theta)d\theta \\
&\leq \chi(v_{i+1},v_i)(\chi_+(h_1'(v_{i+1})(H_1(v_i,v_{i+1})-h_1(v_{i+1})) \\
&+ \chi_-(h_1'(v_i)(H_1(v_i,v_{i+1})-h_1(v_i)) \quad (33)
\end{aligned}$$

Let $\overline{S} \in [\theta_{1h}, \theta_{2h}]$ be such that $h(\overline{S}) = \min(h(\theta_{1h}), h(S))$. If $v_{i+1} \leq \theta_{1h}$, then by the definition of h_1, we get that

$$\begin{array}{lcl} H_1(v_i, v_{i+1}) & = & H(v_i, v_{i+1}) \\ h_1(v_{i+1}) & = & h(v_{i+1}) \geq H(v_{i+1}, v_{i+2}) \end{array}$$

Therefore, we have that $H_1(v_i, v_{i+1}) - h_1(v_{i+1}) \leq |H(v_i, v_{i+1}) - H(v_{i+1}, v_{i+2})|$. If $v_i \in [\theta_{1h}, \theta_{2h}]$ and $v_{i+1} \geq \overline{S}$, then $H(v_i, v_{i+1}) - h_1(v_{i+1}) = 0$. Therefore let $v_{i+1} < \overline{S}$, then

$$\begin{array}{lclcl} H_1(v_i, v_{i+1}) & = & h_1(\max(\theta_{1h}, v_{i+1})) & = & h(\max(\theta_{1h}, v_{i+1})) = H(v_i, v_{i+1}) \\ h_1(v_i) & \geq & H(v_{i-1}, v_i) & \text{if} & v_i \leq \overline{S} \\ h_1(v_i) = h_1(S) & \geq & H(v_{i-1}, v_i) & \text{if} & v_i \geq \overline{S} \end{array}$$

Hence

$$H_1(v_i, v_{i+1}) - h_1(v_{i+1}) \leq |H(v_i, v_{i+1}) - H(v_{i-1}, v_i)|$$

and we have the following,

$$\begin{array}{lcl} \displaystyle\sum_{i=l}^{-2} (z_{1,i} - z_{1,i+1})_+ & \leq & \chi(v_{-1}, v_{-2})\chi_+(h_1'(v_{-1}))(H_1(v_{-2}, v_{-1}) - h_1(v_{-1}) \\ & + & \displaystyle\sum_{-\infty}^{-3} \chi_+(h_1'(v_{i+1}))|H(v_i, v_{i+1}) - H(v_{i+1}, v_{i+2})| \\ & + & \displaystyle\sum_{-\infty}^{-2} \chi_-(h_1'(v_i))|H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| \\ & = & \chi(v_{-1}, v_{-2})\chi_+(h_1'(v_{-1}))(H_1(v_{-2}, v_{-1}) - h_1(v_{-1}) \\ & + & \displaystyle\sum_{-\infty}^{-2} |H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| \end{array}$$

Combining the above with (33) leads to a proof of (25). Next we will prove (28). We have that

$$\begin{array}{lcl} \sum_{i=1}^{l-1} (z_{2,i} - z_{2,i+1})_+ + \sum_{i=1}^{i-1} (z_{2,i} - z_{2,i+1})_- & = & \sum_{i=1}^{l-1} (z_{2,i} - z_{2,i+1}) \\ & = & z_{2,1} - z_{2,l} \end{array}$$

hence,

$$\sum_{i=1}^{l-1} (z_{2,i} - z_{2,i+1})_+ = z_{2,1} - z_{2,l-1} - \sum_{i=1}^{l-1} (z_{2,i} - z_{2,i+1})_-$$

Therefore,

$$\begin{array}{lcl} \sum_{i=1}^{l-1} |z_{2,i} - z_{2,i+1}| & = & \sum_{i=1}^{l-1} (z_{2,i} - z_{2,i+1})_+ - \sum_{i=1}^{l-1} (z_{2,i} - z_{2,i+1})_- \\ & = & z_{2,1} - z_{2,l} - 2\sum_{i=1}^{l-1} (z_{2,i} - z_{2,i+1})_- \end{array}$$

Hence

$$\sum_{i=1}^{\infty} |z_{2,i} - z_{2,i+1}| \leq z_{2,1} - \liminf_{l \to \infty} z_{2,l} - 2\sum_{i=1}^{\infty} (z_{1,i} - z_{1,i+1})_-$$

Now $(z_{2,i} - z_{2,i+1})_+ \neq 0$ implies that $v_{i+1} > v_i$ and from (23) and (24), we get that

$$\begin{array}{rcl} -(z_{2,i} - z_{2,i+1})_- & = & \chi(v_i, v_{i+1})(z_{2,i} - z_{2,i+1}) \\ & = & \chi(v_i, v_{i+1}) \displaystyle\int_{v_i}^{v_{i+1}} |h_2'(\theta)| d\theta \\ & \leq & \chi(v_{i+1}, v_i)(\chi_+(h_2'(v_{i+1})(h_2(v_{i+1} - H_2(v_i, v_{i+1})) \\ & + & \chi_-(h_2'(v_i)(h_2(v_i) - H_2(v_i, v_{i+1})) \end{array} \quad (34)$$

Let $\overline{s} \in [\theta_{1h}, \theta_{2h}]$ be such that $h(\overline{s}) = \max(h(\theta_{2h}), h(s))$. If $v_{i+1} \geq \theta_{2h}$, then by the definition of h_2, we get that

$$\begin{array}{rcll} H_2(v_i, v_{i+1}) & = & H(v_i, v_{i+1}) & \text{if} \quad h(s) \geq h(\theta_{2h}) \\ & \geq & H(v_i, v_{i+1}) & \text{if} \quad h(s) \leq h(\theta_{2h}) \\ h_2(v_{i+1}) & \leq & H(v_{i+1}, v_{i+2}) & \end{array}$$

Therefore, we have that $h_2(v_{i+1}) - H_2(v_i, v_{i+1}) \leq |H(v_i, v_{i+1}) - H(v_{i+1}, v_{i+2})|$. Let $v_i \leq \theta_{2h}$. If $v_{i+1} \leq \overline{s}$, then $h_2(v_i) - H_2(v_i, v_{i+1}) = 0$. Therefore let $v_{i+1} > \overline{s}$, then

$$\begin{array}{lclcl} H_2(v_i, v_{i+1}) & = & \displaystyle\min_{[v_i, v_{i+1}]} h_2 & = & \displaystyle\min_{[v_i, v_{i+1}]} h = H(v_i, v_{i+1}) \\ h_2(v_i) & = & H(v_{i-1}, v_i) & \text{if} & v_{i-1} \leq \overline{s} \\ h_2(v_i) = h_2(s) & \leq & H(v_{i-1}, v_i) & \text{if} & v_{i-1} \leq \overline{s} \end{array}$$

Hence

$$h_2(v_i) - H_2(v_i, v_{i+1}) \leq |H(v_i, v_{i+1}) - H(v_{i-1}, v_i)|$$

This above estimate together with (34) implies that

$$\begin{array}{rcl} -\sum\limits_{i=1}^{\infty} (z_{2,i} - z_{2,i+1})_- & \leq & \chi(v_1, v_2)\chi_-(h_2'(v_1))(h_2(v_1) - H_2(v_1, v_2)) \\ & & + \quad \sum\limits_{i=2}^{\infty} |H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| \end{array}$$

This together with (34) proves (28) thus completing the proof of this crucial variation lemma. ☐

Next we estimate the variation of the defect terms below,

Lemma 6. *Let $h \in \mathbb{F}_3(I)$ and $\{v_i\}_{i \in \mathbb{Z}} \subset I$ be a sequence then the following holds,*

$$\frac{1}{2} \sum_{i=-\infty}^{-2} |z_{3,i} - z_{3,i+1}| \leq \sum_{-\infty}^{-2} |H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| + |h(s) - h(S)| \quad (35)$$

$$\frac{1}{2} \sum_{1}^{\infty} |z_{3,i} - z_{3,i+1}| \leq \sum_{2}^{\infty} |H(v_i, v_{i+1}) - H(v_{i-1}, v_i)| + |h(s) - h(S)| \quad (36)$$

Proof. Let $h \in \mathbb{F}_3^+(I)$ and let $\overline{s}, \overline{S} \in [\theta_{1h}, \theta_{2h}]$ and $\tilde{S} \leq \theta_{1h}, \tilde{s} \geq \theta_{2h}$ be such that,

$$\begin{array}{rcccl} h(\overline{s}) & = & h(\tilde{s}) & = & \max(h(s), h(\theta_{2h})) \\ h(\overline{S}) & = & h(\tilde{S}) & = & \min(h(S), h(\theta_{1h})) \end{array}$$

If $h(s) \geq h(S)$, then h_3 is constant and hence the estimates (35),(36) hold trivially in this case. Hence we assume that $h(s) < h(S)$ and hence $\overline{S} < \overline{s}$ and h_3 is a non-increasing function. Define the following sequences,

$$\begin{array}{rclrcl} w_j &=& \begin{cases} v_j & \text{if} j \leq -1 \\ v_{-1} & \text{if} j \geq -1 \end{cases} & \tilde{w}_j &=& \begin{cases} v_j & \text{if} j \geq 1 \\ v_1 & \text{if} j \leq 1 \end{cases} \\ z_j &=& \psi_3(h, \alpha, w_j) & \tilde{z}_j &=& \psi_3(h, \alpha, \tilde{w}_j) \end{array} \tag{37}$$

then the following holds,

$$\sum_{j=-\infty}^{-2} |z_{3,j} - z_{3,j+1}| = \sum_{j=-\infty}^{+\infty} (z_j - z_{j+1})_+ - \sum_{j=-\infty}^{+\infty} (z_j - z_{j+1})_- \tag{38}$$

$$\sum_{j=1}^{\infty} |z_{3,j} - z_{3,j+1}| = \sum_{j=-\infty}^{+\infty} (\tilde{z}_j - \tilde{z}_{j+1})_+ - \sum_{j=-\infty}^{+\infty} (\tilde{z}_j - \tilde{z}_{j+1})_- \tag{39}$$

Let $w_{j+1} < w_j$ and associated with this, define a non-increasing chain as follows. Let $k \leq j+1, \overline{k} \geq j$ such that $c = \{w_j\}_{\overline{k} \leq j \leq k}$ satisfies,

(i). $w_k \leq w_{k-1} \ldots \leq w_{k-1_1} \leq \overline{S} \leq w_{k-l_1-1} \leq \ldots \leq w_{k-l_2} \leq \tilde{s} \leq w_{k-l_2+1}$.

(ii). $w_{k-l_2-2}, \ldots, w_{\overline{k}} \in (\tilde{s}, S]$.

(iii). $w_k < w_{k+1}, \ldots, w_{\overline{k}-1} \leq \tilde{s}$

By construction, we have for any two chains c_1 and c_2 defined by $\overline{k}_1, k_1$ and $\overline{k}_2, k_2$ respectively with either $[\overline{k}_1, k_1] = [\overline{k}_2, k_2]$ or $[\overline{k}_1, k_1] \cap [\overline{k}_2, k_2] = \phi$. Since h_3 is constant on $[s, \overline{S}]$ and $[\overline{s}, S]$ and hence if $v_{\overline{k}} \leq \overline{S}$ or $v_k \geq \overline{s}$, we have that

$$\sum_{j=\overline{k}}^{k-1} |z_j - z_{j+1}| = 0$$

Hence we define a chain to be non-trivial if we have that $\sum\limits_{j=\overline{k}}^{k-1} |z_j - z_{j+1}| \neq 0$.

Let $c = \{w_j\}_{j=\overline{k}}^{k}$ be a non-trivial chain. Then we have

$$\begin{aligned} \sum_{j=\overline{k}}^{k-1} (z_j - z_{j+1})_+ &= \sum_{j=k-l_2-1}^{k-l_1-1} (z_j - z_{j+1})_+ \\ &= z_{k-l_2-1} - z_{k-l_1} \\ &= h_3(w_{k-l_1}) - h_3(w_{k-l_2-1}) \qquad (40) \\ &= h_3(w_{k-l_1}) - h_3(w_{\overline{k}}) \qquad (41) \end{aligned}$$

Since $h_3(w_{k-l_2-1}) = h_3(w_{\overline{k}}) = h(s)$. As the chain is non-trivial, if $w_{k-l_1} \geq \overline{S}$ or $w_{k-l_1} \leq \overline{S} \leq w_{k-l_1+1}$, from (iii) and (i), we have

$$\begin{aligned} h_3(w_{k-l_1}) &= \max_{[w_{k-l_1}, w_{k-l_1+1}]} h_3 \\ &\leq \max_{[w_{k-l_1}, w_{k-l_1+1}]} h \\ &= H(w_{k-l_1}, w_{k-l_1+1}) \\ &= \sum_{j=\overline{k}}^{k-l_1} (H(w_j, w_{j+1}) - H(w_{j-1}, w_j)) + H(w_{\overline{k}-1}, w_{\overline{k}}) \qquad (42) \end{aligned}$$

Let $\overline{k} > -\infty$. If $w_{\overline{k}} \le \tilde{s}$, since $w_{\overline{k}-1} < w_{\overline{k}}$ and hence we have

$$h_3(w_{\overline{k}}) \ge \min_{[w_{\overline{k}-1}, w_{\overline{k}}]} h_3 \ge \min_{[w_{\overline{k}-1}, w_{\overline{k}}]} h = H(w_{\overline{k}-1}, w_{\overline{k}})$$

If $w_{\overline{k}} > \tilde{s}$, then since $w_{\overline{k}-1} \le \overline{s}$,

$$h_3(w_{\overline{k}}) = h(s) \ge \min_{[w_{\overline{k}-1}, w_{\overline{k}}]} h \ge H(w_{\overline{k}-1}, w_{\overline{k}})$$

Let $\overline{k} = -\infty$, since the chain is non-trivial and hence $\exists \quad w_{j_0}$ in the chain such that $w_{j_0} \ge \overline{S}$. Now suppose $w_l \le \tilde{s}$ for l large implies that $\overline{S} \le w_{j_0} \le w_{j_0-1} \le \ldots \le w_l \le \tilde{s}, \quad \forall l \ge j_0$. Hence for $l \ge j_0 + 1$, we have

$$H(w_{l-1}, w_l) \le h_3(w_l) \tag{43}$$

If $w_l > \tilde{s}$ for l large, then $H(w_{l-1}, w_l) \le h(S)$ and

$$0 \le H(w_{l-1}, w_l) - h_3(w_l) \le h(S) - h(s) \tag{44}$$

Hence from the above estimates, we have that

$$\begin{aligned}
\sum_{\overline{k}}^{k-1} (z_j - z_{j+1})_+ \quad &\le \quad \sum_{j=\overline{k}}^{k-l_1} |H(w_j, w_{j+1}) - H(w_{j-1}, w_j)| + H(w_{\overline{k}-1}, w_{\overline{k}}) - h(w_{\overline{k}}) \\
&\le \quad \sum_{j=\overline{k}}^{k-l_1} |H(w_j, w_{j+1}) - H(w_{j-1}, w_j)| \quad \text{if} \quad \overline{k} > -\infty \\
&\quad\quad \sum_{j=\overline{k}}^{k-l_1} |H(w_j, w_{j+1}) - H(w_{j-1}, w_j)| \\
&\quad\quad\quad + H(S) - h(s) \quad \text{if} \quad \overline{k} = -\infty
\end{aligned} \tag{45}$$

Therefore, we have that

$$\begin{aligned}
\sum_{-\infty}^{\infty} (z_j - z_{j+1})_+ \quad &= \quad \sum_{i=1}^{\infty} \sum_{j \in [\overline{k}_i, k_i - 1]} (z_j - z_{j+1})_+ \\
&\le \quad \sum_{j=-\infty}^{+\infty} |H(w_j, w_{j+1}) - H(w_{j-1}, w_j)| + |h(S) - h(s)|
\end{aligned} \tag{46}$$

Similarly by looking at decreasing chains we get the estimate,

$$-\sum_{-\infty}^{\infty} (z_j - z_{j+1})_- \quad \le \quad \sum_{j=-\infty}^{+\infty} |H(w_j, w_{j+1}) - H(w_{j-1}, w_j)| + |h(S) - h(s)| \tag{47}$$

Hence from (46) and (47), we have that

$$\begin{aligned}
\sum_{j=-\infty}^{-2} |z_{3,j} - z_{3,j+1}| \quad &= \quad \sum_{j=-\infty}^{+\infty} |z_j - z_{j+1}| \\
&= \quad \sum_{j=-\infty}^{+\infty} (z_j - z_{j+1})_+ - \sum_{j=-\infty}^{+\infty} (z_j - z_{j+1})_- \\
&\le \quad 2 \sum_{j=-\infty}^{+\infty} |H(w_j, w_{j+1}) - H(w_{j-1}, w_j)| + |h(S) - h(s)| \\
&= \quad 2 \sum_{j=-\infty}^{+\infty} |H(v_j, v_{j+1}) - H(v_{j-1}, v_j)| + |h(S) - h(s)|
\end{aligned}$$

This proves (35). Similar arguments for $\tilde{z}$ gives (36) and completes the proof of the lemma.

Next ,we will combine the above inequalities in lemma 5 and lemma 6 to obtain a single estimate on the left and the right. Let $h \in \mathbb{F}_3(I)$,$\{v_i\}_{i\in\mathbb{Z}}$ be a sequence in I and let $\{z_{k,i}\}_{i\in\mathbb{Z}}, k = 1, 2, 3$ be as defined above, define

$$L_1(h, v_{-1}) = \begin{cases} \frac{1}{2}(z_{1,-1} - z_{2,-1}) & \text{if} h \in \mathbb{F}_3^+(I) \\ \frac{1}{2}(z_{2,-1} - z_{1,-1}) & \text{if} h \in \mathbb{F}_3^-(I) \end{cases} \tag{48}$$

$$L_1(h, v_{-2}, v_{-1}) = \begin{cases} \chi(v_{-1}, v_{-2})\chi_+(h_1'(v_{-1})(H_1(v_{-2}, v_{-1}) - h_1(v_{-1})) \\ +\chi(v_{-2}, v_{-1})\chi_+(h_2')(v_{-1})(h_2(v_{-1}) - H_2(v_{-2}, v_{-1})) \text{ if} h \in \mathbb{F}_3^+(I) \\ \chi(v_{-2}, v_{-1})\chi_+(h_1'(v_{-1})(h_1(v_{-1}) - H_1(v_{-2}, v_{-1})) \\ +\chi(v_{-1}, v_{-2})\chi_+(h_2'(v_{-1})(H_2(v_{-2}, v_{-1}) - h_2(v_{-1})) \text{ if} h \in \mathbb{F}_3^-(I) \end{cases} \tag{49}$$

$$P_-(h, \{v_i\}) = \frac{1}{2}(\max(\limsup_{l\to-\infty} z_{1,l}, \limsup_{l\to-\infty} z_{2,l}) - \min(\liminf_{l\to-\infty} z_{1,l}, \liminf_{l\to-\infty} z_{2,l})) \tag{50}$$

$$R_1(h, v_1) = \begin{cases} \frac{1}{2}(z_{2,1} - z_{1,1}) & \text{if} h \in \mathbb{F}_3^+(I) \\ \frac{1}{2}(z_{1,1} - z_{2,1}) & \text{if} h \in \mathbb{F}_3^-(I) \end{cases} \tag{51}$$

$$R_2(h, v_1, v_2) = \begin{cases} \chi(v_2, v_1)\chi_-(h_1'(v_1))(H_1(v_1, v_2) - h_1(v_1)) \\ +\chi(v_1, v_2)\chi_-(h_2'(v_1))(h_2(v_1) - H_2(v_1, v_2)) & \text{if} h \in \mathbb{F}_3^+(I) \\ \chi(v_1, v_2)\chi_-(h_1'(v_1)(h_1(v_1) - H_1(v_1, v_2)) \\ +\chi(v_2, v_1)\chi_-(h_2'(v_1))(H_2(v_1, v_2) - h_2(v_1)) & \text{if} h \in \mathbb{F}_3^-(I) \end{cases} \tag{52}$$

$$P_+(h, \{v_i\}) = \frac{1}{2}(\max(\limsup_{l\to\infty} z_{1,l}, \limsup_{l\to\infty} z_{2,l}) - \min(\liminf_{l\to\infty} z_{1,l}, \liminf_{l\to\infty} z_{2,l})) \tag{53}$$

Then by adding (25),(27) and (35) if $h \in \mathbb{F}_3^+(I)$ and (29),(31) and (35) if $h \in F_3^-(I)$ to obtain

$$\begin{aligned} & \tfrac{1}{2}(\sum_{i=-\infty}^{-2} |z_{1,i} - z_{1,i+1}| + \sum_{i=-\infty}^{-2} |z_{2,i} - z_{2,i+1}| + \sum_{i=-\infty}^{-2} |z_{3,i} - z_{3,i+1}|) \\ = \; & L_1(h, v_{-1}) + L_2(h, v_{-2}, v_{-1}) + P_-(h, \{v_i\}) \\ & +|h(s) - h(S)| + 3 \sum_{i=-\infty}^{-2} |H(v_{i-1}, v_i) - H(v_i, v_{i+1})| \end{aligned} \tag{54}$$

Similarly, adding (26),(32) and (35) if $h \in \mathbb{F}_3^+(I)$ and (30),(32) and (5.3.19) if $h \in F_3^-(I)$ to obtain

$$\begin{aligned} & \tfrac{1}{2}(\sum_{i=1}^{\infty} |z_{1,i} - z_{1,i+1}| + \sum_{i=1}^{\infty} |z_{2,i} - z_{2,i+1}| + \sum_{i=1}^{\infty} |z_{3,i} - z_{3,i+1}| \\ = \; & R_1(h, v_{-1}) + R_2(h, v_{-2}, v_{-1}) + P_+(h, \{v_i\}) \\ & +|h(s) - h(S) + 3 \sum_{i=2}^{\infty} |H(v_{i-1}, v_i) - H(v_i, v_{i+1})| \end{aligned} \tag{55}$$

□

Now we will use these estimates in order to obtain the TV bounds for a pair of fluxes. Let (f, g) be an infinity flux pair and (A_0, B_0) be a interface connection vector. Let $f = (f_1, f_2, f_3), g = (g_1, g_2, g_3)$ be the associated flux splittings. For $\alpha, \beta \in I, j = \{1, 2, 3\}$, define the singular mappings by

$$\begin{aligned} \psi_j(u) &= \psi_j(g, \alpha, u) && (56) \\ \phi_j(u) &= \psi_j(f, \beta, u) && (57) \end{aligned}$$

For a sequence $\{u_i\}_{i\neq 0} \subset I$, $j \in \{1, 2, 3\}$, define the sequence $\{z_{ij}\}_{i\neq 0}$ by

$$\begin{aligned} z_{ij} &= \psi_j(u_i) && \text{if} \quad i \leq -1 \\ &= \phi_j(u_i) && \text{if} \quad i \geq 1 \qquad (58) \\ z_i &= z_{i1} + z_{i2} + z_{13} \end{aligned}$$

Then we have the following,

Lemma 7. *Let $F_{A_0B_0}$ be the interface flux associated with the connection (A_0, B_0), then we have*

$$\begin{aligned} \frac{1}{2}TV(z_i) &\leq 3N_{AB}(f, g, \{u_i\}) + E_{AB}(\alpha, \beta, u_{-2}, u_{-1}, u_1, u_2) + C_0 && (59) \\ \text{where } E_{AB} &= |z_{-1} - z_1| + L_1(g, u_{-1}) + L_2(g, u_{-2}, u_{-1}) + R_1(f, u_1) + R_2(f, u_1, u_2) \\ &- 3|G(u_{-2}, u_{-1}) - F_{A_0B_0}(u_{-1}, u_1)| - 3|F(u_1, u_2) \\ &- F_{A_0B_0}(u_{-1}, u_1)| && (60) \\ C_0 &= |f(s) - f(S)| + |g(s) - g(S)| + P_-(g, \{u_i\}) + P_+(f, \{u_i\}) && (61) \end{aligned}$$

Proof. The proof follows immediately by adding (53) and (54). □

We will also need the functions defined below,

$$z_h(x, t) = z_i \quad \forall (x, t) \in [x_{i-1/2}, x_{i+1/2}) \times [t^n, t^{n+1})$$

We have the following lemma giving a TV bound for z_h

Lemma 8. *Let z_h be defined as above, then for all $t > 0$, we have that*

$$\frac{1}{2}TV(z_h(.,t)) \leq 3N_{AB}(f, g, u_0) + C \qquad (62)$$

Proof. The proof follows in a straightforward way from lemma (7) and by observing that E_{AB} and C_0 are bounded in terms of M, s, S and we express the bound as a single constant C. □

We also state another lemma

Lemma 9. *let $u_0, v_0 \in L^\infty(\mathbb{R}, [s, S])$ be such that $N_{A_0B_0}(f, g, u_0) < \infty$ be the initial data for (2) and let u^h, v^h be the corresponding approximate solutions, then the following holds*

$$s \leq u^h(x, t) \leq S, \quad \forall \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+ \qquad (63)$$

$$\int_{\mathbb{R}} |u_h(x, t) - v_h(x, t))dx \leq N_{AB}(f, g, u_0)(2\Delta t + |t - \tau|). \qquad (64)$$

$\forall\ a \leq b$ and $\tau \leq t$, we

$$\int_a^b |u_h(x, t) - v_h(x, t)|dx \leq \int_{a-\frac{1}{\lambda}(t-\tau)}^{b+\frac{1}{\lambda}(t-\tau)} |u^h(x, \tau) - v^h(x, \tau)|dx + 4(S - s)h \qquad (65)$$

Proof: Follows exactly as in [2] (lemma 4.7). As in [2], we need to construct a test function for showing consistency with the interface entropy condition which we do in the following lemma,

Let v_0 be defined as

$$\begin{aligned} v_0(x,\alpha,\beta) &= \alpha \quad \text{if} \quad x<0 \\ &= \beta \quad \text{if} \quad x>0 \end{aligned}$$

Lemma 10. *Fix a connection (A_0, B_0) Let f, g be of the under-compressive type (i.e $N_+(f) = N_-(g) = 1$) and let $(\alpha, \beta) \in [s, S]$, be such that the following holds,*

$$\begin{array}{lll} 1) & \underline{A} < \alpha, & \beta < B \text{ or} \\ 2) & \overline{A}_1 < \alpha < A & B < \beta < \overline{B}_1. \end{array}$$

(for notation refer to section 1). Let $v^h(x,t,\alpha,\beta)$ be the solution of (2) given by the scheme (14) when the initial data is $v_0(x,\alpha,\beta)$. Assume that for a subsequence (again denoted by h). As $h \to 0$, $\quad v^h(x,t,\alpha,\beta) \to v(x,t,\alpha,\beta)$ in $L^\infty_{loc}(\mathbb{R}_+, L^1_{loc}(\mathbb{R}))$. Then

$$\begin{aligned} \lim_{x\to 0^-} v(x,t,\alpha,\beta) &= A \\ \lim_{x\to 0^+} v(x,t,\alpha,\beta) &= B \end{aligned}$$

where $(A_0, B_0) = (A, B)$ is the interface connection vector.

Proof. The proof is similar to one presented in Lemma 4.8 of [2]. For simplicity, we also assume that $g(s) < g(A)$ (see figure (2))). Fix $h > 0$. First we make the following claim regarding the behavior of the approximations V^h at the first time level i.e $n = 1$.

Claim. We have assumed that case (1) holds, then

$$A \le v^1_{-1} < \alpha \text{ and } \beta < v^1_1 \le B \tag{66}$$

Proof. By the scheme (14), we have that

$$\begin{aligned} v^1_{-1} &= \alpha - \lambda((F_{A_0B_0}(\alpha,\beta) - G(\alpha,\alpha)) \\ &= \alpha - \lambda(F_{A_0B_0}(\alpha,\beta) - g(\alpha)) \end{aligned}$$

now by the definition of the interface Godunov flux, it follows that $F_{A_0B_0}(\alpha,\beta) = g(A) = f(B))$ (as data is under-compressive).

$$v^1_{-1} = \alpha - \lambda(g(A) - g(\alpha)) \tag{67}$$

It is easy to see that $g(A) > g(\alpha)$ therefore it follows from the above relation that $v^1_{-1} < \alpha$. Now by using the Lipschitz continuity of the flux, and the CFL condition, we have that

$$\begin{array}{lll} |g(A) - g(\alpha)| & \le & \lambda M|A-\alpha| \le |A-\alpha| \\ v^1_{-1} & \ge & \alpha - |A-\alpha| \ge \alpha - \alpha + A = A \end{array}$$

thus proving the first part of the claim. For second part, from the definition of flux, we have $F(\beta,\beta) = f(\beta)$, therefore

$$v^1_1 = \beta - \alpha(f(\beta) - f(B)) \tag{68}$$

As $\beta \le B$, therefore $f(B) > f(\beta), v^1_1 > \beta$. Again by using the Lipschitz continuity and the CFL condition, we get that $v^1_1 = \beta + \lambda(f(B) - f(\beta)) \le \beta + \lambda M|B-\beta| \le \beta + |B-\beta| \le B$. thus proving the claim. At the n^{th} time level, we have the following claim:

Claim. $\forall n, \{v_j^n\}$ satisfies,

$$A \leq v_{-n}^n \leq \ldots \leq v_{-1}^n < \alpha \tag{69}$$
$$\beta < v_1^n \leq v_2^n \ldots \leq v_n^n \leq B \tag{70}$$

Proof. We prove the above estimates by induction. We have shown that they are true for $n = 1$. Assume that above is true is for $n-1$.

We have that $A \leq v_{-1}^{n-1} \leq \alpha, \quad \beta \leq v_1^{n-1} \leq B$ Therefore by the arguments in the first claim, we get $A \leq v_{-1}^n \leq \alpha, \beta \leq v_1^n \leq B$ Use the fact that the scheme is monotone and consistent for $(j) \geq 2$ to prove the claim.

Letting $h \to 0$, along a subsequence which we still denote by h, we can use the point wise convergence to get that as $h \to 0$, $v^h(x,t,\alpha,\beta)$ satisfies

$$\begin{aligned} A &\leq v^-(t,\alpha,\beta) &=& \lim_{x\to 0^-} v^h(x,t,\alpha,\beta)) \leq \alpha \\ \beta &\leq v^+(t,\alpha,\beta) &=& \lim_{x\to 0^+} v^h(x,t,\alpha,\beta) \leq B. \end{aligned}$$

from the shape of f and g, we get that, $\forall\ \beta$ such that $\theta_{1f} \leq \beta \leq \theta_{2f}, v^h(x,t,\alpha,\beta) = v^h(x,t,\alpha,\theta_{2f}) = v^h(x,t,\alpha,\theta_{1f})$ by the above claim, we have that,

$$A \leq v^-(t,\alpha,\beta) \leq \alpha, \text{ and } B \leq v^+(t,\alpha,\beta) \leq \theta_{2f}.$$

therefore by Rankine Hugoniot condition and the monotonicity of the fluxes, we get that $V^-(t,\alpha,\beta) = A_0$ and $V^+(t,\alpha,\beta) = B_0$ and thus prove the lemma. ☐

In the other over-compressive cases, a similar comparison function can be constructed. Now we state and prove the main convergence theorem of this paper,

Theorem 2. *Assume that the fluxes (f,g) are an infinity - flux pair and fix an interface connection vector (A_0,B_0). Assume that $\lambda, M = \max\{Lipf, Lipg\}$ satisfies the CFL condition $2\lambda M \leq 1$. Let $u_0 \in L^\infty(\mathbb{R})$ such that $s \leq u_0(x) \leq S$ for all $x \in \mathbb{R}$ and $N_{A_0B_)}(f,g,u_0) < \infty$. For $h > 0$, let u_h be the corresponding calculated solution given by (14), Then there exists a subsequence $h_k \to 0$ such that u_{h_k} converges a.e. to a weak solution u of (2) satisfying interior entropy condition (6). Suppose the discontinuities of every limit function u of $\{u_h\}$ is a discrete set of Lipschitz curves; then $u_h \to u$ in $L^\infty_{loc}(\mathbb{R}_+, L^1_{loc}(\mathbb{R}))$ as $h \to 0$, and u satisfies the interface entropy condition (7) relative to the connection (A_0,B_0).*

Proof. Follows exactly as in [2] (proof of theorem 3.1). ☐

Thus we have shown that if the fluxes (f,g) are a infinity flux pair, then for every choice of the interface connection vector (A_0,B_0), the corresponding A_0B_0-entropy solutions exist and are unique. We remark that the above stability result is only valid under the assumptions that the the discontinuities of the limit function form a discrete set of Lipshitz curves. In general, we may not expect such regularity of the solution but we still expect the above result to be true.

5. **The general case.** So far, we have considered the infinity flux case, i.e. fluxes satisfying the hypothesis (H_1, H_2), as most practical applications involve fluxes with at most two extrema as well as the expressions and proofs are much simpler. Also the difficulties encountered in this case are prototypical of the more general case. In this section, we consider fluxes f and g having more than two points of extrema. The configurations of the fluxes can be extremely complicated and their intersections arbitrary. Most of the concepts developed in the infinity flux case can be extended to this case but at the expense of more complicated notation and

lengthier arguments. Hence, we consider this case without giving much detail and emphasizing the points where the analysis is different from that of the infinity flux case.

For simplicity in the presentation, we will consider the following model fluxes f and g (with intersections, the non-intersecting case is much easier) such that they satisfy the following simplifying hypothesis,

$\underline{H}_1$: $f(s) = g(s), \quad f(S) = g(S)$

$\underline{H}_2$: Both f and g have exactly N ($N > 2$) extrema denoted by $\theta_{1f}, \theta_{2f}, \ldots, \theta_{Nf}$ and $\theta_{1g}, \theta_{2g}, \ldots, \theta_{Ng}$ respectively.

$\underline{H}_3$: f and g intersect at exactly $N+1$ points given by $s = \alpha_0, \alpha_1, \ldots, \alpha_N + 1 = S$ and the following hold, $\forall \quad \alpha_i, i = 1, \ldots, N$,

$$\max(\theta_{ig}, \theta_{if}) < \alpha_i < \min(\theta_{(i+1)g}, \theta_{(i+1)f})$$

$\underline{H}_4$:f is increasing (decreasing) and g is decreasing (increasing) at s and at S. Thus implying that the fluxes have opposite behavior at the endpoints.

The above hypotheses imply that the interior points of intersection are between the extrema of the fluxes. This is a model example for the general case. Note that the flux geometry is quite complicated and furthermore, the flux crossings strongly violate the crossing condition of [19] with the points of intersection being of both the under-compressive as well as over-compressive type and hence we expect a combination of effects of both the cases of section 2. We can further classify the fluxes with the above hypotheses into the following types,

Case 5.1: f is increasing and g is decreasing at s and f is decreasing and g is increasing at S.

Case 5.2: f is decreasing and g is increasing at s and f is increasing and g is decreasing at S.

Case 5.3: f is increasing and g is decreasing at both s and S.

Case 5.4: f is decreasing and g is increasing at both s and S.

The shape of the fluxes is given in figure (4). Observe that the above hypotheses imply that we have odd number of extrema $N = 2k + 1$ in Cases (5.1) and (5.2) and even number of intersections $2k + 2$ and we have even number of extrema $N = 2k$ in Cases (5.3) and (5.4) with odd number of intersections $2k + 1$. We will present most of the analysis for this model case. First, we need some definitions-let $\{\alpha_i\}, i = 0, \ldots, N + 1$ be the set of points of intersection of the fluxes, then we define

Definition 11. A point of intersection α_i is said to be of an over-compressive type if f is decreasing and g is increasing at α_i.

Definition 12. A point of intersection α_i is said to be of an under-compressive type if f is increasing and g is decreasing at α_i.

Note that points of intersection of the over-compressive and undercompressive types alternate in case of fluxes satisfying the above hypotheses. Check that there are $k+1$ points of intersection each of the undercompressive as well as the overcompressive type in cases (5.1) and (5.2), k overcompressive and $k+1$ undercompressive points of intersection in case (5.3) and $k+1$ overcompressive and k undercompressive points of intersection in case (5.4). We remark that for each α_i overcompressive, in the region $[\alpha_i, \alpha_{i+2}]$, the flux geometry is of the infinity flux type with undercompressive flux crossings and these intervals divide the domain $[s, S]$ into undercompressive pieces that are connected by overcompressive points of intersection. This observation will play an important role in the analysis of this section. We start with the entropy framework for fluxes of the general case and as in section 1, we have to define a proper concept of interface connection vector which we define below as,

Definition 13. Interface connection Vector:- The vector pair (A_0, B_0) with $A_0 = (A_1, A_2, \dots, A_j)$, $B_0 = (B_1, B_2, \dots, B_j)$,($j$ to be specified later) is said to be an interface connection vector provided that the following holds
1. $g(A_i) = f(B_i)$, $\forall \quad i = 1, \dots j$.
2. Let α_i be a overcompressive point of intersection, then

$$A_i, B_i \in (\alpha_i, \alpha_{i+2})$$

3. f is increasing at B_i and g is decreasing at A_i.

The number of components varies in each case - for example in cases 5.1 and 5.2, the connection vector has k components and in cases 5.3, the connection vector has $k-1$ components and it has $k+1$ components in case 5.4. The interface connection vector is the basis for the definition of the interface entropy condition as in the infinity flux case but in this case, it is not enough to obtain stability and we need another concept - the interface comparison vector

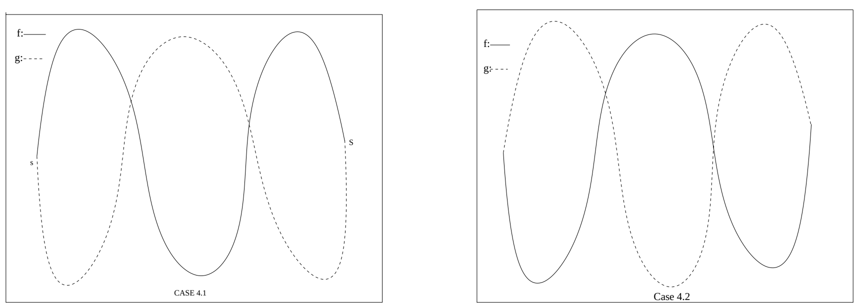

FIGURE 4. Some examples of flux shapes in the general case

which is defined below,

Definition 14. Interface comparison vector: The vector pair (C, D) with $C = (C_1, C_2, \dots, C_l)$, $D = (D_1, D_2, \dots, D_j)$,($l$ to be specified later) is said to be an interface comparison vector provided that the following holds
$\forall \alpha_i$ such that α_i is overcompressive and $\alpha_i \neq \{s, S\}$, then

$$\begin{array}{rclrcl} C_i & = & \theta_{ig} & D_i & = & \theta_{if} \\ C_{i+1} & = & \theta_{(i+1)g} & D_{i+1} & = & \theta_{(i+1)f} \end{array}$$

We have different number of components in different cases i.e $l = 2k$ in cases 5.1 and 5.2, $l = 2(k+1)$ components in case 5.3 and $l = 2(k-1)$ components in case 5.4. The interface comparison vector is necessary in order to patch up the different undercompressive components. First we need some more notations- let $a, b, c, d \in \mathbb{R}$., then let

$$I_{ab}(c, d) = sign(c - a)(g(c) - g(a)) - sign(d - b)(g(d) - g(b))$$

Also denote for given vectors, $\overline{a} = (a_1, a_2, \dots, a_n), \overline{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$,

$$I_{\overline{a}\overline{b}}(c, d) = (I_{a_1 b_1}(c, d), I_{a_2 b_2}(c, d), \dots, I_{a_n b_n}(c, d))$$

We propose the following interface entropy entropy condition,

Definition 15. Interface entropy condition: Let $u \in L^{\infty}_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ is said to satisfy the interface entropy condition if the following holds,
1. $u(0-,t) = u^-(t)$ and $u(0+,t) = u^+(t)$ exists for all t.
2. For any interface connection connection vector A_0B_0 and for the fixed interface comparison vector CD (defined above), we have that

$$\begin{aligned} I_{A_0B_0}(u^-(t), u^+(t)) &\geq 0 \quad a.e \quad t \\ I_{CD}(u^-(t), u^+(t)) &\geq 0 \quad a.e \quad t \end{aligned} \tag{71}$$

As in the infinity flux case, we allow the interface connection vector to be arbitrary. The main differences lie in the definition of the interface connection vector that is now more complicated as well as in the fact that we force the solution to be compared with the interface comparison vector which is new to this case and is essential in the proof of stability. The concept of A_0B_0-entropy solution is similar to one defined in section 2.

Next, we show that for every admissible choice of the interface connection vector A_0B_0, the corresponding A_0B_0 entropy solutions are stable, we have the following theorem,

Theorem 3. *For a given interface connection vector A_0B_0, Let $u, v \in L^{\infty}(\mathbb{R} \times \mathbb{R}_+)$ be two A_0B_0 entropy solutions for (2) with initial data u_0, v_0 respectively, then for any $\overline{M} \geq M = \max\{Lip(f), Lip(g)\}$, $a < 0, b > 0$, $b - a \geq 2\overline{M}t$ the function,*

$$t \mapsto \int\limits_{a+\overline{M}t}^{b-\overline{M}t} |u(x,t) - v(x,t)| dx$$

is non increasing and if $u_0 = v_0$ a.e ,then it follows that $u = v$ a.e

Sketch of the proof: Just as in section 2 (proof of theorem (1)), we have to prove the crucial comparison lemma (lemma (2)). In order to do so, we have to show that

$$I(u,v)(t) \geq 0 \tag{72}$$

where

$$\begin{aligned} I(u,v)(t) \quad = \quad & \text{sign } (u^-(t) - v^-(t))(g(u^-(t)) - g(v^-(t))) \\ & - \text{sign } (u^+(t) - v^+(t))(f(u^+(t) - f(v^+(t))) \end{aligned}$$

We drop t for notational convenience. Observe that there are two possible ways in which $I < 0$ i.e,

$$\text{either} \quad u^- < v^- \quad \text{and} \quad u^+ > v^+ \quad \text{with} \quad f(u^+) > f(v^+)$$
$$\text{or} \quad u^- > v^- \quad \text{and} \quad u^+ < v^+ \quad \text{with} \quad f(u^+) < f(v^+)$$

We have to show that both cases don't occur and by symmetry, it is enough to rule out the first case. As in the proof of lemma (2), we proceed by contradiction. We provide a sketch of the proof. First, it is easy to check that the above case can occur only when g is decreasing at both u^- and v^+ and f is increasing at u^+ and v^+. Thus, $u^- \in (\theta_{ig}, \theta_{(i+1)g})$ with g decreasing in this interval. Consequently, $u^+ \in (\theta_{jg}, \theta_{(j+1)g})$ with f increasing. There are the following possible cases given by,
Case 1: $j = i$

In this case, we compare the solution (u^-, u^+) with the interface connection components (A_i, B_i). Therefore by (71), we get that in this case if $u^- > A_i$, then

$u^+ < B_i$ and since g is decreasing and f is increasing at these points, we will get a contradiction to the interface entropy condition. The same holds if $u^- < A_i$,$u^+ > B_i$ leading to a violation of (71). Therefore, $u^- \equiv A_i$ and $u^+ \equiv B_i$.

Case 2: $j > i$

In this case, there $\theta_{jf} < u^+ < \theta_{(j+1)f}$ with f increasing in this interval. Now we can compare the traces of the solution u^-, u^+ with the interface comparison vector component $(\theta_{jg}, \theta_{(j+1)g})$. As $u^- < \theta_{jg}$ and $u^+ > \theta_{jf}$, we get that $I_{\theta_{jg},\theta_{jf}}(u^-, u^+) < 0$ thus leading to a violation of the interface entropy condition (71) and ruling out this case.

Case 3: $j < i$

As in this case 2, we can rule this case out by using the interface entropy condition (71) and comparing with the component $(\theta_{(j+1)g,}, \theta_{(j+1)f})$ of the interface comparison vector where $B_i \in (\theta_{jf}, \theta_{(j+1)f})$.

In view of the above, we have shown that if u^- is such that g is decreasing at u^- and u^+ such that f is increasing at u^+, then $u^- \equiv A_i$ and $u^+ \equiv B_i$ for some i. Same also holds for v^-, v^+ and by arguments similar to that of lemma(2), it can be shown that $I \geq 0$ and we can use that to prove the theorem. We skip the remaining details.

Next ,we can solve the Riemann problem associated with (2) with the fluxes satisfying the above hypotheses and use them to define the interface numerical flux and the Godunov scheme (14). As in section 4, we perform the convergence analysis of our scheme (14). The only difference from the analysis in the infinity flux case is in the definition of the singular mappings. As in the infinity flux case, we need to split the fluxes in this case as follows

Let $g_1, g_2 : [s, S] \mapsto \mathbb{R}$ be such that for

$$\begin{aligned} g_1(\theta) &= \min(G(\theta, s), G(S, \theta)), \ \forall \quad \theta \in [s, S]. \\ g_2(\theta) &= \max(G(s, \theta), G(\theta, S)), \ \forall \quad \theta \in [s, S]. \end{aligned}$$

similarly we split f as follows, let $f_1, f_2 : [s, S] \mapsto \mathbb{R}$ be such that

$$\begin{aligned} f_1(\theta) &= \min(F(\theta, s)), F(S, \theta))), \ \forall \quad \theta \in [s, S] \\ f_2(\theta) &= \max(F(s, \theta), F(\theta, S)), \ \forall \quad \theta \in [s, S]. \end{aligned}$$

where G and F are Godunov fluxes corresponding to the fluxes g and f respectively. Note that g_1, f_1 represent the "concave-type" parts of the fluxes and g_2, f_2 represent the "convex-type" part of the fluxes. The expressions are exactly the same as in the infinity-flux case. Naturally, we define the singular mappings in terms of these split fluxes as in section 4 . We define

$$\begin{aligned} \psi_1(u) &= \int_\alpha^u |g_1'(\theta)| d\theta \qquad & \phi_1(u) &= \int_\alpha^u |g_2'(\theta) d\theta \\ \psi_2(u) &= \int_\beta^u |f_1'(\theta)| d\theta \qquad & \phi_2(u) &= \int_\beta^u |f_2'(\theta)| d\theta \end{aligned}$$

Clearly, neither $\psi_1 + \psi_2$ nor ϕ_1 are not invertible and we need to consider defect terms like in section 4. In general, we may require up to $N - 1$ defect terms for N extrema. Each defect term consists of a monotone part of the fluxes with constants in other parts (as in h_3 term of section 4) and its variation can be estimated by chain estimates of section 4 (6). We can show that the total variation of the transformed scheme can be controlled in terms of the flux variations (see section 4). This is the key estimate for showing convergence and we just give one part of it below

Lemma 11. *Let* $z_j^n = \psi_1(u_j^n), \quad \forall \quad j \leq -3$*, then we have*

$$(z_j^n - z_{j+1}^n)_+ \leq |G(u_j^n, u_{j+1}^n) - G(u_{j+1}^n, u_{j+2}^n)| + |G(u_{j-1}^n, u_j^n) - G(u_j^n, u_{j+1}^n)| \quad (73)$$

Proof. This estimate (73) is the equivalent of (25) in this case. We drop n for notational convenience and consider 4 points of extrema. First we need some notation - from the flux geometry in this case, denote $s_g \in [\theta_{1g}, \theta_{2g}]$ such that $g(s) = g(s_g)$. Similarly we have $\overline{\theta} \in [\theta_{3g}, \theta_{4g}]$ such that $g(\theta_{2g}) = g(\overline{\theta})$. It is easy to check that in this case g_1 is defined as follows,

$$g_1(\theta) = \begin{cases} g(s) & \text{if} s < \theta \leq s_g \\ g(\theta) & \text{if} s_g \leq \theta \leq \theta_{2g} \\ g(\theta_{2g}) & \text{if} \theta_{2g} \leq \theta \leq \overline{\theta} \\ g(\theta) & \text{if} \overline{\theta} \leq \theta \leq S \end{cases} \quad (74)$$

As in the proof of (25), check that $(z_j - z_{j+1})_+ > 0$ iff $u_j > u_j + 1$. We have to consider the following cases,

Case 1: $u_{j+1} \leq \theta_{4g}$.

In this case, we have the following sub cases,

Case 1.1: $u_j \leq \theta_{2g}$

In this case, it is easy to check that $u_j > s_g$ for $(z_j - z_{j+1})_+ > 0$ and then $G(u_j, u_{j+1}) = g(u_j)$. Similarly we have that,

$$(z_j - z_{j+1}) = g(u_j) - \max(g(u_{j+1}), g(s))$$

for any $u_{j+2} \in [s, S]$, the properties of the Godunov flux and the location of u_{j+1} gives us that $G(u_{j+1}, u_{j+2}) \leq \max(g(u_{j+1}), g(s))$ and combining the above, we get (73) in this case.

Case 1.2: $\theta_{2g} < u_j \leq \theta_{4g}$

We have to consider a few sub cases given by,

Case 1.2.1: $u_{j+1} \leq \theta_{2g}$

In this case, we get that

$$(z_j - z_{j+1}) = \max(g(u_j), g(\theta_{2g}) - \max(g(u_{j+1}), g(s))$$

Similarly, we get that $G(u_j, u_{j+1}) = \max(g(u_j), g(\theta_{2g}))$ and arguing as in case 1.1, we get that $\forall u_{j+2}$, $G(u_{j+1}, u_{j+2}) \leq \max(g(u_{j+1}), g(s))$ and combining the above gives us the required estimate (73).

Case 1.2.2: $\theta_{2g} \leq u_{j+1} \leq \theta_{4g}$

In this case, we have that

$$(z_j - z_{j+1}) = \max(g(u_j), g(\theta_{2g}) - \max(g(u_{j+1}), g(\theta_{2g})$$

and $G(u_j, u_{j+1}) = \max(g(u_j), g(\theta_{2g})$. Similarly, we can check that $u_{j+2} \in [s, S]$, $G(u_{j+1}, u_{j+2}) \leq \max(g(u_{j+1}), g(\theta_{2g})$ and combine the above to get the desired estimate.

Case 1.3: $u_j > \theta_{4g}$

In this case, we have to again consider some sub cases given by,

Case 1.3.1: $u_{j+1} \leq \theta_{2g}$

In this case, we get that

$$(z_j - z_{j+1}) = g(\theta_{4g}) - \max(g(u_{j+1}), g(s)) + g(\theta_{4g}) - g(u_j)$$

Similarly, we get that $G(u_j, u_{j+1}) = \theta_{4g}$ and arguing as in case 1.1, we get that $\forall u_{j+2}$, $G(u_{j+1}, u_{j+2}) \leq \max(g(u_{j+1}), g(s))$. Similarly, we get that $\forall u_{j-1} \in [s, S]$,

$G(u_{j-1}, u_j) \leq g(u_{-1})$ and combining the above gives us the required estimate (73).
Case 1.3.2: $\theta_{2g} \leq u_{j+1} \leq \theta_{4g}$

In this case, we have that

$$(z_j - z_{j+1}) = g(\theta_{4g}) - \max(g(u_{j+1}), g(\theta_{2g}) + g(\theta_{4g}) - g(u_j)$$

and $G(u_j, u_{j+1}) = \theta_{4g}$. Similarly, we can check that $u_{j+2} \in [s, S]$, $G(u_{j+1}, u_{j+2}) \leq \max(g(u_{j+1}), g(\theta_{2g})$ and as in the previous case, we get that $\forall u_{j-1} \in [s, S]$, $G(u_{j-1}, u_j) \leq g(u_{-1})$ and combining the above gives us the required estimate (73).
Case 2: $u_{j+1} > \theta_{4g}$

In this case, we get that,

$$(z_j - z_{j+1}) = g(u_{j+1}) - g(u_j)$$

We can check that $G(u_j, u_{j+1}) = g(u_{j+1})$. By repeating the arguments of case 1.3, we get that $\forall u_{j-1} \in [s, S]$, $G(u_{j-1}, u_j) \leq g(u_{-1})$ and combining the above gives us the required estimate (73) in all cases and we prove the lemma. □

Similarly we can get estimates on the Total variation of the singular mappings in terms of the flux variation as in lemma (7). This enables us to repeat the steps in section 4 and prove the key convergence theorem,

As we had stated in the beginning of this section, we just provided a sketch of the analysis in this more general case. The aim was to illustrate the fact that the results for the infinity flux case can be extended to the more general case and we have indicated the points where the analysis has differed.

Hence, we have established under very general assumptions about the shape of the fluxes f and g that we can extend the notion of AB-entropy solutions of [3] to this case and characterize infinitely many L^1-stable solutions in terms of interface connections. The second step of choosing a particular interface connection and its corresponding semi-group has to be based on the physics of the problem.

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