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THE CAUCHY PROBLEM FOR THE INHOMOGENEOUS POROUS MEDIUM EQUATION

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Abstract. We consider the initial value problem for the filtration equation in an inhomogeneous medium

$$\rho(x)\, u_t = \Delta u^m, \quad m > 1.$$

The equation is posed in the whole space $\mathbb{R}^n$, $n \geq 2$, for $0 < t < \infty$; $\rho(x)$ is a positive and bounded function with a certain behaviour at infinity. We take initial data $u(x, 0) = u_0(x) \geq 0$, and prove that this problem is well-posed in the class of solutions with finite "energy", that is, in the weighted space L^1_ρ, thus completing previous work of several authors on the issue. Indeed, it generates a contraction semigroup.

We also study the asymptotic behaviour of solutions in two space dimensions when ρ decays like a non-integrable power as $|x| \to \infty$: $\rho(x)\,|x|^\alpha \sim 1$, with $\alpha \in (0, 2)$ (infinite mass medium). We show that the intermediate asymptotics is given by the unique selfsimilar solution $U_2(x, t; E)$ of the singular problem

$$\begin{cases} |x|^{-\alpha} u_t = \Delta u^m & \text{in } \mathbb{R}^2 \times \mathbb{R}_+ \\ |x|^{-\alpha} u(x,\, 0) = E\delta(x), & E = \|u_0\|_{L^1_\rho} \end{cases}$$

1. Introduction. Nonlinear diffusion in inhomogeneous media.

This paper is concerned with a model of nonlinear diffusion taking place in an inhomogeneous medium. A main objective of the studies in this area is to show how the theory established in the homogeneous case suffers from qualitative and quantitative changes when inhomogeneity is present in the medium and to develop the tools to answer the relevant questions in the new setting. We take as basis for our study the initial value problem

$$\begin{cases} \rho(x)\, u_t = \Delta u^m & \text{in } Q := \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,\, 0) = u_0 \end{cases} \tag{1}$$

The equation in (1) arises as a simple model in the study of heat propagation in inhomogeneous plasma, as well as in filtration of a liquid or gas through an inhomogeneous porous medium, see the works by Kamin and Rosenau [16], [17] and the references therein. In both cases, the function $\rho(x)$ stands for the properties

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of the material where diffusion of heat or matter takes place. In the case of mass diffusion or filtration in porous media, u is a density, saturation or concentration, and $\rho(x)$ represents the porosity of the medium. In the case of heat propagation, u stands for a temperature and $\rho(x)$ represents the density of the medium. In the sequel, we use the thermal simile for convenience.

The case of a homogeneous medium, *i.e.,* $\rho(x) \equiv 1$ (or a constant), has been extensively studied in the literature since the pioneering work [19]. The basic existence and uniqueness theory is by now well established, as well as further properties of the solutions like propagation properties, smoothing properties and regularity, asymptotic behaviour, and so on. We refer to the surveys [1] and [23] and the quoted literature. The book [25] contains detailed and up-to-dated account on this issue.

The equation with variable $\rho(x)$ (inhomogeneous medium) was first studied in one spatial dimension in [16] and [17]. Thus, in [16], the basic existence and uniqueness results were derived for problem (1) under the assumptions

(i) u_0 is non-negative, smooth and bounded,
(ii) ρ is positive, smooth and bounded.

A main issue of [16] and [17] is the study of the long time behaviour of solutions. It turns out that it strongly depends on the integrability of $\rho(x)$ at infinity. More precisely, according to [16], if $\rho(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$ with $0 < \alpha < 1$ and the initial data are compactly supported, then the solutions decay to zero and behave like a family of explicit solutions $U_1(x, t; E)$, which are the unique selfsimilar solutions to the singular problem

$$\begin{cases} |x|^{-\alpha} u_t = (u^m)_{xx} \quad \text{in } Q \\ |x|^{-\alpha} u(x,\ 0) = E\delta(x) \end{cases} \tag{2}$$

These solutions have the form

$$U_1(x,\ t;\ E) = t^{-\frac{1-\alpha}{1+m(1-\alpha)}} F(\xi); \qquad \xi = |x| t^{-\frac{1}{1+m(1-\alpha)}}, \tag{3}$$

where the profile is given by

$$F(\xi) = C_1 \left[C_2 - \xi^{2-\alpha}\right]_+^{\frac{1}{m-1}},$$

where $C_1 = C_1(m,\ \alpha)$ and $C_2 = C_2(m,\ \alpha,\ E)$. Note that here the dimension is $n = 1$, that

$$E(t) := \|u(\cdot, t)\|_{L^1_\rho} = \int \rho(x) u(x,t)\, dx$$

is an invariant of the evolution, $E(t) = E$, called the "thermal energy", and also that the convergence $|x|^{-\alpha} U_1(x,\ t;\ E) \to E\,\delta(x)$ takes place in the weak sense of measures as $t \to 0$. Note finally that in the case $\alpha = 0$ we recover the homogeneous case, and then the solutions (3) are the famous Barenblatt solutions [4]; the main conclusion we derive is that the homogeneous theory has a nice continuation into this inhomogeneous range. We will call the new solutions also Barenblatt solutions.

Marked differences with the homogeneous case start when $\rho(x) \sim |x|^{-\alpha}$ when $|x| \to \infty$ with $\alpha > 1$ for $n = 1$. Indeed, in [17] it is shown that if $\rho \in L^1(\mathbb{R})$, solutions with bounded data do not decay to zero. Instead, they converge on compact sets to the (spatial) mean of u, that is,

$$u(x,\ t) \to \bar{u} := E/\|\rho\|_{L^1} \qquad \text{as} \quad t \to \infty. \tag{4}$$

When problem (1) is thought of as modelling heat transfer, $\bar{u}$ represents the mean temperature and this phenomenon is known as "isothermalization", and is essentially due to the fact that the thermal energy is preserved in time and is spread out over an infinite medium that has however finite mass. The isothermalization result is extended to the two-dimensional case in Guedda et al. [11], by showing that (4) takes place if $\rho \in L^1(\mathbb{R}^2)$. On the other hand, the one-dimensional result is refined in the recent paper by Galaktionov et al. [12], where the singular self-similar solution representing the long-time behavior is identified. Also in this paper, some estimates of solutions in the critical case $\alpha = 1$ are given. These estimates suggest that the asymptotic behavior in this case is described by a logarithmically contracted version of U_1.

Isothermalization does not take place for the similar problem posed in dimensions $n \geq 3$ when ρ decays fast enough, due to a new feature of the evolution, namely *mass loss,* described in [14]. Moreover, as shown in [9], [10], [15], in dimensions $n \geq 3$ uniqueness is lost in the class of bounded solutions; however, it holds in the narrower class of solutions with certain decay properties that we review in Section 2 for the reader's convenience. Recently, Eidus and Kamin [10] proved existence of solutions in such a class when

$$u_0 \in L^\infty_{loc}(\mathbb{R}^n) \cap L^1_{\rho(x)|x|^{2-n}}(\mathbb{R}^n), \quad u_0 \geq 0.$$

Note that growing data are allowed in this class when $\rho(x)$ decays as $|x| \to \infty$.

We stop here the description of the mathematical problems under investigation and present our contribution that consists of two main results.

• First, we extend the existence theory for equation (1) to the natural class of initial data $u_0 \in L^1_\rho(\mathbb{R}^n)$ with $u_0 \geq 0$ in dimensions $n \geq 2$; some decay restrictions on ρ are needed. This extension requires the a priori estimates that we have recently obtained in [22] by using a new version of the usual technique of Schwartz symmetrization for parabolic equations as developed for instance in [5, 24]. This version is conceived to treat inhomogeneous problems of the present type.

• On the other hand, we settle the question of large time behaviour of these solutions in two space dimensions, in the "infinite mass" case

$$\rho(x)|x|^\alpha \sim c > 0 \quad \text{as } |x| \to \infty, \quad \alpha \in (0, 2).$$

We prove convergence towards the corresponding Barenblatt solutions. This is the correct asymptotics for media with infinite mass.

Organization. The rest of the paper is organized as follows: In Section 2 we present some background material and give the precise statements of our main results, Theorems 1 and 3. Section 3 is devoted to the proof of Theorem 1 concerning well posedness of (1). Finally, in Section 4 we prove our result on the asymptotic behaviour of solutions, Theorem 3.

2. **Preliminaries and statements.**

Given a positive, measurable function ρ defined on $\mathbb{R}^n$, by $L^1_\rho = L^1_\rho(\mathbb{R}^n)$ we denote the weighted Lebesgue space of measurable functions such that

$$\|f\|_{L^1_\rho} := \int_{\mathbb{R}^n} \rho\,|f|\,dx < \infty.$$

Throughout the paper, we will always consider initial data for our evolution problems in the class $u_0 \in \mathrm{L}^+_\rho := \{f \in L^1_\rho : f \geq 0 \ a.e.\}$ where the weight function is the density $\rho(x)$ from (1). We will assume that the weight satisfies

$(\mathbf{H}_\rho)$ $\rho \in C^1(\mathbb{R}^n)$, $\rho > 0$. Moreover, there exist constants $0 < A \leq B$ such that

$$A(1+|x|)^{-\alpha} \leq \rho(x) \leq B(1+|x|)^{-\alpha} \quad \text{on} \quad \mathbb{R}^n \tag{5}$$

with $0 < \alpha < 2$ if $n = 2$ and $0 < \alpha \le 2(n-1)$ if $n \ge 3$.

Due to the degenerate character of the equation in (1), solutions must be understood in a weak sense. We adopt the following definition

Definition 1. A *weak* solution to (1) is a non-negative and continuous in Q function with $u \in C([0, +\infty) : L^1_\rho(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (\tau, +\infty))$, $\nabla u^m \in L^2(\mathbb{R}^n \times (\tau, +\infty))$ for every $\tau > 0$, and such that the identity

$$\iint_Q \{\nabla u^m \cdot \nabla \phi - \rho u \phi_t\}\, dx\, dt = \int_{\mathbb{R}^n} \rho u_0 \phi(x, 0)\, dx$$

holds for every test function $\phi \in C^1(Q) \cap C(\overline{Q})$ with $\phi = 0$ for large t and large $|x|$.

This definition leads to non-uniqueness in dimensions $n \ge 3$. Following [9], [10], [13], [15], we can avoid non-uniqueness by restricting ourselves to solutions satisfying the following extra condition on the average behavior of u^m as $|x| \to \infty$.

$$\lim_{R \to +\infty} R^{1-n} \int_{|x|=R} \int_0^T u^m(x, t)\, dt\, dS = 0 \quad \text{for every } T > 0. \tag{6}$$

Our main result concerning well-posedness reads

Theorem 1. *Let ρ satisfy $(\mathbf{H}_\rho)$ and let $u_0 \in \mathrm{L}^+_\rho$. Then,*

- (i) *If $n = 2$, there exists a unique solution to problem* (1) *in the sense of Definition* 1.
- (ii) *If $n \ge 3$, there exists a unique solution to problem* (1) *in the sense of Definition* 1 *satisfying condition*

$$(\mathbf{C}) \qquad \lim_{R \to +\infty} R^{1-n} \int_{|x|=R} \int_\tau^T u^m(x, t)\, dt\, dS = 0 \quad \textit{for every } 0 < \tau < T.$$

In both cases the maps $S_t : u_0 \mapsto u(t)$ form a semigroup of L^1_ρ-contractions on the set L^+_ρ. The Maximum Principle applies.

Let us make some comments.

1) Theorem 1 extends the existence results in [9], where the data are assumed to be continuous and bounded, as well as those of [10], where the data are assumed locally bounded. Such an extension to the “natural functional space” is not immediate and needs new a priori estimates that we supply.

2) It should be noted, however, that in [10] the growth conditions imposed on the initial data are somewhat weaker for $n \ge 3$ and no decay assumptions on ρ like $(\mathbf{H}_\rho)$ are needed for existence. Indeed, it is well known in the homogeneous theory that well-posedness can be proved in larger classes of solutions not having finite thermal energy, [2, 6]. However, the L^1 theory is a cornerstone of the extended theory in that case, and so is the L^1_ρ theory in our case.

3) The main ingredient for the present extension is the a priori L^∞-estimate of solutions to (1) in terms of $\|u_0\|_{L^1_\rho}$ alone obtained by the authors in [22]. The following is a slightly more general version of Theorem 6.1 of [22].

Theorem 2. *Let $n \ge 2$ and $\rho \in C^1(\mathbb{R}^n)$ satisfy*

$$c\rho_0(x) \le \rho(x) \le \rho_0(x), \tag{7}$$

where $0 < c \leq 1$ *and* ρ_0 *is a bounded, continuous, positive radial function. Let* $s(r)$ *denote the solution of the initial value problem*

$$s^{n-1}\frac{ds}{dr} = \rho_0(r)r^{n-1}; \qquad s(0) = 0$$

and let there exist $K > 0$ *such that*

$$s(r) \geq K\, r\, \rho_0(r)^{1/2}, \quad ds/dr \geq K\, \rho_0(r)^{1/2} \qquad \text{for } r \geq 0. \tag{8}$$

Let $u_0 \in L^1_\rho \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $u_0 \geq 0$ *and let* u *be the unique weak solution to* (1) *according to* ([9]) *(satisfying condition* $(\mathbf{C})$ *if* $n \geq 3$*). Then,*

(*i*) *If* $\int \rho_0(x)\, dx = \infty$, *we have the estimates*

$$u(x,\, t) \leq C\, t^{-n/(n(m-1)+2)}; \qquad \|u(\cdot,\, t)\|_{L^1_\rho} \leq \|u_0\|_{L^1_\rho}, \tag{9}$$

where $C = C(\|u_0\|_{L^1_\rho},\, K,\, c,\, m,\, n)$.

(*ii*) *If* $\int \rho_0(x)\, dx < \infty$, *we have the estimates*

$$u(x,\, t) \leq Ct^{-1/(m-1)}; \qquad \|u(\cdot,\, t)\|_{L^1_\rho} \leq C' t^{-1/(m-1)}, \tag{10}$$

where C *and* C' *depend on* $\|u_0\|_{L^1_\rho}$, K, c, m, *and* n.

Remark 1. Theorem 6.1 of [22] deals with the particular case $\rho_0 = C(1+|x|)^{-\alpha}$. In this case, $(\mathbf{H}_\rho)$ are sufficient conditions for (7), (8) to hold, as shown in Lemma 3.2 of that paper.

Singular problem. We also need a definition of solution to the singular problem

$$\begin{cases} |x|^{-\alpha}u_t = \Delta u^m & \text{in } Q \\ |x|^{-\alpha}u(x,\, 0) = E\delta(x) \end{cases} \tag{11}$$

Definition 2. Let $n = 2$ and $E > 0$. A weak solution to (11) is a non-negative and continuous in Q function with $u \in C((0, +\infty) : L^1_\rho) \cap L^\infty(\mathbb{R}^2 \times (\tau, +\infty))$, $\nabla u^m \in L^2(\mathbb{R}^2 \times (\tau, +\infty))$ for $\tau > 0$, and such that the identity

$$\iint_Q \{\nabla u^m \cdot \nabla\phi - |x|^{-\alpha}u\phi_t\}\, dx\, dt = E\phi(0,\, 0)$$

holds for every test function as in Definition 1.

It can be easily checked that the following Barenblatt solutions

$$U_2(x,\, t;\, E) = t^{-1/m}F(\xi); \qquad \xi = |x|t^{-\frac{1}{m(2-\alpha)}}, \tag{12}$$

and

$$F(\xi) = C_1\left[C_2 - \xi^{2-\alpha}\right]_+^{\frac{1}{m-1}}; \qquad \xi \geq 0,$$

where $C_1 = C_1(m,\, \alpha)$ and $C_2 = C_2(m,\, \alpha,\, E)$, are indeed weak solutions to (2) in the above sense. The following properties of (12) can be easily verified.

i) $\operatorname{supp} U_2(t) = B_{R(t)}$ with $R(t) = C_2^{1/(2-\alpha)}t^{1/m(2-\alpha)}$;

ii) $\|U_2(t)\|_{L^\infty} = C_1 t^{-1/m}$;

iii) The profile F is convex if $m < 2$ and $\alpha \geq 1$ or $m = 2$ and $\alpha > 1$, concave if $m > 2$ and $\alpha \leq 1$ or $m = 2$ and $\alpha < 1$ and linear if $m = 2$, $\alpha = 1$;

iv) $\partial U_2/\partial t \in L^1_{|x|^{-\alpha},\, loc}(Q)$.

As we explained in the Introduction, we prove that for $n = 2$ general solutions to (1) decay to zero, being $U_2(x,\, t;\, E)$ with $E = \|u_0\|_{L^1_\rho}$ the first term in the asymptotic expansion. More precisely, the following holds.

Theorem 3. *Let $n = 2$. Let ρ satisfy $(\mathbf{H}_\rho)$ and let $u_0 \in \mathrm{L}^+_\rho$ with $\|u_0\|_{L^1_\rho} = E > 0$. Assume moreover that*

$$\lim_{|x|\to\infty} \rho(x)|x|^\alpha \to 1 \qquad \text{as} \quad |x| \to \infty. \tag{13}$$

Let $u(x,\, t)$ be the unique solution of Problem (1), according to Theorem 1. Then,

$$\|u(\cdot,\, t) - U_2(\cdot,\, t;\, E)\|_{L^1_\rho} \to 0 \qquad \text{as} \quad t \to \infty. \tag{14}$$

Remark 2. Clearly, the more general assumption $\rho(x)|x|^\alpha \sim c > 0$ can be reduced to (13) by means of the change $t = ct'$. The new energy is then $E' = E/c$.

The proof of Theorem 3 relies on scaling techniques, hence sharp estimates of the solutions are required. Such estimates are a direct consequence of Theorem 2 for $n = 2$. For $n = 1$, such a global estimate is false, as shown in [22]. Indeed, the asymptotic result in [16] takes place on expanding sets of the form $\{|x| \le Ct^\beta\}$. On the other hand, for $n \ge 3$ the estimate given by Theorem 2 does not hold uniformly for the rescaled solutions, see Section 4. This explains the choice $n = 2$.

3. **Well posedness.** This section is devoted to prove Theorem 1.

• Let us first deal with the existence question. In [9], [10], for $n \ge 3$ and $u_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, a solution to (1) is constructed as the monotone limit of solutions to the initial-boundary problems

$$\begin{cases} \rho(x)\, u_t = \Delta u^m & \text{in} \;\; Q_R := B_R \times \mathbb{R}_+ \\ u(x,\, 0) = u_0 & \text{in} \;\; B_R \\ u(x,\, t) = 0 & \text{for} \;\; |x| = R, \end{cases} \tag{15}$$

where $B_R = \{x : |x| < R\}$. More precisely, denoting by u_R the unique solution to (15) (which is in turn constructed by means of an approximation procedure, see [3], [9]), there exists $u := \lim_{R\to+\infty} u_R$ *a.e.* in Q and it is a weak solution to (1) in the sense of Definition 1. This construction also works for $n = 2$, but this case is not considered in [9], [10] since their main concern are non-uniqueness phenomena occurring for $n \ge 3$. The main point that we want to stress from this construction is that the solution obtained is *minimal*, *i.e.* $u \le v$ for any other solution v according to Definition 1. The case $n = 2$ is considered in [11], but their approach is somewhat different.

In order to extend the existence theory to data $u_0 \in \mathrm{L}^+_\rho$, we need some estimates for the minimal solutions. First of all, weak solutions to the problem (15) generate a semigroup of contractions in $L^1_\rho(B_R)$. More precisely, if u_1 and u_2 denote two solutions with initial data u_{01} and u_{02} respectively, we have

$$\|\{u_1(\cdot,\, t) - u_2(\cdot,\, t)\}_+\|_{L^1_\rho(B_R)} \le \|\{u_{01} - u_{02}\}_+\|_{L^1_\rho(B_R)} \tag{16}$$

for all $t > 0$. Here $\{s\}_+ = \max\{s,\, 0\}$. Interchanging the solutions in (16) and adding the results, we obtain

$$\|u_1(\cdot,\, t) - u_2(\cdot,\, t)\|_{L^1_\rho(B_R)} \le \|u_{01} - u_{02}\|_{L^1_\rho(B_R)} \tag{17}$$

The contraction results (16) and (17) can be proved exactly as in [3] for $\rho \equiv 1$; see also [21] for variable ρ. The presence of ρ here is irrelevant, since it is bounded from below by some positive constant on each B_R.

As a consequence of these results, there is at most one weak solution to (15) and we have a comparison result: if we denote by u_R, ($\tilde{u}_R$) the solution to (15) with initial data u_0 (resp. $\tilde{u}_0$) then $u_0 \le \tilde{u}_0$ implies $u_R(x,\, t) \le \tilde{u}_R(x,\, t)$ for

$(x,\,t)\in Q_R$. If we take $\tilde u_0=\|u_0\|_{L^\infty}$ and we pass to the limit $R\to\infty$ we get $u(x,\,t)\le\|u_0\|_{L^\infty}$ for the minimal solution to (1). If moreover $u_0\in \mathrm{L}^+_\rho$ it follows from (17) with $u_{02}=0$ that $u(\cdot,\,t)\in\mathrm{L}^+_\rho$ and $\|u(\cdot,\,t)\|_{L^1_\rho}\le\|u_0\|_{L^1_\rho}$.

Given two initial data $u_{01},\,u_{02}\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\cap\mathrm{L}^+_\rho$, we can pass to the limit $R\to\infty$ in the estimate (17), which is valid for the approximations u_{1R} and u_{2R} and then we have

$$\|u_1(\cdot,\,t)-u_2(\cdot,\,t)\|_{L^1_\rho}\le\|u_{01}-u_{02}\|_{L^1_\rho} \tag{18}$$

for all $t>0$. Convergence of the norms follows by the dominated convergence theorem. Indeed, $u_{0i}\chi_{B_R}\to u_{0i}$ *a.e.* in $\mathbb{R}^n$ for $i=1,\,2$ and moreover $|u_{01}-u_{02}|\chi_{B_R}\le|u_{01}|+|u_{02}|\in\mathrm{L}^+_\rho$ for every R. The same argument applies to the left hand side.

The following estimate is obtained in [9] assuming that $u_0\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and $\rho\in L^1(\mathbb{R}^n)$. It also holds if $u_0\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\cap\mathrm{L}^+_\rho$, as it can be easily verified. See also [11].

$$\int_\tau^T\int_{\mathbb{R}^n}|\nabla u^m|^2\,dx\,dt+\frac{1}{m+1}\int_{\mathbb{R}^n}\rho(x)u^{m+1}(x,\,T)\,dx\le\frac{1}{m+1}\int_{\mathbb{R}^n}\rho(x)u^{m+1}(x,\,\tau)\,dx \tag{19}$$

for any $0\le\tau<T$.

Let now $u_0\in\mathrm{L}^+_\rho\cap L^\infty$. Take a sequence $\{u_{0k}\}\subset\mathrm{L}^+_\rho\cap L^\infty(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ such that $u_{0k}\to u_0$ in L^1_ρ and $\|u_{0k}\|_{L^\infty}\le\|u_0\|_{L^\infty}$, $\|u_{0k}\|_{L^1_\rho}\le\|u_0\|_{L^1_\rho}$ and let u_k denote the corresponding solution. It follows from (18), applied to u_{0k} and u_{0m} that $\{u_k\}$ is a Cauchy sequence in $C([0,\,+\infty):\,L^1_\rho)$, with a limit u in this space.

On the other hand, by comparison we have $u_k\le\|u_{0k}\|_{L^\infty}\le\|u_0\|_{L^\infty}$. Moreover, by estimate (19) with $\tau=0$ we have $\|\nabla u_k^m\|_{L^2(Q)}\le C\|u_0\|^m_{L^\infty}\|u_0\|_{L^1_\rho}$ with a constant $C>0$ independent of k. Therefore ∇u_k^m converges weakly in $L^2(Q)$ to ∇u^m and the limit function u satisfies the integral identity in Definition (1). According to the regularity theory [8], $\{u_k\}$ is locally equicontinuous and $u_k\to u$ in $C_{loc}(Q)$ for some subsequence (not relabelled). Thus u is a weak solution of (1).

Finally, observe that all the estimates above hold in the limit. Clearly, $u\le\|u_0\|_{L^\infty}$. Since $u_{0k}\to u_0$ and $u_k(t)\to u(t)$ in L^1_ρ for each $t>0$, we can pass to the limit in (18) and it holds for any two such constructed solutions. As a consequence of the lower semicontinuity of the norm in the weak topology, estimate (19) holds in the limit. Finally, we also note that the estimates (9) and (10) from Theorem 2 hold with constants depending only on $\|u_0\|_{L^1_\rho}$.

Assume now $u_0\in\mathrm{L}^+_\rho$. Let $\{u_{0k}\}\subset\mathrm{L}^+_\rho\cap L^\infty(\mathbb{R}^n)$ be such that $u_{0k}\to u_0$ in L^1_ρ and $\|u_{0k}\|_{L^1_\rho}\le\|u_0\|_{L^1_\rho}$. Denote by u_k the corresponding solution, according to the previous step. It is at this stage where Theorem 2 plays a prominent role. Combining the estimate (19) with $\tau>0$ and (9), (10) we obtain

$$\int_\tau^T\int_{\mathbb{R}^n}|\nabla u_k^m|^2\,dx\,dt\le C\tau^{-\sigma}, \tag{20}$$

where $\sigma=\sigma(m,\,n)>0$ and $C>0$ depends only on $\|u_0\|_{L^1_\rho}$. As in the previous step, we conclude that $\{u_k\}$ is a Cauchy sequence in $C([0,\,\infty):\,L^1_\rho))$ converging to a limit u in this space. Thanks to (20), (9) and (10) and the regularity results of [8], it follows that u is a weak solution to (1) with data u_0 and estimates (18), (20), (9) and (10) hold in the limit. The construction is complete.

• It remains to verify condition **(C)** for $n \geq 3$. To this end, consider for each $t \geq 0$ the potential function $v_R(x,\, t)$ solving

$$\begin{cases} -\Delta v &= \rho\, u_R \qquad \text{in} \quad B_R \\ \quad v &= 0 \qquad\quad \text{on} \quad \partial B_R \end{cases}$$

where u_R represents the approximated solution to (15) introduced above. Denoting by G_R the Green function of the Laplace operator in B_R, we have $v_R = G_R{*}(\rho u_R)$ and the function $w_R := \partial_t v_R = G_R * (\rho \partial_t u_R) = G_R * (\Delta u_R^m)$ verifies

$$-\Delta w_R = -\Delta(G_R * \Delta u_R^m) = \Delta u_R^m.$$

Therefore, $h_R := w_R + u_R^m$ is a harmonic function on B_R with $h_R = 0$ on ∂B_R, hence $h_R \equiv 0$ on B_R. We conclude that $\partial_t v_R = -u_R^m$. Integrating on $[\tau,\, T]$ with $0 < \tau < T$ we have

$$v_R(y,\, T) + \int_\tau^T u_R^m(y,\, t)\, dt = v_R(y,\, \tau) \leq G * (\rho u_R(\tau)),$$

where G denotes the fundamental solution of the Laplace equation. When $u_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap \mathrm{L}^+_\rho$, it follows from [10] that we can pass to the limit $R \to +\infty$, thus obtaining

$$\int_\tau^T u^m(y,\, t)\, dt \leq G * (\rho u(\tau)) \tag{21}$$

even with $\tau = 0$. If $u_0 \in \mathrm{L}^+_\rho$, (21) holds for the approximations u_k and it is retained in the limit if $\tau > 0$, since $u_k \to u$ uniformly on $\{y\} \times [\tau,\, T]$ and $u_k(t) \to u(t)$ in $L^1_\rho(\mathbb{R}^n)$ for each $t \geq 0$. Condition **(C)** then follows from (21) and the following lemma with $\mu = \rho u(\tau)$.

Lemma 1. ([EK], Lemma A.4 in [7]) *If* $n \geq 3$, $\mu \in L^\infty_{loc}(\mathbb{R}^n)$ *and* $\mu |x|^{2-n} \in L^1(\mathbb{R}^n)$, *then the function* $F(y) = G * \mu$ *satisfies*

$$R^{1-n} \int_{|y|=R} F(y)\, dS \longrightarrow 0 \qquad \textit{as} \quad R \to +\infty.$$

• Let us now turn our attention to the uniqueness question. First of all, observe that if $u(x,\, t)$ denotes the above constructed solution with $u_0 \in \mathrm{L}^+_\rho$, then for any $\tau > 0$ the function $u_\tau(x,\, t) := u(x,\, t + \tau)$ is a solution in the sense of [9] with $u_{\tau,0} = u(x,\, \tau) \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap \mathrm{L}^+_\rho$. Moreover, u_τ satisfies condition (6), since

$$\int_0^T u_\tau^m(x,\, t)\, dt = \int_\tau^{T+\tau} u^m(x,\, t)\, dt$$

and (6) follows from **(C)**. Consequently, the above constructed solution is the unique solution in the sense of [9] after arbitrarily small time $\tau > 0$.

With this in mind, uniqueness follows easily. Let $u_0 \in \mathrm{L}^+_\rho$ and let $u_1,\, u_2$ be two different solutions to (1). Then, there exists $T,\, \varepsilon > 0$ such that $\|u_1(\cdot,\, T) - u_2(\cdot,\, T)\|_{L^1_\rho} > \varepsilon$. According to the definition, we can choose τ with $0 < \tau < T$ so small, that

$$\|u_i(\cdot,\, \tau) - u_0(\cdot)\|_{L^1_\rho} < \varepsilon/4; \qquad i = 1,\, 2$$

and therefore $\|u_1(\cdot,\, \tau) - u_2(\cdot,\, \tau)\|_{L^1_\rho} < \varepsilon/2$. For $t > \tau$, both $u_1,\, u_2$ are uniquely determined by the above token. In particular, they can be obtained by means of the minimal construction above, thus enjoying the L^1-contraction property (18). Then, for $t > \tau$,

$$\|u_1(\cdot,\, t) - u_2(\cdot,\, t)\|_{L^1_\rho} \leq \|u_1(\cdot,\, \tau) - u_2(\cdot,\, \tau)\|_{L^1_\rho} < \varepsilon/2.$$

This gives rise to a contradiction at $t = T$. Uniqueness is proved. $\square$

Next, we prove that for $n = 2$ the total energy is preserved. More precisely, the following holds.

Theorem 4. *Let $n = 2$ and let $\rho(x)$ satisfy $(\mathbf{H}_\rho)$. Let $u(x, t)$ be the unique weak solution of (1) constructed above, with data $u_0 \in \mathrm{L}^+_\rho$ and $\|u_0\|_{L^1_\rho} = E$. Then,*

$$\|u(\cdot, t)\|_{L^1_\rho} = E \qquad \textit{for any } \ t \geq 0. \tag{22}$$

Proof. The proof is rather standard. It relies on the following finite propagation property, which is interesting by itself.

Lemma 2. *Let the hypotheses of Theorem 4 hold. Let $u_0 \in L^\infty(\mathbb{R}^2)$ and compactly supported. Then* supp $u(\cdot, t) \subset B_{C(t+1)^\gamma}$ *for some constant C depending on the data.*

Proof. By our assumption on ρ, there exists the family of Barenblatt solutions $U_2(x, t; E)$. Moreover, we have $\rho(x) \geq A|x|^{-\alpha}$ for $|x| > 1$. Without loss of generality, we may assume $A = 1$; otherwise we perform the change $t = Ct'$ with suitable C. Let $\widetilde{U}(x, t) = U_2(x, t + 1; \widetilde{E})$. Let $\Omega := \{\widetilde{U}_t > 0\}$. In the set $\Omega \cap \{|x| > 1\}$ we have

$$\rho(x)\widetilde{U}_t \geq |x|^{-\alpha}\widetilde{U}_t = \Delta\widetilde{U}^m. \tag{23}$$

Choose $\widetilde{E}$ large enough, such that $u_0(x) \leq \widetilde{U}(x, 0)$ in $\mathbb{R}^2$. A simple computation shows that $\Omega = \{\xi > \xi_0\}$, where $\xi_0^{2-\alpha} = k(\alpha, m)C_2$ with $k < 1$. On the surface $\partial\Omega = \{\xi = \xi_0\}$ we have

$$\widetilde{U}(t) = c(\alpha, m)C_2(t + 1)^{-1/m}.$$

Choosing, if necessary, a larger $\widetilde{E}$, we will have $\Omega \subset \{|x| > 1\}$ and $u \leq \widetilde{U}$ on $\partial\Omega$. This is feasible since C_2 grows with $\widetilde{E}$ and

$$u \leq \min\{\|u_0\|_\infty, Ct^{-1/m}\}$$

by estimates in Section 3.

Given $T > 0$, choose $R = R(T)$ large enough, such that supp $\widetilde{U}(t) \subset B_R$ for $t \in [0, T]$. Let u_R denote the solution to the approximating problem (15) from Section 3. Then we have

$$\begin{aligned} u_R(x, 0) = u_0(x) \leq \widetilde{U}(x, 0); \\ u_R = \widetilde{U} = 0 \quad \text{for } \ |x| = R, \ \ t \in [0, T); \\ u_R \leq u \leq \widetilde{U} \quad \text{on } \ \partial\Omega \cap \{0 \leq t \leq T\}. \end{aligned} \tag{24}$$

From (23), (24) and the comparison principle it follows that $u_R \leq \widetilde{U}$ in the region

$$\Omega_{R,T} := \Omega \cap \{|x| \leq R\} \cap \{0 \leq t \leq T\}$$

In the limit $T, R \to \infty$, $u \leq \widetilde{U}$ in Q. In particular, supp $u(\cdot, t) \subset$ supp $\widetilde{U}(\cdot, t) = B_{C(t+1)^\gamma}$ for all $t \geq 0$. $\square$

Then, by Lemma in [14] p. 119 (which holds in any dimension), we conclude that (22) holds for this class of data. For general data, we argue by approximation, using the fact that convergence takes place in the space $C([0, +\infty) : L^1_\rho)$. $\square$

Remark 3. Note that the L^1_ρ-norm of the solutions is not preserved for ρ satisfying $(\mathbf{H}_\rho)$ if $n \geq 3$, as it follows from [14] and from the second estimate in (10).

Remark 4. Theorem 4 is proved in [11] for solutions with $u_0 \in L^\infty \cap \mathrm{L}^+_\rho$, without any decay restriction on ρ.

4. **Asymptotic behaviour.** This section is devoted to the proof of Theorem 3. It consists of several steps.

Step 1: Rescaling. Define the rescaled versions of $u(x,\,t)$:

$$u_\lambda(x,\,t) = \lambda^\beta u(\lambda^\gamma x,\,\lambda t); \qquad \lambda > 0, \tag{25}$$

where

$$\beta = \frac{1}{m}, \qquad \gamma = \frac{1}{m(2-\alpha)}. \tag{26}$$

It is easy to check that u_λ is a solution of

$$\begin{cases} \rho_\lambda(x)\,u_t = \Delta u^m, \\ u(x,\,0) = u_{0\lambda} \end{cases} \tag{27}$$

with $\rho_\lambda(x) = \lambda^{\alpha\gamma}\rho(\lambda^\gamma x)$ and $u_{0\lambda} = \lambda^\beta u_0(\lambda^\gamma x)$. Besides, we have

$$\int \rho_\lambda(y) u_{0\lambda}\,dy = E \qquad \text{for } \lambda > 0. \tag{28}$$

Step 2: Uniform estimates and compactness. By virtue of $(\mathbf{H}_\rho)$, $\rho(x)$ and $\rho_\lambda(x)$ satisfy hypothesis (7) with $\rho_0 = B(1+|x|)^{-\alpha}$, respectively $\rho_{0\lambda} = \lambda^{\alpha\gamma}\rho_0(\lambda^\gamma x)$. In both cases we have $c = B/A$. By Remark 1, $(\mathbf{H}_\rho)$ also guarantees the existence of K such that (8) holds. Moreover, as it can be easily checked, this condition is met by $\rho_0(x)$ and $\rho_{0\lambda}(x)$ with the same value of K. This is a crucial point in the proof.

Therefore, by Theorem 2 (more precisely, by virtue of its extension to solutions with general $u_0 \in L^1_\rho$, see Section 3), all u_λ satisfy the estimates

$$u_\lambda(x,\,t) \le Ct^{-1/m}; \qquad \|u_\lambda(\cdot,\,t)\|_{L^1_{\rho_\lambda}} \le \|u_{\lambda 0}\|_{L^1_{\rho_\lambda}} = E \quad \text{for } t > 0 \tag{29}$$

with a constant $C(E,\,A,\,B,\,\alpha,\,m)$ independent of λ. The above decay rate is sharp, since it is attained by the Barenblatt solutions (3).

We also need a uniform L^2-estimate for ∇u_λ^m. (20) and (29) entail

$$\int_\tau^{+\infty}\int_{\mathbb{R}^2} |\nabla u_\lambda^m|^2\,dx\,dt \le C\tau^{-1}, \tag{30}$$

with C independent of λ. By virtue of (29), (30) and the fact that, on each compact subset of Q, the equation for u_λ satisfies the ellipticity condition uniformly in λ, we can apply the results in [8] to conclude that the family $\{u_\lambda\}$ is relatively compact in $L^\infty_{loc}(Q)$. By means of diagonal extraction, there exists a subsequence $\lambda_n \to \infty$ such that u_{λ_n} converges uniformly on compacts of Q to some $U \in C(Q)$. By (30), we can also assume that $\nabla u_{\lambda_n}^m \to \nabla U^m$ weakly in $L^2(\mathbb{R}^2 \times (\tau,\,+\infty))$ for each $\tau > 0$.

Step 3: Passage to the limit. The convergences above allow to pass to the limit in the integral identity in Definition 1. It is also clear that $U \in L^\infty(\mathbb{R}^2 \times (\tau,\,+\infty))$ for $\tau > 0$, and satisfies (29) with the same constant C. The lower semicontinuity of the norm in the weak topology implies that the estimate (30) holds in the limit.

Step 4: Identification of the limit. It is convenient to start with compactly supported data. In this case, Theorem 4 applies and

$$\|u_\lambda(t)\|_{L^1_{\rho_\lambda}} = \|u_{0\lambda}\|_{L^1_{\rho_\lambda}} = \|u_0\|_{L^1_\rho} \tag{31}$$

by (28). Moreover, applying Lemma 2 to $u_\lambda(\cdot,\,t)$ we have

$$\text{supp } u_\lambda(\cdot,\,t) = \lambda^{-\gamma}\text{supp } u(\cdot,\,\lambda t) \subset B_{C(t+1/\lambda)^\gamma}. \tag{32}$$

Therefore, in the limit $\lambda_n \to \infty$ we have $\operatorname{supp} U \subset B_{Ct^\gamma}$ for $t > 0$ and the limit $u_{\lambda_n} \to U$ from Step 3 takes place not only locally in Q, but also on sets of the form $[\tau_1, \tau_2] \times \mathbb{R}^2$ with $0 < \tau_1 < \tau_2$. For each $t > 0$, convergence takes place in every $L^p(\mathbb{R}^2)$ $(1 \le p \le \infty)$ and also in $L^1_{|x|^{-\alpha}}$, since $\alpha \in (0, 2)$. It is also clear that $U \in C((0, +\infty) : L^1_\rho)$ and $\nabla U^m \in L^2(\mathbb{R}^2 \times (\tau, \infty))$.

Moreover, the uniform estimates (29), (31), (32), the fact that $\rho_\lambda \le C|x|^{-\alpha} \in L^1_{loc}(\mathbb{R}^2)$ and the Lebesgue dominated convergence theorem imply that for each $t > 0$ we have

$$\int |x|^{-\alpha} U(x, t)\, dx = \lim_{n\to\infty} \int \rho_{\lambda_n}(x) u_{\lambda_n}(x, t)\, dx = E.$$

Since $\operatorname{supp} U(\cdot, t)$ shrinks to $\{0\}$ as $t \to 0$, we have $|x|^{-\alpha} U(x, t) \to E\delta(x)$ in $\mathcal{D}'(\mathbb{R}^2)$ as $t \to 0$. The same arguments allow passing to the limit in the integral identity from Definition 1 with $\rho_\lambda u_{0\lambda}$ replaced by $E\delta(x)$, thus obtaining the integral identity in Definition 2. The details of this line of argumentation are given in [25] for $\rho = 1$. Thus U is a weak solution of the singular problem (2) with $E = \|u_0\|_{L^1_\rho}$.

Next, we prove the following

Lemma 3. *For any weak solution with* $u_0 \in C_c^\infty$ *and* $\|u_0\|_{L^1_\rho} = E$,

$$\lim u_{\lambda_n}(x, t) = U_2(x, t; E)$$

for all convergent subsequences $\{u_{\lambda_n}\}$.

Proof. We borrow from [16]. It is enough to prove that, given $F \in C_c^\infty(Q)$ and $\varepsilon > 0$, there exist small enough $\tau > 0$ and large enough $\lambda > 0$ such that

$$\left| \iint_Q \rho_\lambda(x)[u_\lambda(x, t) - U_2(x, t + \tau; E)] F(x, t)\, dx\, dt \right| < \varepsilon. \tag{33}$$

It is clear that solutions in the sense of Definition 1 are solutions in the weaker sense of [9], [16], *i.e.*, are such that the identity

$$\iint_Q \{\rho u \phi_t + u^m \Delta\phi\}\, dx\, dt + \int_{\mathbb{R}^2} \rho u \phi(x, 0)\, dx = 0 \tag{34}$$

holds for any test function $\phi \in C^{2,1}_{x,t}(\overline{Q})$ vanishing for large t and large $|x|$. The same applies to solutions of the singular problem (2). Subtracting the corresponding integral identities and setting $U_{2,\tau} = U_2(t + \tau)$ for short, we get

$$\begin{aligned} \iint_Q \rho_\lambda (u_\lambda - U_{2,\tau})[\phi_t + a_{\lambda,\tau}(x, t)\Delta\phi]\, dx\, dt = \iint_Q [|x|^{-\alpha} - \rho_\lambda(x)] U_{2,\tau} \phi_t\, dx\, dt \\ + \int [|x|^{-\alpha} U_{2,\tau}(x, 0) - \rho_\lambda(x) u_\lambda(x, 0)]\phi(x, 0)\, dx, \end{aligned} \tag{35}$$

where

$$a_{\lambda,\tau}(x, t) := \begin{cases} \dfrac{u_\lambda^m - U_{2,\tau}^m}{\rho_\lambda(u_\lambda - U_{2,\tau})} & \text{if } u_\lambda \ne U_{2,\tau} \\ \dfrac{m}{\rho_\lambda} U_{2,\tau}^{m-1} & \text{if } u_\lambda = U_{2,\tau} \end{cases}$$

Observe that $a_{\lambda,\tau} \ge 0$ and $a_{\lambda,\tau} \in L^\infty(Q)$ with $\|a_{\lambda,\tau}\|_{L^\infty(Q)}$ depending on λ, τ. Choose a sequence $\{a_{\lambda,\tau,n}\} \subset C^\infty(Q)$ such that

$$\begin{aligned} & n^{-2} \le a_{\lambda,\tau,n} \le \|a_{\lambda,\tau}\|_{L^\infty(Q)} + n^{-2}; \\ & \frac{a_{\lambda,\tau} - a_{\lambda,\tau,n}}{\sqrt{a_{\lambda,\tau,n}}} \to 0 \quad \text{in } L^2_{loc}(Q) \quad \text{as } n \to \infty. \end{aligned} \tag{36}$$

Assume that $\operatorname{supp} F \subset B_{R_0} \times (0, T)$ and consider the solution $\phi_{\lambda,\tau,n,R}$ of the backwards linear problem

$$\begin{cases} \phi_t + a_{\lambda,\tau,n}\Delta\phi = F \quad \text{in} \quad Q_R := B_R \times [0, T) \\ \phi(x,T) = 0, \quad \phi(x,\, t) = 0 \quad \text{on} \quad \partial B_R \times [0, T] \end{cases} \tag{37}$$

with $R > R_0$. Clearly, the problem (37) is uniformly parabolic. Hence, it has a unique solution $\phi_{\lambda,\tau,n,R} \in C^{2,1}_{x,t}(Q_R) \cap C(\overline{Q_R})$. The following estimates are standard, see [3].

$$|\phi_{\lambda,\tau,n,R}| \le C_1; \qquad \iint_{B_R\times[0,\,T]} a_{\lambda,\tau,n}(\Delta\phi_{\lambda,\tau,n,R})^2\, dx\, dt \le C_2, \tag{38}$$

where C_1, C_2 do not depend on λ, τ, n, R.

In order to produce an admissible test, we introduce a function $\eta : [0, +\infty) \to \mathbb{R}$ with the properties *a)* $\eta \in C^2([0, +\infty))$ and *b)* $0 \le \eta \le 1$; $\eta(r) = 1$ for $r \in [0, 1/2]$, $\eta(r) = 0$ for $r \in [1, +\infty)$. Let $\eta_R(x) := \eta(|x|/R)$ for $R > R_0$. The function $\widetilde{\phi}_{\lambda,\tau,n,R}(x,t) = \phi_{\lambda,\tau,n,R}(x,t)\cdot\eta_R(x)$ is clearly in $C^{2,1}_{x,t}(\overline{Q})$ and its support is contained in $B_R \times [0, T]$. Plugging this function in the integral identity (35) and taking into account (37) we obtain

$$\iint_Q \rho_\lambda(u_\lambda – U_{2,\tau})F\, dx\, dt = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &:= \int [|x|^{-\alpha}U_{2,\tau}(x,\, 0) - \rho_\lambda u_\lambda(x,\, 0)]\phi(x,0)\eta_R(0)\, dx; \\ I_2 &:= \iint_Q [|x|^{-\alpha} - \rho_\lambda]U_{2,\tau}\phi_t\eta_R\, dx\, dt; \\ I_3 &:= \iint_Q (U^m_{2,\tau} - u^m_\lambda)[2\nabla\phi\cdot\nabla\eta_R + \phi\Delta\eta_R]\, dx\, dt; \\ I_4 &:= \iint_Q \rho_\lambda(u_\lambda - U_{2,\tau})(a_{\lambda,\tau} - a_{\lambda,\tau,n})\eta_R\Delta\phi\, dx\, dt. \end{aligned}$$

By (32) and property (i) of U_2, we can choose $R_1 > 1$ large enough, such that

$$(\operatorname{supp} u_\lambda \cup \operatorname{supp} U_{2,\tau}) \cap \{0 \le t \le T\} \subset B_{R_1} \times [0, T]$$

for all $\tau < 1$ and $\lambda > 1$ and fix $R = 2R_1$. Then $I_3 = 0$. Since $\eta_R(0) = 1$ and both $\rho_\lambda u_\lambda(x,\, 0)$ and $|x|^{-\alpha}U_{2,\tau}(x,\, 0)$ converge to $E\delta(x)$ in $\mathcal{D}'$ (and also in the sense of measures) as $\lambda \to \infty$ and $\tau \to 0$ respectively, we can choose $\lambda_0 > 1$ large and $\tau_0 < 1$ small such that $|I_1| < \varepsilon/3$ if $\lambda > \lambda_0$ and $\tau = \tau_0$.

Having fixed R and τ, we take $\lambda_1 > \lambda_0$ large such that $|I_2| < \varepsilon/3$ for $\lambda > \lambda_1$. This is possible, since integrating I_2 by parts we have

$$I_2 = -\iint_Q [|x|^{-\alpha} - \rho_\lambda]\frac{\partial U_{2,\tau}}{\partial t}\,\phi\,\eta_R\, dx\, dt + \iint_Q [|x|^{-\alpha} - \rho_\lambda]U_{2,\tau}(x,\, 0)\,\phi\,\eta_R\, dx.$$

Now, by property (iv) of U_2, the first estimate in (38) and the fact that $\rho_\lambda \to |x|^{-\alpha}$ point-wise, it follows that both integrals converge to zero as $\lambda \to \infty$.

Once λ, τ and R are fixed, we fix n large enough such that

$$|I_4| \le C(\lambda,\, \tau,\, R)\left\|\frac{a_{\lambda,\tau} - a_{\lambda,\tau,n}}{\sqrt{a_{\lambda,\tau,n}}}\right\|_{L^2(B_R\times(0,T))} < \varepsilon/3,$$

where use of the second estimate in (38) and the second property of $\{a_{\lambda,\tau,n}\}$ in (36) has been made. The proof is concluded. □

As a consequence, $\lim_{\lambda\to\infty} u_\lambda(x,t) = U_2(x,t;E)$. In particular, for $t=1$ we have

$$\|u_\lambda(\cdot,1) - U_2(\cdot,1;E)\|_{L^1_{|x|^{-\alpha}}} \to 0 \qquad \text{as } \lambda \to +\infty.$$

Recalling the definition of u_λ and using the scaling invariance of U we obtain the desired result. Observe that we can replace the weight $|x|^{-\alpha}$ by the weight $\rho(x)$.
General data: Assume now that $u_0 \in \mathrm{L}^+_\rho$ and denote by u the corresponding solution according to Theorem 1, with $E = \|u(t)\|_{L^1_\rho}$. We use a density argument. Given $\varepsilon > 0$, choose $u_0' \in C_c^\infty(\mathbb{R}^2)$ such that

$$\|u_0 - u_0'\|_{L^1_\rho} \le \varepsilon.$$

If we denote by u' the solution with data u_0', and $E' = \|u'(t)\|_{L^1_\rho}$, we have

$$\begin{aligned}\|u(\cdot,t) - U_2(\cdot,t;E)\|_{L^1_\rho} &\le \|u(\cdot,t) - u'(\cdot,t)\|_{L^1_\rho} + \|u'(\cdot,t) - U_2(\cdot,t;E')\|_{L^1_\rho} \\ &\quad + \|U_2(\cdot,t;E') - U_2(\cdot,t;E)\|_{L^1_\rho}.\end{aligned}$$

Clearly, the functions U_2 are ordered: $U_2(x,t;E_1) \le U_2(x,t;E_2)$ on Q if $E_1 \le E_2$. Therefore,

$$\|U_2(\cdot,t;E') - U_2(\cdot,t;E)\|_{L^1_\rho} = |E - E'| \le \|u_0 - u_0'\|_{L^1_\rho} \le \varepsilon.$$

Moreover, by the L^1_ρ -contraction property (18),

$$\|u(\cdot,t) - u'(\cdot,t)\|_{L^1_\rho} \le \|u_0 - u_0'\|_{L^1_\rho} \le \varepsilon.$$

Therefore,

$$\|u(\cdot,t) - U_2(\cdot,t;E)\|_{L^1_\rho} \le 2\varepsilon + \delta(t),$$

where $\delta(t) \to 0$ as $t \to \infty$, according to our previous result. Passing to the limit $t \to \infty$ the result follows from the arbitrariness of ε. □

Remark 5. The fact that for $n=2$ the estimate (9) holds uniformly in λ for the problems (27) with fixed initial data allows to construct an existence and uniqueness theory of solutions to the singular problem (2) with data $u_0 \in \mathrm{L}^+_\rho$ by approximation. It is enough to take $\rho_1(x) \in C^1$ with $\rho_1(x) = |x|^{-\alpha}$ for $|x| \ge 1$ and pass to the limit as $\lambda \to \infty$ in the sequence of problems (1) with $\rho = \rho_{1\lambda}$. All the estimates hold for such solutions, and Theorem 3 remains valid for the singular problem.

As a consequence of the previous remark and Theorem 3, we have the following uniqueness result.

Corollary 1. *Let $n=2$. Then for each $E>0$ problem* (2) *admits a unique self-similar weak solution, namely the Barenblatt-type solution U_2.*

Proof. Suppose that $V(x,t) = t^{-\beta}G(xt^{-\gamma})$ is a self-similar solution to (2). First, observe that the values of β and γ are uniquely determined. Indeed, plugging V in the equation in (2), we get the relation

$$(2-\alpha)\gamma + (m-1)\beta = 1. \tag{39}$$

On the other hand, it is clear that $\widetilde{V}(x,t) = V(x,t+\tau)$ is a solution to (2) in the sense of Definition 1 with ρ replaced by $|x|^{-\alpha}$ for any $\tau > 0$. By Theorem 4, $\|\widetilde{V}(t)\|_{L^1_\rho}$, and hence $\|V(t)\|_{L^1_\rho}$, is constant. This gives a second relation:

$$(2-\alpha)\gamma = \beta. \tag{40}$$

From (39) and (40) we get the values in (26). Moreover, $\|V(t)\|_{L^1_\rho} = E$.

Consider problem (2) with $u_0(x) = \widetilde{V}(x,\,0)$. By uniqueness, its weak solution is $\widetilde{V}(x,\,t)$. Denote by $\widetilde{V}_\lambda$ its rescaled versions, according to formula (25).

Replacing t by λ in the asymptotic formula (14), performing the change of variables $x = \lambda^\gamma y$ in the integral and recalling (40), the definition of $\widetilde{V}_\lambda$ and the invariance of U_2, we conclude

$$\widetilde{V}_\lambda(1) = V_\lambda(1 + 1/\lambda) \longrightarrow U_2(1;\,E) \tag{41}$$

in $L^1_{|x|^{-\alpha}}$ as $\lambda \to \infty$. By the triangle inequality, and taking into account the self-similarity of V,

$$\begin{aligned} \|V(1) - U_2(1;\,E)\| &\le \|V(1) - V(1+1/\lambda)\| + \|V(1+1/\lambda) - U_2(1;\,E)\| \\ &= \|V(1) - V(1+1/\lambda)\| + \|V_\lambda(1+1/\lambda) - U_2(1;\,E)\|. \end{aligned} \tag{42}$$

(all norms are in $L^1_{|x|^{-\alpha}}$). Given $\varepsilon > 0$, according to (41) we can choose now λ large enough, such that

$$\|V_\lambda(1+1/\lambda) - U_2(1;\,E)\|_{L^1_{|x|^{-\alpha}}} \le \varepsilon. \tag{43}$$

On the other hand, since $V \in C((0,\,+\infty) : L^1_\rho)$,

$$\|V(1) - V(1+1/\lambda)\|_{L^1_{|x|^{-\alpha}}} \le \varepsilon, \tag{44}$$

again for λ large. Since $\varepsilon > 0$ is arbitrary, it follows from (42), (43) and (44) that $V(1) = U_2(1;\,E)$, thus $G = F$ and $V = U_2$ in Q. □

We can easily prove another uniqueness result.

Theorem 5. *Let $n = 2$. Then for each $E > 0$ problem* (2) *admits a unique radial weak solution, namely the Barenblatt-type solution U_2.*

Proof. If $u(x,\,t) = U(r,\,t)$ with $r = |x|$ is a radial solution to (2) and $s = Cr^{1-\alpha/2}$ with $C = 2/(2-\alpha)$, then the function

$$\widetilde{u}(y,\,t) = \widetilde{U}(s,\,t) := U(r,\,t); \qquad s = |y|$$

is a radial solution of the problem

$$\left\{ \begin{array}{l} u_t = \Delta_y u^m \quad \text{in } Q \\ u(y,\,0) = C^{-1}E\delta(y) \end{array} \right. \tag{45}$$

The solution to (45) is unique [20], [18], even without the radiality assumption. This proves the assertion. □

Remark 6. The above change of variables is, up to a constant, the one used in [22] in order to compare solutions to inhomogeneous problems with solutions to related homogeneous problems. In dimensions $n \ge 3$, the corresponding change does not lead to the homogeneous porous medium equation. The theory of the singular problem in dimensions $n \ge 3$, will be presented in a forthcoming paper.

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