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OPTIMAL TRAFFIC DISTRIBUTION AND PRIORITY COEFFICIENTS FOR TELECOMMUNICATION NETWORKS

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Abstract. The aim of this paper is to optimize traffic distribution coefficients in order to maximize the trasmission speed of packets over a network. We consider a macroscopic fluidodynamic model dealing with packets flow proposed in [10], where the dynamics at nodes (routers) is decided by a routing algorithm depending on traffic distribution (and priority) coefficients. We solve the general problem for a node with m incoming and n outgoing lines and explicit the optimal parameters for the simple case of two incoming and two outgoing lines.

1. **Introduction.** There are some recent works on car traffic flow on networks, see [7, 8, 11], that relie on macroscopic description via car densities and other conserved quantities [3, 12, 13]. To treat a telecommunication network, we look at an intermediate time scale, thus assume that packets transmission happens at a faster level but the equilibria of the whole network are reached only as asymptotic.

A network is formed by a finite collection of transmission lines and nodes (or routers), each packet is seen as a particle on the network and it is assumed that each packet travels on the network with fixed speed and assigned final destination. Moreover it is assumed that routers receive, process and then forward packets. Packets may be lost with a probability increasing with the number of packets to be processed. Each lost packet is sent again.

Hence, on a single straight transmission line each router sends packets to the following one a first time and lost packets are sent a second time and so on until they reach next router. Looking at intermediate time scale we assume conservation of packets and get the following simple model consisting of a single conservation law:

$$\rho_t + f(\rho)_x = 0, \tag{1}$$

where ρ is the packet density, v is the velocity and $f(\rho) = v\rho$ is the flux.

Since the speed on the line is assumed constant, an average transmission speed among routers can be derived considering the amount of packets that may be lost along with an assigned loss probability function (see [10]). From the average transmission speed one gets a velocity function and thus a flux function.

In order to consider complex networks, one needs a way of solving dynamics at nodes in which many lines intersect. For this, we consider the routing algorithm:

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(RA) Packets are processed by arrival time and are sent to outgoing lines in order to maximize the flux.

A key role is played by Cauchy problems with initial data constant on each transmission line called Riemann problems at the node. In order to determine unique solutions to Riemann problems, some additional parameters are introduced, called respectively priority parameters and traffic distribution parameters. The theory for this model is developed in [10].

In this paper we focus on a simple network formed of a single node with m incoming and n outgoing lines. We assume that packets flow from m initial nodes to n final ones. We assign the packet quantities flowing from initial to final nodes and compute the final equilibrium as function of the traffic distribution (and priority) parameter. Such equilibrium should belong to the admissible region for the final fluxes. The strategy we use (see [10]) is to project the equilibrium point on the admissible region which is a convex set in $\mathbb{R}^N$ with $N = n, m$. We take this projected point as solution to the Riemann Problem.

Next, from the solution to the Riemann Problem, we determine the average speeds at which packets travel on the network and we define three functionals measuring:

- the speed of the packets travelling on the lines,
- the average travel time,
- the speed of the packets, weighted with their quantity, travelling on the lines, i.e. the fluxes.

We will see that the third functional does not depend on the the traffic distribution and priority parameters. The aim is to optimize the choice of the coefficients in order to maximize the first functional and to minimize the second one.

A key point is that from different choices of the projection on the admissible region we get different solutions to the Riemann Problem. In the simple case of $m = n = 2$, we deal with the projection of a point on a segment in $\mathbb{R}^1$. In this case there is only one reasonable choice and we are able to completely solve the problem giving the optimal values as function of the packets densities.

It is interesting to notice that in many cases there is a set of optimal values (with the extreme case of functional not depending on the parameter) of the traffic distribution and priority parameters.

The paper is organized as follows. Section 2 describes the dynamics of packet density on a single transmission line based on a prescribed packet loss probability. Then basic definitions and notations for telecommunication networks are given. Section 3 illustrates the routing algorithm for Riemann problems at nodes. In Section 4 we indicate the optimal parameters for the dynamic of packet density given in Section 2 and for a simple network with m incoming and n outgoing lines. Finally in Section 5 we exactly compute the optimal parameters for the simple case $n = m = 2$.

2. **Packets flow on a telecommunication network.** Each transmission line, represented by a real interval I, consists of many edges and nodes. Each node corresponds to a server sending and receiving packets. To determine the dynamics on I we need to describe the effect of packets loss on the velocity of transmission function. We assume that each node N_k sends again packets that are lost by the following node N_{k+1}. More precisely, we assume that there exists a function $p : [0, \rho_{max}] \mapsto [0, 1]$ that assigns the packet loss probability as function of the packet

density. Suppose that δ is the distance between the nodes N_k and N_{k+1}. Let Δt_0 be the transmission time of packets from node N_k to node N_{k+1} in the case in which they are sent with success at the first attempt, and Δt_{av} the average transmission time when some packets are lost by N_{k+1} and they are sent again by N_k. Let us denote with $\bar{v} = \frac{\delta}{\Delta t_0}$ and $v = \frac{\delta}{\Delta t_{av}}$ the packets velocity, respectively, in the two cases. Therefore at the first attempt the packets sent by node N_k reach with success node N_{k+1} with probability $(1-p)$ and they are lost by node N_{k+1} and sent again by node N_k with probability p. At the second attempt there are p packets to be sent again and $(1-p)p$ packets are sent with success while p^2 are lost. Going on at the n-th attempt $(1-p)p^{n-1}$ packets are sent successfully and p^n are lost. The average transmission time is equal to

$$\Delta t_{av} = \sum_{n=1}^{+\infty} n \Delta t_0 (1-p) p^{n-1} = \frac{\Delta t_0}{1-p},$$

from which we get that the transmission velocity is given by $v = \frac{\delta}{\Delta t_{av}} = \frac{\delta}{\Delta t_0}(1-p) = \bar{v}(1-p)$.

In this paper we assume that the following packets loss probability is assigned:

$$p(\rho) = \frac{\rho + \bar{v} - 1}{\bar{v}}.$$

Then the transmission velocity is equal to $v(\rho) = \bar{v}(1 - p(\rho)) = 1 - \rho$, and, since $f(\rho) = v(\rho)\rho$, the flux function is:

$$f(\rho) = \rho(1-\rho). \tag{2}$$

Other packets loss probability may be assumed and analogous results in terms of optimal traffic distribution and priority coefficients may be found. For simplicity, we suppose that the maximal packet density is $\rho_{\max} = 1$.

Next we give some basic definitions and notations for telecommunication networks. We model a telecommunication network by a finite set of intervals $I_i = [a_i, b_i] \subset \mathbb{R}, i = 1, ..., N, a_i < b_i$, on which we consider the equation (1). Hence the datum is given by a finite set of functions ρ_i defined on $[0, +\infty[\times I_i$.

On each transmission line I_i we want ρ_i to be a weak entropic solution, that is for every function $\varphi : [0, +\infty[\times I_i \mapsto \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left(\rho_i \frac{\partial \varphi}{\partial t} + f(\rho_i) \frac{\partial \varphi}{\partial x} \right) dx dt = 0, \tag{3}$$

and for every $k \in \mathbb{R}$ and every $\tilde{\varphi} : [0, +\infty[\times I_i \mapsto \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left(|\rho_i - k| \frac{\partial \tilde{\varphi}}{\partial t} + sgn(\rho_i - k) (f(\rho_i) - f(k)) \frac{\partial \tilde{\varphi}}{\partial x} \right) dx dt \geq 0. \tag{4}$$

It is well known that, for equation (1) on $\mathbb{R}$ and for every initial data in L^∞, there exists a unique weak entropic solution depending in a continuous fashion from the initial data in L^1_{loc}. Moreover, for initial data in $L^\infty \cap L^1$ we have Lipschitz continuous dependence in L^1, see [5, 6].

We assume that the transmission lines are connected by some nodes. Each node J is given by a finite number of incoming transmission lines and a finite number of outgoing transmission lines, thus we identify J with $((i_1, ..., i_m), (j_1, ... j_n))$ where the first m-tuple indicates the set of incoming transmission lines and the second n-tuple indicates the set of outgoing transmission lines. Each transmission line can be incoming transmission line at most for one node and outgoing at most for one node. Hence the complete model is given by a couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I} = \{I_i : i = 1, ..., N\}$ is the collection of transmission lines and $\mathcal{J}$ is the collection of nodes. For boundaries of transmission lines not connected to nodes we can use the theory of [1, 2, 4].

3. **Riemann problems at nodes.** Now we discuss the solution at nodes. If $\rho = (\rho_1, ..., \rho_{m+n})$ is a weak solution at the node such that each $x \mapsto \rho_i(t, x)$ has bounded variation, then ρ satisfies the Rankine-Hugoniot condition at the node J, namely

$$\sum_{\varphi=1}^{m} f(\rho_\varphi(t, b_\varphi^-)) = \sum_{\psi=m+1}^{m+n} f(\rho_\psi(t, a_\psi^+)), \tag{5}$$

for almost every $t > 0$.

For a scalar conservation law a Riemann problem is a Cauchy problem for an initial data of Heavyside type, that is piecewise constant with only one discontinuity. One looks for centered solutions, i.e. $\rho(t, x) = \phi(\frac{x}{t})$, which are the building blocks to construct solutions to the Cauchy problem via wave front tracking algorithm. These solutions are formed by continuous waves called rarefactions and by traveling discontinuities called shocks. The speed of waves are related to the values of f', see [5, 9]. Analogously, we call Riemann problem for a node the Cauchy problem corresponding to an initial data which is constant on each transmission line.

To solve Riemann problems according to (RA) we need some additional parameters called priority and traffic distribution parameters. We have only m priority parameter $p \in]0, 1[$ and n traffic distribution parameter $\alpha \in]0, 1[$. We denote with $\rho_\varphi(t, x)$, $\varphi = 1, \dots, m$ and $\rho_\psi(t, x)$, $\psi = m+1, \dots, m+n$ the traffic densities, respectively, on the incoming transmission lines and on the outgoing ones and by $(\rho_{\varphi,0}, \rho_{\psi,0})$ the initial data. We denote by $\tau : [0, 1] \to [0, 1]$ the function that associates to every density the other density with the same flux. By equation (2) we have that $\sigma = \frac{1}{2}$ and $\tau(\rho) = 1 - \rho$. Since the speed of waves must be negative on incoming lines and positive on outgoing ones, we want to determine a unique $(m + n)$-tuple $(\hat{\rho}_1, ..., \hat{\rho}_{m+n}) \in [0, 1]^{m+n}$ such that

$$\hat{\rho}_\varphi \in \begin{cases} \{\rho_{\varphi,0}\} \cup]\tau(\rho_{\varphi,0}), 1], & \text{if } 0 \le \rho_{\varphi,0} \le \sigma, \\ [\sigma, 1], & \text{if } \sigma \le \rho_{\varphi,0} \le 1, \end{cases} \tag{6}$$

$\varphi = 1, ..., m$, and

$$\hat{\rho}_\psi \in \begin{cases} [0, \sigma], & \text{if } 0 \le \rho_{\psi,0} \le \sigma, \\ \{\rho_{\psi,0}\} \cup [0, \tau(\rho_{\psi,0})[, & \text{if } \sigma \le \rho_{\psi,0} \le 1, \end{cases} \tag{7}$$

$\psi = m+1, ..., m+n$, and, on each incoming line I_φ, $\varphi = 1, ..., m$, the solution consists of the single wave $(\rho_{\varphi,0}, \hat{\rho}_\varphi)$, while, on each outgoing line I_ψ, $\psi = m+1, ..., m+n$, the solution consists of the single wave $(\hat{\rho}_\psi, \rho_{\psi,0})$.

Define $\gamma_\varphi^{\max}$ and $\gamma_\psi^{\max}$ as follows:

$$\gamma_\varphi^{\max} = \begin{cases} f(\rho_{\varphi,0}), & \text{if } \rho_{\varphi,0} \in [0, \sigma], \\ f(\sigma), & \text{if } \rho_{\varphi,0} \in]\sigma, 1], \end{cases} \quad \varphi = 1, \dots, m, \tag{8}$$

and

$$\gamma_\psi^{\max} = \begin{cases} f(\sigma), & \text{if } \rho_{\psi,0} \in [0, \sigma], \\ f(\rho_{\psi,0}), & \text{if } \rho_{\psi,0} \in]\sigma, 1], \end{cases} \quad \psi = m+1, \dots, m+n. \tag{9}$$

The quantities $\gamma_\varphi^{\max}$ and $\gamma_\psi^{\max}$ represent the maximum flux that can be obtained by a single wave solution on each transmission line. In order to maximize the number of packets through the node over incoming and outgoing lines we define

$$\Gamma = \min\{\Gamma_{in}, \Gamma_{out}\},$$

where $\Gamma_{in} = \sum_{\varphi=1}^{m} \gamma_\varphi^{\max}$ and $\Gamma_{out} = \sum_{\psi=m+1}^{m+n} \gamma_\psi^{\max}$. One can easily see that, to solve the Riemann problem, it is enough to determine the fluxes $\hat{\gamma}_\varphi = f(\hat{\rho}_\varphi), \varphi = 1,\dots,m$, and $\hat{\gamma}_\psi = f(\hat{\rho}_\psi), \psi = m+1,\dots,m+n$. Let us determine $\hat{\gamma}_\varphi, \varphi = 1,\dots,m$. We have to distinguish two cases:

I: $\Gamma_{in} = \Gamma$,

II: $\Gamma_{in} > \Gamma$.

In the first case we set $\hat{\gamma}_\varphi = \gamma_\varphi^{\max}, \varphi = 1,\dots,m$. Let us analyse the second case in which we use the priority parameters $p_1,\dots,p_m$ where $0 < p_\varphi < 1$ and $\sum_{\varphi=1}^{m} p_\varphi = 1$. Not all packets can enter the node, so let C be the amount of packets that can go through. Then $p_\varphi C$ packets come from the φ–st incoming line. Consider the space $(\gamma_1,\dots,\gamma_m)$ and denote by P the point with coordinates $\gamma_\varphi = p_\varphi \Gamma$. Recall that the final fluxes should belong to the region:

$$\Omega = \left\{(\gamma_1,\dots,\gamma_m) : 0 \le \gamma_\varphi \le \gamma_\varphi^{\max}, \varphi = 1,\dots,m\right\}.$$

We distinguish two cases:

a) P belongs to Ω,

b) P is outside Ω.

In the first case we set $(\hat{\gamma}_1,\dots,\hat{\gamma}_m) = P$, while in the second case we set $(\hat{\gamma}_1,\dots,\hat{\gamma}_m) = Q$, with $Q = proj(P)$, where $proj$ is some projection on $\hat{\Omega} = \Omega \cap \{\sum_{\varphi=1}^{m} \gamma_\varphi = \Gamma\}$. From the choice of this projection the analysis and the choice of the parameters $p_1,\dots,p_m$ can be very different. The most natural projection to take is the projection on a convex set. For $n = m = 2$, since $\hat{\Omega}$ is a one dimensional set, one essentially has a unique reasonable projection that maximizes the fluxes γ_1 and γ_2. This case will be treated in Section 5 where a detailed description of the optimal choices of the parameters p_φ is given.

Let us now determine $\hat{\gamma}_\psi, \psi = m+1,\dots,m+n$. As for the incoming transmission lines we have to distinguish two cases :

I: $\Gamma_{out} = \Gamma$,

II: $\Gamma_{out} > \Gamma$.

In the first case $\hat{\gamma}_\psi = \gamma_\psi^{\max}, \psi = m+1,\dots,m+n$. Let us determine $\hat{\gamma}_\psi$ in the second case in which we use the traffic distribution parameters $\alpha_{m+1},\dots,\alpha_{m+n}$ where $\alpha_\psi \in]0,1[$ and $\sum_{\psi=m+1}^{n} \alpha_\psi = 1$. Since not all packets can go on the outgoing transmission lines, we let C be the amount that goes through. Then $\alpha_\psi C$ packets go on the outgoing line I_ψ. Consider the space $(\gamma_{m+1},\dots,\gamma_{m+n})$ and denote by P the point with coordinates: $\gamma_\psi = \alpha_\psi \Gamma$. Recall that the final fluxes should belong to the region:

$$\Omega = \left\{(\gamma_{m+1},\dots,\gamma_{m+n}) : 0 \le \gamma_\psi \le \gamma_\psi^{\max}, \psi = m+1,\dots,m+n\right\}.$$

We distinguish two cases:

a) P belongs to Ω,

b) P is outside Ω.

In the first case we set $(\hat{\gamma}_{m+1},\dots,\hat{\gamma}_{m+n}) = P$, while in the second case we set $(\hat{\gamma}_{m+1},\dots,\hat{\gamma}_{m+n}) = Q$, where $Q = proj(P)$.

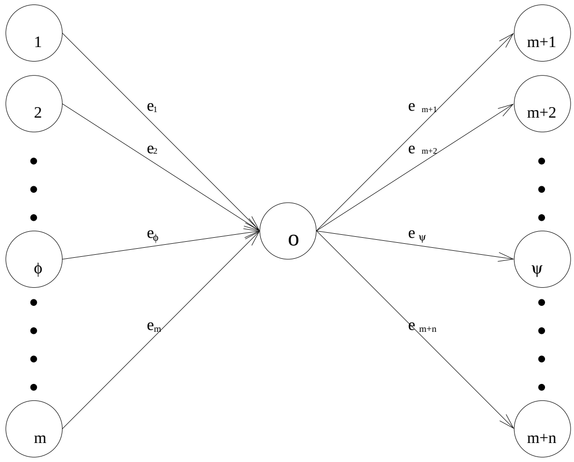

FIGURE 1. A single node network. The circles on the left and on the right represent the sources and the destination respectively. The circle in the middle o represents the node. The segments e_φ and e_ψ represent the lines incoming from the sources and outgoing to the destinations respectively.

4. **Optimization of a simple network.** We focus on a network as in figure 1 comprised of only one node o. There are packets from sources to destinations passing through the node o and running on lines. We denote by $e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}$ the lines from sources $\{1, \dots, m\}$ to the node o and from o to destinations $\{m+1, \dots, m+n\}$ and by $c_{\varphi\psi}$ with $\varphi \in \{1, \dots, m\}$ and $\psi \in \{m+1, \dots, m+n\}$ the number of packets running from source φ to destination ψ first on line e_φ, then through the node o and finally on line e_ψ. We define the packets densities running on the lines as follows

- ρ_φ from e_φ to o: $\rho_\varphi = \sum_{\psi=m+1}^{m+n} c_{\varphi\psi}$;
- ρ_ψ from o to e_ψ: $\rho_\psi = \sum_{\varphi=1}^{m} c_{\varphi\psi}$.

Our aim is to solve the RP for the node at o. Then we want to compute the velocity, the average transmission time and the flux over the network as function of the parameters α_ψ and p_φ. Therefore we introduce the following costs:

$$J_1 = \sum_{\varphi=1}^{m} \sum_{\psi=m+1}^{m+n} V_{\varphi\psi}, \quad J_2 = \sum_{\varphi=1}^{m} \frac{1}{v_\varphi} + \sum_{\psi=m+1}^{m+n} \frac{1}{v_\psi}, \quad J_3 = \sum_{\varphi=1}^{m} \sum_{\psi=m+1}^{m+n} c_{\varphi\psi} V_{\varphi\psi},$$

with $V_{\varphi\psi} = v_\varphi + v_\psi$, $v_\varphi = v(\hat\rho_\varphi)$ and $v_\psi = v(\hat\rho_\psi)$ are the velocities on the lines e_φ, e_ψ, and $\hat\rho$ is the solution to the RP with initial data $(\rho_{\varphi,0}, \rho_{\psi,0})$. We define γ_φ^{max} (resp. γ_ψ^{max}) as in equation (8) (resp. equation (9)) and consider the points described by the following systems:

$$\left\{ \; \gamma_\varphi = \Gamma p_\varphi \right. \tag{10}$$

where the p_φ's are the priority parameters with $\sum_{\varphi=1}^{m} p_\varphi = 1$, and

$$\left\{ \; \gamma_\psi = \Gamma \alpha_\psi \right. \tag{11}$$

where the α_ψ's are the traffic distribution parameters with $\sum_{\psi=m+1}^{m+n} \alpha_\psi = 1$.

We introduce the following conditions:

Aψ: $\gamma_\psi = \alpha_\psi \Gamma_{in} \leq \gamma_\psi^{max}$.

Let now $\Gamma = \Gamma_{in}$ and denote by P_{in} the polyhedron given by the intersection of the simplex $T_{in} = \{\alpha \in \mathbb{R}^n : 0 \leq \alpha_\psi \leq 1, \ \psi = m+1, \dots, m+n, \sum \alpha_\psi = 1\}$ and the cube $C_{in} = \{\alpha \in \mathbb{R}^n : \alpha_\psi \leq \frac{\gamma_\psi^{\max}}{\Gamma}, \ \psi = m+1, \dots, m+n\}$ and *proj* the projection of a point in T_{in} on the boundary of P_{in}. We do not give a precise description of such a projection so as to maintain the description of the solution as general as possible. Indeed different choices of the projection lead to different optimal subsets of T_{in}.

The solutions to the RP are the following:

- $(\gamma_1^{max}, \dots, \gamma_m^{max}, \Gamma\alpha_{m+1}, \dots, \Gamma\alpha_{m+n})$ if all Aψ are satisfied,
- $(\gamma_1^{max}, \dots, \gamma_m^{max}, \Gamma proj(\alpha_{m+1}, \dots, \alpha_{m+n}))$ if any of the Aψ is not satisfied.

Notice that the case where all **A**ψ are false is not possible since otherwise it would be $\Gamma_{in} > \Gamma_{out}$.

Consider the following conditions:

Bφ: $\gamma_\varphi = p_\varphi \Gamma_{out} \leq \gamma_\varphi^{max}$.

Now, if $\Gamma = \Gamma_{out}$ we denote by P_{out} the polyhedron given by the intersection of the simplex $T_{out} = \{p \in \mathbb{R}^m : 0 \leq p_\varphi \leq 1, \ \varphi = 1, \dots, m, \ \sum p_\varphi = 1\}$ and the cube $C_{out} = \{p \in \mathbb{R}^m : p \leq \frac{\gamma_\varphi^{\max}}{\Gamma}, \ \varphi = 1, \dots, m\}$ and *proj* the projection of a point in T_{out} on the boundary of P_{out}. In this case the solutions given to the RP are the following:

- $(\Gamma p_1, \dots, \Gamma p_m, \gamma_{m+1}^{max}, \dots, \gamma_{m+n}^{max})$ if all Bφ are satisfied,
- $(\Gamma proj(p_1, \dots, p_m), \gamma_{m+1}^{max}, \dots, \gamma_{m+n}^{max})$ if any of the Bφ is not satisfied.

Notice that the case where all **B**φ are false is not possible since otherwise it would be $\Gamma_{out} > \Gamma_{in}$. To compute the cost we observe that $\hat\rho_\varphi = f^{-1}(\hat\gamma_\varphi)$ ($\hat\rho_\psi = f^{-1}(\hat\gamma_\psi)$ resp.) and $\hat\gamma_\varphi$ is either γ_φ^{max} or Γp_φ or $\Gamma(proj(p))_\varphi$ ($\hat\gamma_\psi$ is either γ_ψ^{max} or $\Gamma\alpha_\psi$ or $\Gamma(proj(\alpha))_\psi$ resp..)

In what follows we will determine the cost functions and optimize them on the parameters α_ψ and p_φ such that conditions **A**ψ and **B**φ are satisfied.

Remark. Indeed, for the general case of m incoming and n outgoing lines, the projection *proj* has not been fixed. However, we notice that when some of the conditions are not satisfied, the cost functions are evaluated on a projected point $\Gamma proj(\alpha)$ ($\Gamma proj(p)$ resp.) and do not depend on the parameters α_ψ's (p_φ's resp.) anymore. Hence the cost functions are constant in the set $proj^{-1}(\alpha)$ ($proj^{-1}(p)$ resp..) If α^{opt} (p^{opt} resp.) optimizes a cost function when restricted to P_{in} (P_{out} resp.), then $proj^{-1}(\alpha^{opt})$ ($proj^{-1}(p^{opt})$ resp.) optimizes the same cost function non restricted. This procedure will be made in detail in the case $m = n = 2$.

4.1. **Optimal choice for flux (2).** Recall that we set $\rho_{max} = 1$, hence $v_{max} = 1$, $v(\rho) = 1 - \rho$ and $f(\rho) = \rho(1-\rho)$. We want to solve $\hat\rho(1-\hat\rho) = \hat\gamma$. Hence, by solving $\hat\rho^2 - \hat\rho + \hat\gamma = 0$, we get $\hat\rho = \frac{1}{2}(1 \pm \sqrt{\Delta(\hat\gamma)})$ where $\Delta(\hat\gamma) = 1 - 4\hat\gamma$ and $v(\hat\rho_\varphi) = (1-\hat\rho_\varphi) = \frac{1}{2}(1 - s_\varphi\sqrt{\Delta(\hat\gamma_\varphi)})$ ($v(\hat\rho_\psi) = (1-\hat\rho_\psi) = \frac{1}{2}(1 - s_\psi\sqrt{\Delta(\hat\gamma_\psi)})$ resp.), with:

incoming lines:

$$s_\varphi = \begin{cases} -1 & \text{if } \rho_{\varphi,0} \leq \sigma \text{ and } \Gamma = \Gamma_{in}, \\ & \text{or } \rho_{\varphi,0} \leq \sigma, \ p_\varphi\Gamma = \gamma_\varphi^{max} \text{ and } \Gamma = \Gamma_{out}; \\ +1 & \text{if } \rho_{\varphi,0} > \sigma, \\ & \text{or } \rho_{\varphi,0} \leq \sigma, \ p_\varphi\Gamma < \gamma_\varphi^{max} \text{ and } \Gamma = \Gamma_{out}; \end{cases}$$

outgoing lines:

$$s_\psi = \begin{cases} +1 & \text{if } \ \rho_{\psi,0} \geq \sigma \text{ and } \Gamma = \Gamma_{out}, \\ & \text{or } \ \rho_{\psi,0} \geq \sigma, \ \alpha_\psi \Gamma = \gamma_\psi^{max} \text{ and } \Gamma = \Gamma_{in}; \\ -1 & \text{if } \ \rho_{\psi,0} < \sigma, \\ & \text{or } \ \rho_{\psi,0} \geq \sigma, \ \alpha_\psi \Gamma < \gamma_\psi^{max} \text{ and } \Gamma = \Gamma_{in}. \end{cases}$$

Then

$$\begin{aligned} V_{\varphi\psi} &= \frac{1}{2}(2 - s_\varphi \sqrt{\Delta(\hat{\gamma}_\varphi)} - s_\psi \sqrt{\Delta(\hat{\gamma}_\psi)}), \\ J_1 &= \sum_{\varphi=1}^{m} \sum_{\psi=m+1}^{m+n} V_{\varphi\psi} \\ &= \frac{1}{2}\left(2mn - \left(n \sum_{\varphi=1}^{n} s_\varphi \sqrt{\Delta(\hat{\gamma}_\varphi)} + m \sum_{\psi=m+1}^{m+n} s_\psi \sqrt{\Delta(\hat{\gamma}_\psi)}\right)\right), \end{aligned}$$

and

$$J_2 = \sum_{\varphi=1}^{m} \frac{1}{v_\varphi} + \sum_{\psi=m+1}^{m+n} \frac{1}{v_\psi} = \sum_{\varphi=1}^{m} \frac{2}{1 - s_\varphi \sqrt{\Delta(\hat{\gamma}_\varphi)}} + \sum_{\psi=m+1}^{m+n} \frac{2}{1 - s_\psi \sqrt{\Delta(\hat{\gamma}_\psi)}}.$$

For J_3 we get:

$$\begin{aligned} J_3 &= \sum_{\varphi=1}^{m} \sum_{\psi=m+1}^{m+n} c_{\varphi\psi} V_{\varphi\psi} = \sum_{\varphi=1}^{m} \rho_\varphi(1-\rho_\varphi) + \sum_{\psi=m+1}^{m+n} \rho_\psi(1-\rho_\psi) \\ &= \frac{1}{4} \sum_{\varphi=1}^{m} (1 + s_\varphi \sqrt{\Delta(\hat{\gamma}_\varphi)})(1 - s_\varphi \sqrt{\Delta(\hat{\gamma}_\varphi)}) \\ &\quad + \frac{1}{4} \sum_{\psi=m+1}^{m+n} (1 + s_\psi \sqrt{\Delta(\hat{\gamma}_\psi)})(1 - s_\psi \sqrt{\Delta(\hat{\gamma}_\psi)}) \\ &= \frac{1}{4} \sum_{\varphi=1}^{m} (1 - \Delta(\hat{\gamma}_\varphi)) + \frac{1}{4} \sum_{\psi=m+1}^{m+n} (1 - \Delta(\hat{\gamma}_\psi)) \\ &= \sum_\varphi \hat{\gamma}_\varphi + \sum_\psi \hat{\gamma}_\psi. \end{aligned}$$

We notice that J_3 is constant. Indeed, if $\Gamma = \Gamma_{in}$ and all $\mathbf{A}\psi$ are satisfied,

$$J_3 = \sum_\varphi \gamma_\varphi^{max} + \sum_\psi \alpha_\psi \Gamma_{in} = \sum_\varphi \gamma_\varphi^{max} + \Gamma_{in};$$

if $\Gamma = \Gamma_{in}$ and any $\mathbf{A}\psi$ is not satisfied,

$$J_3 = \sum_\varphi \gamma_\varphi^{max} + \Gamma_{in} a,$$

where a is some constant depending on $proj(\alpha)$; if $\Gamma = \Gamma_{out}$ and all $\mathbf{B}\varphi$ are satisfied,

$$J_3 = \sum_\varphi p_\varphi \Gamma_{out} + \sum_\psi \gamma_\psi^{max} = \Gamma_{out} + \sum_\psi \gamma_\psi^{max};$$

if $\Gamma = \Gamma_{out}$ and any $\mathbf{B}\varphi$ is not satisfied,

$$J_3 = \sum_\psi \gamma_\psi^{max} + \Gamma_{out} b,$$

where b is some constant depending on $proj(p)$.

Finally we want to maximize the cost J_1 and to minimize the cost J_2 with respect to the parameters α_ψ and p_φ.

4.2. **Case** $\Gamma_{in} = \Gamma_{out}$**.** Assume first that $\Gamma = \Gamma_{in} = \Gamma_{out}$. Then all $\mathbf{A}\varphi$ are satisfied if and only if $\gamma_\psi = \gamma_\psi^{max}$ for all ψ. Indeed if $\gamma_\psi < \gamma_\psi^{max}$ for some ψ then $\Gamma_{in} \sum \alpha_\psi = \sum \gamma_\psi < \sum \gamma_\psi^{max} = \Gamma_{out}$ and we get a contradiction. In this case

$$\hat{\gamma} = (\gamma_1^{max}, \dots, \gamma_n^{max}, \gamma_{m+1}^{max}, \dots, \gamma_{m+n}^{max}),$$

thus J_1 and J_2 do not depend neither on the parameters α_ψ nor on the parameters p_φ.

4.3. **Case** $\Gamma_{in} < \Gamma_{out}$**.** Assume now that $\Gamma = \Gamma_{in} < \Gamma_{out}$ and all $\mathbf{A}\psi$ satisfied. In this case we have:

$$\hat{\gamma} = (\gamma_1^{max}, \dots, \gamma_m^{max}, \alpha_{m+1}\Gamma_{in}, \dots, \alpha_{m+n}\Gamma_{in}),$$

hence:

$$\begin{aligned} 2J_1 &= 2mn - n \sum s_\varphi \sqrt{\Delta(\hat{\gamma}_\varphi)} - m \sum s_\psi \sqrt{\Delta(\hat{\gamma}_\psi)} \\ &= 2mn - n \sum_\varphi s_\varphi \sqrt{1 - 4\gamma_\varphi^{max}} \qquad (12) \\ &\quad -m \sum_\psi s_\psi \sqrt{1 - 4\alpha_\psi \Gamma_{in}}, \qquad (13) \\ \frac{1}{2} J_2 &= \sum \frac{1}{1 - s_\varphi \sqrt{\Delta(\hat{\gamma}_\varphi)}} + \sum \frac{1}{1 - s_\psi \sqrt{\Delta(\hat{\gamma}_\psi)}} \\ &= \sum_\varphi \frac{1}{1 - s_\varphi \sqrt{1 - 4\gamma_\varphi^{max}}} \qquad (14) \\ &\quad + \sum_\psi \frac{1}{1 - s_\psi \sqrt{1 - 4\alpha_\psi \Gamma_{in}}}. \qquad (15) \end{aligned}$$

Now the part of the cost in (12) (resp. in (14)) does not depend on the α_ψ's and maximizing J_1 (resp. minimizing J_2) is equivalent to maximizing expression (13) (resp. minimizing expression (15).) Since we are in the case $\Gamma = \Gamma_{in} < \Gamma_{out}$, for the s_ψ's we always have $s_\psi = -1$ for all $\psi = m+1, \dots, m+n$ apart the case $s_\psi = +1$ when $\rho_{\psi,0} \geq \sigma$ and $\alpha_\psi \Gamma = \gamma_\psi^{max}$.

Remark. In what follows, we will always consider $s_\psi = -1$. If $\rho_{\psi,0} \geq \sigma$ then the functionals J_1 and J_2 are discontinuous at $\alpha_\psi = \gamma_\psi^{max}/\Gamma$ and their values must be computed separately. We will give all the details in the case $m = n = 2$.

Finally, setting $s_\psi = -1$, we have to maximize the expression

$$\hat{J}_1 = \sum_\psi \sqrt{1 - 4\alpha_\psi \Gamma_{in}} \qquad (16)$$

and to minimize the expression

$$\hat{J}_2 = \sum_\psi \frac{1}{1 + \sqrt{1 - 4\alpha_\psi \Gamma_{in}}}. \qquad (17)$$

Since the parameters α_ψ are restricted to the condition $\sum \alpha_\psi = 1$ we substitute the expression $\alpha_{m+n} = 1 - \sum_{\psi=m+1}^{m+n-1} \alpha_\psi$ in $\hat{J}_1$ and in $\hat{J}_2$ and, by slight abuse of notation, we also denote by $\hat{J}_1$ and $\hat{J}_2$ the resulting functions of $\alpha_{m+1}, \dots, \alpha_{m+n-1}$.

We begin with the analysis of $\hat{J}_1$ as function of $\alpha_{m+1}, \dots, \alpha_{m+n-1}$. The partial derivatives of $\hat{J}_1$ are given by

$$\frac{\partial}{\partial \alpha_i} \hat{J}_1(\alpha) = 2\Gamma_{in} \frac{-\sqrt{1-4(1-\sum_{\psi=m+1}^{m+n-1} \alpha_\psi)\Gamma_{in}} + \sqrt{1-4\alpha_i \Gamma_{in}}}{\sqrt{1-4\alpha_i\Gamma_{in}}\sqrt{1-4(1-\sum_{\psi=m+1}^{m+n-1} \alpha_\psi)\Gamma_{in}}}.$$

We then get that $\frac{\partial}{\partial \alpha_i} \hat{J}_1(\alpha) \geq 0$ for $\alpha_i \leq 1 - \sum_{\psi=m+1}^{m+n-1} \alpha_\psi$. We obtain that $\hat{J}_1$ has a critical point in $\alpha_{m+1} = \dots = \alpha_{m+n-1} = \alpha_{m+n} = \frac{1}{n}$.

We observe that $\hat{J}_1$, as function of $\alpha_{m+1}, \dots, \alpha_{m+n-1}$, and its level surfaces, on the $n-1$–dimensional space with coordinates $\alpha_{m+1}, \dots, \alpha_{m+n-1}$, are symmetric with respect to the line $\alpha_{m+1} = \dots = \alpha_{m+n-1}$. The Hessian of $\hat{J}_1$ is negative definite. Indeed

$$\begin{aligned}
\frac{\partial^2}{\partial \alpha_i^2} \hat{J}_1 &= -4\Gamma^2 \left(\frac{1}{(1-4(1-\sum \alpha_\psi)\Gamma)^{3/2}} + \frac{1}{(1-4\alpha_i\Gamma)^{3/2}} \right) \\
\frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \hat{J}_1 &= -4\Gamma^2 \frac{1}{(1-4(1-\sum \alpha_\psi)\Gamma)^{3/2}}, \quad \text{for } \ i \neq j \\
\alpha^T Hess(\hat{J}_1)\alpha &= -4\Gamma^2 \sum_{i=m+1}^{m+n-1} \left(\frac{1}{(1-4\alpha_i\Gamma)^{3/2}} \alpha_i^2 \right) \\
&\quad -4\Gamma^2 \frac{1}{(1-4(1-\sum \alpha_\psi)\Gamma)^{3/2}} \left(\sum_{i,j=m+1}^{m+n-1} \alpha_i \alpha_j \right).
\end{aligned}$$

Then $\hat{J}_1$ is concave and has a maximum in $\alpha_{m+1} = \dots = \alpha_{m+n-1} = \alpha_{m+n} = \frac{1}{n}$.

Now if such a point of maximum does not satisfy the constraints , i.e. does not belong to P_{in}, we have to find the maximum of $\hat{J}_1$ on P_{in}. The points of P_{in} candidate to maximize $\hat{J}_1$ are those lying on the boundary of P_{in} that is having coordinates $\gamma_\psi = \gamma_\psi^{max}$ for some $\psi \in \{m+1, \dots, m+n\}$. Then we fix ψ and consider the restriction of $\hat{J}_1$ on the hiperplane $\Pi_\psi = \{\alpha \in \mathbb{R}^{n-1} : \ \alpha_\psi = \frac{\gamma_\psi^{max}}{\Gamma}\}$. The critical point of $\hat{J}_1$ restricted to Π_ψ is given by the points $\alpha \in \Pi_\psi$ such that $\nabla \hat{J}_1 \perp \Pi_\psi$, i.e. $\nabla \hat{J}_1 \parallel \mathbf{n}$, where $\mathbf{n}$ is the versor normal to Π_ψ. Therefore the critical points of $\hat{J}_1$ on Π_ψ are given by the points that satisfy the following condition:

$$\nabla \hat{J}_1 - (\nabla \hat{J}_1 \cdot \mathbf{n})\mathbf{n} = 0. \tag{18}$$

Now for $\psi \in \{m+1, \dots, m+n-1\}$, $\mathbf{n} = \bar{e}_\psi$ (the canonical versor having all components zero apart from the ψ–th which is equal to 1) and equation (18) means that

$$\frac{\partial}{\partial \alpha_i} \hat{J}_1 = 0,$$

for all $i \neq \psi, \ i = m+1, \dots, m+n-1$. In particular the points where $\frac{\partial}{\partial \alpha_i} \hat{J}_1 = 0$ are those for which $\alpha_i = 1 - \sum_{j=m+1}^{m+n-1} \alpha_j$.

If $\psi = m+n$ we have that $\alpha_{m+n} = 1 - \sum_{j=m+1}^{m+n-1} \alpha_j = \frac{\gamma_{m+n}^{max}}{\Gamma}$, hence we get that $\mathbf{n} = \frac{1}{\sqrt{n-1}}(1, \dots, 1)^T$ and equation (18) means that

$$(n-2)\frac{\partial}{\partial \alpha_i} \hat{J}_1 - \sum_{j \neq i} \frac{\partial}{\partial \alpha_j} \hat{J}_1 = 0$$

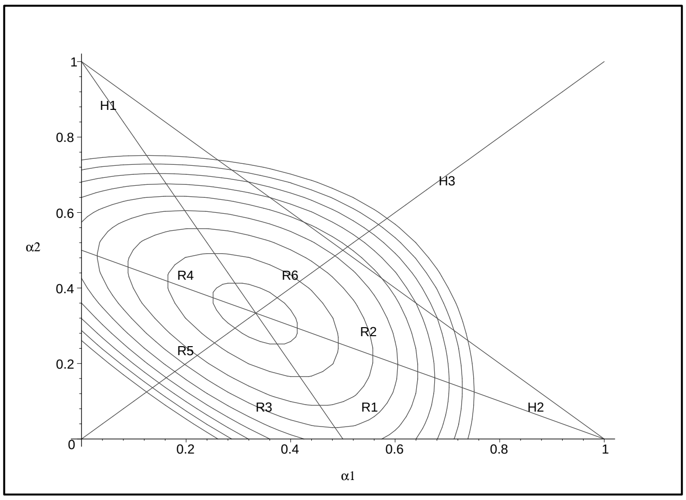

FIGURE 2. The level curves and the regions for J_1 when $n = 3$.

for all $i = m+1, \ldots, m+n-1$. Formally these are $n-1$ conditions which are however linearly dependent. Indeed they can be described by a linear system of rank $n-2$.

We denote by $\mathbf{p}_\psi$ the point of Π_ψ which satisfies equation (18) and by π the canonical projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\pi(\alpha_{m+1}, \ldots, \alpha_{m+n-1}, \alpha_{m+n}) = (\alpha_{m+1}, \ldots, \alpha_{m+n-1})$. There are three possibilities for such points:

1: there exists one and only $\psi \in \{m+1, \ldots, m+n\}$ such that the point $\mathbf{p}_\psi$ belongs to $\pi(P_{in})$;

2: there exists more than one $\psi \in \{m+1, \ldots, m+n\}$ such that the point $\mathbf{p}_\psi$ belongs to $\pi(P_{in})$;

3: the point $\mathbf{p}_\psi$ does not belong to $\pi(P_{in})$ for all $\psi \in \{m+1, \ldots, m+n\}$.

If case 1. holds $\mathbf{p}_\psi$ is the point the maximizes $\hat{J}_1$. If case 2. holds there are a finite number of points that are candidate to maximize $\hat{J}_1$. It is sufficient to check all of them to find out the maximum. Finally, if case 3. holds, it means that the space is divided in regions by the hiperplanes H_i where $\frac{\partial}{\partial \alpha_i} \hat{J}_1 = 0$ and $\pi(P_{in})$ is entirely contained in one of these regions. In this case, for each region, there exists one vertex of $\pi(P_{in})$ which maximizes $\hat{J}_1$. For $n = 3$ we give a complete description of the regions and the maximizing vertices. See figure 2 for a picture of the level curves of J_1 and the different regions.

For $n = 3$ and $\psi = m+1$, condition (18) gives $\frac{\partial}{\partial \alpha_{m+2}} \hat{J}_1 = 0$ which is satisfied by α such that $\alpha_{m+1} = \frac{\gamma_{m+1}^{max}}{\Gamma}$ and $\alpha_{m+2} = \frac{1-\alpha_{m+1}}{2}$, i.e.

$$\mathbf{p}_{m+1} = \left(\frac{\gamma_{m+1}^{max}}{\Gamma}, \frac{1}{2} \left(1 - \frac{\gamma_{m+1}^{max}}{\Gamma} \right) \right).$$

For $\psi = m+2$, condition (18) gives $\frac{\partial}{\partial \alpha_{m+1}} \hat{J}_1 = 0$ which is satisfied by α such that $\alpha_{m+2} = \frac{\gamma_{m+2}^{max}}{\Gamma}$ and $\alpha_{m+1} = \frac{1-\alpha_{m+2}}{2}$, i.e.

$$\mathbf{p}_{m+2} = \left(\frac{1}{2} \left(1 - \frac{\gamma_{m+2}^{max}}{\Gamma} \right), \frac{\gamma_{m+2}^{max}}{\Gamma} \right).$$

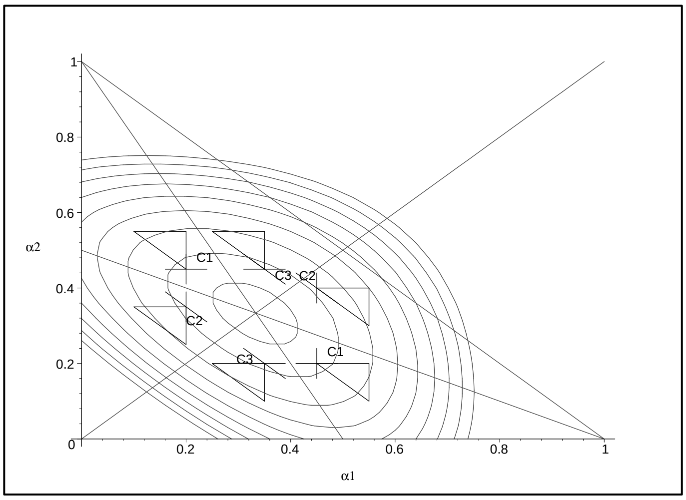

FIGURE 3. The cones C_i in the different regions identify the optimal vertices.

Finally for $\psi = m+3$, condition (18) gives $\frac{\partial}{\partial \alpha_{m+1}}\hat{J}_1 - \frac{\partial}{\partial \alpha_{m+2}}\hat{J}_1 = 0$ which is satisfied by α such that $\alpha_{m+3} = \frac{\gamma_{m+3}^{max}}{\Gamma}$, $\alpha_{m+1} = \alpha_{m+2}$ and $\alpha_{m+1} = \frac{1-\alpha_{m+3}}{2}$, i.e.

$$\mathbf{p}_{m+3} = \left(\frac{1}{2}\left(1 - \frac{\gamma_{m+3}^{max}}{\Gamma}\right), \frac{1}{2}\left(1 - \frac{\gamma_{m+3}^{max}}{\Gamma}\right)\right).$$

If we are in case 3. we introduce the following three lines

$$\begin{aligned} H_1 &= \{\alpha = (\alpha_{m+1}, \alpha_{m+2}) : \alpha_{m+1} = 1 - \alpha_{m+1} - \alpha_{m+2}\} \\ &= \{\alpha = (\alpha_{m+1}, \alpha_{m+2}) : \alpha_{m+2} = 1 - 2\alpha_{m+1}\}, \\ H_2 &= \{\alpha = (\alpha_{m+1}, \alpha_{m+2}) : \alpha_{m+2} = 1 - \alpha_{m+1} - \alpha_{m+2}\} \\ &= \{\alpha = (\alpha_{m+1}, \alpha_{m+2}) : \alpha_{m+2} = \frac{1}{2}(1 - \alpha_{m+1})\}, \\ H_3 &= \{\alpha = (\alpha_{m+1}, \alpha_{m+2}) : \alpha_{m+2} = \alpha_{m+1}\}. \end{aligned}$$

H_i are the lines where $\frac{\partial}{\partial \alpha_i}\hat{J}_1 = 0$ and these lines divide the region of $\mathbb{R}^2$: $\{\alpha \in \mathbb{R}^2 : \alpha_{m+1} + \alpha_{m+2} \leq 1\}$ in 6 regions. We denote by

R_1: the region comprised between H_1 and H_2 below H_3,
R_2: the region comprised between H_2 and H_3 below H_3,
R_3: the region comprised between H_3 and H_1 below H_3,
R_4: the region comprised between H_1 and H_2 above H_3,
R_5: the region comprised between H_2 and H_3 above H_3,
R_6: the region comprised between H_3 and H_1 above H_3.

Now, in the regions R_1 and R_4 the tangent space to the level curves is generated by a vector which belongs to the cone C_1 generated by the vectors $(1,0), (0,1)$ (see figure 3.)

In the regions R_2 and R_5 the tangent space to the level curves is generated by a vector which belongs to the cone C_2 generated by the vectors $(0,1), (1,-1)$.

In the regions R_3 and R_6 the tangent space to the level curves is generated by a vector which belongs to the cone C_3 generated by the vectors $(1,-1), (1,0)$.

We denote by v_i the vertex of $\pi(P_{in})$ such that each line of C_i passing through v_i, separates $\pi(P_{in})$ from the point $(\frac{1}{3}, \frac{1}{3})$.

We state that the maximum of $\hat{J}_1$ on $\pi(P_{in})$ is reached on the vertex v_i in the regions R_i and R_{3+i}. Indeed the level curves passing through the other vertices are farther from $(\frac{1}{3}, \frac{1}{3})$ than those passing through v_i.

Next we show that $\hat{J}_2$ behave in a fashion similar to $\hat{J}_1$. For the partial derivatives, by denoting $\widetilde{\alpha} = \widetilde{\alpha}(\alpha_{m+1}, \dots, \alpha_{m+n-1}) = (1 - \sum \alpha_\psi)$, we have:

$$\frac{\partial}{\partial \alpha_i}\hat{J}_2(\alpha) = 2\Gamma_{in}\frac{\sqrt{1-4\widetilde{\alpha}\Gamma_{in}}\left(1+\sqrt{1-4\widetilde{\alpha}\Gamma_{in}}\right)^2 - \sqrt{1-4\alpha_i\Gamma_{in}}\left(1+\sqrt{1-4\alpha_i\Gamma_{in}}\right)^2}{\sqrt{1-4\alpha_i\Gamma_{in}}\left(1+\sqrt{1-4\alpha_i\Gamma_{in}}\right)^2\sqrt{1-4\widetilde{\alpha}\Gamma_{in}}\left(1+\sqrt{1-4\widetilde{\alpha}\Gamma_{in}}\right)^2}.$$

We have that $\frac{\partial}{\partial \alpha_i}\hat{J}_2(\alpha) \geq 0$ for $\alpha_i \geq 1 - \sum_{\psi=m+1}^{m+n-1}\alpha_\psi$. We obtain that $\hat{J}_2$ has a critical point in $\alpha_{m+1} = \cdots = \alpha_{m+n-1} = \alpha_{m+n} = \frac{1}{n}$.

The Hessian of $\hat{J}_2$ is positive definite. Indeed

$$\begin{aligned}
\frac{\partial^2}{\partial \alpha_i^2}\hat{J}_2 &= 4\Gamma^2\left(\frac{1}{(1+\sqrt{1-4\alpha_i\Gamma})\sqrt{1-4\alpha_i\Gamma}}\right)^2\left(\frac{2}{(1+\sqrt{1-4\alpha_i\Gamma})}+\frac{1}{\sqrt{1-4\alpha_i\Gamma}}\right)\\
&\quad +4\Gamma^2\left(\frac{1}{(1+\sqrt{1-4\widetilde{\alpha}\Gamma})\sqrt{1-4\widetilde{\alpha}\Gamma}}\right)^2\left(\frac{2}{(1+\sqrt{1-4\widetilde{\alpha}\Gamma})}+\frac{1}{\sqrt{1-4\widetilde{\alpha}\Gamma}}\right)\\
\frac{\partial^2}{\partial \alpha_i \partial \alpha_j}\hat{J}_2 &= 4\Gamma^2\left(\frac{1}{(1+\sqrt{1-4\widetilde{\alpha}\Gamma})\sqrt{1-4\widetilde{\alpha}\Gamma}}\right)^2\left(\frac{2}{(1+\sqrt{1-4\widetilde{\alpha}\Gamma})}+\frac{1}{\sqrt{1-4\widetilde{\alpha}\Gamma}}\right),
\end{aligned}$$

and, finally

$$\begin{aligned}
&\alpha^T Hess(\hat{J}_2)\alpha = \\
&4\Gamma^2\sum_{i=m+1}^{m+n-1}\left(\frac{1}{(1+\sqrt{1-4\alpha_i\Gamma})\sqrt{1-4\alpha_i\Gamma}}\right)^2\left(\frac{2}{(1+\sqrt{1-4\alpha_i\Gamma})}+\frac{1}{\sqrt{1-4\alpha_i\Gamma}}\right)\alpha_i^2\\
&+4\Gamma^2\left(\frac{1}{(1+\sqrt{1-4\widetilde{\alpha}\Gamma})\sqrt{1-4\widetilde{\alpha}\Gamma}}\right)^2\left(\frac{2}{(1+\sqrt{1-4\widetilde{\alpha}\Gamma})}+\frac{1}{\sqrt{1-4\widetilde{\alpha}\Gamma}}\right)\left(\sum_{i,j=m+1}^{m+n-1}\alpha_i\alpha_j\right).
\end{aligned}$$

Then $\hat{J}_2$ is convex and has a minimum in $\alpha_{m+1} = \cdots = \alpha_{m+n} = \frac{1}{n}$.

It worths noticing that, up to exchanging the maximum with the minimum, $\hat{J}_2$ behaves exactly as $\hat{J}_1$. Indeed $\frac{\partial}{\partial \alpha_i}\hat{J}_1 = 0$ and $\frac{\partial}{\partial \alpha_i}\hat{J}_2 = 0$ are verified on the same hiperplanes. Therefore we have the same points $\mathbf{p}_\psi$, the same regions and the same optimal vertices if case 3. holds. See figure 4 for the level curves and the regions of J_2 when $n = 3$.

4.4. **Case $\Gamma_{out} < \Gamma_{in}$.** Assume now that $\Gamma = \Gamma_{out} < \Gamma_{in}$. In this case we have:

$$\hat{\gamma} = (p_1\Gamma_{out}, \dots, p_m\Gamma_{out}, \gamma_1^{max}, \dots, \gamma_n^{max}),$$

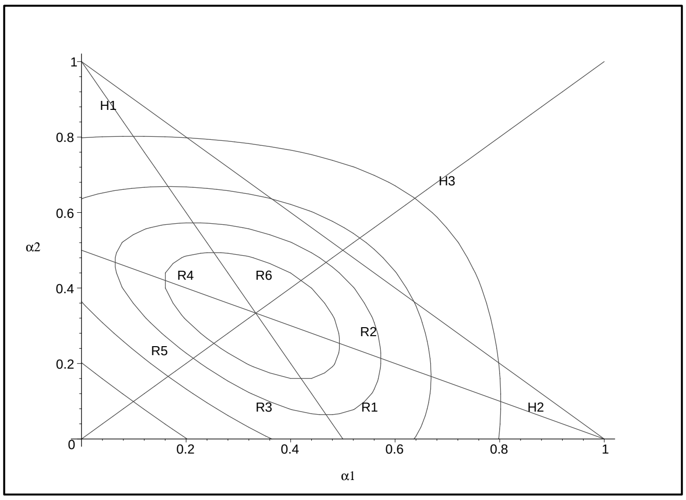

FIGURE 4. The level curves and the regions of J_2 when $n = 3$.

hence:

$$\begin{aligned} 2J_1 &= 2mn - n\sum s_\varphi\sqrt{\Delta(\hat\gamma_\varphi)} - m\sum s_\psi\sqrt{\Delta(\hat\gamma_\psi)} \\ &= 2mn - n\sum_\varphi s_\varphi\sqrt{1-4p_\varphi\Gamma_{out}} \qquad (19)\\ &\quad -m\sum_\psi s_\psi\sqrt{1-4\gamma_\psi^{max}}, \qquad (20)\end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}J_2 &= \sum\frac{1}{1-s_\varphi\sqrt{\Delta(\hat\gamma_\varphi)}} + \sum\frac{1}{1-s_\psi\sqrt{\Delta(\hat\gamma_\psi)}} \\ &= \sum_\varphi \frac{1}{1-s_\varphi\sqrt{1-4p_\varphi\Gamma_{out}}} \qquad (21)\\ &\quad +\sum_\psi\frac{1}{1-s_\psi\sqrt{1-4\gamma_\psi^{max}}}. \qquad (22)\end{aligned}$$

Now the part of the cost in (20) (resp. in (22)) does not depend on p and maximizing J_1 (resp. minimizing J_2) is equivalent to maximizing expression (19) (resp. minimizing expression (21).) Since we are in the case $\Gamma = \Gamma_{out} < \Gamma_{in}$, for the s_φ's we only have the case: $s_\varphi = +1$ for all $\varphi = 1,\dots,m$, apart the case $s_\varphi = -1$ when $\rho_{\varphi,0} \le \sigma$ and $p_\varphi\Gamma = \gamma_\varphi^{max}$.

Remark. As done for the case $\Gamma = \Gamma_{in}$, we only consider $s_\varphi = +1$. Then one should compare the optimal value of the functionals evaluated taking $s_\varphi = +1$ with the value for $s_\varphi = -1$ and $p_\varphi = \gamma_\varphi^{max}/\Gamma$. This will be done in detail for the case $m = n = 2$.

Finally we have to maximize the expression

$$\hat J_1 = -\sum_\varphi \sqrt{1-4p_\varphi\Gamma_{out}} \qquad (23)$$

and to minimize the expression

$$\hat{J}_2 \quad = \quad \sum_{\varphi} \frac{1}{1 - \sqrt{1 - 4p_\varphi \Gamma_{out}}}. \tag{24}$$

Notice that we get the same expression for $\hat{J}_1$ that we obtained in equations (23) with opposite sign and p in the place of α. Hence now $\hat{J}_1$ is convex and we have to find its maximal value when restricted to $\pi(P_{out})$. Such maximal value will fall on a vertex of $\pi(P_{out})$ and there are only a finite number of vertices to be checked.

For $\hat{J}_2$ we have

$$\frac{\partial}{\partial p_i} \hat{J}_2(p) =$$
$$2\Gamma_{in} \frac{\sqrt{1-4(1-\sum_{\varphi=1}^{m-1} p_\varphi)\Gamma_{out}} \ 1-\sqrt{1-4(1-\sum_{\varphi=1}^{m-1} p_\varphi)\Gamma_{out}}^{\ 2} - \sqrt{1-4p_i\Gamma_{out}}\left(1-\sqrt{1-4p_i\Gamma_{out}}\right)^2}{\sqrt{1-4p_i\Gamma_{out}}\left(1-\sqrt{1-4p_i\Gamma_{out}}\right)^2 \sqrt{1-4(1-\sum_{\varphi=1}^{m-1} p_\varphi)\Gamma_{out}} \ 1-\sqrt{1-4(1-\sum_{\varphi=1}^{m-1} p_\varphi)\Gamma_{out}}^{\ 2}}.$$

and

$$\begin{aligned} & p^T Hess(\hat{J}_2)p \\ = \quad & 4\Gamma^2 \sum_{i=1}^{m-1} \left(\frac{1}{\left(1 - \sqrt{1-4p_i\Gamma}\right)\sqrt{1-4p_i\Gamma}} \right)^2 \left(\frac{2}{\left(1 - \sqrt{1-4p_i\Gamma}\right)} + \frac{1}{\sqrt{1-4p_i\Gamma}} \right) p_i^2 \\ & +4\Gamma^2 \left(\frac{1}{\left(1 - \sqrt{1-4(1-\sum_{\varphi=1}^{m-1} p_\varphi)\Gamma}\right)\sqrt{1-4(1-\sum_{\varphi=1}^{m-1} p_\varphi)\Gamma}} \right)^2 \\ & \left(\frac{2}{\left(1 - \sqrt{1-4(1-\sum_{\varphi=1}^{m-1} p_\varphi)\Gamma}\right)} + \frac{1}{\sqrt{1-4(1-\sum_{\varphi=1}^{m-1} p_\varphi)\Gamma}} \right) \left(\sum_{i,j=1}^{m} p_i p_j \right). \end{aligned}$$

Hence $\hat{J}_2$ is concave and has a maximum in $p_1 = \cdots = p_m = \frac{1}{m}$. Since we want to find the minimum value of $\hat{J}_2$ restricted to $\pi(P_{out})$ it is sufficient to analyse the value of $\hat{J}_2$ on the vertices of $\pi(P_{out})$.

5. **Optimization of a simple network with $m = n = 2$.** We focus here on the case $n = m = 2$, i.e. on a simple network with one node o, two sources $\{1, 2\}$, two destinations $\{3, 4\}$ and four lines $\{a, b, c, d\}$, where a and b are the incoming lines to the node o and c and d are the outgoing lines from the node o. There are packets from sources $\{1, 2\}$ to destinations $\{3, 4\}$ passing through o and running on lines a, b, c, d: $c_{\varphi\psi}$ with $\varphi \in \{a, b\}$ and $\psi \in \{c, d\}$. We define the packets densities running on the lines as follows

- ρ_a from a to o: $\rho_a = c_{ac} + c_{ad}$;
- ρ_b from b to o: $\rho_b = c_{bc} + c_{bd}$;
- ρ_c from o to c: $\rho_c = c_{ac} + c_{bc}$;
- ρ_d from o to d: $\rho_d = c_{ad} + c_{bd}$.

Our aim is to solve the RP for the node at o. Then we want to measure the average transmission over the network as function of the parameters α and p. We consider the following costs:

$$J_1 = V_{ac} + V_{ad} + V_{bc} + V_{bd}$$
$$J_2 = \frac{1}{v_a} + \frac{1}{v_b} + \frac{1}{v_c} + \frac{1}{v_d}$$
$$J_3 = c_{ac}V_{ac} + c_{ad}V_{ad} + c_{bc}V_{bc} + c_{bd}V_{bd}.$$

with $V_{\varphi\psi} = v_\varphi + v_\psi$, $v_\varphi = v(\hat{\rho}_\varphi)$, $v_\psi = v(\hat{\rho}_\psi)$ and where $\hat{\rho}$ is the solution to the RP. The systems (10) and (11) are satisfied by the points $(p\Gamma, (1-p)\Gamma)$ for the incoming lines and $(\alpha\Gamma, (1-\alpha)\Gamma)$ for the outgoing lines, respectively. For these to be the solutions to the RP for the outgoing lines, the following conditions must be satisfied:

A1: $\gamma_c = \alpha\Gamma_{in} \leq \gamma_c^{max}$
A2: $\gamma_d = (1-\alpha)\Gamma_{in} \leq \gamma_d^{max}$.

Otherwise, we consider the only reasonable projection which gives:
$proj(\alpha\Gamma, (1-\alpha)\Gamma) = (\gamma_c^{max}, \Gamma - \gamma_c^{max})$ if A1 is not satisfied and A2 is satisfied and $proj(\alpha\Gamma, (1-\alpha)\Gamma) = (\Gamma - \gamma_d^{max}, \gamma_d^{max})$ if A1 is satisfied and A2 is not satisfied.

Therefore if $\Gamma = \Gamma_{in}$ the solutions given to the RP are the following:

- $(\gamma_a^{max}, \gamma_b^{max}, \alpha\gamma_c, (1-\alpha)\gamma_d)$ if both A1 and A2 are satisfied.
- $(\gamma_a^{max}, \gamma_b^{max}, \gamma_c^{max}, \Gamma_{in} - \gamma_c^{max})$ if A1 is not satisfied and A2 is satisfied.
- $(\gamma_a^{max}, \gamma_b^{max}, \Gamma_{in} - \gamma_d^{max}, \gamma_d^{max})$ if A1 is satisfied and A2 is not satisfied.

Notice that the case of both **A1, A2** false is not possible since otherwise it would be $\Gamma_{in} \geq \Gamma_{out}$.

For the incoming lines, the conditions read:

B1: $\tilde{\gamma}_a = p\Gamma_{out} \leq \gamma_a^{max}$
B2: $\tilde{\gamma}_b = (1-p)\Gamma_{out} \leq \gamma_b^{max}$.

Otherwise, we consider the following reasonable projection on the admissible set:
$proj(p\Gamma, (1-p)\Gamma) = (\gamma_a^{max}, \Gamma - \gamma_a^{max})$ if B1 is not satisfied and B2 is satisfied and $proj(p\Gamma, (1-p)\Gamma) = (\Gamma - \gamma_b^{max}, \gamma_b^{max})$ if B1 is satisfied and B2 is not satisfied.

Thus, if $\Gamma = \Gamma_{out}$ the solutions to the RP are the following:

- $(p\gamma_a, (1-p)\gamma_b, \gamma_c^{max}, \gamma_d^{max})$ if both B1 and B2 are satisfied.
- $(\gamma_a^{max}, \Gamma_{out} - \gamma_a^{max}, \gamma_c^{max}, \gamma_d^{max})$ if B1 is not satisfied and B2 is satisfied.
- $(\Gamma_{out} - \gamma_b^{max}, \gamma_b^{max}, \gamma_c^{max}, \gamma_d^{max})$ if B1 is satisfied and B2 is not satisfied.

Also now the case of both **B1, B2** false is not possible since otherwise it would be $\Gamma_{out} \geq \Gamma_{in}$.

Once fixed ρ_φ and ρ_ψ, $\varphi \in \{a, b\}$ and $\psi \in \{c, d\}$, we can find for which α and p conditions **A1**, **A2**, **B1**, **B2** are satisfied as follows. If $\Gamma = \Gamma_{in}$, let

$$\gamma_c' = \Gamma - \gamma_d^{max}, \quad \gamma_d' = \Gamma - \gamma_c^{max}, \quad \beta^- = \frac{\gamma_d'}{\gamma_c^{max}}, \quad \beta^+ = \frac{\gamma_d^{max}}{\gamma_c'}$$

then, for $\alpha \geq \frac{1}{1+\beta^-}$, **A1** is false and **A2** is true, for $\alpha \leq \frac{1}{1+\beta^+}$, **A1** is true and **A2** is false and finally, for $\frac{1}{1+\beta^+} \leq \alpha \leq \frac{1}{1+\beta^-}$, both **A1** and **A2** are true.

If otherwise $\Gamma = \Gamma_{out}$, let

$$\gamma_a' = \Gamma - \gamma_b^{max}, \quad \gamma_b' = \Gamma - \gamma_a^{max}, \quad q^- = \frac{\gamma_b'}{\gamma_a^{max}}, \quad q^+ = \frac{\gamma_b^{max}}{\gamma_a'}$$

then, for $p \geq \frac{1}{1+q^-}$, **B1** is false and **B2** is true, for $p \leq \frac{1}{1+q^+}$, **B1** is true and **B2** is false and finally, for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$, both **B1** and **B2** are true.

To compute the cost we observe that $\hat{\rho}_\varphi = f^{-1}(\hat{\gamma}_\varphi)$ ($\hat{\rho}_\psi = f^{-1}(\hat{\gamma}_\psi)$ resp.) and $\hat{\gamma}_\varphi$ is either $p\gamma_\varphi$, $(1-p)\gamma_\varphi$, γ_φ^{max} or $\Gamma - \gamma_\varphi^{max}$ ($\hat{\gamma}_\psi$ is either $\alpha\gamma_\psi$, $(1-\alpha)\gamma_\psi$, γ_ψ^{max} or $\Gamma - \gamma_\psi^{max}$ resp.).

Substituting all the possible solutions to the RP in the expression of J_3 we obtain that $J_3 = 2\Gamma$. Therefore, as for the general case of m incoming lines and n outgoing lines, we consider the flux of equation (2), set $v_{max} = \rho_{max} = 1$ and maximize J_1 and minimize J_2.

Now if $\Gamma = \Gamma_{in} = \Gamma_{out}$. Then both **A1** and **A2** are satisfied if and only if $\beta^- = \beta^+ = \frac{\gamma_d^{max}}{\gamma_c^{max}}$, hence $\alpha = \frac{\gamma_c^{max}}{\gamma_c^{max}+\gamma_d^{max}} = \frac{\gamma_c^{max}}{\Gamma}$. In this case

$$\hat{\gamma} = (\gamma_a^{max}, \gamma_b^{max}, \gamma_c^{max}, \gamma_d^{max}),$$

and J_1 and J_2 do not depend neither on α nor on p.

5.1. **Case $\Gamma_{in} < \Gamma_{out}$.** Assume now that $\Gamma = \Gamma_{in} < \Gamma_{out}$ and both **A1** and **A2** are satisfied. In this case we have:

$$\hat{\gamma} = (\gamma_a^{max}, \gamma_b^{max}, \alpha(\gamma_a^{max} + \gamma_b^{max}), (1-\alpha)(\gamma_a^{max} + \gamma_b^{max})).$$

and maximizing J_1 and minimizing J_2 is equivalent to maximizing the expression (13) and minimizing the expression (15).

Since we are in the case $\Gamma = \Gamma_{in} < \Gamma_{out}$ and **A1** and **A2** true, for s_c and s_d we have: $s_c = s_d = -1$. Hence we have to maximize the expression

$$\hat{J}_1 \quad = \quad \sqrt{1 - 4\alpha\Gamma_{in}} + \sqrt{1 - 4(1-\alpha)\Gamma_{in}} \tag{25}$$

and to minimize the expression

$$\hat{J}_2 \quad = \quad \frac{1}{1 + \sqrt{1 - 4\alpha\Gamma_{in}}} + \frac{1}{1 + \sqrt{1 - 4(1-\alpha)\Gamma_{in}}}. \tag{26}$$

Now the case $\rho_a = \rho_b = \frac{1}{2}$ cannot happen since we would have $\gamma_a^{max} = \gamma_b^{max} = \frac{1}{4}$, and $\Gamma = \frac{1}{2}$. But the maximal value of Γ_{out} is $\frac{1}{2}$ which fact contradicts the assumption that $\Gamma_{in} < \Gamma_{out}$. Assume then that not both ρ_a and ρ_b are equal to $\frac{1}{2}$. By the expressions of the first and second derivatives of J_1 (J_2 resp.) we get that J_1 is concave (J_2 is convex) and has a maximum (minimum) in $\bar{\alpha}$ where $\bar{\alpha} = \frac{1}{2}$.

For the α's such that **A2** is satisfied but **A1** is not and viceversa, we have that J_1 and J_2 do not depend on α and, in particular the values of the $\hat{J}_1$ and $\hat{J}_2$ are

$$\hat{J}_1 = \begin{cases} \sqrt{1 - 4\gamma_\psi^{max}} + \sqrt{1 - 4(\Gamma - \gamma_\psi^{max})} & \text{if } s_c = s_d = -1 \text{ and } \alpha_\psi\Gamma \geq \gamma_\psi^{max}, \\ -\sqrt{1 - 4\gamma_\psi^{max}} + \sqrt{1 - 4(\Gamma - \gamma_\psi^{max})} & \text{if } s_\psi = +1 \text{ and } \alpha_\psi\Gamma \geq \gamma_\psi^{max}, \end{cases}$$

$$\hat{J}_2 = \begin{cases} \frac{1}{1+\sqrt{1-4\gamma_\psi^{max}}} + \frac{1}{1+\sqrt{1-4(\Gamma-\gamma_\psi^{max})}} & \text{if } s_c = s_d = -1 \text{ and } \alpha_\psi\Gamma \geq \gamma_\psi^{max}, \\ \frac{1}{1-\sqrt{1-4\gamma_\psi^{max}}} + \frac{1}{1+\sqrt{1-4(\Gamma-\gamma_\psi^{max})}} & \text{if } s_\psi = +1 \text{ and } \alpha_\psi\Gamma \geq \gamma_\psi^{max}, \end{cases}$$

where we used the notation $\psi = c, d$, $\alpha_\psi = \alpha, 1-\alpha$.

5.1.1. *Optimal choices of α.* First we give the optimal choice of α when $s_c = s_d = -1$. Then we will treat the case when either s_c or s_d or both are equal to $+1$. We can collect the informations of the previous section as follows:

- $\beta^- \leq 1 \leq \beta^+$ then J_1 is constant for $0 \leq \alpha \leq \frac{1}{1+\beta^+}$ then it increases for $\frac{1}{1+\beta^+} \leq \alpha \leq \frac{1}{2}$ then it decreases for $\frac{1}{2} \leq \alpha \leq \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \leq \alpha \leq 1$.

  The optimal value of J_1 is for $\alpha = \frac{1}{2}$.

- $\beta^- \le \beta^+ \le 1$ then J_1 is constant for $0 \le \alpha \le \frac{1}{1+\beta^+}$ then it decreases for $\frac{1}{1+\beta^+} \le \alpha \le \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \le \alpha \le 1$.
  The optimal values of J_1 are for $\alpha \in [0, \frac{1}{1+\beta^+}]$.
- $1 \le \beta^- \le \beta^+$ then J_1 is constant for $0 \le \alpha \le \frac{1}{1+\beta^+}$ then it increases for $\frac{1}{1+\beta^+} \le \alpha \le \frac{1}{1+\beta^-}$ and then itis again constant for $\frac{1}{1+\beta^-} \le \alpha \le 1$.
  The optimal values of J_1 are for $\alpha \in [\frac{1}{1+\beta^-}, 1]$.

Finally for J_2 we have the following cases:

- $\beta^- \le 1 \le \beta^+$ then J_2 is constant for $0 \le \alpha \le \frac{1}{1+\beta^+}$ then it decreases for $\frac{1}{1+\beta^+} \le \alpha \le \frac{1}{2}$ then it increases for $\frac{1}{2} \le \alpha \le \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \le \alpha \le 1$.
  The optimal value of J_2 is for $\alpha = \frac{1}{2}$.
- $\beta^- \le \beta^+ \le 1$ then J_2 is constant for $0 \le \alpha \le \frac{1}{1+\beta^+}$ then it increases for $\frac{1}{1+\beta^+} \le \alpha \le \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \le \alpha \le 1$.
  The optimal values of J_2 are for $\alpha \in [0, \frac{1}{1+\beta^+}]$.
- $1 \le \beta^- \le \beta^+$ then J_2 is constant for $0 \le \alpha \le \frac{1}{1+\beta^+}$ then it decreases for $\frac{1}{1+\beta^+} \le \alpha \le \frac{1}{1+\beta^-}$ and then itis again constant for $\frac{1}{1+\beta^-} \le \alpha \le 1$.
  The optimal values of J_2 are for $\alpha \in [\frac{1}{1+\beta^-}, 1]$.

Now we treat the case when either s_c or s_d or both are equal to $+1$. For $\hat{J}_1$ it is fundamental to recall that it is a measure of the velocity which is a decreasing function of the density $v = 1 - \rho$ hence taking the solution of ρ_ψ with $s_\psi = +1$ corresponds to taking a lower velocity. We distinguish the following cases.

If $\beta^- \le 1 \le \beta^+$, then the maximal value of J_1 does not change by exchanging $s_\psi = -1$ with $s_\psi = +1$ (with the optimal value for $\alpha = \frac{1}{2}$.)

If $\beta^- \le \beta^+ \le 1$ and $s_d = -1$ then the maximal value of J_1 remains invaried with the optimal choice being $\alpha \in [0, \frac{1}{1+\beta^+}]$.

If $\beta^- \le \beta^+ \le 1$ and $s_d = +1$ then the maximal value of J_1 does not exists. One can chose $\alpha = \frac{1}{1+\beta^+} + \epsilon$.

If $1 \le \beta^- \le \beta^+$ and $s_c = -1$ then the maximal value of J_1 remains invaried with the optimal choice being $\alpha \in [\frac{1}{1+\beta^-}, 1]$.

If $1 \le \beta^- \le \beta^+$ and $s_c = +1$ then the maximal value of J_1 does not exists. One can chose $\alpha = \frac{1}{1+\beta^-} - \epsilon$.

Analogously, for $\hat{J}_2$, since it is a measure of the time, which is an increasing function of the density $\frac{1}{v} = \frac{1}{1-\rho}$, taking the solution of ρ_ψ with $s_\psi = +1$ corresponds to taking a bigger time value. Hence we distinguish the following cases:

If $\beta^- \le 1 \le \beta^+$, then the minimal value of J_2 does not change by exchanging $s_\psi = -1$ with $s_\psi = +1$ (with the optimal value for $\alpha = \frac{1}{2}$.)

If $\beta^- \le \beta^+ \le 1$ and $s_d = -1$ then the maximal value of J_1 remains invaried with the optimal choice being $\alpha \in [0, \frac{1}{1+\beta^+}]$.

If $\beta^- \le \beta^+ \le 1$ and $s_d = +1$ then the maximal value of J_1 does not exists. One can chose $\alpha = \frac{1}{1+\beta^+} + \epsilon$.

If $1 \le \beta^- \le \beta^+$ and $s_c = -1$ then the maximal value of J_1 remains invaried with the optimal choice being $\alpha \in [\frac{1}{1+\beta^-}, 1]$.

If $1 \le \beta^- \le \beta^+$ and $s_c = +1$ then the maximal value of J_1 does not exists. One can chose $\alpha = \frac{1}{1+\beta^-} - \epsilon$.

5.2. **Case** $\Gamma_{out} < \Gamma_{in}$. Assume now that $\Gamma = \Gamma_{out} < \Gamma_{in}$ and **B1** and **B2** are satisfied. In this case we have:

$$\hat{\gamma} = (p\Gamma, (1-p)\Gamma, \gamma_c^{max}, \gamma_d^{max}),$$

and maximizing J_1 and minimizing J_2 is equivalent to maximizing the expression (19) and minimizing the expression (21).

Since we are in the case $\Gamma = \Gamma_{out} < \Gamma_{in}$ and **B1** and **B2** true, for s_a and s_b we only have the case: $s_c = s_d = +1$. Hence we have to maximize the expression

$$\hat{J}_1 = -(\sqrt{1-4p\Gamma} + \sqrt{1-4(1-p)\Gamma}) \tag{27}$$

and to minimize the expression

$$\hat{J}_2 \quad = \quad \frac{1}{1-\sqrt{1-4p\Gamma_{out}}} + \frac{1}{1-\sqrt{1-4(1-p)\Gamma_{out}}}. \tag{28}$$

Notice that in this case we have the same expression of $\hat{J}_1$ that we obtained in equations (25) with opposite sign and p in the place of α. For J_2, as in the general case, we obtain that it is a concave function of p and has a maximum in $p = \frac{1}{2}$. Similarly to the case $\Gamma = \Gamma_{in}$ we obtain that when conditions **B1** is false and **B2** is true or viceversa then the cost functions J_1 and J_2 are constant with respect to p. In particular the values of $\hat{J}_1$ and $\hat{J}_2$ are the following:

$$\hat{J}_1 = \begin{cases} -\sqrt{1-4\gamma_\varphi^{max}} - \sqrt{1-4(\Gamma-\gamma_\varphi^{max})} & \text{if } s_a = s_b = +1 \text{ and } p_\varphi\Gamma \geq \gamma_\varphi^{max}, \\ +\sqrt{1-4\gamma_\varphi^{max}} - \sqrt{1-4(\Gamma-\gamma_\varphi^{max})} & \text{if } s_\varphi = -1 \text{ and } p_\varphi\Gamma \geq \gamma_\varphi^{max}, \end{cases}$$

$$\hat{J}_2 = \begin{cases} \frac{1}{1-\sqrt{1-4\gamma_\varphi^{max}}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma_\varphi^{max})}} & \text{if } s_a = s_b = +1 \text{ and } p_\varphi\Gamma \geq \gamma_\varphi^{max}, \\ \frac{1}{1+\sqrt{1-4\gamma_\varphi^{max}}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma_\varphi^{max})}} & \text{if } s_\varphi = -1 \text{ and } p_\varphi\Gamma \geq \gamma_\varphi^{max}, \end{cases}$$

where we have used the notation $\varphi = a, b$, $p_\varphi = p, 1-p$.

5.2.1. *Optimal choice of* p. First we give the optimal choice of p when $s_a = s_b = +1$. Then we will treat the case when either s_a or s_b or both are equal to -1. We can collect the above informations in the following way:

- $q^- \leq 1 \leq q^+$ then J_1 is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it decreases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{2}$ then it increases for $\frac{1}{2} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$.
  We distinguish three cases: $\frac{1}{2} - \frac{1}{1+q^+} > \frac{1}{1+q^-} - \frac{1}{2}$, $\frac{1}{2} - \frac{1}{1+q^+} = \frac{1}{1+q^-} - \frac{1}{2}$ and $\frac{1}{2} - \frac{1}{1+q^+} < \frac{1}{1+q^-} - \frac{1}{2}$. Simplifying we obtain the three cases: $q^-q^+ > 1$, $q^-q^+ = 1$ and $q^-q^+ < 1$. In the first case we have that the optimal values of J_1 are for $p \in [0, \frac{1}{1+q^+}]$, in the second case the optimal values of J_1 are for $p \in [0, \frac{1}{1+q^+}[\cup]\frac{1}{1+q^-}, 1]$, and in the third case the optimal values of J_1 are for $p \in [\frac{1}{1+q^-}, 1]$,
- $q^- \leq q^+ \leq 1$ then J_1 is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it increases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$.
  The optimal values of J_1 are for $p \in [\frac{1}{1+q^-}, 1]$.
- $1 \leq q^- \leq q^+$ then J_1 is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it decreases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$.
  The optimal values of J_1 are for $p \in [0, \frac{1}{1+q^+}]$.

For J_2 we have the following cases:

- $q^- \leq 1 \leq q^+$ then J_2 is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it increases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{2}$ then it decreases for $\frac{1}{2} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$.
  We distinguish three cases: $\frac{1}{2} - \frac{1}{1+q^+} > \frac{1}{1+q^-} - \frac{1}{2}$, $\frac{1}{2} - \frac{1}{1+q^+} = \frac{1}{1+q^-} - \frac{1}{2}$ and $\frac{1}{2} - \frac{1}{1+q^+} < \frac{1}{1+q^-} - \frac{1}{2}$. Simplifying we obtain the three cases: $q^-q^+ > 1$, $q^-q^+ = 1$ and $q^-q^+ < 1$. In the first case we have that the optimal values of J_2 are for $p \in [0, \frac{1}{1+q^+}]$, in the second case the optimal values of J_2 are for $p \in [0, \frac{1}{1+q^+}[\cup]\frac{1}{1+q^-}, 1]$, and in the third case the optimal values of J_2 are for $p \in [\frac{1}{1+q^-}, 1]$,
- $q^- \leq q^+ \leq 1$ then J_2 is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it decreases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$.
  The optimal values of J_2 are for $p \in [\frac{1}{1+q^-}, 1]$.
- $1 \leq q^- \leq q^+$ then J_2 is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it increases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$.
  The optimal values of J_2 are for $p \in [0, \frac{1}{1+q^+}]$.

We treat now the case where $s_a = s_b = -1$. Concerning J_1 we compare the following quantities:

$$\sqrt{1-4\gamma_a^{max}} - \sqrt{1-4(\Gamma-\gamma_a^{max})} \leq \sqrt{1-4\gamma_b^{max}} - \sqrt{1-4(\Gamma-\gamma_b^{max})}. \qquad (29)$$

Assume first that $q^- \leq 1 \leq q^+$. Then $\gamma_a, \gamma_b \geq \Gamma/2$ and

$$\sqrt{1-4\gamma_\varphi^{max}} - \sqrt{1-4(\Gamma-\gamma_\varphi^{max})} \leq 0.$$

Therefore we get that the inequality (29) is satisfied if:

$$2-4\Gamma-2\sqrt{1-4\gamma_b^{max}}\sqrt{1-4(\Gamma-\gamma_b^{max})} \leq 2-4\Gamma-2\sqrt{1-4\gamma_a^{max}}\sqrt{1-4(\Gamma-\gamma_a^{max})},$$

that is if

$$1-16(\gamma_a^{max})^2 - 4\Gamma + 16\Gamma\gamma_a^{max} \leq 1-16(\gamma_b^{max})^2 - 4\Gamma + 16\Gamma\gamma_b^{max},$$

i.e. when

$$\gamma_a^{max}(\Gamma-\gamma_a^{max}) \leq \gamma_b^{max}(\Gamma-\gamma_b^{max}),$$

which is true if γ_a^{max} is farther than γ_b^{max} from $\Gamma/2$. Hence, in the case $q^- \leq 1 \leq q^+$, the optimal value of J_1 is attained in the opposite intervals that maximize J_1 when both s_a and s_b are equal to $+1$.

If $q^- \leq q^+ \leq 1$ then it means that $\gamma_a \geq \Gamma/2 \geq \gamma_b$ hence

$$\sqrt{1-4\gamma_a^{max}} - \sqrt{1-4(\Gamma-\gamma_a^{max})} \leq 0 \leq \sqrt{1-4\gamma_b^{max}} - \sqrt{1-4(\Gamma-\gamma_b^{max})}$$

and J_1 is maximized for $p \in [0, \frac{1}{1+q^+}]$.

If $1 \leq q^- \leq q^+$ then it means that $\gamma_a, \leq \Gamma/2 \leq \gamma_b$ hence

$$\sqrt{1-4\gamma_a^{max}} - \sqrt{1-4(\Gamma-\gamma_a^{max})} \geq 0 \geq \sqrt{1-4\gamma_b^{max}} - \sqrt{1-4(\Gamma-\gamma_b^{max})}$$

and J_1 is maximized for $p \in [\frac{1}{1+q^-}, 1]$.

We consider now the case where $s_a = -1$ and $s_b = +1$ and compare the following quantities:

$$+\sqrt{1-4\gamma_a^{max}} - \sqrt{1-4(\Gamma-\gamma_a^{max})} \geq -\sqrt{1-4\gamma_b^{max}} - \sqrt{1-4(\Gamma-\gamma_b^{max})}. \quad (30)$$

If $q^- \geq 1$ then (30) is trivially satisfied since its left handside is positive. If otherwise $q^- \leq 1$ we get that

$$2-4\Gamma-2\sqrt{1-4\gamma_a^{max}}\sqrt{1-4(\Gamma-\gamma_a^{max})} \leq 2-4\Gamma+2\sqrt{1-4\gamma_b^{max}}\sqrt{1-4(\Gamma-\gamma_b^{max})}$$

which is always verified. Hence, in this case, the optimal value of J_1 is attained for $p \in [\frac{1}{1+q^-}, 1]$.

Finally we treat the case $s_a = +1$ and $s_b = -1$ and compare the quantities:

$$-\sqrt{1-4\gamma_a^{max}} - \sqrt{1-4(\Gamma-\gamma_a^{max})} \leq +\sqrt{1-4\gamma_b^{max}} - \sqrt{1-4(\Gamma-\gamma_b^{max})}. \quad (31)$$

If $q^+ \leq 1$ then (31) is trivially satisfied since its right handside is positive. If otherwise $q^+ \geq 1$ we get

$$2-4\Gamma+2\sqrt{1-4\gamma_a^{max}}\sqrt{1-4(\Gamma-\gamma_a^{max})} \geq 2-4\Gamma-2\sqrt{1-4\gamma_b^{max}}\sqrt{1-4(\Gamma-\gamma_b^{max})}$$

which is always verified. Hence, in this case, the optimal value of J_1 is attained for $p \in [0, \frac{1}{1+q^+}]$.

Concerning J_2, if $s_a = s_b = -1$, we have to compare the following quantities:

$$\frac{1}{1+\sqrt{1-4\gamma_a^{max}}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma_a^{max})}} \leq \frac{1}{1+\sqrt{1+4\gamma_b^{max}}} + \frac{1}{1+\sqrt{1+4(\Gamma+\gamma_b^{max})}}. \quad (32)$$

Let

$$\Phi(\gamma) = \frac{1}{1+\sqrt{1-4\gamma}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma)}}$$

and notice that

$$\frac{d}{d\gamma}\Phi(\gamma) = \frac{2}{(1+\sqrt{1-4\gamma})^2\sqrt{1-4\gamma}} + \frac{2}{(1-\sqrt{1-4(\Gamma-\gamma)})^2\sqrt{1-4(\Gamma-\gamma)}} \geq 0.$$

Hence $\Phi(\gamma)$ is a strictly increasing function of γ. Therefore the inequality (32) is satisfied if and only if $\gamma_a^{max} \leq \gamma_b^{max}$ that is, if and only if

$$\frac{\Gamma-\gamma_a^{max}}{\gamma_a^{max}} = q^- \geq \frac{1}{q^+} = \frac{\Gamma-\gamma_b^{max}}{\gamma_b^{max}}.$$

Finally $q^+q^- > 1$ implies that J_2 is minimized for $p \in [\frac{1}{1+q^-}, 1]$, $q^+q^- < 1$ implies that J_2 is minimized for $p \in [0, \frac{1}{1+q^-}]$ and $q^+q^- = 1$ implies that J_2 is minimized for $p \in [\frac{1}{1+q^-}, 1] \cup [0, \frac{1}{1+q^-}]$. This means that the intervals that optimize J_2 are reversed with respect to the case where both $s_a = s_b = +1$.

Next we consider the case where $s_a = -1$ and $s_b = +1$ and compare:

$$\frac{1}{1+\sqrt{1-4\gamma_a^{max}}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma_a^{max})}} \leq \frac{1}{1-\sqrt{1-4\gamma_b^{max}}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma_b^{max})}}. \quad (33)$$

We denote by

$$\Psi(\gamma) = \frac{1}{1-\sqrt{1-4\gamma}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma)}}.$$

and we notice that Ψ is a concave function simmetric with respect to $\Gamma/2$. Moreover we have

$$\delta(\gamma) = \Psi(\gamma) - \Phi(\gamma) = \frac{1}{1-\sqrt{1-4\gamma}} - \frac{1}{1+\sqrt{1-4\gamma}} = \frac{\sqrt{1-4\gamma}}{2\gamma} \geq 0.$$

Now if $\gamma_a \leq \gamma_b$, that is if $q^+q^- \geq 1$, then, since Φ is increasing, $\Phi(\gamma_a) \leq \Phi(\gamma_b) \leq \Psi(\gamma_b)$, i.e. the inequality (33) is satisfied, and J_2 is minimized for $p \in [\frac{1}{1+q^-}, 1]$.

If $\gamma_a \geq \gamma_b$, which is equivalent to say that $q^+q^- \leq 1$, then J_2 is minimized by $p \in [\frac{1}{1+q^-}, 1]$ when $s_a = s_b = +1$, a fortiori these are the optimal parameters when $s_a = -1$ which correspond to taking a smaller time.

Finally, if $s_a = +1$ and $s_b = -1$, we compare:

$$\begin{aligned} & \frac{1}{1-\sqrt{1-4\gamma_a^{max}}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma_a^{max})}} \\ \geq \; & \frac{1}{1-\sqrt{1-4\gamma_b^{max}}} + \frac{1}{1-\sqrt{1-4(\Gamma-\gamma_b^{max})}}. \end{aligned} \tag{34}$$

We have that the inequality (34) is satisfied if $\gamma_b \leq \gamma_a$ which means that $q^+q^- \leq 1$. Hence, in this case, J_2 is minimized for $p \in [0, \frac{1}{1+q^+}]$. If otherwise $\gamma_b \geq \gamma_a$, i.e. if $q^+q^- \geq 1$, then $p \in [0, \frac{1}{1+q^+}]$ is the optimal choice when $s_a = s_b = +1$. A fortiori it is the optimal choice if we take $s_b = -1$, that is a smaller time.

REFERENCES

[1] D. Amadori, Initial-boundary value problems for systems of conservation laws, *NoDEA*, 4 (1997), 1–42.

[2] F. Ancona and A. Marson, Scalar non-linear conservation laws with integrable boundary data, *Nonlinear Analysis*, 35 (1999), 687–710.

[3] A. Aw, A. Klar, T. Materne and M. Rascle, Derivation of continuum traffic flow models from microscopic follow-the-leader models, *SIAM J. Appl. Math.*, (1) 63 (2002), 259–278.

[4] C. Bardos and A.Y. Le Roux and J.C. Nedelec, First Order Quasilinear Equations with Boundary Conditions, *Commun. Partial Differential Equations*, 4 (1979), 1017–1034.

[5] A. Bressan, Hyperbolic Systems of Conservation Laws - The One-dimensional Cauchy Problem, Oxford Univ. Press, 2000.

[6] A. Bressan, G. Crasta and B. Piccoli, Well-Posedness of the Cauchy Problem for $n \times n$ Systems of Conservation Laws, *Memoirs of the American Mathematical Society*, (694) 146 (2000).

[7] Y. Chitour and B. Piccoli, Traffic circles and timing of traffic ligths for cars flow, *Discrete and Continuous Dynamical Systems-Series B*, 5 (2005), 599–630.

[8] G. Coclite, M. Garavello and B. Piccoli, Traffic Flow on Road Networks, *SIAM Journal on Mathematical Analysis*, (6) 36 (2005), 1862–1886.

[9] C. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Springer-Verlag, 1999.

[10] C. D'Apice, R. Manzo and B. Piccoli, Packets flows on a telecommunication network, preprint IAC.

[11] M. Garavello and B. Piccoli, Traffic flow on a road network using the Aw-Rascle model, *Communications PDE*, (to appear).

[12] M.J. Lighthill and G.B. Whitham, On kinetic waves. II. Theory of Traffic Flows on Long Crowded Roads, Proc. Roy. Soc. London Ser. A, 229 (1955), 317–345.

[13] P.I. Richards, Shock Waves on the Highway, *Oper. Res.*, 4 (1956), 42–51.

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