COUPLING CONDITIONS FOR GAS NETWORKS GOVERNED BY THE ISOTHERMAL EULER EQUATIONS

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Abstract. We investigate coupling conditions for gas transport in networks where the governing equations are the isothermal Euler equations. We discuss intersections of pipes by considering solutions to Riemann problems. We introduce additional assumptions to obtain a solution near the intersection and we present numerical results for sample networks.

1. **Introduction.** There has been intense research on gas networks in the last decade. Different models for transient flow have been proposed and refer to [21, 4] and the publications of Pipeline Simulation Interest Group [20] for an overview of several approaches. Numerical methods have been proposed [26, 12] for simulating gas networks and optimization problems have been investigated [8, 18, 22, 25]. Today a variety of models and optimization problems related to gas transport in networks exist.

Our starting point for describing transient gas flow are the isothermal Euler equations. Using the properties of solutions to Riemann problems for those equations, we derive coupling conditions for an intersection of pipes with variable temperatures. The coupling of the pipes is such that the arising waves can change their directions. This is a major difference to most of the models proposed in the above references. The Ansatz is similar to the work done in [9, 14] for deriving coupling conditions of a hyperbolic system for traffic flow.

The present paper is a companion paper to [3]. In [3] a network of pipelines with constant temperatures (or to be more precise, equal sound speeds on all considered

pipes) and coupling conditions based on the equality of pressure at the junctions has been investigated. In Section 4 of the present paper several theorems used in [3] are proven using a demand and supply analysis of the situation at the junctions. In Section 5 different coupling conditions are introduced. Imposing additional assumptions on the distribution of the gas flow and the structure of the arising waves (equality of pressure at the junction or subsonic flow, respectively) we point out the construction of a solution near the intersection for pipes having different temperatures. Restricting ourselves to the case of a simple intersection of pipes with one ingoing and one outgoing pipe we prove the existence of solutions. For the case of general junctions, i.e. tee fittings, with multiple ingoing and outgoing pipelines but constant temperature for all pipes we refer to [3]. Finally, we describe in Section 6 a numerical method used to solve the isothermal Euler equations on the network before we present examples on sample problems.

2. **The isothermal Euler equations.** The isothermal Euler equations are a simplification of the Euler equations. They are obtained from the isentropic equations which in turn are derived from the Euler equations under the assumption that the energy equation is redundant. A further assumption for pipe networks is that there is negligible wall expansion or contraction under pressure loads; i.e. pipes have constant cross-sectional area. In the isothermal Euler equations the temperature is constant. The equation of state applied here is

$$p = \frac{Z\mathcal{R}T}{M_g}\rho, \tag{1}$$

where p is the pressure, Z is the natural gas compressibility factor, $\mathcal{R}$ the universal gas constant, T the absolute gas temperature, M_g is the gas molecular weight and ρ the density. For a simplified presentation we finally assume that all pipes have the same diameter D.

We consider the isothermal equations in conservative form with a constant $a^2 = Z\mathcal{R}T/M_g$, i.e.,

$$\partial_t \rho + \partial_x(\rho u) = 0 \tag{2a}$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + a^2\rho) = 0 \tag{2b}$$

Here, u denotes the velocity of a gas and ρu is the flux. The first equation is the conservation of mass and the second states the conservation of momentum.

We briefly recollect the main properties of Riemann problems for the system (2). For a general reference on the solution of hyperbolic systems we refer to [7, 10] and for an application to the isothermal Euler equations to [17]. The understanding of Riemann problems is of importance for the coupling conditions of several pipes later on. We will use the following notation throughout the remaining sections,

$$q := \rho u, \ U := \begin{pmatrix} \rho \\ q \end{pmatrix}, \ F(U) := \begin{pmatrix} q \\ q^2/\rho + a^2\rho \end{pmatrix}. \tag{3}$$

A Riemann problem for (2) is an initial value problem for (2) with Heavi-side initial data, i.e.,

$$\partial_t U + \partial_x F(U) = 0, \ U(x,0) = \begin{cases} U_l & x \leq 0 \\ U_r & x > 0 \end{cases} \tag{4}$$

where U_l and U_r are given constants. It is well–known that the isothermal Euler equations enjoy the following properties: The eigenvalues are

$$\lambda_1(U) = q/\rho - a \text{ and } \lambda_2(U) = q/\rho + a. \tag{5}$$

Both fields are genuinely non–linear, see [17]. A point U can be connected to U_l by a 1-(Lax-)shock, if and only if there exists ξ such that

$$U = U_l + \xi \rho_l \begin{pmatrix} 1 \\ s_1(\xi; U_l) \end{pmatrix} \text{ and } \xi \geq 0. \tag{6}$$

It can be connected to U_l by a 2-(Lax-)shock, if and only if there exists ξ such that

$$U = U_l + \xi \rho_l \begin{pmatrix} 1 \\ s_2(\xi; U_l) \end{pmatrix} \text{ and } \xi \in [-1, 0]. \tag{7}$$

The shock speeds s_1 and s_2 are given by

$$s_{1,2}(\xi; U_l) = q_l/\rho_l \mp a\sqrt{1 + \xi}. \tag{8}$$

The set of all points U that can be connected to U_l by a 1-(Lax-)shock (resp. 2-(Lax-)shock) is said to be the 1-(Lax-)shock curve (resp. 2-(Lax-)shock curve) and it is parameterized by ξ.

Similarly, the set of all points that can be connected to U_l by 1-rarefaction (resp. 2-rarefaction) wave is said to be the 1-rarefaction (resp. 2-rarefaction) curve. A parametrization of the 1-rarefaction (resp. 2-rarefaction) curve is given by

$$U(\xi) = \rho_l \exp(-(\xi - \xi_1)/a) \begin{pmatrix} 1 \\ \xi + a \end{pmatrix}, \xi \geq \xi_1 := \lambda_1(U_l) \tag{9}$$

and

$$U(\xi) = \rho_l \exp((\xi - \xi_2)/a) \begin{pmatrix} 1 \\ \xi - a \end{pmatrix}, \xi \geq \xi_2 := \lambda_2(U_l), \tag{10}$$

respectively. We refer to Figure 1 for an example of the shape of all curves in the $q - \rho-$plane and to [17] for more details.

For the discussion later on, we also state a parametrization of the 1,2-(Lax-)shock curves and 1,2-rarefaction curves for a given right state U_r, i.e., a description of the possible points U_l that can be connected to U_r.

$$\text{1-s.} \qquad U(\xi) = U_r + \xi \rho_r \begin{pmatrix} 1 \\ s_1(\xi; U_r) \end{pmatrix}, \quad -1 \leq \xi \leq 0. \tag{11a}$$

$$\text{2-s.} \qquad U(\xi) = U_r + \xi \rho_r \begin{pmatrix} 1 \\ s_2(\xi; U_r) \end{pmatrix}, \quad \xi \geq 0. \tag{11b}$$

$$\text{1-r.} \quad U(\xi) = \rho_r \exp(-(\xi - \xi_1)/a) \begin{pmatrix} 1 \\ \xi + a \end{pmatrix}, \xi \leq \xi_1 := \lambda_1(U_r). \tag{11c}$$

$$\text{2-r.} \quad U(\xi) = \rho_r \exp((\xi - \xi_2)/a) \begin{pmatrix} 1 \\ \xi - a \end{pmatrix}, \xi \leq \xi_2 := \lambda_2(U_r). \tag{11d}$$

Remark 1. The shock speed s_i for two given states U_l, U_r on the same i-(Lax-)shock curve is given by the slope of the line connecting U_l and U_r. Further, the slope of the 1- and 2-rarefaction curves at a point U is equal to the first and second eigenvalue at this point U, respectively. For the isothermal Euler equations, the 1-(Lax-) shock and 1-rarefaction curves do not coincide. They are connected smoothly up to the second derivative.
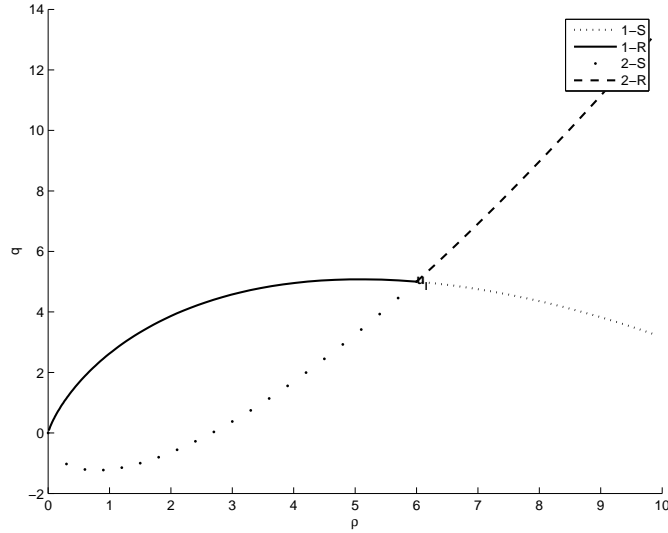
FIGURE 1. 1- and 2-curves in the $q - \rho-$plane for given U_l and $a = 1$.

3. **Pipeline Networks.** Before we discuss a network of pipes, we introduce two main assumptions, that simplify the discussion following.

A0	There are no vacuum states present, i.e., $\rho > 0$.	(12a)
A1	The direction of flow does not change, i.e., $u \geq 0$.	(12b)

At the appropriate places we will point out additional difficulties that arise, if the assumptions (12b) is relaxed.

Due to assumption (12b) we model a network of pipes as a **directed**, finite graph $(\mathcal{J}, \mathcal{V})$ and in addition we may connect edges tending to infinity. Each edge $j \in \mathcal{J}$ corresponds to a pipe. Each pipe j is modelled by an interval $[x_j^a, x_j^b]$. For edges ingoing or outgoing to the network we have $x_j^a = -\infty$ or $x_j^b = +\infty$, respectively. Each vertex $v \in \mathcal{V}$ correspond to an intersection of pipes. For a fixed vertex $v \in \mathcal{V}$ we denote by δ_v^- (δ_v^+) the set of all indices of edges $j \in \mathcal{J}$ ingoing (outgoing) to the vertex v.

On each edge $j \in \mathcal{J}$ we assume that the dynamics is governed by the isothermal Euler equations: for all $x \in [x_j^a, x_j^b]$ and $t \geq 0$,

$$\partial_t \begin{pmatrix} \rho_j \\ q_j \end{pmatrix} + \partial_x \begin{pmatrix} q_j \\ q_j^2/\rho_j + a_j^2 \rho_j \end{pmatrix} = 0, \tag{13a}$$

$$U_j(x,0) := \begin{pmatrix} \rho_j \\ q_j \end{pmatrix}(x,0) = \begin{pmatrix} \rho_j^0 \\ q_j^0 \end{pmatrix}(x), \tag{13b}$$

where a_j is a constant depending on the pipe and $U_j^0 = (\rho_j^0, q_j^0)^T$ is the initial data.

At each vertex $v \in \mathcal{V}$ we have to couple systems of the type (13) by suitable coupling conditions. For given constant initial data we construct a solution $(U_j)_j$ at a single vertex by solving (half-)Riemann problems, see below. The main idea is to construct wave fans consisting of waves of negative (positive) speeds on the ingoing (outgoing) pipes of the intersection. This is done by introducing "intermediate" states at the vertex, similar to the constructions given in [9, 13, 14]. We have one

intermediate state for each connecting pipe and those states have to satisfy the coupling conditions at the vertex.

To be more precise: Consider a single vertex v with ingoing pipes $j \in \delta_v^-$ and outgoing pipes $j \in \delta_v^+$. Assume constant initial data U_j^0 given on each pipe. A family of functions $(U_j)_{j \in \delta_v^- \cup \delta_v^+}$ is called a solution at the vertex, provided that U_j is a weak entropic solution on the pipe j and (for U_j sufficiently regular)

$$\sum_{j \in \delta_v^-} q_j(x_j^b-, t) = \sum_{j \in \delta_v^+} q_j(x_j^a+, t) \; \forall t > 0. \tag{14}$$

This condition resembles Kirchoff's law and is referred to as Rankine-Hugoniot condition at the vertex. It states the **conservation of mass** at the intersection:

$$\mathbf{A2} \qquad \text{At each vertex the total mass is conserved.} \tag{15}$$

Remark 2. It turns out that (A0)-(A2) is not sufficient to obtain a unique solution $(U_j)_j$ at the vertex. Further conditions need to be imposed, compare also with the discussion in [9, 13] for coupling conditions of different hyperbolic systems.

4. **The (half-)Riemann problems.** Before introducing additional conditions to obtain solutions at a vertex, we discuss special Riemann problems at the junctions: At a fixed junction $v \in \mathcal{V}$ we consider the (half-)Riemann problems for a pipe j,

$$\partial_t U_j + \partial_x F_j(U_j) = 0, \tag{16a}$$

$$U_j(x, 0) = \begin{cases} U_j^0 & x < x_j^b \\ \bar{U}_j & x \geq x_j^b \end{cases} \quad j \in \delta_v^-, \tag{16b}$$

$$U_j(x, 0) = \begin{cases} \bar{U}_j & x > x_j^a \\ U_j^0 & x \leq x_j^a \end{cases} \quad j \in \delta_v^+, \tag{16c}$$

with given constant initial data $U_{j,0}$ and $F_j(U_j)$ is defined as in (3).

We determine all possible states $\bar{U}_j, j \in \delta_v^-$, (resp. $j \in \delta_v^+$) such that the solution to the Riemann problem (16a,16b) (resp. (16a,16c)) have negative (resp. positive) speed. By the same reasoning as in [14] we neglect solutions of zero speed (stationary shocks).

Proposition 1. *Given constant initial data $U_j^0 =: U_l$ with $\rho_l > 0$ and $q_l \geq 0$ on an ingoing pipe $j \in \delta_v^-$. Then the (half-)Riemann problem (16a,16b) with $\bar{U}_j =: U_r$ such that $q_r \geq 0, \rho_r > 0$, admits either the constant solution $U_r \equiv U_l$ or solutions with negative wave speed, if and only if*

$$U_r \text{ belongs either to the 1-(Lax)-shock curve (6) or} \tag{17a}$$

$$\text{to the 1-rarefaction curve (9) through } U_l \tag{17b}$$

$$\text{and either } \rho_l \max\{1, \frac{q_l^2}{\rho_l^2 a_j^2}\} \leq \rho_r \tag{17c}$$

$$\text{or } \rho_l \min\{1, \exp(\frac{q_l}{\rho_l a} - 1)\} \leq \rho_r \leq \rho_l. \tag{17d}$$

Remark 3. Due to assumptions (A0) and (A1), the assertion of the previous proposition exclude states U_r which consist either of the vacuum state $\rho = 0$ or states U_r with negative speed $u = q/\rho < 0$. (Half-)Riemann problems with such states are not solved and in particular the previous proposition does not give any information on those problems.

Moreover, there exists points U_r satisfying (17d), only if U_l is *subsonic,* i.e., $q_l/\rho_l \leq a_j$. If U_r satisfies (17d) it is connected to U_l by a 1-rarefaction wave. If equation (17c) is satisfied then U_r is connected to U_l by a 1-(Lax-)shock. The equations (17) allow an interpretation in terms of demand functions, see below and in [13, 16].

Proof. Given a state U_l with $q_l \geq 0, \rho_l > 0$. We discuss the possible states U_r with $\rho_r > 0, q_r \geq 0$ that can be connected by a 1-wave. Assume U_r is connected by a 1-(Lax-)shock to U_l. Then there exists a $\xi \geq 0$ such that the shock speed is given by $s_1(\xi; U_l) = q_l/\rho_l - a_j\sqrt{1 + \xi}$. Hence, U_r can be connected by a 1-(Lax-)shock of negative speed, if and only if

$$\xi \geq \frac{1}{\rho_l a_j^2} \left(\frac{q_l^2}{\rho_l} - \rho_l a_j^2 \right) \ and \ \xi \geq 0.$$

Rephrasing this conditions in terms of $\rho_r(\xi) := q_l + \xi\rho_l$ we obtain (17c). Assume U_r is connected by a 1-rarefaction wave to U_l. Then the wave speed is $\lambda_1(U) = q/\rho - a_j$ and λ_1 is increasing along the 1-rarefaction curve in the $q - \rho-$plane. Hence, if $q_l/\rho_l \geq a_j$, then all points U_r are connected to U_l by a 1-rarefaction wave of positive speed. If $q_l/\rho_l < a_j$, then there exists a point U^* such that $\lambda_1(U^*) = 0$ and U^* and U_l can be connected by a 1-rarefaction wave. We have $\rho^* = \rho_l \exp(q_l/(a_j\rho_l) - 1)$ and this yields (17d).

Now assume, U_l is connected to a state U_m by a 1-wave of negative speed. We consider states U_r that can be reached from U_m by a 2-wave of negative speed. A 2-rarefaction wave connecting any point $U_m \neq U_r$ to a point U_r with $q_r \geq 0$ and $\rho_r > 0$ has always positive speed. Hence, there are no 2-rarefaction waves in the solution to (16a, 16b). Further, if any point $U_m \neq U_r$ is connected to U_r by a 2-(Lax-)shock, then there exists $\xi \in [-1, 0]$, such that $U_r = q_m/\rho_m + \xi\rho_m \begin{pmatrix} 1 \\ s_2 \end{pmatrix}$. The shock speed is $s_2(\xi, U_m) = q_m/\rho_m + a_j\sqrt{1 + \xi}$. Hence, if $q_m \geq 0$, then $s_2 \geq 0$. If U_m is such that $q_m < 0$, then the 2-(Lax-)shock curve lies in the fourth quadrant of the $q - \rho-$plane: Indeed, the curve $\xi \to q_m + \xi\rho_m s_2(\xi; U_m)$ has no zero for $-1 < \xi \leq 0$. Therefore, U_m cannot be connected to a point U_r ($q_r \geq 0, \rho_r > 0$) by a 2-curve of negative speed. $\square$

For the subsequent discussion we introduce the demand function $\rho \to d(\rho; U_l, a_j)$ depending on a given (left) initial datum U_l and the constant of the pipe a_j: for a given pipe and given initial datum U_l, the demand function d is the increasing part of the 1-rarefaction and 1-(Lax-)shock curves through U_l. Further, the supply function $\rho \to s(\rho; U_l, a_j)$ is the decreasing part of the 1-rarefaction and 1-(Lax-)shock curve through U_l, see Figure 2.

If we rewrite the conditions for a point $\bar{U}_j$ given by equations (17) of Proposition 1 in terms of the range for the possible flux $\bar{q}_j$, we obtain the following result.

Proposition 2. *Consider an ingoing pipe $j \in \delta_v^-$ and given constant initial data $U_j^0 =: U_l$. Denote by $d(\rho; U_l, a_j)$ the demand function. Then, for any given flux*

$$0 \leq q^* \leq d(\rho_l; U_l, a_j), \tag{18}$$

there exists a unique state $\bar{U}_j =: U_r$ with $\rho_r > 0$ and $q_r = q^*$ and such that the half-Riemann problem (16a,16b) admits a solution which is either constant or consists of waves with negative speed only.
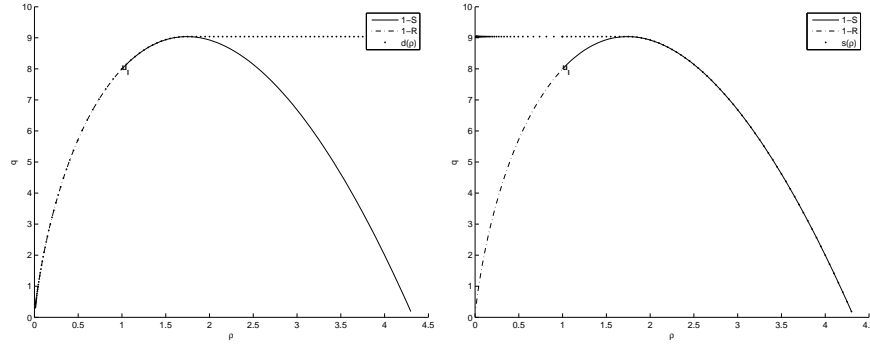
FIGURE 2. Demand function (to the left) and supply function (to the right) for $a_j = 5$ and U_l as indicated.

Proof. If $q^* = q_l$ we set $U_r = U_l$. In the case $q^* \neq q_l$ we distinguish two subcases. First, assume $q_l/\rho_l > a_j$. Then (17d,17c) imply that U_r must be connected by a 1-(Lax-)shock, i.e., $U_r = U(\xi) := U_l + \xi\rho_l \begin{pmatrix} 1 \\ s_1(\xi;U_l) \end{pmatrix}$ for some $\xi > 0$. Denote by $\rho(\xi)$ and $q(\xi)$ the first and second component of the function $U(\xi)$, respectively. Then (17c) yields $\rho_r = \rho(\xi) \geq q_l^2/(\rho_l a_j^2)$ which in turn implies $0 \leq q_r = q(\xi) \leq q_l = d(\rho_l; U_l, a_j)$. Since $s_1(\xi)$ is a strictly monotone decreasing function there exists a unique value ξ_r such that $q(\xi_r) = q^*$. Further, $U_r := U(\xi_r)$ is the corresponding unique state.

Now assume $q_l/\rho_l \leq a_j$. Denote by $\tilde{\rho} := \rho_l \exp(q_l/(\rho_l a_j) - 1)$. Then (17d, 17c) yields $\tilde{\rho} < \rho_l = \rho_l \max\{1, q_l^2/(\rho_l^2 a_j^2)\}$. Note that the 1-rarefaction curve (9) can be rewritten as follows

$$q(\rho; U_l) = \rho\frac{q_l}{\rho_l} - a_j\rho\log\left(\frac{\rho}{\rho_l}\right), \; 0 < \rho \leq \rho_l. \tag{19}$$

Then it is easy to see that the maximal flux along the 1-rarefaction curve for $\tilde{\rho} \leq \rho \leq \rho_l$ is $\tilde{q} = a_j\tilde{\rho} = d(\rho_l; U_l, a_j) > q_l$. Further, as a function of ρ, the 1-rarefaction curve $q(\rho)$ is strictly monotone decreasing and is connected to the (strictly decreasing) 1-(Lax-)shock curve at $\rho = \rho_l$. This implies that there exists U_r such that $q_r = q^*$ for $q^* \leq d(\rho_l; U_l, a_j)$. If $q_r \geq \tilde{q}$, then U_r and U_l are connected by a 1-rarefaction wave and else by a 1-shock wave. $\square$

If we skip assumption (12b), we obtain further states U_r in the fourth quadrant of the $q - \rho-$plane, which can be connected to U_l by waves of negative speed. We give an example of the area of the possible states in Figure 3: For the given U_l the states U_r lie either on the 1-(Lax-)shock curve through U_l $(1 - S)$ or in either of the following areas. The area A_1 is bounded from above by the curve $\{(\rho^*, q^*)\}$ and to the right by the curve $1 - S$. States U_r in A_1 can be connected to U_l by a 2-(Lax-)shock and a 1-(Lax-)shock both of negative speed. The area A_2 is bounded to the left by $1 - S$ and to the right by the line $q = a_j\rho$; points in this area can be connected to U_l by a 2-rarefaction and a 1-(Lax-)shock. Note that the intersection point $\tilde{U}$ of $1 - S$ and $q = a_j\rho$ satisfies $\lambda_2(\tilde{U}) = 0$. Further, an expression of the areas A_i in terms of a demand function (c.f. Proposition 2) is impossible.

Next, we consider an outgoing pipe $j \in \delta_v^+$ and the Riemann problem (16a, 16c) with given right initial data $U_j^0 =: U_r$. We are looking for all states $\bar{U}_j =: U_l$ that
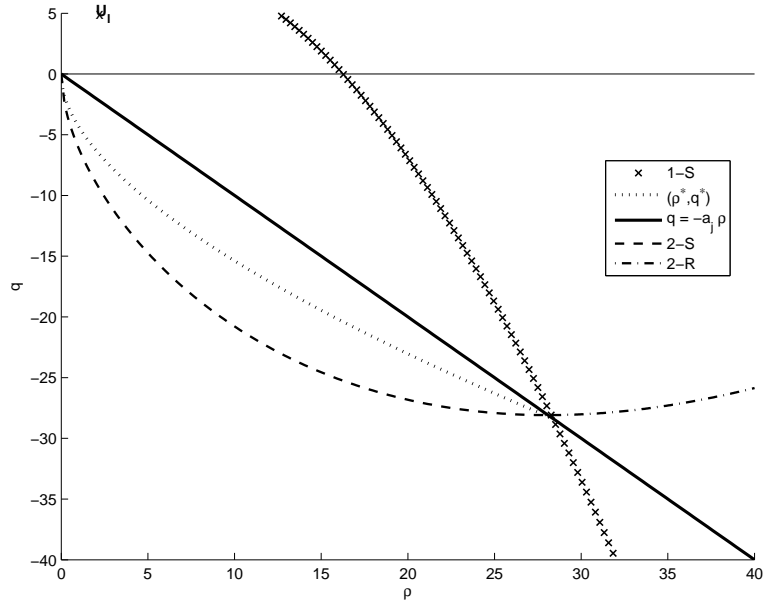
FIGURE 3. Plot of the boundaries of the areas for states U_r that can be connected to U_l by waves of negative speed.

can be connected to U_r by waves of positive speed. A first result uses the notion of the supply function and describes all states U_l that can be connected by 1-waves only.

Proposition 3. *Let $j \in \delta_v^+$. Given a constant initial data $U_j^0 =: U_r$ with $q_r \geq 0, \rho_r > 0$. Denote by $s(\rho; U_r, a_j)$ the supply function through the state U_r. Then,*

$$\forall 0 \leq q^* \leq s(\rho_r; U_r, a_j), \tag{20}$$

there exists a unique state $\bar{U}_j := U_l$ such that $q_l = q^*, \rho_l > 0$ and such that the solution to (16a, 16c) is either constant or consists of a 1-(Lax-)shock or a 1-rarefaction wave of positive speed.

Proof. Assume $q^* \neq q_r$. By the discussion in the previous section, the states U_l that can be connected to U_r by a 1-(Lax-)shock belong to the curve (11a). Therefore, the states $U_l = U(\xi) \equiv \begin{pmatrix} \rho(\xi) \\ q(\xi) \end{pmatrix}$, which can be connected to U_r by a 1-(Lax-)shock of positive speed have to satisfy

$$\rho(\xi) \in [0, \rho_r \min\{1, \frac{q_r^2}{\rho_r^2 a^2}\}], \text{ resp. } q(\xi) \in [0, q_r]. \tag{21}$$

Further, the states U_l that can be connected to U_r by a 1-rarefaction belong to the curve (11c) or equivalently by $\{(\rho, q(\rho))\}$ with $q(\rho) = \rho \frac{q_r}{\rho_r} - a_j \log\left(\frac{\rho}{\rho_r}\right)$. Hence, a state $U = (\rho, q(\rho))$ can be connected to U_r by a 1-rarefaction wave of positive speed, if and only if $q_r < \rho_r a_j$. Then the flux $q(\rho)$ satisfies

$$q_r < q(\rho) \leq q(\rho^*), \tag{22}$$

where $\rho^* = \rho_r \exp\left(\frac{q_r}{\rho_r a_j} - 1\right)$, i.e., $\frac{d}{d\rho} q(\rho^*) = 0$. Note that in this case $s(\rho_r; U_r, a_j) = q(\rho^*)$. Combining (21) and (22) we obtain: If $s(\rho_r; U_r, a_j) \leq q_r$, then there exists a unique state $U(\xi) =: U_l$ such that $q^* = q_l$ connected to U_r by a 1-(Lax-)shock. If $s(\rho_r; U_r, a_j) > q_r$, then there exists a unique state U_l with $q^* = q_l$ connected to U_r by a 1-rarefaction wave. □

The next result completes the discussion of the possible states U_l that can be connected to U_r such that the solution to (16a, 16c) is a juxtaposition of 1-waves and/or 2-waves of positive speed.

Proposition 4. *Given constant initial data $U_j^0 =: U_r$ with $\rho_r > 0$ and $q_r \geq 0$ on an outgoing pipe $j \in \delta_v^+$. Then the (half-)Riemann problem (16a, 16c) with $\bar{U}_j =: U_l$ and $\rho_l > 0$, $q_l \geq 0$ admits either the constant solution $U_l \equiv U_r$ or the solution is a juxtaposition of waves of positive speed provided that*

$$0 \leq q_l \leq s(\rho_m; U_m, a_j) \tag{23}$$

for an arbitrary state $U_m \equiv (\rho_m, q_m)$ with the properties

$$\tilde{\rho} \leq \rho_m < \infty \text{ and} \tag{24a}$$

$$U_m \text{ is either on the 2-rarefaction or} \tag{24b}$$

$$\text{on the 2-(Lax-)shock curve through } U_r, \tag{24c}$$

where $\tilde{\rho}$ is the zero of the function

$$q(\rho) := \rho \frac{q_r}{\rho_r} + a_j \rho \log\left(\frac{\rho}{\rho_r}\right) \tag{25}$$

in the interval $\left[\rho_r \exp\left(-\frac{q_r}{\rho_r a_j} - 1\right), \rho_r\right]$.

Proof. According to (11b), a left state U_m can be connected to U_r by a 2-(Lax-)shock of positive speed, if $\rho_m > \rho_r$. Further, the 2-rarefaction curve (11d) in the $q - \rho -$plane can be equivalently rewritten as $\{(\rho, q) : q = q(\rho)\}$ where $q(\rho)$ is defined in (25). Since the speed of the 2-rarefaction waves generated is equal to the slope $\frac{d}{d\rho} q(\rho)$, then U_m can be connected to U_r by a 2-rarefaction wave of positive wave speed, if and only if

$$\rho_r > \rho_m \geq \rho_r \exp\left(-\frac{q_r}{\rho_r a_j} - 1\right).$$

Note that a state U_m with $q_m < 0$ cannot be connected to a state U_l with $q_l \geq 0$ by a 1-(Lax-)shock wave of positive speed. Indeed, assume the contrary, then the shock speed is given by $\frac{q_l - q_m}{\rho_l - \rho_m} < 0$, since $\rho_l < \rho_m$. Further, a 1-rarefaction connecting $U_m, q_m < 0$ and $U_l, q_l \geq 0$ consists of waves with speed $\lambda_1(U) = q/\rho - a < 0$.

Hence, we obtain the assertion on the possible states U_m. For a fixed state U_m we can connect any left state U_l by a 1-wave of positive speed, provided that $q_l \leq s(\rho_m; U_m; a_j)$ (see Proposition 3). □

Remark 4. As in Remark 3 Riemann problems are not solved, if there is vacuum present or if intermediate states with negative velocity $u = q/\rho$ appear.

To obtain a suitable solution to a network problem we need to reduce the range of possible states U_l. We already imposed the assumptions (A0)-(A2), but we see, that further conditions have to be imposed. The exact formulation of additional

coupling conditions is subject to modelling and we present a possible choice in the next section.

5. **Coupling conditions for pipe intersections.** We present different coupling conditions for pipe to pipe intersections. We consider an intersection of two pipes. The pipes might have different sound speeds, i.e. $a_j \neq a_{j'}$, but as a simplification they are assumed to have the same diameter D. Note that this situation can also be seen as a system of conservation laws having a discontinuous flux function. For a discussion of pipe intersections like tees in the more simpler situation of pipes with the same sound speed, we refer to [3].

We introduce a further condition to obtain a unique solution, namely (A3),

$$\textbf{A3} \qquad q_j/\rho_j \leq a_j \; \forall t > 0, j \in \delta_v^+, \tag{26}$$

or (A3'),

$$\textbf{A3'} \quad \text{The pressure at the vertex is a constant, i.e., } a_i^2 \rho_i = a_j^2 \rho_j, \tag{27}$$

respectively. Furthermore, either of the above conditions need to be complemented by condition (A4) to obtain a unique solution. This assumption is as in [14, 13, 9] and resembles an entropy condition.

$$\textbf{A4} \qquad \text{The flux is maximal at the intersection.} \tag{28}$$

We discuss the modelling and theoretical properties of solutions at an intersection satisfying (A0)-(A2),(A3) and (A4) in Section 5.1 and solutions satisfying (A0)-(A2),(A3') and (A4) in Section 5.2, respectively.

Some remarks are in order. In real–world applications the modelling of intersection is more complex and additional effects are taken into account. In the engineering literature various tables can be found which depending on the geometry and material properties of the intersection describe an approximation of the behavior for coupled pipes. Condition (A3') is a rather simple model which is widely used in the engineering community to model pipe–to–pipe intersections. We call solutions satisfying (A3') therefore "physical solutions." Moreover, solutions satisfying (A0)-(A2),(A3') and (A4) have further interesting mathematical properties, see Section 5.2. But as seen later, condition (A3') *cannot* be evaluated analytically for any initial data and any sound speeds. Therefore, we propose assumption (A3) replacing (A3'). Then, we can analytically compute solutions satisfying (A0)-(A2),(A3) and (A4). *But* condition (A3) can yield *non-physical* solutions and especially, in the case of only two connected pipes having the same sound speed, (A3) yields the physical solution only for particular choices of initial data. We present numerical results showing the difference in the solutions to the different coupling conditions in Section 6.

5.1. **Coupling conditions I.** The first assumption which simplifies the theoretical discussion below is as follows. We restrict ourselves to subsonic solutions on the outgoing pipe, c.f. [24]. This implies to require a solution to satisfy in particular (A3):

$$q_j/\rho_j \leq a_j \; \forall t > 0, j \in \delta_v^+ \tag{29}$$

Using the assumptions (A0)-(A2), (A3) and (A4), we can construct a solution to the pipe–to–pipe fitting using the notion of (half-)Riemann problems and supply/demand functions introduced in the previous sections: Consider a single intersection v with an ingoing pipe $j = 1 \in \delta_v^-$ and an outgoing pipe $j = 2 \in \delta_v^+$

connected at $x_1^b = x_2^a$. Assume constant initial data U_j^0. We determine states $\bar{U}_j$, such that the solution to (16a,16b) and (16a,16c), respectively, consists of waves of non–positive and non–negative speed, only.

We start with a discussion for the outgoing pipe $j = 2$ and construct the sonic point U_m first. Let $U_r := U_2^0$ and for the case $q_r = a_2\rho_r$, we set $U_m := U_r$. Assume $q_r < a_2\rho_r$. Then, in the $q - \rho-$plane, there is a unique intersection point $U_m = (\rho_m, q_m)$ of the line $\{(\rho, q) : q = a_2\rho\}$ and the 2-(Lax-)shock curve through U_r given by (11b). U_m is given by $\rho_m = \rho_r + \xi_m\rho_r, q_m = a_2\rho_m$ where

$$\xi_m = \frac{1}{2}\left(1 - \frac{q_r}{\rho_r a_2}\right)\left(1 - \frac{q_r}{a_2\rho_r} + \sqrt{\left(\frac{q_r}{a_2\rho_r}\right)^2 + 4}\right).$$

In the case $q_r > a_2\rho_r$, there exists a unique intersection point U_m of the line $\{(\rho, q) : q = a_2\rho\}$ and the 2-rarefaction curve through U_r given by (11d). U_m is given by $\rho_m = \rho_r \exp(1 - \frac{q_r}{a_2\rho_r}), q_m = a_2\rho_m$. U_m fulfills $\rho_m > 0$ and $q_m \geq 0$ and $q_m/\rho_m = a_2$. Next and in order to fulfill assumption (A4) we consider the maximization problem:

$$\min_{\tilde{q}\in\mathbb{R}_0^+} \quad \tilde{q} \tag{30a}$$

$$\text{subject to} \quad \tilde{q} \leq d(\rho_1^0; U_1^0, a_1) \tag{30b}$$

$$\text{and to} \quad \tilde{q} \leq s(\rho_m; U_m, a_2), \tag{30c}$$

where $d(\cdot; U_1^0, a_1)$ is the demand function on pipe $j = 1$ for the given (left) state U_1^0 and $s(\cdot; U_m, a_2)$ is the supply function on pipe $j = 2$ for the (right) state U_m constructed above. Note, that in this particular case $s(\rho_m; U_m, a_2) = a_2\rho_m$. The solution to (30) is $q^* := \min\{d(\rho_l; U_1^0, a_1), s(\rho_m; U_m, a_2)\}$. Moreover, due to Proposition 2 there exists a unique state $\bar{U}_1$, such that $\bar{q}_1 = q^*$ and such that the solution $U_1(x, t)$ to the Riemann problem (16a, 16b) with initial data $U_1^0, \bar{U}_1$ satisfies $U_1(x_1^b, t) = \bar{U}_1$ for all $t > 0$. Similarly, by Proposition 3, there exists a unique state $\bar{U}_2$ such that $\bar{q}_2 = q^*$. But, the solution to (16a,16c) may violate (A3). However, if we additionally assume that $U_r = U_2^0$ is subsonic, then we conclude as follows: For every $q^* \leq q_m = a_2\rho_m = s(\rho_m; U_m, a_2)$ there exist a unique point $\bar{U}_2$ on either the 2-(Lax-)shock curve or the 2-rarefaction wave curve through the (right) state U_2^0, such that $\bar{q}_2 = q^*$. Then, the solution $U_2(x, t)$ to the Riemann problem (16a, 16c) with initial data $\bar{U}_2, U_2^0$ satisfies $U_2(x_2^a, t) = \bar{U}_2$ for all $t > 0$ and consists of a 2-wave connecting the states $\bar{U}_2$ to $U_r \equiv U_2^0$. Furthermore, (A0)-(A4) are satisfied. This finishes the construction, refer to Figure 4 for a sketch of the states.

5.2. **Coupling conditions II.** We now discuss solutions satisfying (A0)- (A2), **(A3')** and (A4). We recall condition (A3'): For all $i \in \delta_v^-, j \in \delta_v^+$:

$$\text{The pressure at the vertex is a constant, i.e., } a_i^2\rho_i = a_j^2\rho_j. \tag{31}$$

Before turning to the construction of a solution, we motivate the above choice.

Common modelling of pipe–to–pipe fittings in the engineering literature [19, 6] is as follows: In real–world applications the momentum is **not** conserved at the intersection but reduced by a pressure drop depending on the physical construction of the intersection. Usually the pressure drop f_{ext} at the intersection is approximated as in the incompressible case and modelled by a function f_{ext} depending on the geometry of the intersection, a resistance coefficient ("$K-$factor"), see [6] and the actual flows q_i and pressures $p_i(\rho_i)$ at the intersection. The pressure drop
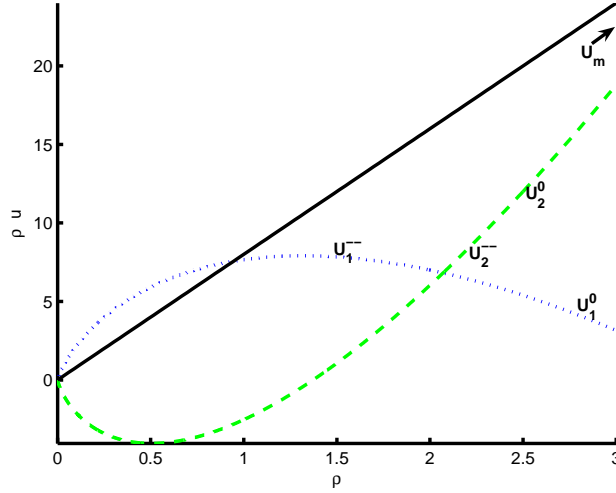
FIGURE 4. Example with given data $U_1^0 = [2.8, 4.8], a_1 = 6$ and $U_2^0 = [2.6, 12], a_2 = 8$. The solid line is $q = a_2\rho$ and the dashed line is the 2-wave curve through the right state U_2^0. The state U_m (not in the picture) is the intersection of the dashed and the solid line. The dotted line is the 1–wave curve through the left state U_1^0. In this particular case $s(\rho_m; U_m, a_2) > d(\rho_1^0; U_1^0, a_1)$. The states $\bar{U}_1 \equiv U_1^-$ and $\bar{U}_2 \equiv U_2^-$ are also depicted and have the same flux. In this example $\bar{U}_1$ and U_1^0 are connected by a 1-rarefaction and $\bar{U}_2$ is connected by is connected to U_2^0 by a 2-rarefaction.

f_{ext} is called minor loss and is not given analytically but in tables, see [6]. Those minor losses are *different* from the friction inside the pipes. If we assume that the pressure loss at the intersection is given by the quantity f_{ext}, then (A3') has to be replaced by the condition

$$a_j^2 \rho_j = a_i^2 \rho_i - f_{ext} \tag{32}$$

for $j \in \delta_v^+, i \in \delta_v^-$. For simplicity we assume in the following $f_{ext} \equiv 0$, i.e., (A3') holds.

Furthermore, in case of intersections with one(!) ingoing and one(!) outgoing pipe *and* $a_2 = a_1$ *and* mass conservation (A2), the condition (A3') implies the conservation of momentum, i.e.,

$$\sum_{i \in \delta_v^-} \frac{q_i^2(x_i^b-, t)}{\rho_i(x_i^b-, t)} + a_i^2 \rho_j(x_i^b-, t) = \sum_{j \in \delta_v^+} \frac{q_j^2(x_j^a+, t)}{\rho_j(x_j^a+, t)} + a_j^2 \rho_j(x_j^a+, t). \tag{33}$$

Mathematically, the conservation of momentum and mass (14) implies that a solution $U_j, j = 1, 2$ is also a weak solution in the sense of [14, 13]: The functions U_j satisfy

$$\sum_{j=1}^{2} \int_0^\infty \int_{x_j^a}^{x_j^b} U_j \cdot \partial_t \phi_j + F_j(U_j) \cdot \partial_x \phi_j dx dt - \int_{x_j^a}^{x_j^b} U_j^0 \cdot \phi_j(x, 0) dx = 0,$$

for all smooth functions $\phi_j : [0, \infty] \times [x_j^a, x_j^b] \to \mathbb{R}^2$ having compact support in $[x_j^a, x_j^b]$ and being smooth across the vertex

$$\phi_i(x_i^b) = \phi_j(x_j^a) \; \forall i \in \delta_v^-, j \in \delta_v^+.$$

The equivalence of (A3') and (33) is not true for general pipe intersections or intersection of pipes with different sound speeds. For a discussion of (A3') in the case of tee fittings we refer to [3].

Now, we turn to the construction of a solution and proceed similar to Section 5.1. Again, assume an intersection of two pipes $j = 1$ and $j = 2$ connected at $x_1^b = x_2^a$ with possibly different sound speeds a_j and constant initial data U_1^0 and U_2^0. Furthermore, we define $\mathbf{c} := a_1^2/a_2^2$. Analogously to the previous section, we consider a maximization problem for the flux $\tilde{q}$ at the vertex:

$$\max_{\tilde{q} \in \mathbf{R}_0^+} \tilde{q} \qquad \text{subject to} \qquad (34a)$$
$$0 \le \tilde{q} \le d(\rho_1^0; U_1^0, a_1) \quad \text{and} \qquad (34b)$$
$$0 \le \tilde{q} \le s(\rho_m; U_m, a_2) \quad \text{and} \qquad (34c)$$
$$a_2^2 \bar{\rho}_2 = a_1^2 \bar{\rho}_1. \qquad (34d)$$

But in contrast to (30), some quantities in (34) are defined implicitly: Given a fixed flux $\mathbf{q}$ less than the demand, i.e., $\mathbf{q} < d(\rho_1^0; U_1^0)$. Then, $\bar{\rho}_1$ is uniquely defined, such that the state $\bar{U}_1 = (\bar{\rho}_1, \mathbf{q})$ can be connected to U_1^0 by waves of negative speed, c.f. Proposition 1. Next, let $\bar{U}_2 := (\bar{\rho}_2, bq) := (\mathbf{c}\bar{\rho}_1, \mathbf{q})$. Finally, define $U_m = (\rho_m, q_m)$ as the point intersection of the following two curves in the first quadrant of the $\rho - q -$plane: First, the composition of the 1-(Lax-)shock curve and the 1–rarefaction wave curve through the (left) state $\bar{U}_2$ and sound speed a_2. Second, the composition of the 2-(Lax-)shock curve and the 2–rarefaction wave curve through the (right) state U_2^0 and sound speed a_2. Assume $\rho_m > 0$, the if U_m exists, it is unique. Hence, we call a flux $\mathbf{q}$ *admissible* for (34), if the above construction is possible or to be precise, if U_m exists and if $\rho_m > 0$ and if $\mathbf{q} \le s(\rho_m; U_m, a_2)$. Note that it is possible, that *no* flux $\mathbf{q}$ is admissible, see also discussion in Remark 5.

From now on, assume that the set of admissible fluxes $\mathbf{q}$ is non empty. Then, the maximization problem (34) is well-defined and has a unique solution q^*. As described above we construct the states $\bar{U}_1$ and $\bar{U}_2$. Due to (34b) and Proposition 1, the (half-)Riemann problem (16a,16b) with initial data U_1^0 and $\bar{U}_1$ consists of waves of non–positive speed and therefore $U_1(x_1^b-, t) = \bar{U}_1$. Due to (34c) and Proposition 4, the (half-)Riemann problem (16a,16c) with initial data $\bar{U}_2$ and U_2^0 consists of waves of non–negative speed. Its solution $U_2(x, t)$ therefore satisfies $U_2(x_2^a+, t) = \bar{U}_2$. Due to (34d) and due to $\bar{q}_2 = \bar{q}_1$, the solution $U_j, j = 1, 2$ satisfies (A3') and (14). This finishes the construction. We give a few remarks on extensions and related topics.

Remark 5. First, if we want to include the modelling of minor losses, we have to replace (34d) by (32), which implies that $\bar{U}_2 = \left(\mathbf{c} \left(\bar{\rho}_1 - a_2^2 f_{ext} \right), \mathbf{q} \right)$.

Second, in the simpler case $\mathbf{c} = 1$, i.e., $a_2 = a_1(!)$, there exists the following a-priori criterion for the existence of admissible fluxes c.f. [3]: If $\tilde{\rho} \le \tilde{\tilde{\rho}}$, then there exist at least one admissible flux $\mathbf{q}$. Here, $\tilde{\rho}$ is as in Proposition 4 and $(\tilde{\tilde{\rho}}, 0)$ is the point of intersection in the first quadrant of the $\rho - q -$plane of the 1-(Lax-)shock wave curve the (left) state U_1^0 with the line $\{(\rho, q) : \rho > 0, q = 0\}$. This point is unique.

Third, again consider the simpler case $\mathbf{c} = 1$. Then we might also define solutions to the full problem (13) and (14) as restriction of the entropic solution $U^* = (\rho^*, q^*)$

to a (usual) Riemann problem:

$$U_t^* + F(U^*)_x = 0, \ U^*(x,0) = \begin{cases} U_1^0 & x < x_1^b = x_2^a \\ U_2^0 & x > x_1^b \end{cases}. \tag{35}$$

I.e., $U_1(x,t) := U^*$ for $x < x_1^b$ and $U_2(x,t) := U^*$ for $x > x_2^a$. Both, the one given above and the solution constructed above by solving (34) *coincide* provided that $\rho^*(x,t) > 0$ and $u^*(x,t) = q^*(x,t)/\rho^*(x,t) \geq 0$. In the case $u^* < 0$ the assumption (12a) is violated. However, the presented approach holds also true for *different* sound speeds and can be extended to intersections of more than two pipes [3].

Finally, other mathematical models mentioned in the introduction propose the following coupling conditions: As additional assumption the model of [8, 18] the terms $\partial_t q$ and $\partial_x(\rho u^2)$ in (2) are neglected. Therefore, the authors finally deal with a scalar(!) conservation law. Coupling conditions for scalar conservation laws can be found for example in [14].

The model introduced in [12] does neglect the term $\partial_x(\rho u^2)$. Then it has to deal with a linear(!) hyperbolic system and fixed eigenvalues.

In [24] additional coupling conditions for hyperbolic and parabolic gas models are defined: At the boundary the hyperbolic system is linearized to obtain fixed wave directions. This treatment is also common in engineering community when numerically solving the pipe network problem.

6. **Numerical results.** We present results for the isothermal Euler equations with friction inside the pipes. The equation of state applied is (1). For actual operations of pipelines we consider transient states of isothermal flow in pipes with a constant cross-sectional area with the steady state friction in all pipes [18, 26], i.e.,

$$\partial_t U_j + \partial_x F_j(U_j) = r(U_j), \ r(U_j) = \begin{pmatrix} 0 \\ -f_g \frac{q_j|q_j|}{2D\rho_j} \end{pmatrix}, \tag{36}$$

where D is the diameter of the pipe. The friction factor f_g is calculated using Chen's equation [5]:

$$\frac{1}{\sqrt{f_g}} := -2\log\Big(\frac{\varepsilon/D}{3.7065} - \frac{5.0452}{N_{Re}}\log\Big(\frac{1}{2.8257}\big(\frac{\varepsilon}{D}\big)^{1.1098} + \frac{5.8506}{0.8981 N_{Re}}\big)\Big)$$

where N_{Re} is the Reynolds number $N_{Re} = \rho u D/\mu$, μ the gas dynamic viscosity and ϵ the pipeline roughness.

For the numerical results we use a relaxed scheme [15, 2]: We apply a second order MUSCL scheme together with a second order TVD time integration scheme [23, 11]. Other approaches can be similarly applied [1]. A treatment of the source term (pipe friction) is undertaken as discussed in [26]. The integration of the source term is done by solving an ordinary differential equation exactly after splitting (36). In all computations we use a space discretization of $\Delta x = 1/800$ and the time discretization Δt is chosen according to the CFL condition with CFL $= 3/4$. In the following the coupling conditions augmented in the relaxed scheme will be tested. We present results for both coupling conditions (A3) and (A3'), respectively. The maximization problem (34) is solved numerically by direct search.

Example 1: The first example is similar to the one discussed in Figure 4: Given two connected pipes $j = 1, 2$ with constant initial data $U_1^0 = [2, 8]$ and $U_2^0 = [2, 12]$. The sound speeds are $a_1 = 6$ and $a_2 = 8$. We prescribe inflow data at $x = x_1^a = 0$

and outflow $x = x_2^b = 1$. The pipes are connected at $x = x_1^b = x_2^a = \frac{1}{2}$. We consider the case of no friction inside the pipes. Furthermore, we consider the same initial data but apply condition (A3). Snapshots of the solution are given in Figure 5. Next, we consider condition (A3') at the intersection and present snapshots of the numerical solution for times $t \in [0, 1]$ in Figure 6. In both cases the flux is continuous through the intersection, but in the first case (subsonic) the pressure is not due to the different coupling (A3). In the second case the pressure is continuous through the intersection and hence fulfilling (A3'). In all simulations the arising shock waves are well captured by our second order scheme.
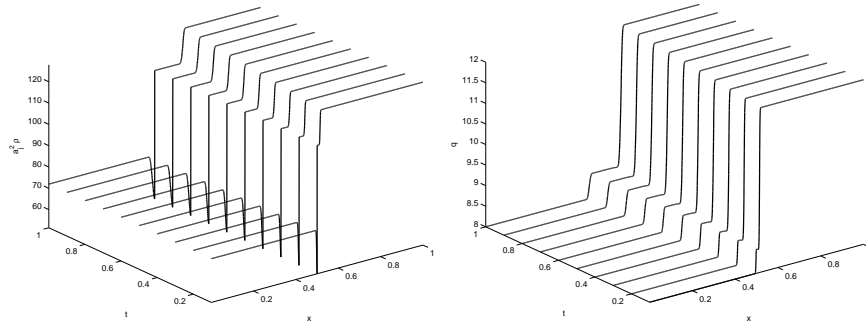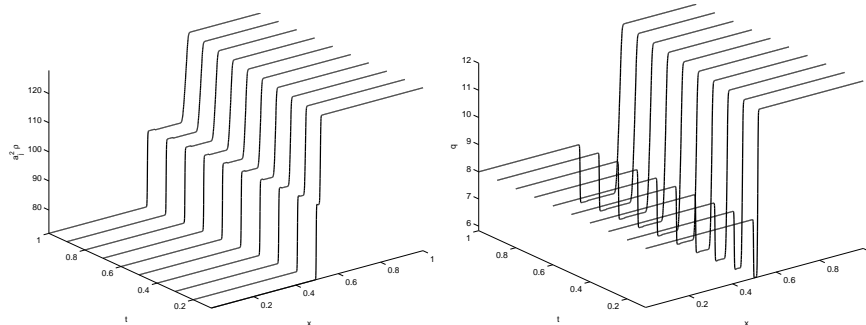

FIGURE 5. Example 1: Snapshots of the solution U_j to the problem of two connected pipes at different times t. The intersection is at $x = \frac{1}{2}$. The densities ρ_j are shown to the left and to the right q_j is shown. The condition used at the intersection is (A3).

FIGURE 6. Example 1: Snapshots of the solution U_j to the problem of two connected pipes at different times t. The intersection is at $x = \frac{1}{2}$. The densities ρ_j are shown to the left and to the right q_j is shown. The condition used at the intersection is (A3').

Example 2: The second example is similar to the closed valve example [26]. We consider an incoming wave on pipe $j = 1$, which cannot pass the intersection, due to a given low flux profile on the outgoing pipe $j = 2$. We assume to have friction in the pipes present. We use the subsonic condition (A3) at the interface. The sound speeds are $a_i = 360$ m/s. On each pipe we assume a friction factor of $f_g = 10^{-3}$ and a pipe diameter of $D = 0.1$ m. We prescribe an inflow of $q_{in} = 70 \; kgm^{-2}s^{-1}$

at $x = x_1^a$ and an outflow of $q_{out} = 1$ at $x = x_2^b$. Both pipes have the same initial condition $U_j^0 = [\frac{1}{360}, 1]$. In Figure 7 we give a plot of the snapshots of the density evolution. We observe that the inflow profile moves on pipe 1 until it reaches the intersection. Since the maximal flow on the outgoing pipe is $q^* = 1$, we obtain a backwards moving shock wave on pipe 1 and an increase in the density near the intersection.
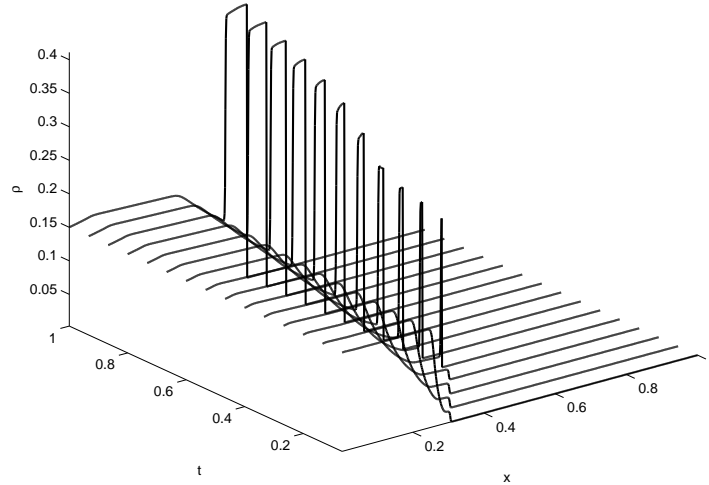

FIGURE 7. Example 2: Snapshots of the solution ρ_j to the problem of two connected pipes at different times t. The intersection is at $x = \frac{1}{2}$. The condition used at the intersection is (A3).

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Example 3: The next example deals again with two connected pipes and again we use the subsonic condition (A3) for the coupling, which yields a simple bound for maximum possible flow on the outgoing pipe. We start with an equilibrium situation at the vertex and consider a rarefaction wave as initial data on the ingoing pipe. Parts of the wave exceed the maximal flow on the outgoing pipe and therefore the flow passes the intersection until the outgoing pipe is filled. Then, we observe a backwards moving shock wave on the ingoing pipe. We consider the case of no friction. The pipes are parameterized as before. The rarefaction wave on the ingoing pipe is given by the solution to the Riemann problem with the data $U_1^l = [0.15, 70], U_1^r = [6 \times 10^{-3}, 10]$. On the outgoing pipe we have $U_2^0 = [0.3, 10]$. The sound speeds are $a_1 = 360$ and $a_2 = 75$. The states and corresponding 1- and 2-curves are depicted in Figure 8. Further, we present a plot of the flow $q_2(x = \frac{1}{2}, t) = q_1(x = \frac{1}{2}, t)$ together with the pressures $a_2^2 \rho_2(x = \frac{1}{2}, t)$ and $a_1^2 \rho_1(x = \frac{1}{2}, t)$ in Figure 8. We observe the sudden appearance of the shock wave on pipe $j = 1$, once the maximal flow for the outgoing pipe ($q^* = 10$) is reached. Finally, we give a contour plot of ρ_j in the $x - t-$plane. One observes the ingoing rarefaction as well as the shock wave.

Example 4: This example uses the coupling condition (A3'). We start with an equilibrium situation and then have a rarefaction wave incoming to the intersection with the same flow as in the previous example. The initial data is as follows:
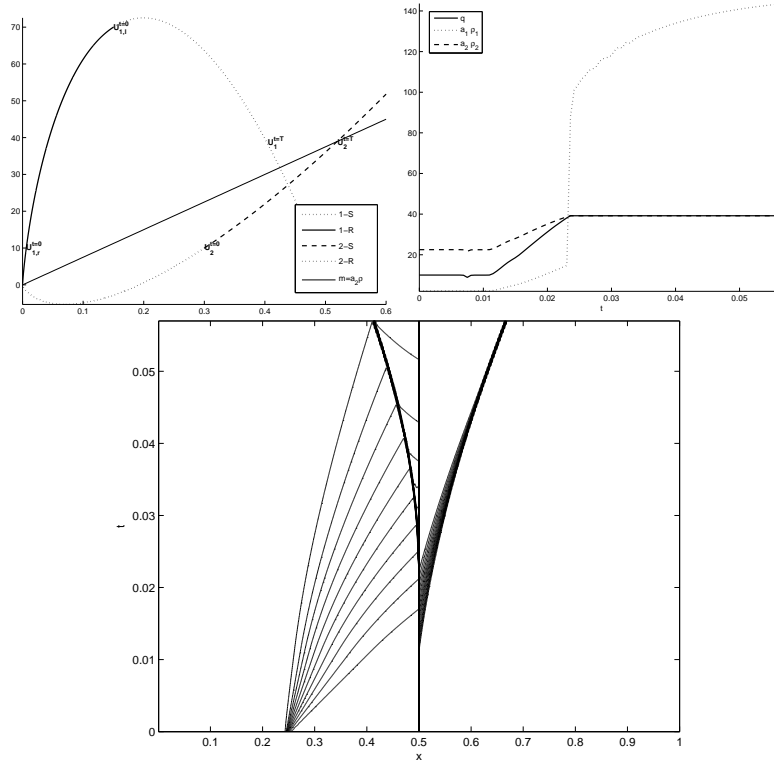
FIGURE 8. Example 3: Example of an incoming rarefaction wave which partly passes the intersection and which is reflected as soon as the maximal flow on the outgoing pipe is reached. At the intersection we use condition (A3).

U_1^l, U_1^r as in Example 3 and $U_2^0 = [1.25 \times 10^{-3}, 10]$ with $a_1 = 360$ and $a_2 = 800$. The intersection is at $x = \frac{1}{2}$. Plots of the contour lines of the pressure $a_j^2 \rho_j$ and the flux q_j are given in Figure 9. Here, the incoming flow passes the vertex and pressure and flux are continuous through the vertex.

Example 5: The last example deals with a row of three connected pipes $j = 1, 2, 3$. We give an example where an incoming flow partly passes the first intersection and is thereafter reflected at the second intersection. The arising backwards moving wave on the middle pipe finally reaches again the first intersection and causes another backwards moving rarefaction wave on the first pipe. Each pipe is parameterized as before and we assume A3 as condition at the intersection. No friction is assumed. The initial data on the pipes $j = 1, 2$ is the same as in the previous example; they are depicted in Figure 10 where we also see the intermediate states in the full solution. On pipe $j = 3$ we assume $U_3^0 = [0.3, 3]$. The sound speeds are $a_1 = 360, a_2 = 110$ and $a_3 = 20$ and we use condition (A3). A plot of the pressure $a_j^2 \rho_j$ in the $x - t$−plane is given in Figure 10. The intersections are at $x = 1/3$ and $x = 2/3$.

7. **Summary.** We introduced different coupling conditions for the isothermal Euler equations. We gave general conditions that a solution near an intersection of pipes
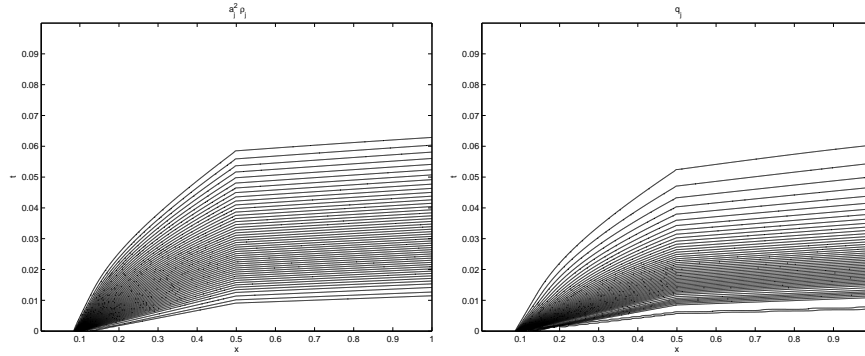
FIGURE 9. Example 4: Example of an incoming rarefaction wave which passes through the vertex and where we use condition (A3'). We give contour plots of the pressure $a_j^2 \rho_j$ (left) and flux q_j.
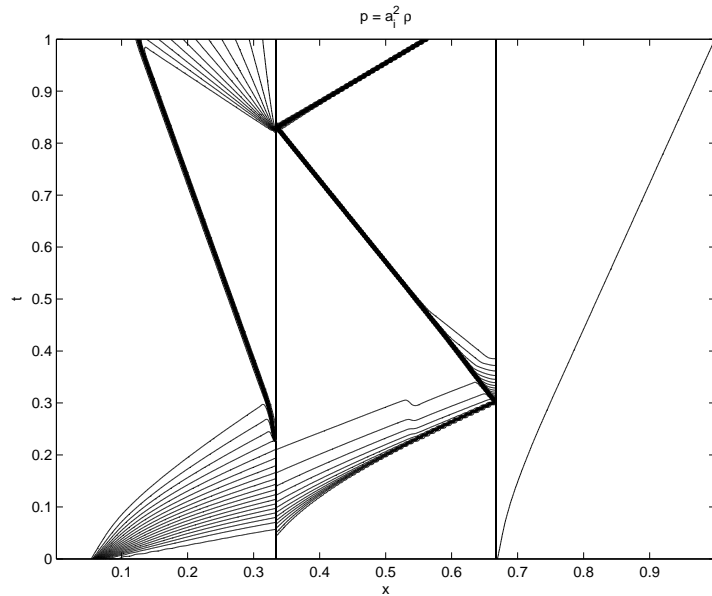

FIGURE 10. Example 5: Example for a double reflection on the middle of three pipes. Condition (A3) is used.

has to satisfy and introduced the notion of demand and supply. We give additional assumptions and construct solutions to the network problem. We presented a numerical method using the coupling conditions as boundary values at the intersections and presented results for sample intersections. The numerical scheme could resolve well all arising shock and rarefaction waves. More complicated intersections like tees are considered in [3]. Additional elements (e.g. valves, compressors) and the incorporation of realistic pressure loss models are considered in a forthcoming publication. Since the presented approach holds for different sound speeds on each

road, we will then also provide a comparison with the work on conservations laws with discontinuous coefficients.

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