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OPTIMIZATION CRITERIA FOR MODELLING INTERSECTIONS OF VEHICULAR TRAFFIC FLOW

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Abstract. We consider coupling conditions for the "Aw–Rascle" (AR) traffic flow model at an arbitrary road intersection. In contrast with coupling conditions previously introduced in [10] and [7], all the moments of the AR system are conserved **and** the total flux at the junction is maximized. This nonlinear optimization problem is solved completely. We show how the two simple cases of merging and diverging junctions can be extended to more complex junctions, like roundabouts. Finally, we present some numerical results.

1. **Introduction.** Traffic flow models have been under investigation for a long time. We are particularly interested in macroscopic traffic flow models based on hyperbolic conservation laws. Models of this type have been considered for example in [14, 6, 15, 2, 1, 8]. In the following, we focus on the "Aw–Rascle" (AR) model. This (class of) "second–order" model(s) consists of a nonlinear, coupled system of conservation laws, introduced in [2] and independently in [16]. Those models describe the behavior of traffic density and velocity where different cars can have a different response to local traffic situations, e.g., the model distinguishes trucks and cars. Recently, first extensions of theses models to a traffic network have been proposed [7, 10]. The crucial point is the modelling of coupling conditions at junctions. Typically, one has to introduce further assumptions to show that the problem is well-defined and admits a unique solution, see also the discussion in the scalar case [4, 11, 9]. In this paper we propose new coupling conditions for the AR-system. In contrast with [7], those conditions conserve all moments of the system and in contrast with [10] the derived conditions maximize the flux at the junctions without any further constraint. Furthermore, we present a numerical algorithm to solve the problem and to construct the intermediate states of the homogenized solution.

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2. **Preliminary discussion.** We first give a brief summary of the properties of the AR–model and advise the reader to consult [2, 10] for more details.

A road network is modelled as a finite, directed graph $(\mathcal{I}, \mathcal{N})$ (with $|\mathcal{I}| = \mathbf{I}$ and $|\mathcal{N}| = \mathbf{N}$) wherein each arc $i = 1, \dots, \mathbf{I}$ corresponds to a road and each vertex $n \in \mathcal{N}$ to a junction. For a fixed junction n the set δ_n^- contains all the indices k of incoming roads to n. Similarly, δ_n^+ denotes the indices j of outgoing roads. We skip the subindex n whenever the situation is clear. Each road i is modelled by an interval $I_i := [a_i, b_i]$ where we allow either $a_i = -\infty$ or $b_i = +\infty$ for incoming or outgoing roads in the whole network. We require the AR–equations (1) to hold on each arc $i \in \mathcal{I}$ of the network:

$$\partial_t \rho_i + \partial_x(\rho_i v_i) = 0 \tag{1a}$$

$$\partial_t(\rho_i w_i) + \partial_x(\rho_i v_i w_i) = 0 \tag{1b}$$

$$w_i = v_i + p_i(\rho_i) \tag{1c}$$

where, for each i, $\rho_i \mapsto p_i(\rho_i)$ is a known function ("traffic pressure") with the following properties

$$\forall \rho_i, \rho_i p_i''(\rho_i)) + 2p_i'(\rho_i) > 0 \text{ and e.g. } p_i(\rho_i) \sim \rho_i^\gamma \text{ at } \rho_i = 0, \tag{2}$$

and where $\gamma > 0$. ρ_i and v_i respectively, describe the density and velocity of traffic on road i.

The conservative form of (1) is

$$\partial_t \begin{pmatrix} \rho_i \\ y_i \end{pmatrix} + \partial_x \begin{pmatrix} y_i - \rho_i p_i(\rho_i) \\ (y_i - \rho_i p_i(\rho_i)) y_i / \rho_i \end{pmatrix} = 0, \tag{3}$$

where $y_i := \rho_i w_i = \rho_i(v_i + p_i(\rho_i))$.

We now recall some basic facts on the solution of the Riemann Problem for (1), i.e. to the initial value problem with constant data for $\pm x > 0$.

The system is strictly hyperbolic if $\rho_i > 0$. The eigenvalues are

$$\lambda_{1,i}(U) = v_i - \rho_i p_i'(\rho_i) \text{ and } \lambda_{2,i}(U) = v_i \tag{4}$$

The first characteristic field is *genuinely nonlinear*. The second one is *linearly degenerate* and therefore associated with a *contact discontinuity*. Moreover, the *1–shock* and *1–rarefaction* curves *coincide*, see [5, 2]. We recall that of course they are associated with braking and acceleration waves, respectively. For each fixed i, the Riemann invariants are

$$\mathbf{w}_i(U) = v_i + p_i(\rho_i) \text{ and } \mathbf{v}_i(U) = v_i. \tag{5}$$

We refer to [10, 7, 11] for a derivation of the necessary conditions at the junction, i.e. the coupling conditions. First, we define weak solutions of the network problem in the following sense. A set of functions $\{U_i = (\rho_i, \rho_i v_i)\}_{i \in \mathcal{I}}$ is called a weak solution of (1) if and only if

$$\sum_{i=1}^{\mathbf{I}} \int_0^\infty \int_{a_i}^{b_i} \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} \cdot \partial_t \phi_i + \begin{pmatrix} \rho_i v_i \\ \rho_i v_i w_i \end{pmatrix} \cdot \partial_x \phi_i \, dx dt$$
$$+ \int_{a_i}^{b_i} \begin{pmatrix} \rho_{i,0} \\ \rho_{i,0} w_{i,0} \end{pmatrix} \cdot \phi_i(x,0) dx = 0 \tag{6}$$

holds for any set of smooth functions $\{\phi_i\}_{i \in \mathcal{I}} : [0, +\infty[\times I_i \to \mathbb{R}^2$ having compact support and also smooth across a junction n, i.e.

$$\phi_k(b_k) = \phi_j(a_j) \; \forall k \in \delta_n^- \text{ and } \forall j \in \delta_n^+. \tag{7}$$

Herein, $U_{i,0}(x) = \big(\rho_{i,0}(x), (\rho_{i,0}v_{i,0})(x)\big)$ is the initial data. Furthermore, the set of functions U_i satisfies for all i the relation

$$\mathbf{w}_i(x,t) = v_i(x,t) + p_i^{\dagger}(\rho_i(x,t)), \tag{8}$$

where the function $p_i^{\dagger}(\cdot)$ is initially unknown. On an outgoing road its explicit form depends on the mixture of the cars. On any incoming roads $k \in \delta^-$ it holds $p_k^{\dagger} \equiv p_k$.

From now on, we consider a **single** junction. Then, from (6), (8) we derive the Rankine–Hugoniot condition for piecewise smooth solutions

$$\sum_{k\in\delta^-} (\rho_k v_k)(b_k^-, t) = \sum_{j\in\delta^+} (\rho_j v_j)(a_j^+, t), \tag{9a}$$

$$\sum_{k\in\delta^-} (\rho_k v_k w_k)(b_k^-, t) = \sum_{j\in\delta^+} (\rho_j v_j w_j)(a_j^+, t). \tag{9b}$$

These properties, respectively, correspond to conservation of mass and (pseudo)-"momentum". Note that in [7] the pseudo-"momentum" is not conserved and the proposed solution is **not** a weak solution in the above sense.

In the remaining part we consider the case of initial data constant on each road:

$$\begin{cases} (\rho_{k,0}, \rho_{k,0}v_{k,0}) = U_{k,0} = \text{const}_k, & \forall\ k \in \delta^-, \\ (\rho_{j,0}, \rho_{j,0}v_{j,0}) = U_{j,0} = \text{const}_j, & \forall\ j \in \delta^+. \end{cases} \tag{10}$$

We discuss the construction of weak solutions in the sense of (6) for initial data constant on each road. For each vertex or junction (say located at $x = x_0$), we consider the Riemann problem

$$\partial_t \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} + \partial_x \begin{pmatrix} \rho_i v_i \\ \rho_i v_i w_i \end{pmatrix} = 0,\ U_i(x,0) = \begin{pmatrix} U_i^- & x < x_0 \\ U_i^+ & x > x_0 \end{pmatrix}, \tag{11}$$

for each $i \in \delta^- \cup \delta^+$.

Depending on the road, only one of the Riemann data is defined for $t = 0$:

$$\text{If } i \in \delta^- : U_i^- = U_{i,0}\,,\ x_0 = b_i \ \text{ and if } i \in \delta^+ : U_i^+ = U_{i,0}\,,\ x_0 = a_i. \tag{12}$$

We construct an (entropy) solution to (11-8) such that all generated waves have non-positive (if $x < x_0$ i.e. $i \in \delta^-$) or non-negative (if $x > x_0$ i.e. $i \in \delta^+$) speed. For each road the remaining **unknown** state U_i^+ when $i \in \delta^-$ (resp. U_i^- when $i \in \delta^+$) has to be determined in such way that the coupling conditions (9a) and (9b) are satisfied. Then we solve each of the problems (11-8) and obtain a weak solution in the sense of (6). This solution $(U_i(x,t))$ is restricted to $U_k(x,t)$ with $k \in \delta^-$, when $x < x_0$ and $U_j(x,t)$ with $j \in \delta^+$, if $x > x_0$.

Summarizing, **depending** on the road, we only know a part of the initial data for (11), namely (12).

First, we denote by α_{jk} the percentage of cars on road k willing to go (and actually going, see below) to road j. The corresponding matrix $A := (\alpha_{jk})_{j\in\delta^+,k\in\delta^-}$ is assumed to be **known**, see [4, 7, 10]. By definition we have

$$\sum_{j\in\delta^+} \alpha_{jk} = 1\ \forall k \in \delta^-. \tag{13}$$

Next, let

$$q_k(t) := \rho_k v_k(b_k-, t),\ q_j(t) := \rho_j v_j(a_j+, t)$$

denote the (initially unknown) total first component of the flux (i.e. the mass–flux) on the incoming road k (resp. on the outgoing road j). Furthermore, let us introduce the (initially unknown) flux q_{jk} of cars actually going from road k to road j and let $\beta_{jk} := q_{jk}/q_j$, which is also initially unknown. Then, by the above definitions,

$$\alpha_{jk} = \frac{q_{jk}}{q_k} \text{ and } \sum_{k\in\delta^-} \beta_{jk} = 1.$$

As a final preparation, we describe the construction of the demand and supply functions (on an arbitrary road) for a given level curve of $\{\mathbf{w}(U) = c\}$, $c \geq 0$. Recall, $\mathbf{w}(U) = v + p(\rho)$ and its level curve is a concave function in the $(\rho, \rho v)$-plane with a unique maximum. As in the case of first-order models, e.g. [13, 7], in the $(\rho, \rho v)$ plane the demand function $d(\rho; \mathbf{w}, c)$ is an extension of the **non–decreasing** part of this level curve $\{\mathbf{w}(U) = c\}$ for $\rho \geq 0$ and the supply function $s(\rho; \mathbf{w}, c)$ is an extension of the **non–increasing** part of this curve $\{\mathbf{w}(U) = c\}$ and $\rho \geq 0$. We denote by $d_k := d(\rho_k; \mathbf{w}, c_k)$ the demand on an *incoming* road k and by $s_j := s(\rho_j, \mathbf{w}, c_j)$ the supply on an *outgoing* road j.

3. **Solving the problem at a junction.** We first recall the following results of [10]: The Riemann invariant of (1), $\{\mathbf{v}(U_i) = v_i\}$ is a straight line with slope v_i passing through the origin. Consider the curve $\{\mathbf{w}(U_i) := v_i + p_i(\rho_i) = w_i\}$, where $w_i \in \mathbb{R}$ denotes a constant. By assumption (2) on p_i this curve is strictly concave and passes through the origin. Furthermore, if $w_i > 0$, then the curve $\{\mathbf{w}(U_i) = w_i\}$ lies in the first quadrant of the $(\rho_i, \rho_i v_i)$ plane for ρ_i between 0 and a maximal value $\bar{\rho}_i \in]0, 1]$. The maximal value $\bar{\rho}_i$ depends on w_i and $p_i(\cdot)$. Due to the strict concavity, there exists a unique point (i.e. the "sonic point") $\sigma(\mathbf{w}, w_i)$ (with $0 < \sigma(\mathbf{w}, w_i) \leq 1$, depending on w_i and the function $p_i(\cdot)$) which maximizes the flux $\rho_i v_i$ on $\{\mathbf{w}(U_i) = w_i\}$. Moreover, we introduce the functions $r_i(\rho_i; \mathbf{w}, w_i)$ and $u_i(\rho_i; \mathbf{w}, w_i)$ below. Assume $w_i > 0$. Then for all $\rho_i \in [0, \bar{\rho}_i]$ there exists a unique v_i such that $\mathbf{w}(\rho_i, \rho_i v_i) = w_i$. Moreover, there exists a unique pair (r_i, u_i) such that

$$\mathbf{w}(r_i, r_i\, u_i) = \mathbf{w}(\rho_i, \rho_i\, v_i), \tag{14a}$$
$$r_i\, u_i = \rho_i\, v_i, \tag{14b}$$
$$r_i \neq \rho_i \text{ except for } \rho = \sigma(\mathbf{w}, w_i). \tag{14c}$$

Therefore, for each curve $\{\mathbf{w}(U_i) = w_i\}$ with $w_i > 0$ there exists two unique functions $\rho_i \to r_i(\rho_i; \mathbf{w}, w_i)$ and $\rho_i \to u_i(\rho_i; \mathbf{w}, w_i)$ satisfying (14) for all $\rho_i \in [0, \bar{\rho}_i]$.

Proposition 1. *Let $U^- = (\rho^-, \rho^- v^-) \neq (0, 0)$ be the initial value on an incoming road. Let the 1-curve through U^- be $\mathbf{w}(U) = v + p(\rho) = w^-$ with $w^- := \mathbf{w}(U^-)$. Then the "admissible" states $U^+ = (\rho^+, \rho^+ v^+)$ for the Riemann problem (11) must belong to that curve, i.e., $\mathbf{w}(U^+) = w^-$ and $\rho^+ v^+ \geq 0$. Depending on U^- we distinguish two cases:*

1. *$\rho^- < \sigma(\mathbf{w}, w^-)$: U^+ is admissible if and only if $\rho^+ > r(\rho^-; \mathbf{w}, w^-)$ or if $U^+ \equiv U^-$.*
2. *$\rho^- \geq \sigma(\mathbf{w}, w^-)$: U^+ is admissible if and only if $\sigma(\mathbf{w}, w^-) \leq \rho^+ \leq 1$.*

If $U^- = (0, 0)$ then the admissible state is $U^+ \equiv U^-$.

In all cases the maximal possible flux associated with any admissible state U^+ is $d(\rho^-; \mathbf{w}, w^-)$.

Proposition 2. *Consider the initial state $U^+ \neq (0, 0)$ on an outgoing road and the level curve of the first Riemann invariant $\{\mathbf{w}(U) = c\}$ with an arbitrary non–negative constant c.*

Let $U^\dagger = (\rho^\dagger, \rho^\dagger v^\dagger)$be the point of intersection, if it exists, of the two Riemann invariants $\{\mathbf{v}(U) = v^+\}$ and $\{\mathbf{w}(U) := v + p(\rho) = c\}$ with $\rho > 0$ and $v > 0$. Then the "admissible" states U^- for the Riemann problem (11) satisfying $\mathbf{w}(U^-) = c$ and $\rho^- v^- \geq 0$ are given by the two cases:

1. *$\rho^\dagger \leq \sigma(\mathbf{w}, c)$: U^- is admissible if and only if $0 \leq \rho^- \leq \sigma(\mathbf{w}, c)$.*
2. *$\rho^\dagger > \sigma(\mathbf{w}, c)$: U^- is admissible if and only if $0 \leq \rho^- < r(\rho^\dagger; \mathbf{w}, c)$ or if $U^- \equiv U^\dagger$.*

Note that the set of admissible states U^- depends on the existence of the point $U^\dagger$. Now assume that either $U^+ = (0,0)$ or there is no such point $U^\dagger$ with $\rho^\dagger, v^\dagger > 0$. Then we set $U^\dagger = (0,0)$ and as in Case 1, U^- is admissible, if and only if $0 \leq \rho^- \leq \sigma(\mathbf{w}, c)$.

In all cases the maximal possible flux associated with any "admissible" state U^- is $s(\rho^\dagger; \mathbf{w}, c)$.

An example is given in Figure 3 for incoming and in Figure 2 for outgoing roads, respectively. In the latter case the constant c is given by $c = \mathbf{w}_2(U_2^\dagger) = \mathbf{w}_2(U_2^*)$.

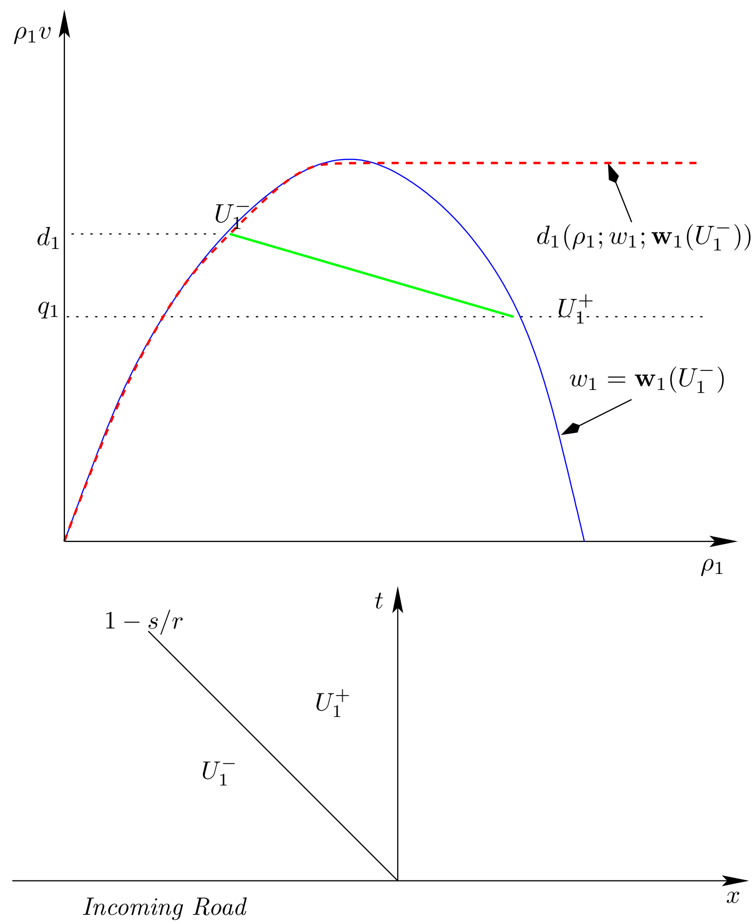

FIGURE 1. (Half-)Riemann Problem on an incoming road.

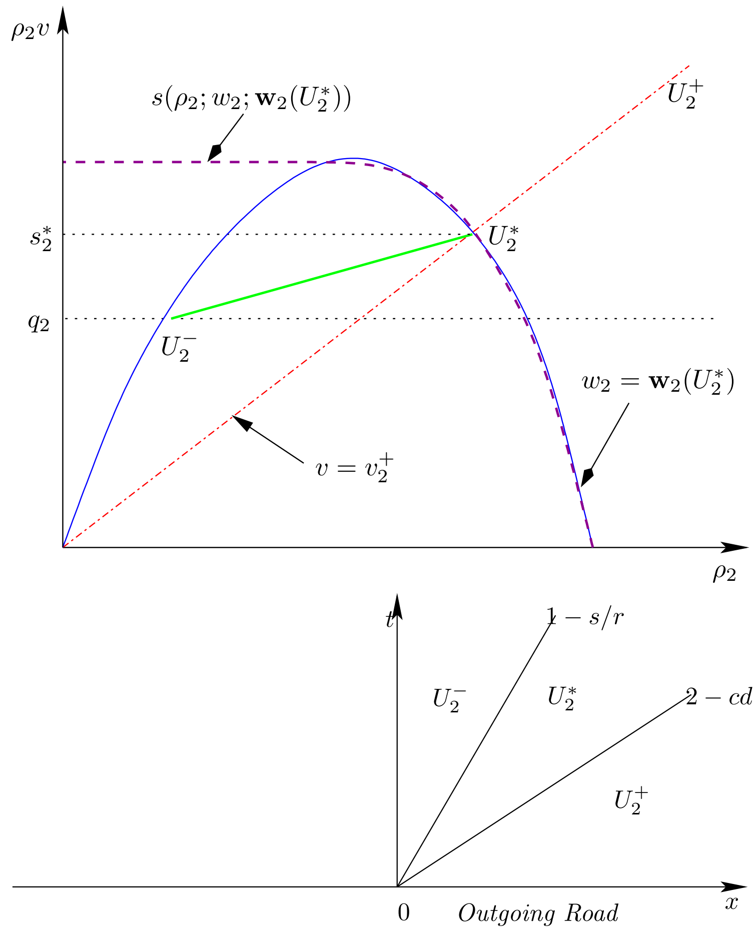

FIGURE 2. (Half-)Riemann Problem on an outgoing road.

Next, we describe how to construct a solution for a single junction and constant initial data $(U_{k,0}, U_{j,0})_{k\in\delta^-,\ j\in\delta^+}$. We assume that the weak solution in the sense of (6) satisfies the already imposed condition,

$$q_j \equiv (\rho_j v_j)(a_j+, t) = \sum_{k\in\delta^-} q_{jk} = \sum_{k\in\delta^-} \alpha_{jk}(\rho_k v_k)(b_k-, t)\ \forall j \in \delta^+. \tag{15}$$

As already noted in [10, 7] the condition (15) is not sufficient to construct a unique solution $\{U_i\}_{i\in\delta^-\cup\delta^+}$ to (11-8). In [10] we introduced a further assumption on the distribution of the cars. Here, and in contrast with [10] and with [7] we present a *new* approach to solve the problem: We prove that solving the maximization of the total incoming (mass) flux $\sum_{k\in\delta^-} \rho_k v_k(b_k-, t) = \sum_{k\in\delta^-} q_k$ (or equivalently total outgoing (mass) flux $\sum_{j\in\delta^+} \rho_j v_j(a_j+, t) = \sum_{j\in\delta^+} q_j$) is sufficient to obtain a **unique** solution. Furthermore, this solution will satisfy assumptions (15) for a given matrix A with the property (13). In contrast with [7], **no** additional assumptions on A is needed. However, we maximize the total (mass) flux on a smaller set of admissible states. On the other hand, in contrast with [10], **no** additional assumption on β_{ji} (i.e. the mixture of the cars) has to be imposed.

At this point, using Theorem 7.1 in [10] we want to define admissible solutions for a general junction.

Definition 1. Consider a junction with m incoming and n outgoing roads, with constant initial data $U_{i,0} = (\rho_{i,0}, \rho_{i,0}v_{i,0})_{i\in\delta^-\cup\delta^+}$ under assumptions (13).
We say that the family $\{U_i(x,t)\}_{i\in\delta^-\cup\delta^+}$ is an admissible solution of the Riemann problems (11-12) if and only if it satisfies:

(C1) $\forall\ i \in \delta^- \cup \delta^+$, $U_i(x,t)$ is a weak entropy solution in the sense of (6) of the network problem (11-8), where $p_i^\dagger \equiv p_i, \forall i \in \delta^-$.
On an outgoing road $j \in \delta^+$, the solution $U_j(x,t)$ is constructed as in [10]: in the triangle $\{(x,t); a_j < x < a_j + tv_{j,0}\}$, U_j is the homogenized solution defined below with $p_j^\dagger \equiv p_j^*$, whereas for $x > a_j + tv_{j,0}$, $p_j^\dagger \equiv p_j$.

(C2) The flux distribution satisfies (15).

(C3) The sum of the incoming fluxes $\sum_{k\in\delta^-} \rho_k v_k(b_k-,t) = \sum_{k\in\delta^-} q_k$ (or equivalently the sum of outgoing fluxes $\sum_{j\in\delta^+} \rho_j v_j(a_j+,t) = \sum_{j\in\delta^+} q_j$) is maximal subject to (C1) and (C2).

Remark 1. We recall that in [7] the maximization problem involves different cost functions and mostly a larger set of admissible states, since relation (9b) is not imposed. In contrast, in [10] no such maximization criterion is imposed, but the proportions between all the incoming fluxes are fixed.

Next, we describe the homogenization to define p_j^*. For a motivation and a detailed discussion of the homogenization, see [3] and Section 6 in [10]. Recall that for each $k \in \delta^-$, $p_k^\dagger \equiv p_k$ and $\mathbf{w}_k(U) = v + p_k(\rho)$ are well-defined. First, we define the (initially unknown) homogenized value for each outgoing road $j \in \delta^+$

$$\bar{w}_j := \sum_{k\in\delta^-} \beta_{jk}\mathbf{w}_k(U_{k,0}). \tag{16}$$

Then, for each $j \in \delta^+$, $p_j^*(\cdot)$ is defined as in [10]. Namely, we first define the function

$$P_j(\tau) := p_j(1/\tau), \tag{17}$$

where $\tau = \frac{1}{\rho}$ is the specific volume, see [1, 3].

Now, we consider the function:

$$v \longmapsto \tau := \sum_{k\in\delta^-} \frac{q_{jk}}{q_j} P_j^{-1}(\mathbf{w}_k(U_{k,0}) - v) = \sum_{k\in\delta^-} \beta_{jk} P_j^{-1}(\mathbf{w}_k(U_{k,0}) - v). \tag{18}$$

Then, we *choose* to define a new invertible function P_j^* by rewriting (18) under the form

$$\tau := \left(P_j^*\right)^{-1}(\bar{w}_j - v), \tag{19}$$

which we **only** use with the particular value $\bar{w}_j$ defined by (16), see [3] for more details.

Finally, we set

$$p_j^\dagger(\rho) := p_j^*(\rho) := P_j^*(1/\rho), \tag{20a}$$

$$\mathbf{w}_j^\dagger(U) := v + p_j^\dagger(\rho). \tag{20b}$$

This construction is perfectly well–defined once the proportions $\beta_{jk} = q_{jk}/q_j$ are known. In [10] we have assumed that we knew these proportions a priori, see also

Remark 1. Here, in contrast, we show that the proportions β_{jk} can be determined by solving a maximization problem stated below. Unfortunately, so far, the problem is only tractable for particular types of junctions. With all the previous remarks in mind, we conclude: there exists a unique solution $\{U_i\}_{i\in\delta^-\cup\delta^+}$ in the sense of Definition 1 if the following maximization problem:

$$\max \sum_{j\in\delta^+} q_j \text{ subject to} \tag{21a}$$

$$\forall k \in \delta^-,\ 0 \le q_k \le d_k(\rho_{k,0}, \mathbf{w}_k, \mathbf{w}_k(U_{k,0})), \tag{21b}$$

$$\forall j \in \delta^+,\ 0 \le q_j \le s_j(\rho_{j,0}, \mathbf{w}_j^\dagger, w_j^*), \tag{21c}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ \beta_{jk} = \frac{q_{jk}}{q_j}, \tag{21d}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ \alpha_{jk} = \frac{q_{jk}}{q_k}, \tag{21e}$$

$$\forall j \in \delta^+,\ \sum_{k\in\delta^-} \beta_{jk} = 1, \tag{21f}$$

$$\forall j \in \delta^+,\ q_j = \sum_{k\in\delta^-} \alpha_{jk} q_k, \tag{21g}$$

$$\forall j \in \delta^+,\ \sum_{k\in\delta^-} q_{jk} = q_j, \tag{21h}$$

$$\forall k \in \delta^-,\ \sum_{j\in\delta^+} q_{jk} = q_k,, \tag{21i}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ 0 \le \beta_{jk} \le 1. \tag{21j}$$

has a unique solution, with $\mathbf{w}_j^\dagger(U) \equiv v + p_j^\dagger(\rho)$ and $w_j^* \equiv \sum_{k\in\delta^-} \beta_{jk}\mathbf{w}_k(U_{k,0})$

Remark 2. The functions $\mathbf{w}_k$ for $k \in \delta^-$ and the values $\mathbf{w}_k(U_{k,0})$ and α_{jk} are initially known. As already noted, β_{jk} is initially unknown and depends on the solution $\{U_k,\ U_j\}_{k\in\delta^-, j\in\delta^+}$, as well as the function $\mathbf{w}_j^\dagger$ and the fluxes q_{jk} and q_j. In particular the maximization problem contains the implicit constraints (21c), for all $j \in \delta^+$.

In (21) some equations are redundant and a more compact equivalent reformulation is:

$$\max \sum_{j\in\delta^+} q_j \text{ subject to} \tag{22a}$$

$$\forall k \in \delta^-,\ 0 \le q_k \le d_k(\rho_{k,0}; \mathbf{w}_k, \mathbf{w}_k(U_{k,0})), \tag{22b}$$

$$\forall j \in \delta^+,\ 0 \le q_j \le s_j(\rho_{j,0}, \mathbf{w}_j^\dagger, w_j^*), \tag{22c}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ \beta_{jk} q_j = \alpha_{jk} q_k, \tag{22d}$$

$$\forall j \in \delta^+,\ \sum_{k\in\delta^-} \beta_{jk} = 1, \tag{22e}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ 0 \le \beta_{jk} \le 1. \tag{22f}$$

We now move to the first type of junctions considered here (a merge).

4. **Two incoming roads and one outgoing road.** We write $k = 1, 2$ for incoming roads, $j = 3$ for the outgoing road and give simplifications for (22) in this

case. Furthermore, let $\beta_1 := \beta_{31}$ and $\beta_2 := \beta_{32}$ and $d_1 := d(\rho_{1,0}; \mathbf{w}_1, \mathbf{w}_1(U_{1,0}))$ and $d_2 := d(\rho_{2,0}; \mathbf{w}_2, \mathbf{w}_2(U_{2,0}))$. By assumption (13), $\alpha_{31} = \alpha_{32} = 1$.

The crucial point in solving (22) is to determine the supply s_j. We briefly describe the homogenization leading to s_j, before describing the solution.

Let $U := (\rho, \rho v)$. In this particular case, the homogenization process described in Section 3 can be rewritten as follows.

First, on each incoming road $k = 1, 2$, the curve $\{\mathbf{w}_k(U) = \mathbf{w}_k(U_{k,0})\}$ becomes in Lagrangian coordinates:

$$\{U(\tau, v);\quad v + P_k(\tau) = \mathbf{w}_k(U_{k,0})\}.$$

Now, on the outgoing road 3, the general equations (16) to (20) become

$$\bar{w} = \beta_1 \mathbf{w}_1(U_{1,0}) + (1 - \beta_1)\mathbf{w}_2(U_{2,0}) \tag{23}$$

and

$$\tau = \beta_1 P_1^{-1}(\mathbf{w}_1(U_{1,0}) - v) + (1 - \beta_1) P_2^{-1}(\mathbf{w}_2(U_{2,0}) - v), \tag{24}$$

where β_1 is still unknown.

As a prototype, we treat the case where $p_i(\rho) = \rho^\gamma$ (or $P_i(\tau) = 1/\tau^\gamma$) for $i = 1, 2, 3$, with $\gamma = 1$. Then, (24) becomes

$$\tau_3 = \frac{\beta_1}{w_1 - v} + \frac{(1 - \beta_1)}{w_2 - v}, \tag{25}$$

where w_k is the *constant* $w_k := \mathbf{w}_k(U_{k,0})$, $k = 1, 2$.

This homogenized relation (25) implies

$$\rho_3 v = \frac{(w_2 - v)(w_1 - v)v}{\beta_1(w_2 - w_1) + w_1 - v}. \tag{26}$$

Combining (26), (25), problem (22) is equivalent to solving the following maximization problem

$$\max q_3 \text{ subject to} \tag{27a}$$

$$0 \le q_3 \le \frac{d_1}{\beta_1}; \tag{27b}$$

$$0 \le q_3 \le \frac{d_2}{(1 - \beta_1)}; \tag{27c}$$

$$0 \le q_3 \le s_3(U_{3,0}, \mathbf{w}_3^\dagger, \mathbf{w}_3^*); \tag{27d}$$

$$0 \le \beta_1 \le 1. \tag{27e}$$

We set $v_3 := v_{3,0}$. Then, for each given β_1, we denote by v_c the velocity corresponding to the maximal flux on the outgoing road, according to the supply i.e, v_c is obtained by solving $\frac{d(\rho_3 v)}{dv} = 0$ for any *fixed* β_1. The supply $s_3(U_{3,0}, \mathbf{w}_3^\dagger, \mathbf{w}_3^*)$ is then:

$$s_3 = \begin{cases} \dfrac{(w_2 - v_3)(w_1 - v_3)v_3}{\beta_1(w_2 - w_1) + w_1 - v_3} & if\;\; v_3 \le v_c; \\ \dfrac{(w_2 - v_c)(w_1 - v_c)v_c}{\beta_1(w_2 - w_1) + w_1 - v_c} & if\;\; v_3 > v_c. \end{cases} \tag{28}$$

For any fixed v_3, we note that the function $\beta_1 \longmapsto s_3(v_3, \beta_1)$ is non–decreasing if $w_1 > w_2$, non–increasing if $w_1 < w_2$ and constant if $w_1 = w_2$, in which case the homogenization problem is trivial.

Now, we can solve the maximization Problem (27). Since the problem is symmetric with respect to the incoming roads, it suffices to consider the case $w_1 \geq w_2$, see below.

4.1. **The case $w_1 > w_2$.** The optimal solution of the Problem (27) is reached only in one of the following cases:

4.1.1. *Case 1:* $q_3 < s_3(v_3, \beta_1)$.
Subcase 1.1. In this case the constraint (27d) is not saturated. Therefore two constraints (27b) and (27c) must be saturated.

$$\begin{cases} q_1 = d_1; \\ q_2 = d_2; \end{cases} \quad \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 = \frac{d_2}{1-\beta_1}; \end{cases}$$

An example of optimal solution of the problem (27) in the (β_1, q_3) plane is shown in Figure 4.1.1.

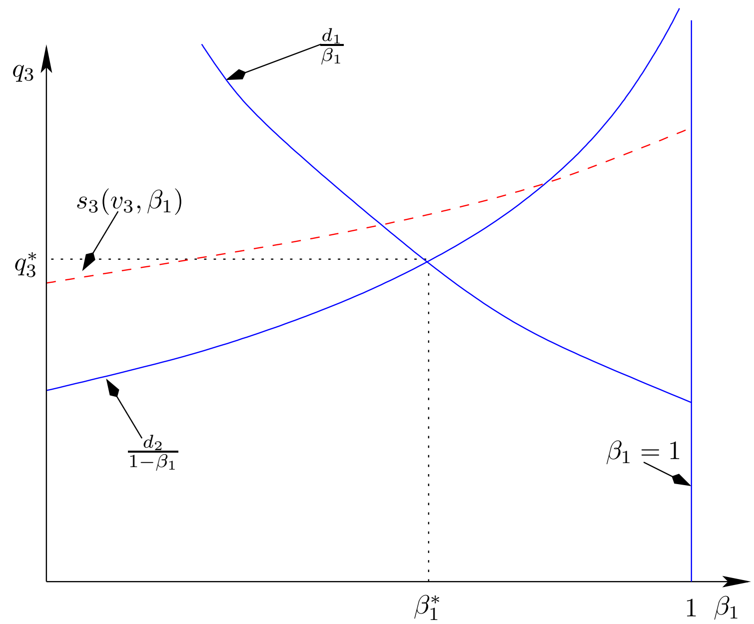

FIGURE 3. Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 1.1 - Case$w_1 > w_2$).

4.1.2. *Case 2:* $q_3 = s_3(v_3, \beta_1)$..
Subcase 2.1.

$$\begin{cases} q_1 < d_1; \\ q_2 < d_2; \end{cases} \quad \Leftrightarrow \begin{cases} q_3 < \frac{d_1}{\beta_1}; \\ q_3 < \frac{d_2}{1-\beta_1}; \end{cases}$$

We draw in Figure 4 an example of optimal solution of the problem (27) in the (β_1, q_3) plane.

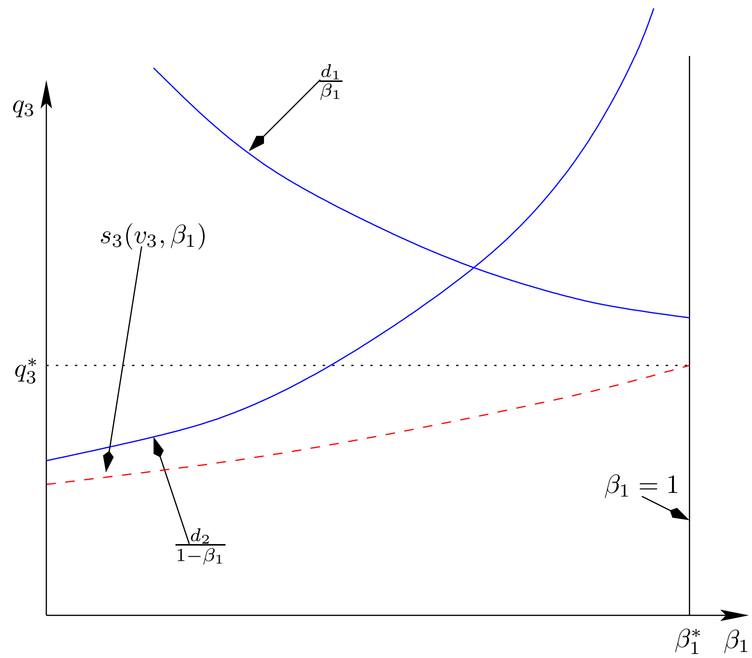

FIGURE 4. Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.1 - Case $w_1 > w_2$).

Subcase 2.2.

$$\begin{cases} q_1 = d_1; \\ q_2 < d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 < \frac{d_2}{1-\beta_1}; \end{cases}$$

An example of optimal solution of the problem (27) is drawn is Figure 5.
Subcase 2.3.

$$\begin{cases} q_1 < d_1; \\ q_2 = d_2; \end{cases}$$

In this subcase $q_1 < d_1$. Then, since $w_1 > w_2$, the drivers on road 1 are "more agressive" than those on road 2, so the flux on road 1 would strictly increase while the road 3 is not saturated (i. e. $q_3 = s_3(v_3, \beta_1)$). So, in this case we have necessarily $\beta_1 = 1$, i.e. $q_2 = (1 - \beta_1)q_3 = 0 =$. Since $q_2 = d_2$ then $d_2 = 0$. Therefore the road 2 is empty. This situation is in fact the case of one road with two different traffic conditions [2].
Subcase 2.4.

$$\begin{cases} q_1 = d_1; \\ q_2 = d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 = \frac{d_2}{1-\beta_1}; \end{cases}$$

In this subcase, the problem (27) has a unique solution. We give in Figure 6 an example of optimal solution of the problem (27).

4.2. **The case $w_1 = w_2$.** The optimal solution of the problem (27) is reached only in one of the following cases:

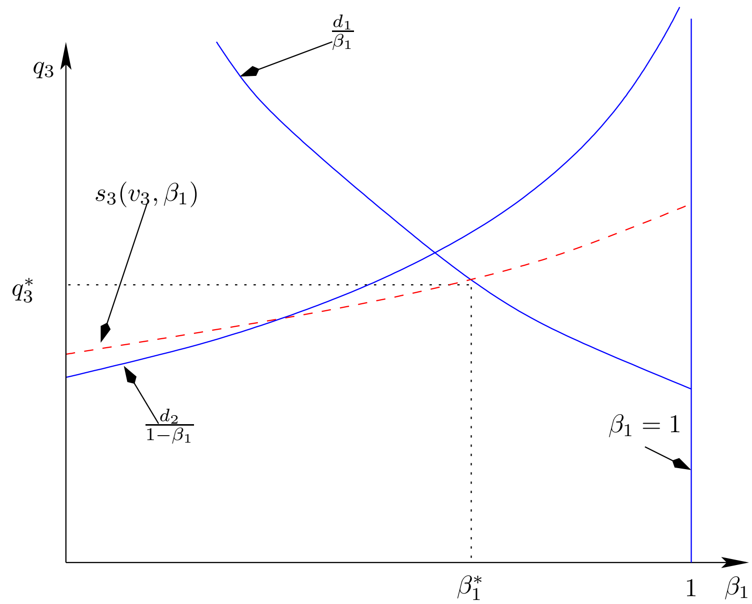

FIGURE 5. Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.2 - Case $w_1 > w_2$).

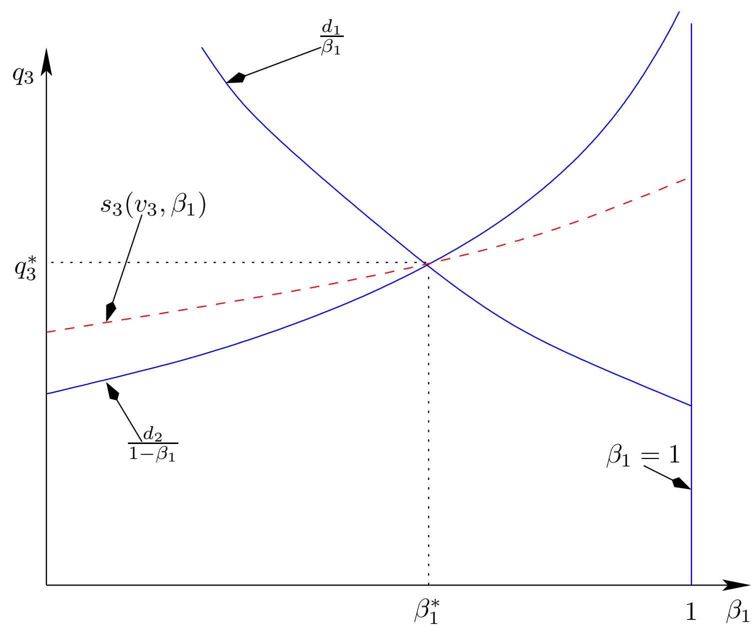

FIGURE 6. Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.4 - Case $w_1 > w_2$)).

4.2.1. *Case 1:* $q_3 < s_3(v_3, \beta_1)$.

Subcase 1.1.

$$\begin{cases} q_1 = d_1; \\ q_2 = d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 = \frac{d_2}{1-\beta_1}; \end{cases}$$

In this subcase the problem (27) has a unique solution.

An example of optimal solution of the problem (27) is drawn is Figure 7.

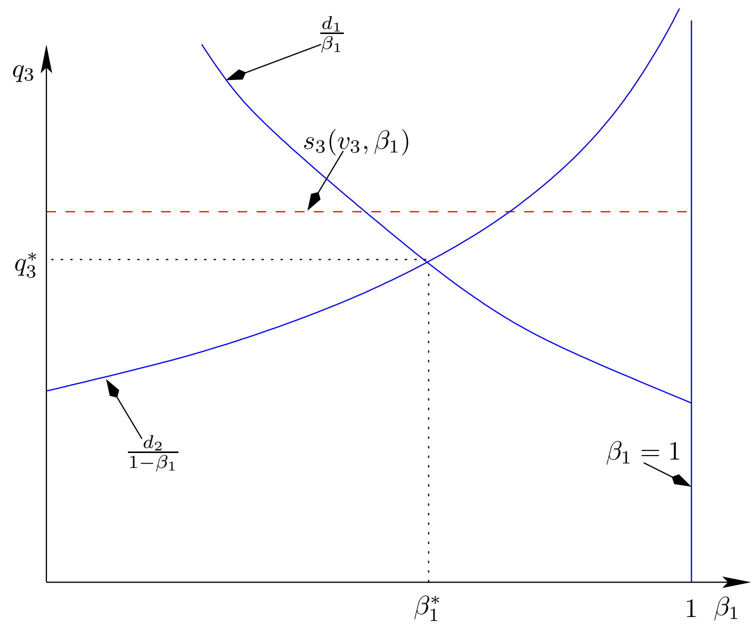

FIGURE 7. Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 1.1 - Case $w_1 = w_2$).

4.2.2. *Case 2:* $q_3 = s_3(v_3, \beta_1)$.

Subcase 2.1.

$$\begin{cases} q_1 = d_1; \\ q_2 = d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 = \frac{d_2}{1-\beta_1}; \end{cases}$$

As in the subcase 1.1 above, the problem (27) has a unique solution.
An example of optimal solution of the problem (27) is drawn is Figure 8.

Subcase 2.2.

$$\begin{cases} q_1 \leq d_1; \\ q_2 \leq d_2; \\ \text{with}(q_1, q_2) \neq (d_1, d_2) \end{cases}$$

In this subcase the problem (27) has **an infinity** of solutions.

An example of optimal solution of the problem (27) is drawn is Figure 9.

In the Appendix we give a numerical algorithm which solves (27) when the solution is unique. There, for simplicity, when there is **no** uniqueness, we fix $\beta_1 = \beta_1^{**} := \frac{d_1}{d_1+d_2}$ as **additional** assumption.

Finally, we summarize the discussion above in the following proposition.

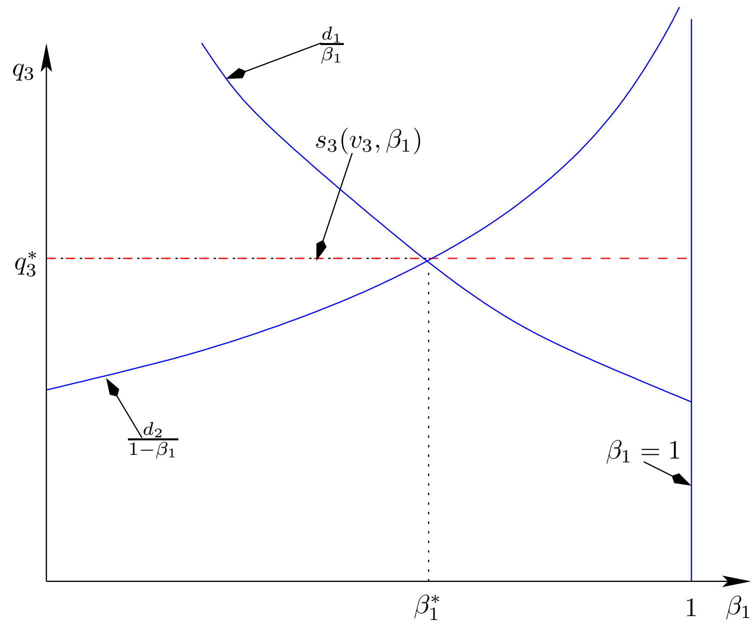

Figure 8. Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.1 - Case $w_1 = w_2$).

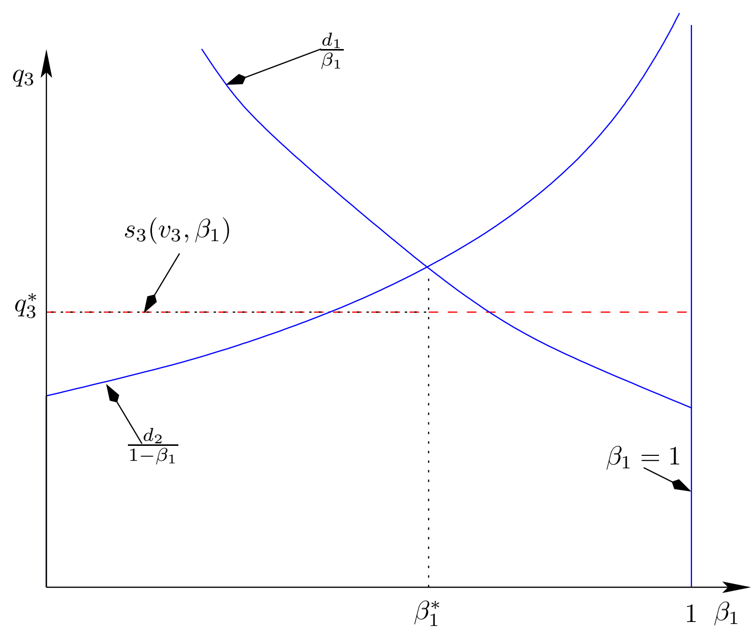

Figure 9. Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.2 - Case $w_1 = w_2$).

Proposition 3. *Consider two incoming roads* $1, 2$ *and one outgoing road* 3 *with* $a_1 = a_2 = -\infty, b_1 = b_2 = a_3$ *and* $b_3 = \infty$ *and constant initial data* $U_{i,0} =$

$(\rho_{i,0}\rho_{i,0}v_{i,0}), i = 1,2,3$. *Assume* $\mathbf{w}_1(U_{1,0}) \neq \mathbf{w}_2(U_{2,0})$. *Then there exists a unique solution* $\{U_i(x,t)\}_{i=1,2,3}$ *of the Riemann problem at the junction (11) and (12) with the following properties:*

1. $\{U_i(x,t)\}_{i=1,2,3}$ *is a weak solution (in the sense of (6)) of the network problem (11-8), where* $p_i^\dagger \equiv p_i$ *for the incoming roads* $i = 1,2$.
   For the outgoing road $i = 3$, *we obtain two different expressions for* $p_3^\dagger$, *depending on the position* (x,t)*:*
   In the $x-t$ *plane, in a triangle near the junction, we consider the* **homogenized solution** *described above. Therefore,* $p_i^\dagger(\cdot) := p_i^*(\cdot)$ *is given by the general relations (16), (18), (19) and (20)and more precisely by formulas (23)–(26). The triangle is bounded at any fixed time* $t > 0$ *by* $x = a_3$ *and* $x = a_3 + tv_{3,0}$. *In the remaining part of the outgoing road we have* $p_3^\dagger \equiv p_3$.
2. *The Rankine-Hugoniot conditions (9a-9b) are satisfied, with* $\rho_i(x,t)v_i(x,t) \geq 0\ ,\ 1 \leq i \leq 3$. *In particular* $U_3(a_3^+,t)$ *satisfies*
   $$\mathbf{w}_3^\dagger(U_3(a_3^+,t)) := \mathbf{w}_3^*(U_3(a_3^+,t)) := v_3(a_3^+,t) + p_3^*(\rho_3(a_3^+,t)) = \bar{w},$$
   where $\bar{w}$ *is the* homogenized value *given by (16).*
3. *The incoming fluxes are maximal subject to the other conditions.*

We now consider a second type of junction (a diverge).

5. **One incoming road and two outgoing roads.** Here, we follow the presentation of [10]. The results are also recovered by the presentation in [7]: they are just recalled for sake of completeness.

In this case, $k = 1$ for the incoming road and $j = 2,3$ for the outgoing roads. For notational convenience we set $\alpha_{21} = \alpha$ and $\alpha_{31} = (1-\alpha)$. Furthermore, we set $w_1 := \mathbf{w}_1(U_{1,0})$.

Again, we simplify the general maximization problem (21). From equation (22d), we obtain

$$\beta_{j1} = \frac{\alpha_{j1}q_1}{q_j}, \qquad j = 2,3.$$

Since there is only one incoming road, $q_j = \alpha_{j1}q_1$, $j = 2,3$ and therefore $\beta_{21} = \beta_{31} = 1$.

Obviously, here, **no** homogenization is needed, since there is a single incoming road: all the cars have kept the same value w_1 ("color") when passing the junction. Hence, $p_j^\dagger \equiv p_j$ for $j = 2,3$ and

$$\mathbf{w}_j(U) = v + p_j(\rho_j) = v + P_j(\tau_j) = w_1 \quad j = 2,3. \tag{29}$$

As above, we assume that for each $i = 1,2,3$, $p_i(\rho) = \rho^\gamma$ (or equivalently $P_i(\tau) = 1/\tau^\gamma$) and $\gamma = 1$.

Now, we discuss the possible supplies $s_j(U_{j,0})$ and then finally solve (21). By equation (29) we have

$$\rho_j v_j = v_j(w_1 - v_j),\ j = 2,3.$$

Let $v_{j,c}$ the velocity corresponding to the maximal flux on the outgoing road j, i.e.,

$$v_{jc} = \{v_j; \quad \frac{d(\rho_j v_j)}{dv_j} = 0\} = \frac{w_1}{2} \quad j = 2,3$$

Therefore, the supplies $s_j(U_{j,0};\mathbf{w}_j,w_1)$ are

$$s_j(U_{j,0};\mathbf{w}_j,w_1)=\begin{cases} v_{j,0}(w_1-v_{j,0}) & if \;\; v_{j,0}\le \frac{w_1}{2};\\ \dfrac{w_1}{2}(w_1-\dfrac{w_1}{2}) & if \;\; v_{j,0}>\dfrac{w_1}{2}.\end{cases} \tag{30}$$

Finally, problem (22) reduces to

$$\max q_1 \text{ subject to} \tag{31a}$$
$$0\le q_1\le d_1, \tag{31b}$$
$$0\le \alpha q_1\le s_2(U_{2,0};\mathbf{w}_2,w_1), \tag{31c}$$
$$0\le (1-\alpha)q_1\le s_3(U_{3,0};\mathbf{w}_3,w_1), \tag{31d}$$

and its optimal solution is $q_1^*=\min\{d_1,\frac{s_2(U_{2,0};\mathbf{w}_2,w_1)}{\alpha},\frac{s_3(U_{3,0};\mathbf{w}_3,w_1))}{1-\alpha}\}$.

As before, the above discussion can be summarized in the following proposition, c.f. Proposition 4.1 [10].

Proposition 4. *Consider three roads $i=1,2,3$ with $a_1=-\infty, b_1=a_2=a_3$ and $b_2=b_3=\infty$ and constant initial data $U_{i,0}=(\rho_{i,0},\rho_{i,0}v_{i,0}), i=1,2,3$. Let $0\le\alpha\le 1$ be given. Then there exists a unique solution $\{U_i(x,t)\}_{i=1,2,3}$ of the Riemann problem at the junction (11) and (12) with the following properties:*

1. *$\{U_i(x,t)\}_{i=1,2,3}$ is a weak solution (in the sens of (6)) of the network problem (11-8) with $p_i^\dagger\equiv p_i$. Therefore, the Rankine-Hugoniot conditions (9a-9b) are satisfied, and*

$$\rho_i(x,t)v_i(x,t)\ge 0,\quad i=1,2,3.$$

2. *For all $t>0$ the flux is distributed in proportions α and $1-\alpha$ between roads 2 and 3, c.f. equations (21e):*

$$\alpha(\rho_1v_1)(b_1^-,t)=(\rho_2v_2)(a_2^+,t), \tag{32a}$$
$$(1-\alpha)(\rho_1v_1)(b_1^-,t)=(\rho_3v_3)(a_3^+,t), \tag{32b}$$

3. *The flux $(\rho_1v_1)(b_1^-,t)$ is maximal at the interface, subject to the above conditions, c.f. equation (21a).*

6. **Extensions and numerical results.** So far, we have presented a model to deal with a general junction with n incoming and m outgoing roads. Due to the strong non–linearities arising in particular in equation (21c), the general maximization problem (21) is rather complex. However, most of the traffic intersections with n incoming and m outgoing roads can be seen and (in fact are designed) as roundabouts. For this type of junctions, see Figure 6, all the *conflict points* (i.e. points of intersections of roads) are either $2\mapsto 1$ or $1\mapsto 2$ junctions. Therefore, we can model a general junction with a combination of these two types of junctions, for which we have explicitly solved the maximization problem in Sections 4 and 5.

Now, we present an example of numerical results for a $2\mapsto 1$ junction. We use the algorithm presented in the Appendix to solve the maximization problem (27) with constant initial data U_k^- (for incoming roads) and U_j^+ (for outgoing roads) and in particular to obtain the optimal β_1. The numerical simulation uses a standard first–order relaxation scheme [12], with a fixed discretization size $\Delta x=1/800$ and the time step is chosen according to the CFL condition. With $U=(\rho,\rho v)$, we set initial data $U_1^-=(3,5)$, $U_2^-=(2,3)$ and $U_3^+=(3,7)$ with the same function $p_i(\rho)=p(\rho)$ on all roads $i=1,2,3$. For these initial data, the optimal value of β_1 is $\beta^*=1$ and the maximal flux at the interface is $q_3^*=49/9$. Since $q_3^*=q_1^*+q_2^*$

and $q_1^* = \beta^* q_3^*$, we have $U_1^+ = (7/3, 49/9) \approx (2.33, 5.44)$, $U_2^+ = (7/2, 0) = (3.5, 0)$ and $U_3^- = U_3^* = (7/3, 49/9)$. The corresponding values of the Lagrangian marker w on the incoming roads are: $w_1(U_1^-) = 14/3$ and $w_2(U_2^-) = 7/2$. In other words, the drivers on road 1 are assumed to be more "aggressive", in the sense that $w_1(U_1^-) > w_2(U_2^-)$. Furthermore, the supply on road 3 is not sufficient to accept *both* inflows but is exactly sufficient for road 1. Hence, the flux on road $j = 2$ vanishes, whereas the maximal flux on road $j = 1$ passes through the junction: $q_1^+ = q_3^-$. In fact, in this very particular case, we have

$$d_1(U_1^-, \mathbf{w}_1, \mathbf{w}_1(U_1^-)) = s_3(U_3^*, \mathbf{w}_3^*, \mathbf{w}_3^*(U_3^*)) = s_3(U_3^*, \mathbf{w}_1, \mathbf{w}_1(U_1^-)),$$

since there is no mixture of the incoming fluxes: $\beta_1 = \beta^* = 1$. Therefore $q_3^- = q_3^* = \frac{d_1}{\beta_1} = d_1$. This situation corresponds to the Subcase 2.2 of the Section 4.1.2, in the very particular case where the intersection of the curves $s_3(v_3, \beta_1)$ and $\frac{d_1}{\beta_1}$ corresponds exactly to $\beta_1 = 1$.

On the incoming road $j = 1$ we have U_1^- connected to U_1^+ with a 1–rarefaction wave of negative speed, see Figure 6. Here, since the drivers are more "aggressive" than those on road $j = 2$, they accelerate to enter the intersection and take all the available supply on road $j = 3$.

Furthermore, on the other incoming road $j = 2$ we connect U_2^- and U_2^+ with a 1–shock wave of negative speed, see Figure 6. This 1–shock wave corresponds to a braking of the cars on road $j = 2$, which are here completely stopped: $q_2^+ = 0$.

On the outgoing road, we have $q_3^- = q_3^* = q_1^+$ therefore $U_3^- = U_3^*$, we only connect U_3^* and U_3^+ through a 2–contact discontinuity. The flux has increased up to the maximal possible flux on this road due to the maximization (27). The corresponding solution on the outgoing road is depicted in Figure 6.

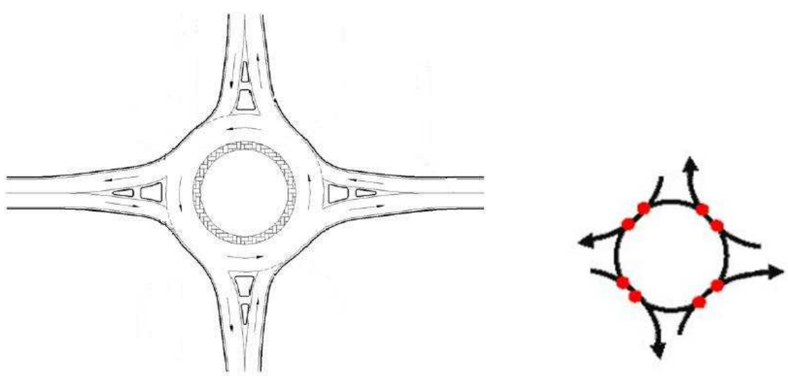

FIGURE 10. Roundabouts for a 4–4 junction

Note that the flux $\rho_2^+ v_2^+ = q_2^+ = q_2^* = 0$. Due to the numerical diffusion of (the used) first order scheme [12], the shock is smoothed out, and even appears like a rarefaction fan, on the top figure. Obviously, a second order scheme would give sharper results, but that was not our main concern in this work.

7. **Summary.** We have extended the results in [10] and obtained a general formulation for suitable coupling conditions at an intersection, *without* imposing a fixed

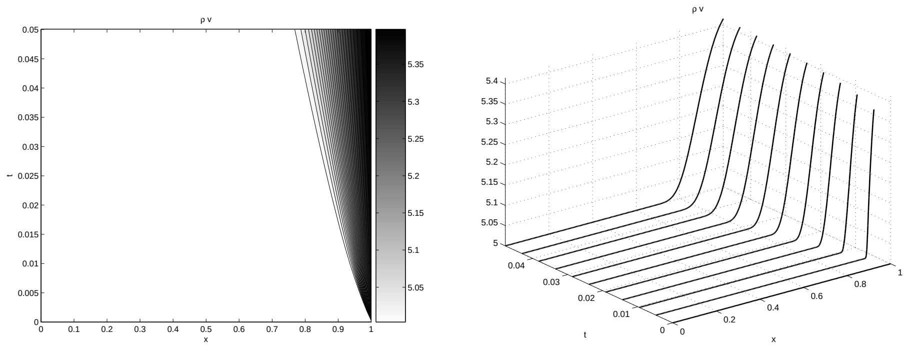

FIGURE 11. Plots of the level curves of the flux $\rho_1 v_1$ on road 1 in the (x, t) plane (right) and snapshots of the corresponding flux for different times t_n (left).

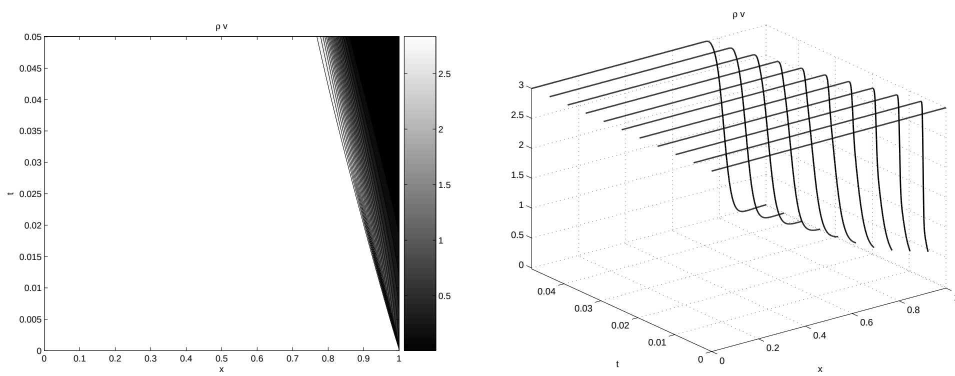

FIGURE 12. Plots of the level curves of the flux $\rho_2 v_2$ on road 2 in the (x, t) plane (right) and snapshots of the corresponding flux for different times t_n (left).

mixture principle. The solution conserves mass and (pseudo-)momentum and additionally *maximizes the total (mass) flux* at an intersection. This problem has been completely solved in the case of $1 \mapsto 2$ (diverge) and $2 \mapsto 1$ (merge) junctions and we have given an example of numerical results for (the more interesting) latter case.

Of course, the algorithm allows to solve all the cases described in Section 4. Moreover, as we already said, in principle any roundabout can be reduced to a combination of $1 \mapsto 2$ and $2 \mapsto 1$ junction, see Figure 6.

Naturally, due to the strong non-linearity in the homogenized supply, solving the optimization problem is more complicated than for a first order model, but it is still tractable even in the more difficult case of $2 \mapsto 1$ junction (merge). A sketch of the optimization algorithm is given in the Appendix.

Appendix

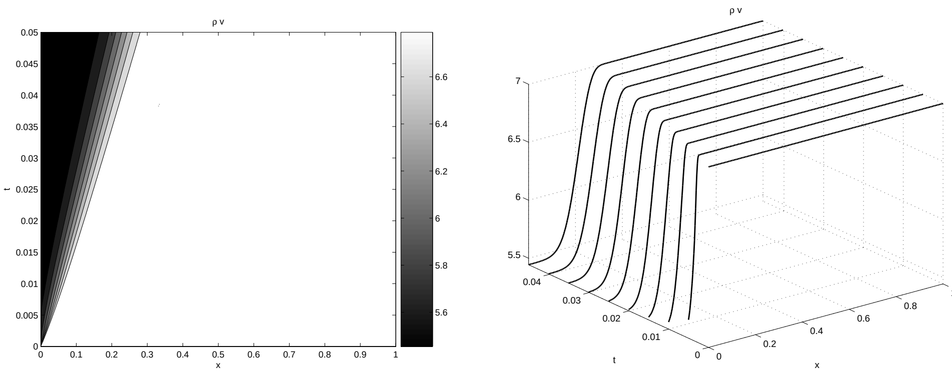

FIGURE 13. Plots of the level curves of the flux $\rho_3 v_3$ on road 3 in the (x, t) plane (right) and snapshots of the corresponding flux $\rho_3 v_3$ for different times t_n (left).

Algorithm for solving (27)

Begin
If $w_1 > w_2$ **then**
  If $s_3(v_3, 1) \leq d_1$ **then**
    $q_3^* := s_3(v_3, 1);$
    $\beta_1^* := 1;$
  Else
    $\beta_1^{**} := \{\beta_1 / \ \frac{d_1}{\beta_1} = \frac{d_2}{1-\beta_1}\};\ (\beta_1^{**} := \frac{d_1}{d_1+d_2})$
    $\beta_1^* := \{\beta_1 / \ \ s_3(v_3, \beta_1) = \frac{d_1}{\beta_1}\};$
    If $\beta_1^* \geq \beta_1^{**}$ **then**
      $q_3^* := s_3(v_3, \beta_1^*);$
    Else
      $q_3^* := s_3(v_3, \beta_1^{**});$
      $\beta_1^* := \beta_1^{**};$
    EndIf
  EndIf
Else $(w_1 \leq w_2)$
  If $w_1 < w_2$ **then**
    If $s_3(v_3, 0) \leq d_2$ **then**
      $q_3^* := s_3(v_3, 0);$
      $\beta_1^* := 0;$
    Else
      $\beta_1^{**} := \{\beta_1 / \ \frac{d_1}{\beta_1} = \frac{d_2}{1-\beta_1}\};\ (\beta_1^{**} := \frac{d_1}{d_1+d_2})$
      $\beta_1^* := \{\beta_1 / \ \ s_3(v_3, \beta_1) = \frac{d_2}{1-\beta_1}\};$
      If $\beta_1^* \leq \beta_1^{**}$ **then**
        $q_3^* := s_3(v_3, \beta_1^*)$
      Else
        $q_3^* := s_3(v_3, \beta_1^{**});$
        $\beta_1^* := \beta_1^{**};$
      EndIf
    EndIf
  Else $(w_1 = w_2)$
    $\beta_1^{**} := \{\beta_1 / \ \frac{d_1}{\beta_1} = \frac{d_2}{1-\beta_1}\};(\beta_1^{**} := \frac{d_1}{d_1+d_2})$
    If $s_3(v_3, \beta_1^{**}) > \frac{d_1}{\beta_1^{**}}$ **then** (or $s_3(v_3, \beta_1^{**}) > \frac{d_2}{1-\beta_1^{**}}$)
      $\beta_1^* := \beta_1^{**};$
      $q_3^* := \frac{d_1}{\beta_1^{**}};$ (or $q_3^* := \frac{d_2}{1-\beta_1^{**}}$)
    Else
      $\beta_1^* := \beta_1^{**};$
      $q_3^* := s_3(v_3, \beta_1^{**})$
    EndIf
  EndIf
EndIf
RETURN$(q_3^*,\ \beta_1^*)$
End

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