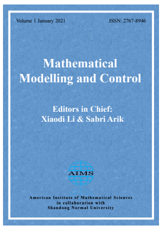
Mathematical Modelling and Control

https://www.aimspress.com/journal/mmc



MMC, 4(3): 286–296.
DOI: 10.3934/mmc.2024023
Received: 28 December 2023
Revised: 27 March 2024
Accepted: 04 June 2024
Published: 02 September 2024

Research article

Results on non local impulsive implicit Caputo-Hadamard fractional differential equations

K. Venkatachalam[1], **M. Sathish Kumar**[2,*] **and P. Jayakumar**[2]

[1] Department of Mathematics, Nandha Engineering College (Autonomous), Erode 638052, Tamil Nadu, India

[2] Department of Mathematics, Paavai Engineering College (Autonomous), Namakkal 637018, Tamil Nadu, India

* **Correspondence:** Email: msksjv@gmail.com.

Abstract: The results for a new modeling integral boundary value problem using Caputo-Hadamard impulsive implicit fractional differential equations with Banach space are investigated, along with the existence and uniqueness of solutions. The Krasnoselskii fixed-point theorem, Schaefer's fixed point theorem and the Banach contraction principle serve as the basis of this unique strategy, and are used to achieve the desired results. We develop the illustrated examples at the end of the paper to support the validity of the theoretical statements.

Keywords: fractional differential equation; Caputo-Hadamard; existence; uniqueness; fixed point theorems

1. Introduction

In engineering, physics, chemistry, control theory, signal, image processing, and biology, the study of fractional differential equations (FDEs) (see e.g., [1–3]), which is connected to fractional calculus, is significant. Integer-order derivatives are less helpful and useful for characterizing the memory and heredity characteristics of various materials and processes compared to fractional derivatives and integrals of arbitrary order (see [4–11]) .

The investigation of integral boundary value problems (BVPs) has advanced in the past few decades. It has also been extremely useful to develop a variety of applied mathematical models of actual processes in applied sciences and engineering. Liu et al. [12], an averaging principle for Caputo-Hadamard (C-H) fractional stochastic differential equations was established. Under appropriate conditions, we demonstrate that the mild solution of the original equation is approximately equivalent to that of the reduced averaged equation without impulses in [13]. In [14], Kahouli et al. demonstrated the existence and uniqueness of the solution to a class of Hadamard fractional Itô-Doob stochastic integral equations of order $\phi \in (0, 1)$ via a fixed point technique. Hyers-Ulam stability was investigated for Hadamard fractional Itô-Doob stochastic integral equations according to the Gronwall inequality. Recently, it has been noted that many of the materials on the subject focus on FDEs of the Caputo and Riemann-Liouville types with various situations, including time delays, impulses, and boundary value conditions [15–18].

Along with the Riemann-Liouville and Caputo derivatives, another kind of fractional derivatives that is mentioned in the literature is the Hadamard fractional derivatives, which first appeared in 1892; see e.g., [19]. It differs from the previous ones in that it includes an arbitrary logarithm function; see [20–22] for additional details.

The most common numerical methods used in [23], the spectral element method (SEM), has both high accuracy and a lower computational cost when compared with finite-element or finite-difference methods, SEM is not widely utilized in the modeling of boundary value problems in electromagnetics. In [24], SEM was recently utilized in some branches of electromagnetics as waveguides and photonic structures for the sake of accuracy. The numerical approximation to the set of the partial differential equations governing a typical magnetostatic problem was presented using SEM. Legendre polynomials and Gauss-Legendre-Lobatto grids are employed in the current study as test functions and meshing of the elements. The domain decomposition based on the application reasoning of the perfectly matched layer was studied using the SEM for the first time in order to solve near and far electromagnetic fields without requiring substantial computational resources in [25]. The fundamental fractional calculus theorem was subsequently included in the C-H derivative in [26], where they also suggested a Caputo-type version of the Hadamard fractional derivatives. Impulsive differential equations with Hadamard and C-H derivatives have been the focus of recent studies (see [27–29] and the references therein).

Now a days, problems for FDEs and the Caputo-Hadamard fractional derivative (C-HFD) with initial and boundary conditions (BCs) have been concentrated on by many authors as discussed below.

In [30], Mahmudov et al. investigated the integral boundary conditions (IBCs) for C-HFD's of the form:

$$\begin{aligned} &{}^{CH}\mathcal{D}^{\alpha_1}\wp(\mathfrak{I}) = \mathscr{F}(\mathfrak{I}, \wp(\mathfrak{I})), \ \ 1 < \alpha_1 \leq 2, \\ &\mathfrak{u}(\mathfrak{a}) = 0, \ \ \mathfrak{u}(\mathscr{T}) = \nu \int_a^\eta \mathfrak{u}(\mathtt{s})\mathfrak{ds}, \ \ \mathfrak{a} < \eta < \mathscr{T}, \nu \in \mathbb{R}, \end{aligned}$$

where ${}^{C}\mathcal{D}^{\alpha_1}$ is the C-HFD and the function is continuous.

In [31], Arioua et al. studied the C-HFDEs involving BCs:

$$\begin{aligned} &{}^{C}\mathcal{D}^{\alpha_1}_{1^+}\mathfrak{u}(\mathfrak{I}) + \mathscr{F}(\mathfrak{I}, \mathfrak{u}(\mathfrak{I})) = 0, \quad 1 < \mathfrak{I} < e, \quad 2 < \alpha_1 \leq 3, \\ &\mathfrak{u}(1) = \mathfrak{u}'(1) = 0, ({}^{C}\mathcal{D}^{\alpha_1-1}_{1^+}\mathfrak{u})(e) = ({}^{C}\mathcal{D}^{\alpha_1-1}_{1^+}\mathfrak{u})(e) = 0, \end{aligned}$$

where ${}^{C}\mathcal{D}^{\alpha_1}$-C-HFD and $\mathscr{F}$ is a continuous function.

In [32], Benhamida et al. investigated the main results for C-HFDEs of the BVP:

$$ {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I}) = \mathscr{F}(\mathfrak{I}, \mathscr{Y}(\mathfrak{I})), \ \ (\mathfrak{I}) \in [1, \mathscr{T}], \ \ 0 < \alpha_1 \leq 1, $$

$$ \mathfrak{a}\wp(1) + \mathfrak{b}\wp(\mathscr{T}) = \mathfrak{c}, $$

where ${}^{C}\mathcal{D}^{\alpha_1}$-C-H derivative, the function is continuous, and $\mathfrak{a} + \mathfrak{b}$ is not equal to zero.

In [33], Ardjounia et al. studied a nonlocal conditions for FDE's by using the C-H derivative:

$$ {}^{CH}\mathcal{D}^{\alpha_1}_{1}\mathscr{Y}(\mathfrak{I}) = \mathscr{F}(\mathfrak{I}, \mathscr{Y}(\mathfrak{I}), {}^{CH}\mathcal{D}^{\alpha_1}_{1}\mathscr{Y}(\mathfrak{I})), $$

$$ \mathfrak{I} \in [1, \mathfrak{I}], 0 < \alpha_1 < 1, $$

$$ \wp(1) + \mathfrak{g}(\mathfrak{x}) = \wp_0, $$

where $\mathscr{F}$-nonlinear continuous functions, $\mathcal{D}^{\alpha_1}_{1}$-C-H derivative.

In [34], Irguedi et al. discussed the BVP for functional impulsive FDEs of the form

$$\begin{aligned} {}^{CH}D^{r}y(t) &= f(t, y_t), \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \\ \Delta y'|_{t=t_k} &= \bar{I}_k(y(t_k^-)), \\ y(a) &= \Phi(t), \\ y'(T) &= \int_a^T h(s, y(s))ds. \end{aligned}$$

In [35], Graef et al. studied impulsive C-HFDEs with IBCs of the form

$$\begin{aligned} {}^{CH}D^{r}y(t) &= f(t, y(t)), \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \\ \Delta y'|_{t=t_k} &= \bar{I}_k(y(t_k^-)), \\ y(1) &= \int_1^T g(s, y(s))ds, \\ y'(T) &= \int_1^T h(s, y(s))ds. \end{aligned}$$

In [36], Akhter studied the new BCs for the CFD

$$\begin{aligned} &{}^{C}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I}) = \mathscr{F}(\mathfrak{I}, \mathscr{Y}(\mathfrak{I})), \ \mathfrak{I} \in [0, 1], \ \ 1 < \alpha_1 \leq 2, \\ &\wp(0) = 0, \ \ \mathfrak{a}\mathscr{Y}'(\xi_1) + \mathfrak{b}\mathscr{Y}'(\xi_2) = \beta \int_0^\eta \mathscr{Y}(\mathfrak{s})\mathfrak{ds}, \end{aligned}$$

where ${}^{C}\mathcal{D}^{\alpha_1}$- are Caputo fractional derivatives, $\mathscr{F}$ is a continuous function, and $\mathfrak{a}, \mathfrak{b}, \beta \in \mathbb{R}$.

Inspired by the above performance, we investigate the C-H impulsive implicit FDEs with integral BC of the form

$$ {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I}) = \mathscr{F}(\mathfrak{I}, \mathscr{Y}(\mathfrak{I}), {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I})), \tag{1.1} $$

for $\mathfrak{I} \in [0, 1], 1 < \alpha_1 \leq 2$,

$$ \mathscr{Y}(\mathfrak{I}_k^+) = \mathscr{Y}(\mathfrak{I}_k^-) + \mathscr{Y}_k, \ \ \mathscr{Y}_k \in \mathbb{R}, \ \ k=1,\ldots,m, \tag{1.2} $$

$$ \mathscr{Y}(0) = 0, \ \ \mathfrak{a}\mathscr{Y}'(\xi_1) + \mathfrak{b}\mathscr{Y}'(\xi_2) = \epsilon \int_0^\eta \mathscr{Y}(\mathtt{s})\mathfrak{ds}, \tag{1.3} $$

where ${}^{CH}\mathcal{D}^{\alpha_1}$ is C-HFD of order α_1,

$$ \mathscr{F} : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} $$

is a continuous function, $\mathfrak{a}, \mathfrak{b}, \epsilon \in \mathbb{R}, 0 < \eta < 1$ and

$$0 = \mathfrak{I}_0 < \mathfrak{I}_1 < \mathfrak{I}_2 < ... < \mathfrak{I}_{\mathfrak{m}} = 1,$$

$$\Delta\mathscr{Y}|_{\mathfrak{I}=\mathfrak{I}_k} = \mathscr{Y}(\mathfrak{I}_k^+) - \mathscr{Y}(\mathfrak{I}_k^-)$$

and

$$\mathscr{Y}(\mathfrak{I}_k^+) = \lim_{\mathfrak{h}\to 0^+} \mathscr{Y}(\mathfrak{I}_k + \mathfrak{h}),$$

$$\mathscr{Y}(\mathfrak{I}_k^-) = \lim_{\mathfrak{h}\to 0^-} \mathscr{Y}(\mathfrak{I}_k + \mathfrak{h})$$

denote the limits of the right and left hand $\mathscr{Y}(\mathfrak{I})$ at

$$\mathfrak{I} = \mathfrak{I}_k.$$

The linear combination of the BC values ξ_1 and ξ_2 is the nonlocal position of an unknown function and is proportional to the continuous distribution of an unknown function of an arbitrary length η.

Motivations:

(1) This study uses the C-HFD to develop a new class of implicit C-HIFI-DE with BCs.
(2) We additionally verify the existence and uniqueness of the solutions to Eqs (1.1)–(1.3) using the Banach contraction principle and Krasnoselskii fixed-point theorem respectively.
(3) We extend the C-HFD, nonlinear integral terms, and implicit conditions to the results discussed in [36].

The rest is organized as follows: In Section 2, we discuss the basic concepts and lemmas that will be used to support findings. In Section 3, we prove the uniqueness of solutions (1.1)–(1.3) and the existence of the system under suitable assumptions. Applications are then presented in Section 4.

2. Preliminary notes

Let the space

$$\mathscr{PC}(\mathcal{J}, \mathbb{R}) = \{\mathscr{Y} : \mathrm{J} \to \mathbb{R} : \mathscr{Y} \in \mathscr{C}(\mathfrak{I}_k, \mathfrak{I}_{k+1}], \mathbb{R}\}$$

be continuous-everywhere for $\mathfrak{I}_k$ at $\mathscr{Y}(\mathfrak{I}_k^-)$ and $\mathscr{Y}(\mathfrak{I}_k^+)$, where

$$\mathscr{Y}(\mathfrak{I}_k^-) = \mathscr{Y}(\mathfrak{I}_k^+)$$

exists, and

$$\|\mathscr{Y}\|_{\mathscr{PC}} = \sup\left\{\left|\mathscr{Y}(\mathfrak{I})\right| : 0 \le \mathfrak{I} \le 1\right\}.$$

Definition 2.1. *[36] The Caputo derivative of $\mathscr{F}$ is a continuous function*

$$^{C}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I}) = \frac{1}{\Gamma(\mathfrak{n}-\alpha_1)} \int_0^{\mathfrak{I}} (\mathfrak{I} - \mathfrak{s})^{\mathfrak{n}-\alpha_1-1} \mathscr{F}^{\mathfrak{n}}(\mathfrak{s}) d\mathfrak{s},$$

where

$$\mathfrak{n} - 1 < \alpha_1 < \mathfrak{n}, \;\; \mathfrak{n} = [\alpha_1] + 1.$$

Definition 2.2. *[36] The Hadamard fractional integral of $\mathscr{F}$ is a continuous function*

$$_{\mathcal{H}}\mathcal{I}^{\alpha_1}\mathscr{Y}(\mathfrak{I}) = \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}} \left(\log \frac{(\mathfrak{I})}{\mathfrak{s}}\right)^{\alpha_1-1} \frac{\mathscr{F}(\mathfrak{s})}{\mathfrak{s}} \mathfrak{d}\mathfrak{s},$$

where $\alpha_1 > 0$.

Definition 2.3. *[36] The Hadamard fractional derivative of $\mathscr{F}$ is given as*

$$_{\mathcal{H}}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I}) = \frac{1}{\Gamma(\mathfrak{n}-\alpha_1)} \left(\frac{\mathfrak{d}}{\mathfrak{d}\mathfrak{I}}\right)^{\mathfrak{n}} \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\mathfrak{n}-\alpha_1-1} \frac{\mathscr{F}(s)}{\mathfrak{s}} \mathfrak{d}\mathfrak{s}.$$

Definition 2.4. *[36] For the $\mathfrak{n}^{\mathfrak{th}}$ order differentiable function $\mathscr{F}$, the C-HFD of order α_1 is*

$$^{C\mathcal{H}}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I}) = \frac{1}{\Gamma(\mathfrak{n}-\alpha_1)} \int_1^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\mathfrak{n}-\alpha_1-1} \vartheta^{\mathfrak{n}}\mathscr{F}(\mathfrak{s}) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}.$$

Lemma 2.1. *Let $\beta > \alpha_1 > 0$, $\mathscr{F} \in L_1[a, b]$. Then,*

$$^{C}\mathcal{D}^{\alpha_1}\mathcal{I}^{\beta} = \mathcal{I}^{\beta-\alpha_1}\mathscr{F}(\mathfrak{I}), \quad \forall \mathfrak{I} \in [a, b].$$

Lemma 2.2. *Let $\mathscr{Y}$ be an impulsive solution of IBC with*

$$\mathfrak{g} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

given by

$$^{C\mathcal{H}}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I}) = \mathfrak{g}(\mathfrak{I}), \quad 1 < \alpha_1 \le 2, \tag{2.1}$$

$$\mathscr{Y}(\mathfrak{I}_k^+) = \mathscr{Y}(\mathfrak{I}_k^-) + \mathscr{Y}(\mathfrak{I}), \quad k = 1, ..., m, \tag{2.2}$$

$$\mathscr{Y}(0) = 0, \;\; \mathfrak{a}\mathscr{Y}'(\xi_1) + \mathfrak{b}\mathscr{Y}'(\xi_2) = \epsilon \int_0^{\eta} \mathscr{Y}(\mathfrak{s})\mathfrak{d}\mathfrak{s}, \tag{2.3}$$

if and only if

$$
\mathscr{Y}(\mathfrak{I})=\begin{cases}
\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
+\frac{\mathfrak{I}}{\mathfrak{B}}\Big\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{s}\\
\frac{1}{\Gamma(\alpha_1-1)}+\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{r}\\
-\frac{\epsilon}{\Gamma(\alpha_1)}\int_0^{\eta}\left(\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}}\mathfrak{d}\mathfrak{u}\right)\mathfrak{d}\mathfrak{s}\Big\}\\
for\ \mathfrak{I}\in[0,\mathfrak{I}_1),\\
\mathscr{Y}_1+\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}d\mathfrak{s}\\
+\frac{\mathfrak{I}}{\mathfrak{B}}\Big\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{s}\\
+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{r}\\
-\frac{\epsilon}{\Gamma(\alpha_1)}\int_0^{\eta}\left(\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}}\mathfrak{d}u\right)\mathfrak{d}\mathfrak{s}\Big\}\\
for\ \mathfrak{I}\in(\mathfrak{I}_1,\mathfrak{I}_2),\\
\mathscr{Y}_1+\mathscr{Y}_2+\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{t}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
+\frac{\mathfrak{I}}{\mathfrak{B}}\Big\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{s}\\
+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{r}\\
-\frac{\epsilon}{\Gamma(\alpha_1)}\int_0^{\eta}\left(\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}}\mathfrak{d}\mathfrak{u}\right)\mathfrak{d}\mathfrak{s}\Big\}\\
for\ \mathfrak{I}\in(\mathfrak{I}_2,\mathfrak{I}_3),\\
\vdots\\
\sum_{k=1}^{m}\mathscr{Y}_i+\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
+\frac{\mathfrak{I}}{\mathfrak{B}}\Big\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{s}\\
+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{r}\\
-\epsilon\int_0^{\eta}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}}\mathfrak{d}\mathfrak{u}\right)\mathfrak{d}\mathfrak{s}\Big\}\\
for\ \mathfrak{I}\in(\mathfrak{I}_{\mathbf{k}},\mathfrak{I}_{\mathbf{k}+1}),
\end{cases}
\tag{2.4}
$$

where

$$
\mathfrak{B}=\left(\epsilon\frac{\eta^2}{2}-\mathfrak{a}\xi_1-\mathfrak{b}\xi_2\right)\neq 0.
$$

Proof. Assume $\mathscr{Y}$ satisfies (2.1) and (2.2). If $\mathfrak{I}\in[0,\mathfrak{I}_1)$, then

$$
\begin{aligned}
&{}^{C\mathcal{H}}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I})=\mathfrak{g}(\mathfrak{I}),\quad \mathfrak{I}\in(0,\mathfrak{I}_1],\quad \mathscr{Y}(0)=0,\\
&\mathfrak{a}\mathscr{Y}'(\xi_1)+\mathfrak{b}\mathscr{Y}'(\xi_2)=\epsilon\int_0^{\eta}\mathscr{Y}(\mathfrak{s})\mathfrak{d}\mathfrak{s}.
\end{aligned}
\tag{2.5}
$$

From Lemma 2.1, we have

$$
\begin{aligned}
\mathscr{Y}(\mathfrak{I})=&\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
&+\frac{\mathfrak{I}}{\mathfrak{B}}\Big\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{s}\\
&+\frac{1}{\Gamma(\alpha_1-1}\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{r}\\
&-\frac{\epsilon}{\Gamma(\alpha_1)}\int_0^{\eta}\left(\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}}\mathfrak{d}\mathfrak{u}\right)\mathfrak{d}\mathfrak{s}\Big\}.
\end{aligned}
$$

If $\mathfrak{I}\in(\mathfrak{I}_1,\mathfrak{I}_2)$, then

$$
{}^{C\mathcal{H}}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I})=\mathfrak{g}(\mathfrak{I}),\quad \mathfrak{I}\in(\mathfrak{I}_1,\mathfrak{I}_2]
$$

with

$$
\mathscr{Y}(\mathfrak{I}_{\mathbf{k}}^{+})=\mathscr{Y}(\mathfrak{I}_{\mathbf{k}}^{-})+\mathscr{Y}_{\mathbf{k}},
$$

and then

$$
\begin{aligned}
\mathscr{Y}(\mathfrak{I})=&\mathscr{Y}(\mathfrak{I}_1^{+})-\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}_1}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
&+\frac{\mathfrak{I}}{\mathfrak{B}}\Big\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{s}\\
&+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{r}\\
&-\frac{\epsilon}{\Gamma(\alpha_1)}\int_0^{\eta}\left(\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}}\mathfrak{d}\mathfrak{u}\right)\mathfrak{d}\mathfrak{s}\Big\}\\
=&\mathscr{Y}(\mathfrak{I}_1^{+})+\mathscr{Y}_1-\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}_1}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
&+\frac{\mathfrak{I}}{\mathfrak{B}}\Big\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{s}\\
&+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{r}\\
&-\epsilon\int_0^{\eta}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}}\mathfrak{d}\mathfrak{u}\right)\mathfrak{d}\mathfrak{s}\Big\}\\
=&\mathscr{Y}_1+\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}}\mathfrak{d}\mathfrak{s}\\
&+\frac{\mathfrak{I}}{\mathfrak{B}}\Big\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{s}\\
&+\frac{1}{\Gamma(\alpha_1-1}\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}}\mathfrak{d}\mathfrak{r}\\
&-\frac{1}{\Gamma(\alpha_1)}\epsilon\int_0^{\eta}\left(\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1-1}\frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}}\mathfrak{d}\mathfrak{u}\right)\mathfrak{d}\mathfrak{s}\Big\}.
\end{aligned}
$$

If $\mathtt{t} \in (\mathfrak{I}_2, \mathfrak{I}_3)$, then from (2.1),

$$\begin{aligned}
\mathscr{Y}(\mathfrak{I}) =& \mathscr{Y}(\mathfrak{I}_2^+) - \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}_2} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1 - 1} \frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}} \mathfrak{d}\mathfrak{s} \\
&+ \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1 - 1} \frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}} \mathfrak{d}\mathfrak{s} \\
&+ \frac{\mathfrak{I}}{\mathfrak{B}} \Bigg\{ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_1} \left(\log \frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1 - 2} \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}} \mathfrak{d}\mathfrak{s} \\
&+ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_2} \left(\log \frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1 - 2} \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}} \mathfrak{d}\mathfrak{r} \\
&- \epsilon \int_0^{\eta} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1 - 1} \frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}} \mathfrak{d}\mathfrak{u} \right) \mathfrak{d}\mathfrak{s} \Bigg\} \\
=& \mathscr{Y}(\mathfrak{I}_2^+) + \mathscr{Y}_2 - \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}_1} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1 - 1} \frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}} \mathfrak{d}\mathfrak{s} \\
&+ \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1 - 1} \frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}} \mathfrak{d}\mathfrak{s} \\
&+ \frac{\mathfrak{I}}{\mathfrak{B}} \Bigg\{ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_1} \left(\log \frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1 - 2} \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}} \mathfrak{d}\mathfrak{s} \\
&+ \frac{1}{\Gamma(\alpha_1 - 1} \int_0^{\xi_2} \left(\log \frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1 - 2} \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}} \mathfrak{d}\mathfrak{r} \\
&- \frac{\epsilon}{\Gamma(\alpha_1)} \int_0^{\eta} \left(\int_0^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1 - 1} \frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}} \mathfrak{d}\mathfrak{u} \right) \mathfrak{d}\mathfrak{s} \Bigg\} \\
=& \mathscr{Y}_1 + \mathscr{Y}_2 + \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1 - 1} \frac{\mathfrak{g}(\mathfrak{s})}{\mathfrak{s}} \mathfrak{d}\mathfrak{s} \\
&+ \frac{\mathfrak{I}}{\mathfrak{B}} \Bigg\{ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_1} \left(\log \frac{\xi_1}{\mathfrak{r}}\right)^{\alpha_1 - 2} \frac{\mathfrak{g}(\mathtt{r})}{\mathfrak{r}} \mathfrak{d}\mathfrak{s} \\
&+ \frac{1}{\Gamma(\alpha_1 - 1} \int_0^{\xi_2} \left(\log \frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1 - 2} \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}} \mathfrak{d}\mathfrak{r} \\
&- \epsilon \int_0^{\eta} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{r}}\right)^{\alpha_1 - 1} \frac{\mathfrak{g}(\mathfrak{u})}{\mathfrak{u}} \mathfrak{d}\mathfrak{u} \right) \mathfrak{d}\mathfrak{s} \Bigg\}.
\end{aligned}$$

If $\mathfrak{I} \in (\mathfrak{I}_{\mathtt{k}}, \mathfrak{I}_{\mathtt{k}+1})$ and we again try to apply (2.1), we get the other equation of (2.1). □

3. Main results

We require the following assumptions to show the major results:

($\mathtt{A}_1$) For some constants $\mathscr{L}_{\mathfrak{f}} > 0$ and $0 < \mathscr{M}_{\mathfrak{f}} < 1 \ni$,

$$\left|\mathscr{F}(\mathfrak{I}, \mathfrak{u}, \mathfrak{v}) - \mathscr{F}(\mathfrak{I}, \overline{\mathfrak{u}}, \overline{\mathfrak{v}})\right| \le \mathscr{L}_{\mathfrak{f}} \left|\mathfrak{u} - \overline{\mathfrak{u}}\right| + \mathscr{M}_{\mathfrak{f}} \left|\mathfrak{v} - \overline{\mathfrak{v}}\right|$$

for $\mathfrak{I} \in \mathcal{J}$, $\mathfrak{u}, \overline{\mathfrak{u}}, \mathfrak{v}, \overline{\mathfrak{v}} \in \mathbb{R}$.

($\mathtt{A}_2$) $\exists\, \mathfrak{a}_{\mathfrak{f}}, \mathfrak{b}_{\mathfrak{f}}, \mathfrak{c}_{\mathfrak{f}} \in \mathscr{C}(\mathcal{J}, \mathbb{R})$, where

$$\mathfrak{c}_{\mathfrak{f}}^* = \sup_{\mathfrak{I} \in \mathcal{J}} \mathfrak{c}_{\mathfrak{f}}(\mathfrak{I}) < 1 \ni,$$

$$\left|\mathscr{F}(\mathfrak{I}, \mathfrak{u}, \mathfrak{v})\right| \le \mathfrak{a}_{\mathfrak{f}} + \mathfrak{b}_{\mathfrak{f}}(\mathfrak{I}) \left|\mathfrak{u}\right| + \mathfrak{c}_{\mathfrak{f}}(\mathfrak{I}) \left|\mathfrak{v}\right|,$$

for $\mathfrak{I} \in \mathcal{J}$, $\mathfrak{u}, \mathfrak{v} \in \mathfrak{r}$.

($\mathtt{A}_3$) $\exists$ a constant $K > 0 \ni$

$$|f(t, x, y)| \le K, \quad \text{for a.e } t \in J.$$

Theorem 3.1. *Assume assumptions ($\mathtt{A}_1$) holds. Then we have the following inequality:*

$$\begin{aligned}
\frac{\mathscr{L}_{\mathfrak{f}}}{1 - \mathscr{M}_{\mathfrak{f}}} \Bigg[\frac{(\log \mathscr{T})^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{\mathfrak{I}}{|\mathfrak{B}|} \Bigg(\frac{(\log \xi_1)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \\
+ \frac{(\log \xi_2)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} - \frac{\epsilon (\log \eta)^{\alpha_1 + 1}}{\Gamma(\alpha_1 + 2)} \Bigg) \Bigg] < 1.
\end{aligned} \tag{3.1}$$

Then Eqs (1.1)–(1.3) have a unique solution on $[0, 1]$.

Proof. The operator

$$\mathscr{M} : \mathscr{PC}(\mathcal{J}, \mathbb{R}) \to \mathscr{PC}(\mathcal{J}, \mathbb{R})$$

is defined as

$$\begin{aligned}
&\mathscr{M}(\wp(\mathfrak{I})) \\
&= \sum_{i=1}^{m} \mathscr{Y}_i + \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1 - 1} \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1} \wp(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&+ \frac{\mathfrak{I}}{\mathfrak{B}} \Bigg\{ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_1} \left(\log \frac{\xi_1}{\mathfrak{s}}\right)^{\alpha_1 - 2} \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1} \wp(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&+ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_2} \left(\log \frac{\xi_2}{s}\right)^{\alpha_1 - 2} \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1} \wp(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&- \epsilon \int_0^{\eta} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{u}}\right)^{\alpha_1 - 1} \mathscr{F}(\mathtt{u}, \wp(\mathfrak{u}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}(\mathfrak{u})) \frac{\mathfrak{d}\mathfrak{u}}{\mathfrak{u}} \right) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \Bigg\}.
\end{aligned} \tag{3.2}$$

Clearly, the fixed points of operator $\mathscr{M}$ are solutions of problems (1.1)–(1.3). For any $\wp, \mathscr{Y} \in \mathscr{PC}(\mathcal{J}, \mathbb{R}$ and $\mathfrak{I} \in \mathcal{J}$, we have

$$\begin{aligned}
&\left|\mathscr{M}(\wp(\mathfrak{I})) - \mathscr{M}(\mathscr{Y}(\mathfrak{I}))\right| \\
&\le \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1 - 1} \Big| \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1} \wp(\mathtt{s})) \\
&- \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1} \wp(\mathfrak{s})) \Big| \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&+ \frac{\mathfrak{I}}{\mathfrak{B}} \Bigg\{ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_1} \left(\log \frac{\xi_1}{\mathfrak{s}}\right)^{\alpha_1 - 2} \Big| \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1} \wp(\mathfrak{s})) \\
&- \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1} \wp(\mathfrak{s})) \Big| \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&+ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_2} \left(\log \frac{\xi_2}{\mathfrak{s}}\right)^{\alpha_1 - 2} \Big| \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1} \wp(\mathfrak{s}))
\end{aligned}$$

$$- \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{CH}\mathcal{D}^{\alpha_1}(\mathfrak{s}))\Big|\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$- \epsilon \int_0^{\eta} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{u}} \right)^{\alpha_1 - 1} \Big| \mathscr{F}(\mathfrak{u}, \wp(\mathfrak{u}), {}^{CH}\mathcal{D}^{\alpha_1}\wp(\mathfrak{u})) \right.$$
$$\left. - \mathscr{F}(\mathfrak{u}, \wp(\mathfrak{u}), {}^{CH}\mathcal{D}^{\alpha_1}\wp(\mathfrak{u}))\Big| \frac{\mathfrak{d}\mathfrak{u}}{\mathfrak{u}} \right) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \Bigg\},$$

where
$$\mathfrak{g}(\mathfrak{I}) = \mathscr{F}(\mathfrak{I}, \wp(\mathfrak{I}), \mathfrak{g}(\mathfrak{I}))$$
and
$$\mathfrak{h}(\mathfrak{I}) = \mathscr{F}((\mathfrak{I})\mathscr{Y}(\mathfrak{I}), \mathfrak{h}(\mathfrak{I})).$$
From (A_1),
$$\Big|\mathfrak{g}(\mathfrak{I}) - \mathfrak{h}(\mathfrak{I})\Big| = \Big|\mathscr{F}(\mathfrak{I}, \wp(\mathfrak{I}), \mathfrak{g}(\mathfrak{I})) - \mathscr{F}(\mathfrak{I}, \mathscr{Y}(\mathfrak{I}), \mathfrak{h}(\mathfrak{I}))\Big|$$
$$\leq \mathscr{L}_{\mathfrak{f}} \Big|\wp(\mathfrak{I}) - \mathscr{Y}(\mathfrak{I})\Big| + \mathscr{M}_{\mathfrak{f}} \Big|\mathfrak{g}(\mathfrak{I}) - \mathfrak{h}(\mathfrak{I}))\Big|,$$
and
$$\Big|\mathfrak{g}(\mathfrak{I}) - \mathfrak{h}(\mathfrak{I})\Big| \leq \frac{\mathscr{L}_{\mathfrak{f}}}{1 - \mathscr{M}_{\mathfrak{f}}} \Big|\wp(\mathfrak{I}) - \mathscr{Y}(\mathfrak{I})\Big|.$$
We have
$$\Big|\mathscr{M}(\mathfrak{x})(\mathfrak{I}) - \mathscr{M}(\mathfrak{x})(\mathfrak{I})\Big|$$
$$\leq \frac{\mathscr{L}_{\mathfrak{f}}}{1 - \mathscr{M}_{\mathfrak{f}}} \left[\frac{(\log \mathscr{T})^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{\mathfrak{I}}{|\mathfrak{B}|} \left(\frac{(\log \xi_1)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \right.\right.$$
$$\left.\left. + \frac{(\log \xi_2)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} - \frac{\epsilon (\log \eta)^{\alpha_1 + 1}}{\Gamma(\alpha_1 + 2)} \right) \right] |\mathfrak{x} - \mathfrak{y}|.$$

The above equation is less than one, and therefore $\mathscr{M}$ is a contraction. The problem is stated in (1.1)–(1.3) has a unique solution on $[0, 1]$. □

Theorem 3.2. *Suppose that condition* $\mathtt{A}_2$ *holds. Then, problems (1.1)–(1.3) have at least one solution on* $[0, 1]$.

Proof. Consider
$$\mathtt{B}_{\mathfrak{r}} = \{\wp \in \mathscr{PC}(\mathcal{J}, \mathbb{R}) : |\wp| \leq \mathfrak{r}\}.$$
Let $\mathscr{A}$ and $\mathscr{B}$ be the two operators explained on $\mathtt{B}_{\mathfrak{r}}$ by
$$(\mathscr{A}\wp)(\mathfrak{I}) := \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}} \right)^{\alpha_1 - 1} \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), {}^{CH}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
and
$$(\mathscr{B}\wp)(\mathfrak{I}) := \frac{\mathfrak{I}}{\mathfrak{B}} \Bigg\{ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_1} \left(\log \frac{\xi_1}{\mathfrak{s}} \right)^{\alpha_1 - 2} \mathscr{F}(\mathfrak{s}, \mathscr{Y}(\mathfrak{s}), {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$+ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_2} \left(\log \frac{\xi_2}{\mathfrak{s}} \right)^{\alpha_1 - 2} \mathscr{F}(\mathfrak{s}, \mathscr{Y}(\mathfrak{s}), {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$- \epsilon \int_0^{\eta} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{u}} \right)^{\alpha_1 - 1} \mathscr{F}(\mathfrak{u}, \mathscr{Y}(\mathfrak{u}), {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{u})) \frac{\mathfrak{d}\mathfrak{u}}{\mathfrak{u}} \right) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \Bigg\}$$
$$+ \sum_{i=1}^{m} \mathscr{Y}_i.$$

Note that $\mathfrak{x}, \mathfrak{y} \in \mathtt{B}_{\mathfrak{r}}$. Then,
$$\mathscr{A}\wp + \mathscr{B}\mathscr{Y} \in \mathtt{B}_{\mathfrak{r}},$$
and checking the inequality in the above equation,
$$|\mathscr{A}\wp + \mathscr{B}\mathscr{Y}| \leq |\frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}} \right)^{\alpha_1 - 1} \mathscr{F}(\mathfrak{s}, \mathscr{Y}(\mathfrak{s}), {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$+ \frac{\mathfrak{I}}{\mathfrak{B}} \Bigg\{ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_1} \left(\log \frac{\xi_1}{\mathfrak{s}} \right)^{\alpha_1 - 2} \mathscr{F}(\mathfrak{s}, \mathscr{Y}(\mathfrak{s}), {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$+ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_2} \left(\log \frac{\xi_2}{s} \right)^{\alpha_1 - 2} \mathscr{F}(\mathfrak{s}, \mathscr{Y}(\mathfrak{s}), {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s})) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$- \epsilon \int_0^{\eta} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{u}} \right)^{\alpha_1 - 1} \mathscr{F}(\mathfrak{u}, \mathscr{Y}(\mathfrak{u}), {}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{u})) \frac{\mathfrak{d}\mathfrak{u}}{\mathfrak{u}} \right) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \Bigg\}$$
$$+ \sum_{i=1}^{m} \mathscr{Y}_i|$$
$$\leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{I}} \left(\log \frac{\mathfrak{I}}{\mathfrak{s}} \right)^{\alpha_1 - 1} |\mathscr{F}(\mathfrak{s})| \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$+ \frac{\mathfrak{I}}{\mathfrak{B}} \Bigg\{ \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^{\xi_1} \left(\log \frac{\xi_1}{\mathfrak{r}} \right)^{\alpha_1 - 2} |\mathscr{F}(\mathfrak{s})| \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$+ \int_0^{\xi_2} \frac{1}{\Gamma(\alpha_1 - 1)} \left(\log \frac{\xi_2}{\mathfrak{s}} \right)^{\alpha_1 - 2} |\mathscr{F}(\mathfrak{s})| \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}$$
$$- \epsilon \int_0^{\eta} \left(\frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{u}} \right)^{\alpha_1 - 1} |\mathscr{F}(\mathfrak{u})| \frac{\mathfrak{d}\mathfrak{u}}{\mathfrak{u}} \right) \frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} + \sum_{i=1}^{m} \mathscr{Y}_i \Bigg\},$$

and by $(\mathtt{A}_2)$, we have
$$\Big|\mathfrak{g}(\mathfrak{I})\Big| = \Big|\mathscr{F}(\mathfrak{I}, \wp(\mathfrak{I}), \mathfrak{g}(\mathfrak{I}))\Big|$$
$$\leq \mathfrak{a}_{\mathfrak{f}(\mathfrak{I})} + \mathfrak{b}_{\wp(\mathfrak{I})} \Big|\wp(\mathfrak{I})\Big| + \mathfrak{c}_{\mathfrak{f}(\mathfrak{I})} \Big|\mathfrak{g}(\mathfrak{I})\Big|$$
$$\leq \mathfrak{a}^*_{\mathfrak{f}} + \mathfrak{b}^*_{\mathfrak{f}} \Big|\wp(\mathfrak{I})\Big| + \mathfrak{c}^*_{\mathfrak{f}} \Big|\mathfrak{g}(\mathfrak{I})\Big|,$$
and
$$\Big|\mathfrak{g}(\mathfrak{I})\Big| \leq \frac{\mathfrak{a}^*_{\mathfrak{f}} + \mathfrak{b}^*_{\mathfrak{f}} \Big|\wp(\mathfrak{I})\Big|}{(1 - \mathfrak{c}^*_{\mathfrak{f}})},$$
where
$$\mathfrak{a}^*_{\mathfrak{f}} = \sup_{\mathfrak{I} \in \mathcal{J}} \mathfrak{a}_{\mathfrak{f}(\mathfrak{I})}$$
and
$$\mathfrak{b}^*_{\mathfrak{f}} = \sup_{\mathtt{t} \in \mathcal{J}} \mathfrak{b}_{\mathfrak{f}(\mathfrak{I})},$$

$$
\begin{aligned}
|\mathscr{A}\wp+\mathscr{B}\mathscr{Y}| \leq & \frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{a}_{\mathfrak{f}}^*+\mathfrak{b}_{\mathfrak{f}}^*|\wp(\mathfrak{I})|}{(1-\mathfrak{c}_{\mathfrak{f}}^*)}\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
& +\frac{\mathfrak{I}}{\mathfrak{B}}\left\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{s}}\right)^{\alpha_1-2}\frac{\mathfrak{a}_{\mathfrak{f}}^*+\mathfrak{b}_{\mathfrak{f}}^*\|\wp(\mathfrak{I})\|}{(1-\mathfrak{c}_{\mathfrak{f}}^*)}\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}\right. \\
& +\int_0^{\xi_2}\frac{1}{\Gamma(\alpha_1-1}\left(\log\frac{\xi_2}{\mathfrak{r}}\right)^{\alpha_1-2}\frac{\mathfrak{a}_{\mathfrak{f}}^*+\mathfrak{b}_{\mathfrak{f}}^*|\wp(\mathfrak{I})|}{(1-\mathfrak{c}_{\mathfrak{f}}^*)}\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
& -\epsilon\int_0^{\eta}\left(\int_0^{\mathfrak{s}}\frac{1}{\Gamma(\alpha_1)}\left(\log\frac{\mathfrak{s}}{\mathfrak{u}}\right)^{\alpha_1-1}\frac{\mathfrak{a}_{\mathfrak{f}}^*+\mathfrak{b}_{\mathfrak{f}}^*|\wp(\mathfrak{I})|}{(1-\mathfrak{c}_{\mathfrak{f}}^*)}\frac{\mathfrak{d}\mathfrak{u}}{\mathfrak{u}}\right)\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
& \left.+\sum_{i=1}^{m}\mathscr{Y}_i\right\} \\
\leq & \left[\frac{1}{\Gamma(\alpha_1+1)}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1}+\frac{\mathfrak{I}}{|\mathfrak{B}|}\left[\frac{1}{\Gamma(\alpha_1)}\left(\log\frac{\xi_1}{\mathfrak{s}}\right)^{\alpha_1-1}\right.\right. \\
& +\frac{1}{\Gamma(\alpha_1)}\left(\log\frac{\xi_2}{\mathfrak{s}}\right)^{\alpha_1-1} \\
& \left.\left.-\frac{\epsilon}{\Gamma(\alpha_1+2)}\left(\log\frac{\eta}{\mathfrak{u}}\right)^{\alpha_1+1}+\sum_{i=1}^{m}\mathscr{Y}_i\right]\right]\frac{\mathfrak{a}_{\mathfrak{f}}^*+\mathfrak{b}_{\mathfrak{f}}^*|\wp(\mathfrak{I})|}{(1-\mathfrak{c}_{\mathfrak{f}}^*)} \\
\leq & \mathfrak{r}.
\end{aligned}
$$

Thus,

$$\mathfrak{A}\wp+\mathfrak{B}\mathscr{Y}\in \mathtt{B}_{\mathfrak{r}}.$$

It is also clear that $\mathfrak{B}$ is a contraction mapping. From continuity of $\wp$, the operator $(\mathfrak{A}\wp)(\mathfrak{I})$ is continuous in accordance with F. Also we observe that

$$
\begin{aligned}
\left|(\mathfrak{A}\wp)(\mathfrak{I})\right| &\leq \frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}(\log\frac{\mathfrak{I}}{\mathfrak{s}})^{\alpha_1-1}\mathscr{F}(\mathtt{s})\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}. \\
&\leq \frac{1}{\Gamma(\alpha_1+1)}\frac{\mathfrak{a}_{\mathfrak{f}}^*+\mathfrak{b}_{\mathfrak{f}}^*|\wp(\mathfrak{I})|}{(1-\mathfrak{c}_{\mathfrak{f}}^*)}.
\end{aligned}
$$

Hence, $\mathfrak{A}$ is uniformly bounded on $\mathtt{B}_{\mathfrak{r}}$.

Now, we show that $(\mathfrak{A}\wp)(\mathfrak{I})$ is equicontinuous, and $\mathfrak{I}_1,\mathfrak{I}_2\in\mathscr{J}$, $\mathfrak{I}_2\leq\mathfrak{I}_1$, and $\wp\in\mathtt{B}_{\mathfrak{r}}$. The compact set $\mathscr{F}$ is bounded,

$$\sup_{(\mathfrak{I},x,\mathfrak{y})\in\mathscr{J}\times\mathtt{B}_{\mathfrak{r}}}|\mathscr{F}(\mathfrak{s})|:=\mathsf{C}_0<\infty,$$

and we get

$$
\begin{aligned}
&\left|(\mathfrak{A}\wp)(\mathfrak{I}_2)-(\mathfrak{A}\wp)(\mathfrak{I}_1)\right| \\
&=\left|\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathfrak{I}_1}\left(\log\frac{\mathfrak{I}_1}{\mathfrak{s}}\right)^{\alpha_1-1}\mathscr{F}(\mathfrak{s})\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}\right. \\
&\quad\left.-\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathfrak{I}_1}\left(\log\frac{\mathfrak{I}_2}{\mathfrak{s}}\right)^{\alpha_1-1}\mathscr{F}(\mathfrak{s})\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}\right| \\
&\leq\frac{1}{\Gamma(\alpha_1)}\left|\int_1^{\mathfrak{I}_1}\left[\left(\log\frac{\mathfrak{I}_1}{\mathfrak{s}}\right)^{\alpha_1-1}-\left(\log\frac{\mathfrak{I}_2}{\mathfrak{s}}\right)^{\alpha_1-1}\right]\mathscr{F}(\mathfrak{s})\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}\right| \\
&\quad+\int_{\mathfrak{I}_1}^{\mathfrak{I}_2}\left(\log\frac{\mathfrak{I}_2}{\mathfrak{s}}\right)^{\alpha_1-1}\mathscr{F}(\mathfrak{s})\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&\leq\frac{\mathscr{C}_0}{\Gamma(\alpha_1)}\left(\int_1^{\mathfrak{I}_1}\left|\left(\log\frac{\mathfrak{I}_1}{\mathfrak{s}}\right)^{\alpha_1-1}-\left(\log\frac{\mathfrak{I}_2}{\mathfrak{s}}\right)^{\alpha_1-1}\right|\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}\right. \\
&\quad\left.+\int_{\mathfrak{I}_1}^{\mathfrak{I}_2}\left(\log\frac{\mathfrak{I}_2}{\mathfrak{s}}\right)^{\alpha_1-1}\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}\right)\longrightarrow 0 \text{ as } \mathfrak{I}_2\longrightarrow\mathfrak{I}_1.
\end{aligned}
$$

Thus, $\mathfrak{A}(\mathtt{B}_{\mathfrak{r}})$ is relatively compact, and using the Ascoli-Arzela theorem, $\mathfrak{A}$ is compact.

Problems (1.1)–(1.3) have at least one fixed point on $\mathscr{J}$. □

Theorem 3.3. *Suppose that condition* $\mathtt{A}_3$ *holds. Then, problems (1.1)–(1.3) have at least one solution on* $[0,1]$.

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathscr{M}$ defined by (10) has a fixed point. The proof will be given in several steps.

Step 1. $\mathfrak{M}$ is continuous. Let $\wp_n$ be a sequence $\ni\wp_n\to\wp$ in $\mathfrak{P}\mathfrak{C}(J,\mathbb{R})$. Then, for each $\mathfrak{I}\in J$,

$$
\begin{aligned}
&\left\|\left(\mathfrak{M}\wp_n\right)(\mathfrak{I})-(\mathfrak{M}\wp)(\mathfrak{I})\right\| \\
&\leq\frac{1}{\Gamma(\alpha_1)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha_1-1}\left\|\mathscr{F}(\mathfrak{s},\mathscr{Y}_n(\mathfrak{s}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}_n(\mathfrak{s}))\right. \\
&\quad\left.-\mathscr{F}(\mathfrak{s},\mathscr{Y}(\mathfrak{s}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s}))\right\|\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&\quad\frac{\mathfrak{I}}{\mathfrak{B}}\left\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{s}}\right)^{\alpha_1-2}\left\|\mathscr{F}(\mathfrak{s},\mathscr{Y}_n(\mathfrak{s}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}_n(\mathfrak{s}))\right.\right. \\
&\quad\left.-\mathscr{F}(\mathfrak{s},\mathscr{Y}(\mathfrak{s}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s}))\right\|\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&\quad+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}\left(\log\frac{\xi_2}{\mathfrak{s}}\right)^{\alpha_1-2}\left\|\mathscr{F}(\mathfrak{s},\mathscr{Y}_n(\mathfrak{s}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}_n(\mathfrak{s}))\right. \\
&\quad\left.-\mathscr{F}(\mathfrak{s},\mathscr{Y}(\mathfrak{s}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s}))\right\|\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}} \\
&\quad-\epsilon\int_0^{\eta}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{u}}\right)^{\alpha_1-1}\left|\mathscr{F}(\mathfrak{u},\mathscr{Y}(\mathfrak{u}_{\mathfrak{n}}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}_n(\mathfrak{u}))\right.\right. \\
&\quad\left.\left.\left.-\mathscr{F}(\mathfrak{u},\mathscr{Y}(\mathfrak{u}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{u}))\right|\frac{\mathfrak{d}\mathfrak{u}}{\mathfrak{u}}\right)\frac{\mathfrak{d}\mathfrak{s}}{\mathfrak{s}}\right\} \\
&\quad+\sum_{i=1}^{m}\mathscr{Y}_i\times\left\|\mathscr{F}(\mathfrak{s},\mathscr{Y}_n(\mathfrak{s}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}_n(\mathfrak{s}))\right. \\
&\quad\left.-\mathscr{F}(\mathfrak{s},\mathscr{Y}(\mathfrak{s}),{}^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{s}))\right\|.
\end{aligned}
$$

Since $\mathscr{M}$ is continuous, we have

$$\left\|\left(\mathscr{M}\wp_n\right)(\mathfrak{I})-(\mathscr{M}\wp)(\mathfrak{I})\right\|_{\infty}\to 0 \text{ as } n\to\infty.$$

Step 2. $\mathscr{M}$ maps bounded sets into bounded sets in $\mathscr{P}\mathscr{C}(J,\mathbb{R})$.

Now, for any $r > 0$, we take

$$u \in B_r = \{\wp \in \mathscr{PC}(J,\mathbb{R}), \|\wp\|_\infty \le r\}.$$

For $x \in B_r$ and for each $t \in [1, T]$, we have

$$\begin{aligned}
&\left|(\mathfrak{M}\wp)(\mathfrak{I})\right| \frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}} (\log\frac{\mathfrak{I}}{\mathfrak{s}})^{\alpha_1-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&+\frac{\mathfrak{I}}{\mathfrak{B}}\left\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}(\log\frac{\xi_1}{\mathfrak{s}})^{\alpha_1-2}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\right.\\
&+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}(\log\frac{\xi_2}{\mathfrak{s}})^{\alpha_1-2}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&\left.-\epsilon\int_0^{\eta}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{s}}(\log\frac{\mathfrak{s}}{\mathfrak{u}})^{\alpha_1-1}\mathscr{F}(\mathfrak{u},\wp(\mathfrak{u}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{u}))\frac{\mathfrak{du}}{\mathfrak{u}}\right)\frac{\mathfrak{ds}}{\mathfrak{s}}\right\}\\
&+\sum_{i=1}^{m}\mathscr{Y}_i\\
&\le K\left\{\frac{(\log\mathscr{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{\mathfrak{I}}{|\mathfrak{B}|}\left[\frac{(\log\xi_1)^{\alpha_1-1}}{\Gamma(\alpha_1)}+\frac{(\log\xi_2)^{\alpha_1-1}}{\Gamma(\alpha_1)}\right.\right.\\
&\left.\left.-\frac{\epsilon(\log\eta)^{\alpha_1+1}}{\Gamma(\alpha_1+2)}\right]+\sum_{i=1}^{m}\mathscr{Y}_i\right\}\\
&\le r.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(\mathscr{M}\wp)(\mathfrak{I})\| \le& K\left\{\frac{(\log\mathscr{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{\mathfrak{I}}{|\mathfrak{B}|}\left[\frac{(\log\xi_1)^{\alpha_1-1}}{\Gamma(\alpha_1)}\right.\right.\\
&\left.\left.+\frac{(\log\xi_2)^{\alpha_1-1}}{\Gamma(\alpha_1)}-\frac{\epsilon\,(\log\eta)^{\alpha_1+1}}{\Gamma(\alpha_1+2)}\right]+\sum_{i=1}^{m}\mathscr{Y}_i\right\}\\
:=& r.
\end{aligned}$$

Step 3. $\mathscr{M}$ maps bounded sets into equicontinuous sets of $\mathscr{PC}(J,\mathbb{R})$.

Let $\mathfrak{I}_1, \mathfrak{I}_2 \in J, \mathfrak{I}_1 < \mathfrak{I}_2, B_r$ be a bounded set of $\mathscr{PC}(J,\mathbb{R})$ as in Step 2, and let $\wp \in B_r$. Then,

$$\begin{aligned}
&\left\|\mathfrak{M}\wp(\mathfrak{I}_2)-\mathscr{M}\wp(\mathfrak{I}_1)\right\|\\
&\le\frac{1}{\Gamma(\alpha_1)}\int_1^{\mathfrak{I}_1}\left[\left(\log\frac{\mathfrak{I}_2}{s}\right)^{\alpha_1-1}-\left(\log\frac{\mathfrak{I}_1}{s}\right)^{\alpha_1-1}\right]\\
&\quad\times\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{\mathfrak{I}_1}^{\mathfrak{I}_2}\left(\log\frac{\mathfrak{I}_2}{s}\right)^{\mathfrak{I}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&\le\frac{K}{\Gamma(\alpha_1)}\int_1^{\mathfrak{I}_1}\left[\left(\log\frac{\mathfrak{I}_2}{s}\right)^{\alpha_1-1}-\left(\log\frac{\mathfrak{I}_1}{s}\right)^{\alpha_1-1}\right]\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&\quad+\frac{K}{\Gamma(\alpha_1)}\int_{\mathfrak{I}_1}^{\mathfrak{I}_2}\left(\log\frac{\mathfrak{I}_2}{s}\right)^{\alpha_1-1}\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&\le\frac{K}{\Gamma(\alpha_1+1)}\left[(\log\mathfrak{I}_2)^{\alpha_1}-(\log\mathfrak{I}_1)^{\alpha_1}\right],
\end{aligned}$$

which implies

$$\left\|\mathscr{M}\wp(\mathfrak{I}_2)-\mathscr{M}\wp(\mathfrak{I}_1)\right\|_\infty \to 0 \text{ as } \mathfrak{I}_1\to\mathfrak{I}_2,$$

and from Steps 1–3, we can conclude that $\mathscr{M}$ is continuous and completely continuous by the Arzela-Ascoli theorem.

Step 4. A priori bounds.

Now it remains to show that the set

$$\Lambda = \{\wp \in \mathscr{PC}(J,\mathbb{R}) : \wp = \rho\mathscr{M}(\wp) \text{ for some } 0<\rho<1\}$$

is bounded.

For such a $\wp \in \Lambda$, and for each $\mathfrak{I} \in J$, we have

$$\begin{aligned}
\wp(\mathfrak{I}) \le&\rho\left\{\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\right.\\
&+\frac{\mathfrak{I}}{\mathfrak{B}}\left\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{s}}\right)^{\alpha_1-2}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\right.\\
&+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}\left(\log\frac{\xi_2}{s}\right)^{\alpha_1-2}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&\left.-\epsilon\int_0^{\eta}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{u}}\right)^{\alpha_1-1}\mathscr{F}(\mathfrak{u},\wp(\mathfrak{u}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}(\mathfrak{u}))\frac{\mathfrak{du}}{\mathfrak{u}}\right)\frac{\mathfrak{ds}}{\mathfrak{s}}\right\}\\
&\left.+\sum_{i=1}^{m}\mathscr{Y}_i\right\}.
\end{aligned}$$

For $\rho \in [0, 1]$, let $\wp$ be $\ni$ for each $\mathfrak{I} \in J$

$$\begin{aligned}
&\|\mathscr{M}\wp(\mathfrak{I})\|\\
&\le\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{I}}\left(\log\frac{\mathfrak{I}}{\mathfrak{s}}\right)^{\alpha_1-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&\quad+\frac{\mathfrak{I}}{\mathfrak{B}}\left\{\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_1}\left(\log\frac{\xi_1}{\mathfrak{s}}\right)^{\alpha_1-2}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\right.\\
&\quad+\frac{1}{\Gamma(\alpha_1-1)}\int_0^{\xi_2}\left(\log\frac{\xi_2}{s}\right)^{\alpha_1-2}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}\wp(\mathfrak{s}))\frac{\mathfrak{ds}}{\mathfrak{s}}\\
&\quad\left.-\epsilon\int_0^{\eta}\left(\frac{1}{\Gamma(\alpha_1)}\int_0^{\mathfrak{s}}\left(\log\frac{\mathfrak{s}}{\mathfrak{u}}\right)^{\alpha_1-1}\mathscr{F}(\mathfrak{u},\wp(\mathfrak{u}), {}^{\mathcal{CH}}\mathcal{D}^{\alpha_1}(\mathfrak{u}))\frac{\mathfrak{du}}{\mathfrak{u}}\right)\frac{\mathfrak{ds}}{\mathfrak{s}}\right\}\\
&\quad+\sum_{i=1}^{m}\mathscr{Y}_i,\\
&\le K\left\{\frac{(\log\mathscr{T})^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{\mathfrak{I}}{|\mathfrak{B}|}\left[\frac{(\log\xi_1)^{\alpha_1-1}}{\Gamma(\alpha_1)}+\frac{(\log\xi_2)^{\alpha_1-1}}{\Gamma(\alpha_1)}\right.\right.\\
&\quad\left.\left.-\frac{\epsilon\,(\log\eta)^{\alpha_1+1}}{\Gamma(\alpha_1+2)}\right]\right\}+\sum_{i=1}^{m}\mathscr{Y}_i.
\end{aligned}$$

Thus,

$$\|\mathscr{M}\wp(\mathfrak{I})\| \le \infty.$$

Thus, we deduce that $\mathscr{M}$ has a fixed point which is a solution on J of problems (1.1)–(1.3). □

4. Example

Let us consider the C-H implicit impulsive fractional BVP

$$^{CH}\mathcal{D}^{\frac{10}{7}}\mathscr{Y}(\mathfrak{I}) = \frac{1}{15}\left(\frac{1}{1+\left|\mathscr{Y}(\mathfrak{I})\right|} + \left|^{CH}\mathcal{D}^{\frac{10}{7}}\mathscr{Y}(\mathfrak{I})\right|\right), \mathfrak{I} \in [0,1]. \tag{4.1}$$

$$\mathscr{Y}(\mathfrak{I}_{\mathrm{k}}^{+}) = \mathscr{Y}(\mathfrak{I}_{\mathrm{k}}^{-}) + \mathscr{Y}_{\mathrm{k}} + \frac{1}{6}, \tag{4.2}$$

$$\mathscr{Y}(0) = 0, \ \ 3\mathscr{Y}'(\frac{1}{2}) + 4\mathscr{Y}'(\frac{1}{5}) = 2\int_0^{\frac{1}{4}} \mathscr{Y}(\mathfrak{s})\mathfrak{ds}. \tag{4.3}$$

Here,

$$\alpha_1 = \frac{10}{7}, \ \mathfrak{a} = 4, \ \mathfrak{b} = 3, \ \xi_1 = \frac{1}{2}, \ \xi_2 = \frac{1}{5}, \ \epsilon = 2, \ \eta = \frac{1}{4}$$

and

$$\mathscr{F}(\mathfrak{t}, \mathscr{Y}(\mathfrak{I}), ^{CH}\mathcal{D}^{\alpha_1}\mathscr{Y}(\mathfrak{I})) = \frac{1}{15}\left(\frac{1}{1+\left|\mathscr{Y}(\mathfrak{I})\right|} + \left|^{CH}\mathcal{D}^{\frac{10}{7}}\mathscr{Y}(\mathfrak{I})\right|\right)$$

for $\mathfrak{u}, \mathfrak{v}, \overline{\mathfrak{u}}, \overline{\mathfrak{v}} \in \mathbb{R}$.

$$\left|\mathscr{F}(\mathfrak{I}, \mathfrak{u}, \mathfrak{v}) - \mathscr{F}(\mathfrak{I}, \overline{\mathfrak{u}}, \overline{\mathfrak{v}})\right| \leq \frac{1}{15}\left|\mathfrak{u} - \overline{\mathfrak{u}}\right| + \frac{1}{15}\left|\mathfrak{v} - \overline{\mathfrak{v}}\right|,$$

$$\frac{\mathscr{L}_{\mathfrak{f}}}{1-\mathscr{M}_{\mathfrak{f}}}\left[\frac{(\log \mathscr{T})^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{\mathfrak{I}}{|\mathfrak{B}|}\left[\frac{(\log \xi_1)^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{(\log \xi_2)^{\alpha_1-1}}{\Gamma(\alpha_1)} - \frac{\epsilon(\log \eta)^{\alpha_1+1}}{\Gamma(\alpha_1+2)}\right]\right] \approx 0.00367 < 1.$$

Thus, the conditions are satisfied. Then, the above problems (4.1)–(4.3) have a unique solution on $[0, 1]$.

5. Conclusions

We examine a novel integral BVP issue for the fractional derivative of C-H in this research. To look into the existence and uniqueness, fixed point theorems are used. The findings of this work not only extend the scope of earlier findings, but also offer a completely new method in that various fractional derivatives are taken into account, distinct BCs and derivatives are linked, various fixed point theorems are applied, and the C-H fractional derivative is examined. Future directions for this study include using numerical approaches and utilizing various fractional derivative types to solve integral boundary value issues for FDEs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

References

1. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science, 2006. https://doi.org/10.1016/s0304-0208(06)x8001-5

2. I. Podlubny, *Fractional differential equations*, Academic Press, 1999.

3. A. A. Kilbas, Hadamard type fractional calculus, *J. Korean Math. Soc.*, **38** (2001), 1191–1204. https://doi.org/10.4134/JKMS.2001.38.6.1191

4. W. Benhamida, J. R. Graef, S. Hamani, Boundary value problems for Hadamard fractional differential equations with nonlocal multi-point boundary conditions, *Fractional Differ. Calculus*, **8** (2018), 165–176. https://doi.org/10.7153/fdc-2018-08-10

5. A. Boutiara, K. Guerbati, M. Benbachir, Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces, *AIMS Math.*, **5** (2020), 259–272. https://doi.org/10.3934/math.2020017

6. P. Karthikeyan, K. Venkatachalam, Nonlocal multipoint boundary value problems for Caputo-Hadamard fractional integro-differential equations, *Appl. Math.*, **2** (2020), 23–35.

7. R. P. Agarwal, A. Alsaedi, A. Alsharrif, B. Ahmed, On nonlinear fractional order boundary value problems with nonlocal multi-point conditions involving Liouville-Caputo derivatives, *Differ. Equations Appl.*, **9** (2017), 147–160. https://doi.org/10.7153/dea-09-12

8. C. Derbazi, H. Hammouche, Caputo-Hadamard fractional differential equations with nonlocal fractional integro-differential boundary conditions via topological degree theory, *AIMS Math.*, **5** (2020), 2694–2709. https://doi.org/10.3934/math.2020174

9. M. S. Kumar, M. Deepa, J. Kavitha, V. Sadhasivam, Existence theory of fractional order three-dimensional differential system at resonance, *Math. Modell. Control*, **3** (2023), 127–138. https://doi.org/10.3934/mmc.2023012

10. C. V. Bose, R. Udhayakumar, A. M. Elshenhab, M. S. Kumar, J. S. Ro, Discussion on the approximate controllability of Hilfer fractional neutral integro-differential inclusions via almost sectorial operators, *Fractal Fract.*, **6** (2022), 607. https://doi.org/10.3390/fractalfract6100607

11. T. Gayathri, M. S. Kumar, V. Sadhasivam, Hille and Nehari type oscillation criteria for conformable fractional differential equations, *Iraqi J. Sci.*, **62** (2021), 578–587. https://doi.org/10.24996/ijs.2021.62.2.23

12. J. Liu, W. Wei, J. Wang, W. Xu, Limit behavior of the solution of Caputo-Hadamard fractional stochastic differential equations, *Appl. Math. Lett.*, **140** (2023), 108586. https://doi.org/10.1016/j.aml.2023.108586

13. J. Liu, W. Wei, W. Xu, An averaging principle for stochastic fractional differential equations driven by fBm involving impulses, *Fractal Fract.*, **6** (2022), 256. https://doi.org/10.3390/fractalfract6050256

14. O. Kahouli, A. B. Makhlouf, L. Mchiri, H. Rguigui, Hyers-Ulam stability for a class of Hadamard fractional Itô-Doob stochastic integral equations, *Chaos Solitons Fract.*, **166** (2022), 112918. https://doi.org/10.1016/j.chaos.2022.112918

15. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of impulsive differential equations*, Worlds Scientific, 1989.

16. A. M. Samoilenko, N. A. Perestyuk, *Impulsive differential equations*, World Scientific, 1995. https://doi.org/10.1142/9789812798664

17. M. Benchohra, S. Bouriah, J. R. Graef, Boundary value problems for nonlinear implicit Caputo-Hadamard type fractional differential equations with impulses, *Medit. J. Math.*, **14** (2017), 206. https://doi.org/10.1007/s00009-017-1012-9

18. M. Benchohra, J. E. Lazreg, Existence results for nonlinear implicit fractional differential equations with impulse, *Commun. Appl. Anal.*, **19** (2015), 413–426. https://doi.org/10.3934/caa.2015.19.413

19. J. Hadamard, Essai sur létude des fonctions donnees par Leur developpement de Taylor, *J. Math. Pure Appl.*, **8** (1892), 101–186. https://eudml.org/doc/233965

20. P. L. Butzer, A. A. Kilbas, J. J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.*, **270** (2002), 1–15. https://doi.org/10.1016/S0022-247X(02)00066-5

21. P. L. Butzer, A. A. Kilbas, J. J. Trujillo, Fractional calculus in the mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.*, **269** (2002), 1–27. https://doi.org/10.1016/S0022-247X(02)00001-X

22. P. L. Butzer, A. A. Kilbas, J. J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.*, **269** (2002), 387–400. https://doi.org/10.1016/S0022-247X(02)00049-5

23. I. Mahariq, M. Kuzuoğlu, I. H. Tarman, H. Kurt, Photonic nanojet analysis by spectral element method, *IEEE Pho. J.*, **6** (2014), 6802714. https://doi.org/10.1109/JPHOT.2014.2361615

24. I. Mahariq, A. Erciyas, A spectral element method for the solution of magnetostatic fields, *Turk. J. Electr. Eng. Comput. Sci.*, **24** (2017), 2922–2932. https://doi.org/10.3906/elk-1605-6

25. I. Mahario, On the application of the spectral element method in electromagnetic problems involving domain decomposition, *Turk. J. Electr. Eng. Comput. Sci.*, **25** (2017), 1059–1069. https://doi.org/10.3906/ELK-1511-115

26. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equations*, **2012** (2012), 142. https://doi.org/10.1186/1687-1847-2012-142

27. M. Benchohra, J. Henderson, S. K. Ntouyas, *Impulsive differential equations and inclusions*, Hindawi Publishing Corporation, 2006. https://doi.org/10.1155/9789775945501

28. B. Ahmad, G. Wang, A study of an impulsive four-point nonlocal boundary value problem of nonlinear fractional differential equations, *Comput. Math. Appl.*, **62** (2011), 1341–1349. https://doi.org/10.1016/j.camwa.2011.04.033

29. A. Al-Omari, H. Al-Saadi, (ω, ρ)-BVP solution of impulsive Hadamard fractional differential equations, *Mathematics*, **11** (2023), 4370. https://doi.org/10.3390/math11204370

30. N. I. Mahmudov, M. Awadalla, K. Abuassba, Hadamard and Caputo-Hadamard FDE's with three point integral boundary conditions, *Nonlinear Anal. Differ. Equations*, **5** (2017), 271–282. https://doi.org/10.12988/nade.2017.7916

31. Y. Arioua, N. Benhamidouche, Boundary value problem for Caputo-Hadamard fractional differential equations, *Sur. Math. Appl.*, **12** (2017), 103–115.

32. W. Benhamida, S. Hamani, J. Henderson, Boundary value problems for Caputo-Hadamard fractional differential equations, *Adv. Theory Nonlinear Anal. Appl.*, **2** (2018), 138–145.

33. A. Ardjounia, A. Djoudi, Existence and uniqueness of solutions for nonlinear implicit Caputo-Hadamard fractional differential equations with nonlocal conditions, *Adv. Theory Nonlinear Anal. Appl.*, **3** (2019), 46–52. https://doi.org/10.31197/atnaa.501118

34. A. Irguedi, K. Nisse, S. Hamani, Functional impulsive fractional differential equations involving the Caputo-Hadamard derivative and integral boundary conditions, *Int. J. Anal. Appl.*, **21** (2023), 15. https://doi.org/10.28924/2291-8639-21-2023-15

35. J. R. Graef, S. Hamani, Existence results for impulsive caputo-hadamard fractional differential equations with integral boundary conditions, *Dyn. Syst. Appl.*, **32** (2023), 1–16. https://doi.org/10.46719/dsa2023.32.01

36. S. Akhter, Existence of weak solutions for Caputo's fractional derivatives in Banach spaces, *GANIT: J. Bang. Math. Soc.*, **39** (2019), 111–118. https://doi.org/10.3329/ganit.v39i0.44166