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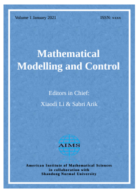

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Research article

On numerical/non-numerical algebra: Semi-tensor product method

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Abstract: A kind of algebra, called numerical algebra, is proposed and investigated. As its opponent, non-numerical algebra is also defined. The numeralization and dis-numeralization, which convert non-numerical algebra to numerical algebra and vise versa, are considered. Product structure matrix (PSM) of a finite dimensional algebra is constructed. Using PSM, some fundamental properties of finite dimensional algebras are obtained. Then a necessary and sufficient condition for a numerical algebra to be a field is presented. Finally, the invertibility of Segre (commutative) quaternion and some related properties of matrices over Segre quaternion are investigated.

Keywords: numerical (non-numerical) algebra; numeralization (dis-numeralization); semi-tensor product (STP) of matrices; commutative quaternion

1. Introduction

It is well known that the Galois extension of a field is of special importance in analyzing various fields, solving polynomial equations, etc. [16]. In addition to field extension, some other extensions of $\mathbb{R}$ to algebras on it are also useful. This paper considers the extension of a field $\mathbb{F}$ to a finite dimensional algebra on $\mathbb{F}$. Such extensions are very general, which contain Galois extension, noncommutative Galois extension [1, 21], commutative quaternions [17, 18], dual and hyperbolic numbers [19, 22], etc. as its special cases.

To make the problem clear, the algebra considered in this paper is defined as follows:

Definition 1.1. *[12, 14] An algebra, $\mathcal{A}$, is a finite dimensional vector space V over a pre-assigned field $\mathbb{F}$ with a bilinear operator, $* : V \times V \to V$, satisfying distributive rule:*

$$X * (aY + bZ) = a(X * Y) + b(X * Z)$$
$$(aX + bY) * Z = a(X * Z) + b(Y * Z),\ X, Y, Z \in V,\ a, b \in \mathbb{F}.$$

Starting from the idea of extension, it is obvious that we need to distinct two kinds of algebras over a given field $\mathbb{F}$, called numerical and non-numerical algebras, which are defined as follows:

Definition 1.2. *An algebra $\mathcal{A}$ over $\mathbb{F}$, denoted by $\mathcal{A} = (V, *)$, is called a numerical algebra, if $\mathbb{F}$ is one dimensional subspace of V. The set of numerical algebras is denoted by NA. Otherwise, it is called a non-numerical algebra. The set of non-numerical algebras is denoted by VA.*

Example 1.3. *(i) Quaternion, denoted by Q, is a numerical algebra, because $\mathbb{F} = \mathbb{R}$ is one dimensional subspace of Q.*

(ii) Cross product over $\mathbb{R}^3$, denoted by $C^r = (\mathbb{R}^3, \vec{\times})$, is a non-numerical algebra.

From another point of view: an algebra extension of $\mathbb{F}$ can also be considered as an extension of $(V, *) \in VA$ by adding $\mathbb{F}$ to it. From this perspective, this paper investigates the algebra extensions of a field by studying the relationship between VA and NA.

The following fundamental problems about an algebra over a given field are considered in this paper:

(i) How to convert a numerical algebra to a non-numerical algebra by removing the "number" dimensional subspace, and how to convert a non-numerical algebra to a numerical algebra by adding a "number" dimensional subspace.
(ii) Some properties of finite algebras, such as commutativity, associativity, invertibility, etc.
(iii) Check whether a numerical algebra is a field.
(iv) Find the set of non-invertible elements of a commutative quaternion Q^S, and investigate the solutions of linear systems over Q^S.

Finally, as an application we apply the above approach to investigate a commutative quaternion Q^S, which was firstly proposed by Segre [20] and received many applications recently [3, 18].

Recently, the semi-tensor product (STP) of matrices has been proposed and used to investigate some algebraic properties, such as cross-dimensional general linear algebra $gl(\mathbb{R}) = \cup_{i=1}^{\infty} gl(n, \mathbb{R})$ [8, 9], Boolean-like algebras [11], etc. The basic tool used in this paper is also STP. Using it, the product structure matrix (PSM) of a given algebra is constructed. The PSM, which completely determines the algebra, is the key issue in our investigation.

The rest of this paper is organized as follows: Section 2 reviews some necessary preliminaries, including (i) STP of matrices; (ii) Structure matrices of binary operators, particularly, PSM of finite dimensional algebras. In Section 3 the numeralization and dis-numeralization of algebras are proposed, algorithms are developed. Section 4 considers base transformation, which provides an essential numerically separable algebra. Some properties of an algebra are investigated via its PSM. In Section 6 we consider when a numerical algebra is a field. Necessary and sufficient conditions are obtained. In Section 7 a commutative quaternion, Q^S is investigated. The set of non-invertible elements is revealed, which is shown to be a zero-measure set. The solution of linear systems over Q^S is discussed. Section 8 gives some brief concluding remarks.

Before ending this section, a list of notations is presented as follows:

1. $\mathcal{A}$: an algebra; $\mathbb{F}$: a field; $\mathbb{R}(\mathbb{C}, \mathbb{Q})$: field of real (complex, rational) numbers.
2. $\mathbb{F}_{m\times n}$: the set of $m \times n$ matrices with all entries in $\mathbb{F}$.
3. $Col(M)$ ($Row(M)$): the set of columns (rows) of matrix M. $Col_i(M)$ ($Row_i(M)$) is the i-th column (row) of matrix M.
4. δ_n^i: the i-th column of identity matrix I_n.
5. $\uplus$: direct sum of vector spaces.

2. Preliminaries

2.1. STP of matrices

This subsection provides a brief survey on semi-tensor product (STP) of matrices. We refer to [6,7] for more details.

Definition 2.1. *[5, 6]: Let $M \in \mathbb{F}_{m\times n}$ and $N \in \mathbb{F}_{p\times q}$ where $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, and $t = \text{lcm}\{n, p\}$ be the least common multiple of n and p. The STP of M and N, denoted by $M \ltimes N$, is defined as*

$$(M \otimes I_{t/n})\left(N \otimes I_{t/p}\right) \in \mathbb{F}_{mt/n\times qt/p}, \tag{2.1}$$

where $\otimes$ is the Kronecker product.

Remark 2.2. *(i) When $n = p$, $M \ltimes N = MN$. That is, the semi-tensor product is a generalization of conventional matrix product. Moreover, it keeps all the properties of conventional matrix product available [6].*
(ii) Throughout this paper the matrix product is assumed to be the STP and the symbol $\ltimes$ is mostly omitted.
(iii) For statement ease, the field $\mathbb{F}$ in this paper is assumed to be of characteristic 0*. Particularly, the reader, who is not familiar with abstract algebra, may consider $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. In fact, the results in this paper are mostly applicable to Galois fields.*

We briefly review some basic properties of STP:

Proposition 2.3. *[5, 6]*
1. (Associative Law) $(F \ltimes G) \ltimes H = F \ltimes (G \ltimes H)$.

2. *(Distributive Law)*

$$F \ltimes (aG \pm bH) = aF \ltimes G \pm bF \ltimes H,$$

$$(aF \pm bG) \ltimes H = aF \ltimes H \pm bG \ltimes H, \quad a, b \in \mathbb{R}.$$

Define a swap matrix $W_{[m,n]} \in \mathcal{M}_{mn\times mn}$ as follows:

$$W_{[m,n]} := \left[I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \cdots, I_n \otimes \delta_m^m\right] \in \mathcal{M}_{mn\times mn}. \quad (2.2)$$

Proposition 2.4. *[5, 6] Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ be two column vectors. Then $W_{[m,n]} x \ltimes y = y \ltimes x$.*

Proposition 2.5. *[5, 6] Let $x \in \mathbb{F}^t$ be a column vector, and A be an arbitrary matrix over $\mathbb{F}$. Then $x \ltimes A = (I_t \otimes A) \ltimes x$.*

2.2. *PSM of a finite dimensional algebra*

Let $\mathcal{A} = (V, *)$. Assume a basis of V is $B = (e_1, e_2, \cdots, e_n)$. Then it is clear that the properties of $\mathcal{A}$ is completely determined by the binary operator $*$, which is briefly called "product". For a fixed basis B, a vector $X \in V$ can be expressed as a vector $X = (x_1, x_2, \cdots, x_n)^T$, which means $X = \sum_{i=1}^{n} x_i e_i$. We introduce a matrix, called the PSM, which is a complete description of the product $*$.

Proposition 2.6. *Let $\mathcal{A} = (V, *)$ be a finite dimensional algebra over $\mathbb{F}$ with a fixed basis $B = (e_1, e_2, \cdots, e_n)$. Then there exists a unique matrix $M_{\mathcal{A}} \in \mathbb{F}_{n\times n^2}$, called the PSM of $\mathcal{A}$, such that $X * Y = M_{\mathcal{A}} XY, \quad X, Y \in V$.*

Proof. Assume $e_\alpha * e_\beta = \sum_{k=1}^{n} \lambda_k^{\alpha\beta} e_k$, and set

$$j = (\alpha - 1)n + \beta, \quad 1 \le \alpha \le n, \ 1 \le \beta \le n. \quad (2.3)$$

Then we define

$$M_{\mathcal{A}}(i, j) := \lambda_i^{\alpha\beta}, \quad i = 1, 2, \cdots, n, \ j = 1, 2, \cdots, n^2. \quad (2.4)$$

It is easy to verify that for $1 \le j \le n^2$, there exists a unique pair (α,β), with $1 \le \alpha \le n$, $1 \le \beta \le n$, such that (2.3) holds. Hence, $M_{\mathcal{A}}$ is completely constructible by (2.4). Straightforward computation shows that $X * Y = M_{\mathcal{A}} XY$. □

Using PSM, we can transform the complex form of an operation into algebraic form. In addition, PSM can transform some qualitative properties into quantitative expressions, which will be mentioned in Section 5.

Example 2.7. *Consider the following algebras:*

(i) Quaternion, Q. Consider the classical (i.e., Hamiltonian) quaternion. Let $(1, i, j, k)$ be its classical basis. Then it is easy to calculate that its PSM is

$$M_Q = \begin{bmatrix} 1&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&-1 \\ 0&1&0&0&1&0&0&0&0&0&0&1&0&0&-1&0 \\ 0&0&1&0&0&0&0&-1&1&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&1&0&0&-1&0&0&1&0&0&0 \end{bmatrix}. \quad (2.5)$$

(ii) Cross product over $\mathbb{R}^3$ (denoted by C^r): Let (I, J, K) be its classical basis. Then its PSM is

$$M_{C^r} = \begin{bmatrix} 0&0&0&0&0&1&0&-1&0 \\ 0&0&-1&0&0&0&1&0&0 \\ 0&1&0&-1&0&0&0&0&0 \end{bmatrix} \quad (2.6)$$

3. Numeralization and dis-numeralization

Definition 3.1. *Assume $\mathcal{A} = (V, *)$ is a non-numerical algebra over $\mathbb{F}$ with a basis as $(e_1, e_2, \cdots, e_n)$. Then we can add a numerical dimension to it to make it a numerical algebra. Precisely speaking, we define a new operator $\oplus$ as*

$$X \oplus Y = \begin{cases} X * Y + c_{i,j}, & X = e_i, \ Y = e_j, \ c_{i,j} \in \mathbb{F}, \\ Y, & X = 1, \\ X, & Y = 1, \\ 1, & X = Y = 1. \end{cases} \quad (3.1)$$

Then $\mathcal{A}^a := (V \uplus \mathbb{F}, \oplus)$ is called a numeralization of $\mathcal{A}$.

If the $c_{i,j} = 0$, $\forall i, j$. Then the numeralization is called a numerically separable numeralization.

Remark 3.2. *(i) In the enlarged vector space, we assume $\sum_{i=1}^n 0e_i = 0$, that is, the zero element in V is identified with $0 \in \mathbb{F}$.*

(ii) By definition, in the enlarged vector space we also have $1 \oplus X = X, \quad X \in V$.

Example 3.3. *(i) Consider C^r, denote a basis of $V = \mathbb{R}^3$ as $B = \{e_1, e_3, e_3\}$. It is a non-numerical algebra. Now we add $\mathbb{R}$ to it and define a new operator as*

$$X \oplus Y = \begin{cases} X \vec{\times} Y + c_{i,j}, & X = e_i, \ Y = e_j, \\ Y, & X = 1, \\ X, & Y = 1, \\ 1, & X = Y = 1, \end{cases}$$

where $c_{i,j} = -1$, *if* $i = j$, *otherwise,* $c_{i,j} = 0$.
Then it is easy to verify that the numeralized algebra $(\mathbb{R}^3 \uplus \mathbb{R}, \oplus)$ *is exactly the quaternion* Q.

(ii) *Consider special linear algebra* $sl(2, \mathbb{R})$, *which is the set of* 2×2 *real matrices with their trace equal to* 0 *[2]. Choose a basis as*

$$e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad e_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}; \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Its PSM is

$$M_{sl(2,\mathbb{R})} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}.$$

Now we add $\mathbb{R}$ *to it and define*

$$X \oplus Y = \begin{cases} [X, Y], & X, Y \in sl(2, \mathbb{R}), \\ rY, & X = r \in \mathbb{R}, \\ rX, & Y = r \in \mathbb{R}, \\ rs, & X = r \in \mathbb{R}, Y = s \in \mathbb{R}. \end{cases}$$

Then we have the numericalized algebra $(sl(2, \mathbb{R}) \uplus \mathbb{R}, \oplus)$. *We calculate its PSM under the basis of* $(1, e_1, e_2, e_3)$

$$M_{sl(2,\mathbb{R}) \uplus \mathbb{R}} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&1&0&-1&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&2&1&0&0&0&0&-2&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&-1&1&0&1&0 \end{bmatrix}. \qquad (3.2)$$

It is a numerically separable numericalization.

Remark 3.4. *In [9] a cross-dimensional general linear algebra is defined as* $gl(\mathbb{R}) := (\{A \mid A \text{ is square}\}, [\cdot, \cdot])$, *where* $[A, B] := A \ltimes B - B \ltimes A$, *which is also applicable to* 1×1 *matrix, that is* $A \in \mathbb{R}$ *or* $B \in \mathbb{R}$. *But it is completely different from* $\oplus$ *defined in previous example.*

Definition 3.5. *Assume* $\mathcal{A} = (V, *)$ *is a numerical algebra over* $\mathbb{F}$.
(i) The subspace $V \backslash \mathbb{F}$ *is called the non-numerical subspace, denoted by* V_a. *It is clear that if* $\dim(V) = n$, *then* $\dim(V_a) = n - 1$.
(ii) If $\mathcal{A}_a := (V_a, *)$ *is a sub-algebra,* $\mathcal{A}$ *is said to be numerically separable.*

Example 3.6. *[17]*

Let $\mathcal{A} = (V, *)$ *be a numerical algebra on* $\mathbb{R}$, *where* $V = \{a + bi \mid a, b \in \mathbb{R}\}$.

(i) *Complex Numbers* ($\mathbb{C}$)*: The product is defined by* $1 * i = i * 1 = i$; $\quad i * i = -1$. *Then we have* $\mathbb{C}$ *as a two dimensional algebra over* $\mathbb{R}$.

(ii) *Dual Numbers* ($\mathbb{D}$)*: Assume the product is defined by* $1 * i = i * 1 = i$; $\quad i * i = 0$. *Then we have* $\mathbb{D}$ *as another two dimensional algebra over* $\mathbb{R}$. *It is easy to see that* $\mathbb{D}$ *is numerically separable.*

(iii) *Hyperbolic Numbers* ($\mathbb{H}$)*: Assume the product is defined by* $1 * i = i * 1 = i$; $\quad i * i = 1$. *Then we know that* $\mathbb{H}$ *is also a two dimensional algebra over* $\mathbb{R}$.

(iv) $e_1 - e_2$ *algebra [17] is a numerical algebra, which is defined as* $\mathcal{A} = (V, *)$, *where* $V = \{a + be_1 + ce_2, \mid a, b, c \in \mathbb{R}\}$, *with the product defined by* $e_i^2 = e_i, \quad i = 1, 2$; *and* $e_1 * e_2 = e_2 * e_1 = 0$.
It is easy to verify that $e_1 - e_2$ *is numerically separable.*

(v) *Consider a numerical algebra* $\mathcal{A} = (V, *)$ *over* $\mathbb{R}$, *with a basis* $B = (e_1 = 1, e_2, e_3, e_4)$. *Corresponding to this basis its PSM is*

$$M_{\mathcal{A}} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0 \\ 0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0 \\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1 \end{bmatrix}. \qquad (3.3)$$

It is ready to verify that this is a numerically separable algebra.

Definition 3.7. *Let* $\mathcal{A} = (V, *)$ *be a numerical algebra. Define a new algebraic structure on* $V_a = V \backslash \mathbb{F}$ *as*

$$X \ominus Y := \Pi_a X * Y, \quad X, Y \in V_a, \qquad (3.4)$$

where Π_a *is the projection from* V *to* V_a.

Proposition 3.8. *Let* $\mathcal{A} = (V, *)$ *be a numerical algebra. Then* $\mathcal{A}_a := (V_a, \ominus)$ *is an algebra, which is called the dis-numeralized algebra of* $\mathcal{A}$.

Proof. It is enough to show that $\ominus$ is distributive. Since the projection Π_a is linear, the conclusion is obvious. □

The process from a numerical algebra to its dis-numeralized algebra is called the dis-numeralization.

Definition 3.9. *A subspace* $V_s \subset V$ *is a closed subspace, if it is closed under* $*$. *That is,* $X * Y \in V_s, \quad \forall X, Y \in V_s$.

The following proposition comes from Definition 3.9.

Proposition 3.10. *Let* $\mathcal{A}$ *be a numerical algebra. Then the following are equivalent:*

(i) $\mathcal{A}$ *is numerically separable;*

(ii) V_a *is a closed subspace;*

(iii) $\mathcal{A}_a = (V_a, *)$ *is an algebra.*

Example 3.11. *Let* $\mathbb{F} = \mathbb{Q}$ *be the field of rational numbers. Consider the extended field* $\mathcal{A} := \mathbb{Q}(\sqrt{2}, \sqrt{3})$*, which is a numerical algebra. Set its basis as* $(e_1 = 1, e_2 = \sqrt{2}, e_3 = \sqrt{3}, e_4 = \sqrt{6})$*. Then its PSM is calculated as in Table 1. Expressing the table into matrix form yields*

$$M_{\mathcal{A}} = \begin{bmatrix} 1&0&0&0&0&2&0&0&0&0&3&0&0&0&0&6 \\ 0&1&0&0&1&0&0&0&0&0&0&3&0&0&3&0 \\ 0&0&1&0&0&0&0&2&1&0&0&0&0&2&0&0 \\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0 \end{bmatrix}. \tag{3.5}$$

It is obvious that this algebra is not numerically separable. A simple computation shows the PSM of $\mathcal{A}_a = (V_a, \circ)$ *is*

$$M_{\mathcal{A}_a} = \begin{bmatrix} 0&0&0&0&0&3&0&3&0 \\ 0&0&2&0&0&0&2&0&0 \\ 0&1&0&1&0&0&0&0&0 \end{bmatrix}.$$

It is obvious that $M_{\mathcal{A}_a}$ *can be obtained from* $M_{\mathcal{A}}$ *by deleting rows and columns of* $M_{\mathcal{A}}$*, which are related to* e_1*.*

Example 3.12. *Recall the* $\mathcal{A}_a$ *in Example 3.11. If we choose*

$$\begin{cases} c_{1,1} = 2,\ c_{2,2} = 3,\ c_{3,3} = 6, \\ c_{i,j} = 0, \quad i \neq j, \end{cases}$$

then $(\mathcal{A}_a)_a = \mathcal{A}$*, where* $(\mathcal{A}_a)_a$ *is a numeralization of non-numerical algebra* $\mathcal{A}_a$*.*

Next, we consider numeralization of vector spaces. A vector space V over $\mathbb{F}$, such as $\mathbb{R}^n$ over $\mathbb{R}$, is not an algebra, because there is no product. A simple way to turn it into an algebra is to add a trivial product, called zero product, to it. That is, define $X * Y = 0, \quad \forall X, Y \in V$. It is ready to verify that $(V, *)$ is a non-numerical algebra.

Definition 3.13. *Let* V *be a vector space over* $\mathbb{F}$*. A numeralization of* V *is a numeralization of* $(V, *)$*, where* $*$ *is zero product on* V*.*

Example 3.14. *Let* $V = \mathbb{R}^n$ *(or* $V = \mathbb{C}^n$*). Consider algebra* $(V, *)$*, where* $*$ *is the zero product. Define*

$$\begin{cases} c_{i,i} = 1, \quad i = 1, 2, \cdots, n, \\ c_{i,j} = 0, \quad i \neq j. \end{cases} \tag{3.6}$$

Then $(V \uplus \mathbb{F}, \oplus)$ *is a numerical algebra.*

Table 1. $M_{\mathcal{A}}$.

$X * Y$	$e_1 * e_1$	$e_1 * e_2$	$e_1 * e_3$	$e_1 * e_4$
e_1	1	0	0	0
e_2	0	1	0	0
e_3	0	0	1	0
e_4	0	0	0	1
$X * Y$	$e_2 * e_1$	$e_2 * e_2$	$e_2 * e_3$	$e_2 * e_4$
e_1	0	2	0	0
e_2	1	0	0	0
e_3	0	0	0	2
e_4	0	0	1	0
$X * Y$	$e_3 * e_1$	$e_3 * e_2$	$e_3 * e_3$	$e_3 * e_4$
e_1	0	0	3	0
e_2	0	0	0	3
e_3	1	0	0	0
e_4	0	1	0	0
$X * Y$	$e_4 * e_1$	$e_4 * e_2$	$e_4 * e_3$	$e_4 * e_4$
e_1	0	0	0	6
e_2	0	0	3	0
e_3	0	2	0	0
e_4	1	0	0	0

Remark 3.15. *Note that an interesting fact is: in the above example* $\oplus|_V$ *is exactly the inner product on* V*. That is, in the numeralized vector space, the inner product becomes a standard vector product.*

Example 3.16. *Let* $V = \mathbb{R}^3$*, and consider* $(\mathbb{R}^3, *)$ *as an algebra with zero product. Now we define*

$$\begin{cases} c_{i,i} = -1/c, \quad i = 1, 2, 3, \\ c_{i,j} = 0, \quad i \neq j, \end{cases} \tag{3.7}$$

where c *is the speed of light. Then numeralized algebra* $(\mathbb{R}^3 \uplus \mathbb{R}, \oplus)$ *is the four dimensional spacious-time space in general relativity, consisting of real world* $\mathbb{R}^3$ *and time* $t \in \mathbb{R}$*. In fact, (3.7) represents the Riemannian coefficients used for general relativity [10].* [*]

4. Basis transformation

Assume $\mathcal{A} = (V, *)$ is an algebra with $B = (e_1, e_2, \cdots, e_n)$ as a basis of V. Moreover, its PSM is M. It is clear that

[*]The idea comes from a private talk with a Physician friend.

M depends on the choice of B. Let $B' = (e'_1, e'_2, \cdots, e'_n)$ be another basis. Under this basis, the PSM is M'. Then their relationship is easily obtained as follows.

Proposition 4.1. *Assume $B' = BT$, where $T \in \mathbb{F}_{n\times n}$ is non-singular. Then*

$$M' = T^{-1}MT(I_n \otimes T) = T^{-1}M(T \otimes T). \tag{4.1}$$

Proof. Let $B = (e_1, e_2, \cdots, e_n)$ and $B' = (e'_1, e'_2, \cdots, e'_n)$. Then

$$\begin{aligned} V_x &= (e_1, e_2, \cdots, e_n)X = (e'_1, e'_2, \cdots, e'_n)X', \\ V_y &= (e_1, e_2, \cdots, e_n)Y = (e'_1, e'_2, \cdots, e'_n)Y'. \end{aligned}$$

Hence $Y' = T^{-1}Y, \quad X' = T^{-1}X$. It follows that

$$\begin{aligned} V_x * V_y &= (e_1, e_2, \cdots, e_n)MXY = (e'_1, e'_2, \cdots, e'_n)M'X'Y' \\ &= (e_1, e_2, \cdots, e_n)TM'T^{-1}XT^{-1}Y \\ &= (e_1, e_2, \cdots, e_n)TM'T^{-1}(I_n \otimes T^{-1})XY. \end{aligned}$$

Since $X, Y \in \mathbb{F}^n$ are arbitrary, it follows that

$$M = TM'T^{-1}(I_n \otimes T^{-1}) = TM'\left(T^{-1} \otimes T^{-1}\right). \tag{4.2}$$

It is ready to verify that (4.1) and (4.2) are equivalent. □

Recall the numerically separability. It is obvious that the definition is basis-depending. A natural question is: If a numerical algebra is not numerically separable, is it possible to turn it into a numerically separable algebra by a proper basis transformation? We give an example for this.

Example 4.2. *Consider a numerical algebra $\mathcal{A} = (V, *)$ over $\mathbb{R}$ with a basis as $B = (e_1 = 1, e_2, e_3, e_4)$. Assume its PSM is*

$$M = \begin{bmatrix} 1&0&0&0&0&6&5&5&0&5&-2&2&0&5&2&0 \\ 0&1&0&0&1&1&3&1&0&3&1&1&0&1&1&1 \\ 0&0&1&0&0&-2&-2&-2&1&-2&3&-1&0&-2&-1&0 \\ 0&0&0&1&0&2&1&1&0&1&0&2&1&1&2&-1 \end{bmatrix}. \tag{4.3}$$

It is easy to see that this is not a numerically separable algebra. For instance, let $x = e_3$. Then $x*x = -2+e_2+3e_3 \notin V_a$. Hence V_a is not closed.

Now we consider a basis transformation $B' = BT$, where

$$T = \begin{bmatrix} 1&2&-2&3 \\ 0&1&0&0 \\ 0&-1&1&-1 \\ 0&0&0&1 \end{bmatrix}.$$

Under this new basis, the PSM becomes

$$M' = T^{-1}M(T \otimes T) = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0 \\ 0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0 \\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1 \end{bmatrix},$$

which is exactly the matrix of (3.3) in Example 3.6. Hence, it is numerically separable.

From above example one sees that the definition of numerically separable needs to be modified. We give the following one.

Definition 4.3. *A numerical algebra is essentially numerically separable, if there exists a basis such that under this basis it is numerically separable.*

The following proposition is obvious.

Proposition 4.4. *A numerical algebra is essentially numerically separable, if and only if, there exists a closed subspace V_a such that*

$$V = V_a \uplus \mathbb{F}.$$

Definition 4.5. *Given two algebras $\mathcal{A}_i = (V_i, *_i)$, $i = 1, 2$ over same $\mathbb{F}$.*

1. *$\mathcal{A}_1$ is said to be homomorphic to $\mathcal{A}_2$, if there exists a mapping $\pi : V_1 \to V_2$ such that*
   - *(i)* $\pi(a_1X_1 +_1 a_2X_2) = a_1\pi(X_1) +_2 a_2\pi(X_2)$;
   - *(ii)* $\pi(X_1 *_1 X_2) = \pi(X_1) *_2 \pi(X_2)$.
2. *$\mathcal{A}_1$ is said to be isomorphic to $\mathcal{A}_2$, if $\mathcal{A}_1$ is homomorphic to $\mathcal{A}_2$, and the homomorphism $\pi : V_1 \to V_2$ is bijective.*
3. *If $\mathcal{A}_1 = \mathcal{A}_2$, and $\pi : V \to V$ is an isomorphism, then π is called an automorphism.*

We have the following result.

Proposition 4.6. *(i) Let $\mathcal{A}_i = (V_i, *_i)$, where $\dim(V_i) = n_i$, $i = 1, 2$. $\pi : V_1 \to V_2$ is a homomorphism, if and only if, there is a matrix $T_\pi \in \mathbb{F}_{n_2\times n_1}$ such that*

$$T_\pi M_{\mathcal{A}_1} - M_{\mathcal{A}_2}(T_\pi \otimes T_\pi) = 0. \tag{4.4}$$

(ii) Let $\mathcal{A}_i = (V_i, *_i)$, where $\dim(V_i) = n$, $i = 1, 2$. $\pi : V_1 \to V_2$ is an isomorphism, if and only if, there is a non-singular matrix $T_\pi \in \mathbb{F}_{n\times n}$ such that

$$T_\pi M_{\mathcal{A}_1} - M_{\mathcal{A}_2}(T_\pi \otimes T_\pi) = 0. \tag{4.5}$$

(iii) Let $\mathcal{A} = (V, *)$, where $\dim(V) = n$. $\pi : V \to V$ is an automorphism, if and only if, there is a non-singular matrix $T_\pi \in \mathbb{F}_{n\times n}$ such that

$$T_\pi M_{\mathcal{A}} - M_{\mathcal{A}} (T_\pi \otimes T_\pi) = 0. \tag{4.6}$$

Proof. We prove (4.6) only. Proofs for (4.4) and (4.5) are similar.

(Necessity) To meet the requirement of (i) of Definition 4.5, it is clear that π must be a linear mapping. Hence there exists a matrix $T_\pi \in \mathbb{F}_{n\times n}$ such that $\pi(X) = T_\pi X, \quad X \in V$. Since π is bijective, T_π must be non-singular.

Now for any $X, Y \in V$ we have

$$\begin{aligned} T_\pi M_{\mathcal{A}} XY &= M_{\mathcal{A}} T_\pi X T_\pi Y \\ &= M_{\mathcal{A}} T_\pi (I_n \otimes T_\pi) XY = M_{\mathcal{A}} (T_\pi \otimes T_\pi) XY. \end{aligned}$$

Since $X, Y \in V$ are arbitrary, (4.6) follows immediately.

(Sufficiency) Verifying the proof of necessity, it is easy to see that each step is necessary and sufficient. The conclusion follows. □

Example 4.7. *Consider a two dimensional NA over $\mathbb{R}$ as $V = \{a + be \mid a, b \in \mathbb{R}\}$, where $1 * e = e * 1 = e; \quad e * e = e$. It seems that we have a new two dimensional algebra over $\mathbb{R}$. But if we let $i = 1 - 2e$, then it is easy to see that $i^2 = 1$. That is, this new two dimensional algebra is isomorphic to the hyperbolic algebra $\mathbb{H}$.*

Using STP, [4] has proved that there are only three two dimensional algebras over $\mathbb{R}$ with their isomorphic algebras.

5. Properties via PSM

Definition 5.1. *$\mathcal{A} = (V, *)$ is a finite dimensional algebra.*

- *(i) $\mathcal{A}$ is commutative, if $*$ is symmetric. That is, $X * Y = Y * X, \quad X, Y \in V$.*
- *(ii) $\mathcal{A}$ is anti-commutative, if $*$ is skew-symmetric. That is, $X * Y = -Y * X, \quad X, Y \in V$.*
- *(iii) $\mathcal{A}$ is associative if*

$$X * (Y * Z) = (X * Y) * Z, \quad X, Y, Z \in V. \tag{5.1}$$

Proposition 5.2. *Let $\mathcal{A} = (V, *)$ be a given finite dimensional algebra, and $M_{\mathcal{A}}$ is its PSM.*

(i) $\mathcal{A}$ is symmetric if and only if,

$$M_{\mathcal{A}} (I_{n^2} - W_{[n,n]}) = 0. \tag{5.2}$$

(ii) $\mathcal{A}$ is skew-symmetric, if and only if,

$$M_{\mathcal{A}} (I_{n^2} + W_{[n,n]}) = 0. \tag{5.3}$$

(iii) $\mathcal{A}$ is associative, if and only if,

$$M_{\mathcal{A}} (I_n \otimes M_{\mathcal{A}}) = M_{\mathcal{A}}^2. \tag{5.4}$$

Proof. (i) Using PSM, it is clear that $X * Y = Y * X$ can be expressed as $M_{\mathcal{A}} XY = M_{\mathcal{A}} YX$. Using swap matrix, the right hand side can be expressed as $M_{\mathcal{A}} XY = M_{\mathcal{A}} W_{[n,n]} XY$. Since $X, Y \in V$ are arbitrary, (5.2) follows. (ii) The proof is similar to (i). (iii) Using PSM, (5.1) can be expressed as

$$M_{\mathcal{A}} X M_{\mathcal{A}} Y Z = M_{\mathcal{A}} (M_{\mathcal{A}} XY) Z.$$

It becomes $M_{\mathcal{A}} (I_n \otimes M_{\mathcal{A}}) XYZ = M_{\mathcal{A}}^2 XYZ$ using Proposition 2.5 . Thus (5.4) follows. □

6. From numerical algebra to field

Assume $\mathcal{A} = (V, *)$ is a numerical algebra over $\mathbb{F}$, we ask when $\mathcal{A}$ is a field? In other words, when $\mathcal{A} \supset \mathbb{F}$ is a finite extension of $\mathbb{F}$ (with $[\mathcal{A} : \mathbb{F}] = n$) [16]?

Since V is a vector space, of course, $(V, +)$ is an abelian group. As for the distribution, it is also ensured by the properties of algebra. Hence we have the following obvious fact.

Lemma 6.1. *If $\mathcal{A} = (V, *)$ be a numerical algebra over $\mathbb{F}$, then $\mathcal{A}$ is a field if and only if, $(V \backslash \{0\}, *)$ is an abelian group.*

According to Lemma 6.1, we have only to check (i) commutativity? (ii) associativity? (iii) invertibility? (i) can be verified by (5.2), and (ii) by (5.4). Hence, we have only to find a way to verify when a numerical algebra is invertible. To this end, we need some preparations.

Definition 6.2. *Let $\mathbb{F}$ be a given field, $A \in \mathbb{F}_{k\times k^2}$. A is said to be jointly non-singular, if for any $0 \neq x \in \mathbb{F}^k$, Ax is non-singular.*

Split A into k square blocks as $A = [A_1, A_2, \cdots, A_k]$, where $A_i = A\delta_k^i \in \mathbb{F}_{k\times k}$. Denote by

$$\mu_{i_1,i_2,\cdots,i_k} := \det\left[\mathrm{Col}_1(A_{i_1})\,\mathrm{Col}_2(A_{i_2})\cdots\mathrm{Col}_k(A_{i_k})\right], \quad i_1,\cdots,i_k = 1,2,\cdots,k.$$

Then we have the following result:

Proposition 6.3. *$A \in \mathbb{F}_{k\times k^2}$ is jointly non-singular, if and only if, the following homogeneous polynomial*

$$\begin{aligned} p(x_1,\cdots,x_k) &= \det(Ax) \\ &= \sum_{i_1=1}^{k}\cdots\sum_{i_k=1}^{k}\mu_{i_1,\cdots,i_k}x_{i_1}\cdots x_{i_k} \neq 0, \quad \forall x \neq 0. \end{aligned} \tag{6.1}$$

Proof. Expanding the determinant $\det(Ax)$ yields this. □

Example 6.4. *(i) Consider $\mathbb{C}$. Using $\{1, i\}$ as a basis of $\mathbb{C}$ over $\mathbb{R}$, it is easy to verify that the PSM of $\mathbb{C}$ is*

$$M_{\mathbb{C}} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \tag{6.2}$$

(ii) Calculating right hand side of (6.1) for $M_{\mathbb{C}}$, we have

$$\mu_{11} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1; \quad \mu_{12} = \det\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = 0;$$
$$\mu_{21} = \det\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0; \quad \mu_{22} = \det\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 1.$$

It follows that $p(x_1, x_2) = x_1^2 + x_2^2$. Hence, $p(x_1, x_2) = 0$, if and only if, $x_1 = x_2 = 0$. It follows that $M_{\mathbb{C}}$ is jointly non-singular.

From above arguments, we have the following result.

Theorem 6.5. *Assume $\mathcal{A} = (V, *)$ is a numerical algebra over $\mathbb{F}$, where V is an n-dimensional vector space with a basis $(e_1 = 1, e_2, \cdots, e_n)$. Moreover, assume the PSM is $M_{\mathcal{A}}$. Then $\mathcal{A}$ is a field, if and only if,*

(i) $M_{\mathcal{A}}^2 = M_{\mathcal{A}}(I_n \otimes M_{\mathcal{A}})$;
(ii) $M_{\mathcal{A}} = M_{\mathcal{A}}W_{[n,n]}$.
(iii) *$M_{\mathcal{A}}$ is jointly non-singular;*

Proof. It was shown in Proposition 5.2 that condition (i) is equivalent to associativity, and condition (ii) is equivalent to commutativity.

Now we consider condition (iii). In fact, it is equivalent to that each $x \neq 0$ has unique inverse.

Let $x_0 \neq 0$. Then we consider the algebraic equation $M_{\mathcal{A}}x_0y = \delta_k^1$. To get unique solution, $M_{\mathcal{A}}x_0$ should be non-singular. The conclusion is obvious. □

Example 6.6. *Consider $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.*

It is easy to see that a basis of E over $\mathbb{Q}$ is: $(1, \sqrt{2}, \sqrt{3}, \sqrt{6})$. Using this basis, the PSM is calculated as in Example 3.11, (see (3.5)).

We verify three conditions in Theorem 6.5. Verifications of conditions (i) and (ii) are straightforward calculations. We verify condition (iii) only. Let $x = (a, b, c, d)^T$. It is easy to calculate that

$$\begin{aligned} \det(M_{\mathcal{A}}x) = a^4 + 4b^4 + 9c^4 + 36d^4 - 4a^2b^2 - 6a^2c^2 \\ - 12a^2d^2 - 12b^2c^2 - 24b^2d^2 - 36c^2d^2 + 48abcd. \end{aligned}$$

Factorizing the above polynomial yields

$$\begin{aligned} & \det(M_{\mathcal{A}}x) \\ = \; & (a + \sqrt{2}b + \sqrt{3}c + \sqrt{6}d)(a + \sqrt{2}b - \sqrt{3}c - \sqrt{6}d) \\ & (a - \sqrt{2}b + \sqrt{3}c - \sqrt{6}d)(a - \sqrt{2}b - \sqrt{3}c + \sqrt{6}d). \end{aligned}$$

Since each factor is a linear combination of the basis elements $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$, we conclude that $\det(M_{\mathcal{A}}x) \neq 0$, $\forall (a, b, c, d)^T \neq \mathbf{0}_4$.

Assume E is a Galois extension over $\mathbb{F}$, $[E, \mathbb{F}] = k$. As a convention, we assume the basis of E is $(e_1 = 1, e_2, \cdots, e_k)$. Then we have the following result:

Proposition 6.7. *Let E be a Galois extension of $\mathbb{F}$, and $[E : \mathbb{F}] = k$. Denote the PSM of E by $M_E = [M_E^1, M_E^2, \cdots, M_E^k] \in \mathbb{F}_{k\times k^2}$. Then M_E satisfies the following conditions:*

(i) $M_E^1 = I_k$.
(ii) $\mathrm{Col}_1(M_E^s) = \delta_k^s, \quad s = 1, 2, \cdots, k.$

Proof. It is clear that in vector form we have $e_i = \delta_k^i$. By definition, $\mathrm{Col}_i(M_E^1) = \delta_k^1 \times_E \delta_k^i = \delta_k^i, \quad i = 1, \cdots, k$. Hence $M_E^1 = I_k$. The reason for the other condition is the same. □

Since we can add i to $\mathbb{R}$ to generate another field $\mathbb{C}$, it is a natural question: Is it possible to add some new numbers to $\mathbb{C}$ to generate a new field $\mathbb{F}$ such that $\mathbb{F}$ is a Galois extension of $\mathbb{C}$? The answer is "No" [†].

As an application of the PSM of finite extension, we prove the following result:

Theorem 6.8. *There is no Galois extension over $\mathbb{C}$.*

[†] W. Li [15] mentioned that in 1861 Weierstrass proved that $\mathbb{C}$ is the only finite field extension over $\mathbb{R}$.

Proof. We prove it by contradiction. Assume there exists a Galois extension $[E : \mathbb{C}] = k$, and let M_E be the PSM of E. Choose $x^* = [a\ \mathbf{1_{k-1}}]^{\mathbf{T}}$. Using equation (6.1) and the first requirement of Proposition 6.7, we have

$$\det(M_E x^*) = a^k + LOT(a) := 0, \tag{6.3}$$

where $LOT(a)$ stands for lower order terms. According to fundamental algebraic theorem, (6.3) has solution a_0. That is, x^* has no unique inverse, which is a contradiction. □

Remark 6.9. *Using similar argument, it is easy to show that there is no $k = 2s+1$ dimensional extension over $\mathbb{R}$, because equation (6.3) surely has real solution. But this method is not applicable to the case of $k = 2s$.*

7. Numerical algorithm for Segre quaternion

7.1. Invertibility of Segre quaternion

The Segre quaternion, denoted by Q^S, is commutative. It is defined as follows: $Q^S = \{x_1 + x_2 I + x_3 J + x_4 K \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$. The product is multi-linear over $\mathbb{R}$ and determined by the following rules:

$$I^2 = -1; \quad J^2 = 1; \quad K^2 = -1$$
$$I * J = J * I = -K,\ I * K = K * I = J,\ J * K = K * J = -I.$$

Then the PSM of Q^S is easily calculated as follows:

$$M_{Q^S} := \begin{bmatrix} 1&0&0&0&0&-1&0&0&0&0&1&0&0&0&0&-1 \\ 0&1&0&0&1&0&0&0&0&0&0&-1&0&0&-1&0 \\ 0&0&1&0&0&0&0&1&1&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&-1&0&0&-1&0&0&1&0&0&0 \end{bmatrix}. \tag{7.1}$$

A straightforward computation shows that M_{Q^S} satisfies the first two requirements of Theorem 6.5. Therefore, Q^S is commutative and associative. Calculating

$$\begin{aligned} &p(x_1, x_2, x_2, x_4) = \det(M_{Q^S} x) \\ =&(x_1^2 - x_3^2)^2 + (x_2^2 - x_4^2)^2 + 2(x_1 x_2 + x_3 x_4)^2 + 2(x_1 x_4 + x_2 x_3)^2. \end{aligned}$$

Hence, $x = x_1 + x_2 I + x_3 J + x_4 K$ is invertible, if and only if, $[x_1\ x_2]^T \neq \pm[x_3\ -x_4]^T$. Then we know that Q^S is almost invertible except a zero-measure set Ω, where $\Omega = \left\{ x^{\mathrm{T}} \in \mathbb{R}^4 \,\middle|\, (x_1, x_2) = \pm(x_3, -x_4) \right\}$.

From above argument, we obtain the following result:

Proposition 7.1. *The commutative quaternion Q^S is: (1) commutative; (2) associative; and (3) invertible over $Q^S \backslash \{\Omega\}$.*

Next, we consider the dis-numeralization of Q^S, the following is obvious:

Proposition 7.2. *The dis-numeralization of Q^S is a symmetric cross product, denoted by $\bar{\times}$, over $\mathbb{R}^3$, which has PSM as*

$$M_{\bar{\times}} = \begin{bmatrix} 0&0&0&0&0&-1&0&-1&0 \\ 0&0&1&0&0&0&1&0&0 \\ 0&-1&0&-1&0&0&0&0&0 \end{bmatrix}$$

7.2. Linear systems over Q^S

Denote by ${Q^S}_{m\times n}$ the set of matrices, which have its entries in Q^S.

Definition 7.3. *Let $M \in {Q^S}_{n\times n}$. M is said to be non-singular (or invertible), if* $\det(M) \notin \Omega$.

Proposition 7.4. *If $M \in {Q^S}_{n\times n}$ is invertible, there exists a matrix $M^{-1} \in {Q^S}_{n\times n}$, such that $M * M^{-1} = M^{-1} M = I_n$.*

Proof. The M^{-1} can be constructed exactly the same as matrices over $\mathbb{R}$ or $\mathbb{C}$. The uniqueness is also trivial. □

In numerical computation it is important to verify if an element in an algebra has inverse. If the answer is "yes", how to calculate it. Recalling the proof of Theorem 6.5, the following result is obvious:

Proposition 7.5. *Assume $\mathcal{A} = (V, *)$ is a k dimensional commutative algebra with $M_{\mathcal{A}}$ as its PSM. Then $x \in V$ is invertible, if and only if, $M_{\mathcal{A}} x$ is invertible. Moreover, $x^{-1} = (M_{\mathcal{A}} x)^{-1} \delta_k^1$.*

Example 7.6. *Recall Example 6.6 again. Then $M_{\mathcal{A}}$ can be used to calculate the inverse of any $x \neq 0$ using the proposition above. For example, let $x = 1 + \sqrt{2} - \sqrt{3} - \sqrt{6}$. In vector form it becomes $x = (1, 1, -1, -1)^T$. Then*

$$x^{-1} = (M_{\mathcal{A}} x)^{-1} \delta_4^1 = (0.5, -0.5, 0.5, -0.5)^T.$$

Back to scalar form, we have $x^{-1} = 0.5(1 - \sqrt{2} + \sqrt{3} - \sqrt{6})$.

Now we can calculate the inverse of a matrix on Q^S.

Example 7.7. *[13] Given*

$$M = \begin{bmatrix} -I & 1-K \\ J & I \end{bmatrix},$$

in vector form we have

$$m_{11} = -I \sim [0,-1,0,0]^T,\ m_{12} = 1-K \sim [1,0,0,-1]^T$$
$$m_{21} = J \sim [0,0,1,0]^T,\ m_{22} = I \sim [0,1,0,0]^T.$$

It is ready to calculate that

$$\begin{aligned} &\det(M) = 1 - I - J \sim [1,-1,-1,0]^T \notin \Omega. \\ &\frac{1}{\det}(M) = (M_{Q^S}\det(M))^{-1}\delta_4^1 \\ &\qquad = [0.2, 0.6, -0.2, -0.4]^T \sim 0.2 + 0.6I - 0.2J - 0.4K. \\ &\qquad M^* = \begin{bmatrix} I & -1+K \\ -J & -I \end{bmatrix}. \end{aligned}$$

Then $M^{-1} = \frac{1}{\det}(M)M^* := B = (b_{ij})$,

$$b_{11} \sim [-0.6, 0.2, -0.4, 0.2]^T,\quad b_{12} \sim [0.2, -0.4, 0.8, 0.6]^T,$$
$$b_{21} \sim [0.2, -0.4, -0.2, 0.6]^T,\quad b_{22} \sim [0.6, -0.2, 0.4, -0.2]^T,$$

which are the same as in [13].

Finally, we consider a linear system $Ax = b$, where $A \in Q^S_{n\times n}$, $b \in Q^S_{n\times 1}$. We have the following result.

Proposition 7.8. *Assume A is non-singular, then system $Ax = b$ has unique solution $x = A^{-1} * b$.*

Example 7.9. *Consider a linear system $Ax = b$, where $A = \left(a_{i,j}\right) \in Q^S_{3\times 3}$ with*

$$a_{11} \sim [1,0,0,1]^T;\ a_{12} \sim [0,-1,0,0]^T;\ a_{13} \sim [0,0,-1,0]^T;$$
$$a_{21} \sim [-1,0,0,0]^T;\ a_{22} \sim [1,0,0,-1]^T;\ a_{23} \sim [0,-1,1,0]^T;$$
$$a_{31} \sim [-1,-1,0,0]^T;\ a_{32} \sim [0,2,0,1]^T;\ a_{33} \sim [1,1,-1,-1]^T.$$

$b = (b_1, b_2, b_3)^T \in Q^S_{3\times 1}$ *with*

$$b_1 \sim [2,0,-1,-1]^T; b_2 \sim [-2,0,0,1]^T; b_3 \sim [0,-1,-1,1]^T.$$

Then it is easy to calculate that

$$\det(A) \sim [1,0,-1,-5]^T \notin \Omega,$$
$$\tfrac{1}{\det}(A) \sim [0.0345, -0.0138, -0.0345, 0.1862]^T.$$

The conjugate matrix of A is $A^* = (b_{i,j})$, where

$$b_{11} \sim [-2,1,-1,0]^T;\ b_{12} \sim [-1,2,-1,3]^T;\ b_{13} \sim [-1,1,1,1]^T;$$
$$b_{21} \sim [0,2,-2,0]^T;\ b_{22} \sim [2,2,-1,1]^T;\ b_{23} \sim [0,2,1,0]^T;$$
$$b_{31} \sim [1,-1,-1,-2]^T; b_{32} \sim [0,-1,-2,-1]^T; b_{33} \sim [2,-1,0,0]^T.$$

Finally, the solution is $x = \frac{1}{\det(A)}A^*b = (x_1, x_2, x_3)^T$, where

$$x_1 \sim [0.2345, -0.8138, 0.1655, -1.0138]^T,$$
$$x_2 \sim [-1.0276, 0.1310, -0.3724, -0.2690]^T,$$
$$x_3 \sim [0.6207, 0.1517, -0.6207, -0.0483]^T.$$

A direct computation shows that the solution is correct.

8. Conclusions

Two classes of algebras called numerical and non-numerical ones, were proposed and investigated. The conversions between these two kinds of algebras were established. Particularly, it was pointed out that the cross product on $\mathbb{R}^3$ (C^r) and the quaternion (Q) are a couple of representatives. That is, C^r is a non-numerical algebra and Q is a numerical algebra. Properly numeralizing C^r yields Q and dis-numeralizing Q yields C^r. Then numerization of vector space was also investigated. It was pointed out that Einstein four dimensional special-time space is of this kind of numerizations. Then the condition for a numerical algebra to be a field is investigated. Finally, as a commutative quaternion, the Segre quaternion Q^S is considered. Its set of non-invertible elements is revealed. Non-singular matrices and solutions of linear systems over Q^S were investigated.

The basic tool in this investigation is STP. Using STP the PSM of an algebra was proposed, which contains all the information about an algebra. In fact, PSM provides a convenient framework for the investigation.

In one word, this paper established a bridge to connect a field with algebras (including vector spaces) over it.

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Conflict of interest

The authors declare that they have no conflicts of interest to this work.

References

1. V. Abramov, Noncommutative galois extension and graded q-differential algebra, *Adv. Appl. Clifford Alg.*, **26** (2016), 1–11.
2. W. Boothby, *An Introduction to Differential Manifold and Riemannian Geometry*, Orlando: Academic Press, 1986.
3. F. Catoni, R. Cannata, P. Zampetti, An introduction to commutative quaternions, *Adv. Appl. Clifford Alg.*, **16** (2005), 1–28.
4. D. Cheng, Some applications of semi-tensor product of matrices in algebra, *Comp. Math. Appl.*, **52** (2005), 1045–1066.
5. D. Cheng, H. Qi, Z. Li, *Analysis and Control of Boolean Networks–A Semi-tensor Product Approach*, London: Springer, 2011.
6. D. Cheng, H. Qi, Y. Zhao, *An Introduction to Semi-tensor Product of Matrices and Its Applications*, Singapore: World Scientific, 2012.
7. D. Cheng, H. Qi, Y. Zhao, Analysis and control of general logical networks–An algebraic approach, *Annu. Rev. Control*, **36** (2012), 11–25.
8. D. Cheng, On equivalence of matrices, *Asian J. Math.*, **23** (2019), 257–348.
9. D. Cheng, *From Dimensional-Free Matrix Theory to Cross-Dimensional Dynamic Systems*, London: Elsevier, 2019.
10. C. Clarke, *Elementary General Relativity*, New York: John Wiley & Sons, 1979.
11. S. Fu, D. Cheng, J. Feng, J. Zhao, Matrix expression of finite Boolean-type algebras, *Applied Math. Comput.*, **395** (2021), 125880.
12. T. Hungerford, *Algebra*, Springer-Verlag, 1974.
13. X. Kong, T. Tong, Ral representation and inverse matrix method of the mixed tyoe comutative quaternion matrix, *J. Northeas Normal Univ. (Natural Sci. Ed.).*, **51** (2019), 15–20 (in Chinese).
14. S. Lang, *Algebra*, Revised Third Edition, New York: Springer-Verlag, 2002.
15. W. Li, *Lecture on History of Mathematics*, Beijing: Higher Edication Press, 1999 (in Chinese).
16. P. Morandi, *Field and Galois Theory*, New York: Springer-Verlag, 1996.
17. S. Pei, J. Chang, J. Ding, Commutative reduced biquaternions and their fourier transform for sigmal and image processing applications, *IEEE Trans. Signal Proc.*, **52** (2004), 2012–2031.
18. D. Pinotsis, Segre quaternions, spectral analysis and a four-dimensional laplace equation. In *Progress in Analysis and its Applications*, M. Ruzhansky and J. Wirth, eds, World Scientific, Singapore, 240-246 (2010).
19. J. Rooncy, On the three types of compex number and planar transformation, *Environ. Plann. B.*, **5** (1978), 89–99.
20. C. Segre, The real representations of complex elements and extension to bicomplex systems, *Math. Ann.*, **40** (1892), 413–467.
21. A. Trovon, Q. Suzuki, Noncommutative galois extensions and ternary clifford analysis, *Adv. Appl. Clifford Alg.*, **27** (2017), 59–70. .
22. J. Yaglom, *Complex Numbers in Geometry*, New York: Academic Press, 1968.