http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(4): 1–43.
DOI:10.3934/mine.2023079
Received: 22 August 2022
Revised: 08 January 2023
Accepted: 08 March 2023
Published: 20 March 2023

Research article

Approximation of elliptic and parabolic equations with Dirichlet boundary conditions[†]

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[†] **This contribution is part of the Special Issue:** PDEs and Calculus of Variations–Dedicated to Giuseppe Mingione, on the occasion of his 50th birthday
Guest Editors: Giampiero Palatucci; Paolo Baroni
Link: www.aimspress.com/mine/article/6240/special-articles

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Abstract: We obtain an approximation result of the weak solutions to elliptic and parabolic equations with Dirichlet boundary conditions. We show that the weak solution can be obtained with a limit of approximations by regularizing the nonlinearities and approximating the domains.

Keywords: nonlinear elliptic equations; nonlinear parabolic equations; approximations

Dedicated to Giuseppe Mingione, on the occasion of his 50th birthday.

1. Introduction

For localized problems, many papers showed that the weak solution of elliptic and parabolic equations can be obtained with a limit of approximations by regularizing the nonlinearities, see for instance [1, 2, 4, 28, 29, 32]. However, as far as we are concerned, it was hard to find a suitable reference for global problems which considered approximations on domains. In this paper, we will show that the weak solution can be obtained with a limit of approximations by regularizing the nonlinearities and approximating the domains for Dirichlet boundary value problems. Also we refer to [19, 20] which used regularization on the nonlinearities and approximation on the convex domains for a class of nonlinear elliptic systems.

For the interested readers, we briefly explain about the mentioned papers in the previous paragraph, which are mainly related to the regularity of elliptic and parabolic problems. Acerbi and Fusco [1] obtained local $C^{1,\gamma}$ for local minimizers of p–energy density, where we refer to [35, 52, 53] for fundamental papers and [27] for generalized elliptic systems. Acerbi and Mingione [2] obtained local $C^{1,\gamma}$ regularity for local minimizers with variable exponents, where we refer to [54] for fundamental paper and [3, 8, 16] for Calderón-Zygmund type estimates. Esposito, Leonetti and Mingione [32, 33] obtained higher integrability results for elliptic equations with p–q growth conditions, where we refer to [10, 18, 24] for the related results and [46, 47] for Lipschitz regularity. Also we refer to [9, 21–23, 25] for double phase problems and [37] for a unified approach of p–q, Orlicz, $p(x)$ and double phase growth conditions. Acerbi and Mingione [4] obtained Calderón-Zygmund type estimate for a class of parabolic systems, and we refer to [11, 15, 17] for the global results and [6] for Lorentz space type estimate. Duzaar and Mingione [28] obtained local Lipschitz regularity for nonlinear elliptic equations and a class of elliptic systems. Also Cianchi and Maz'ya [19, 20] obtained Lipschitz regularity for a class of elliptic systems in convex domains. Duzaar and Mingione [29] obtained Wolff potential type estimate for nonlinear elliptic equations, and we refer to [39–44, 49] for further references and [7] for nonlinear elliptic equations with general growth. We remark that one of the authors obtained [14] based on the techniques of [29, 48].

1.1. Parabolic equations

Suppose that $a : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfies

$$\begin{cases} a(\xi, x, t) \text{ is measurable in } (x, t) \text{ for every } \xi \in \mathbb{R}^n, \\ a(\xi, x, t) \text{ is } C^1\text{-regular in } \xi \text{ for every } (x, t) \in \mathbb{R}^{n+1}, \end{cases} \tag{1.1}$$

and the following ellipticity and growth conditions:

$$\begin{cases} |a(\xi, x, t)| + |D_\xi a(\xi, x, t)|(|\xi|^2 + s^2)^{\frac{1}{2}} \leq \Lambda(|\xi|^2 + s^2)^{\frac{p-1}{2}}, \\ \qquad\qquad \langle D_\xi a(\xi, x, t)\zeta, \zeta\rangle \geq \lambda(|\xi|^2 + s^2)^{\frac{p-2}{2}}|\zeta|^2, \end{cases} \tag{1.2}$$

for every $(x, t) \in \mathbb{R}^{n+1}$, for every $\xi, \zeta \in \mathbb{R}^n$ and for some constants $0 < \lambda \leq \Lambda$ and $s \geq 0$.

To regularize the nonlinearity a, we define $\phi \in C_c^\infty(\mathbb{R}^n)$ as a standard mollifier:

$$\phi(x) = \begin{cases} c_1 \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases} \tag{1.3}$$

where $c_1 > 0$ is a constant chosen so that

$$\int_{\mathbb{R}^n} \phi(x)\, dx = 1. \tag{1.4}$$

Under the assumptions (1.1) and (1.2), let $a_\epsilon(\xi, x, t)$ be a regularization of $a(\xi, x, t)$:

$$a_\epsilon(\xi, x, t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\,(\xi - \epsilon y, x - \epsilon z, t)\, \phi(y)\phi(z)\, dydz \qquad (0 < \epsilon < 1). \tag{1.5}$$

Then $a_\epsilon(\xi, x, t)$ satisfies the ellipticity and growth conditions and it is smooth enough, precisely,

$$\begin{cases} a_\epsilon(\xi, x, t) \text{ is } C^\infty\text{-regular in } \xi \in \mathbb{R}^n \text{ for every } (x, t) \in \mathbb{R}^{n+1}, \\ a_\epsilon(\xi, x, t) \text{ is } C^\infty\text{-regular in } x \in \mathbb{R}^n \text{ for every } \xi \in \mathbb{R}^n \text{ and } t \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} |a_\epsilon(\xi, x, t)| + |D_\xi a_\epsilon(\xi, x, t)|(|\xi|^2 + s_\epsilon^2)^{\frac{1}{2}} \le c\,\Lambda(|\xi|^2 + s_\epsilon^2)^{\frac{p-1}{2}}, \\ \qquad |D_x^m a_\epsilon(\xi, x, t)| + |D_\xi^m a_\epsilon(\xi, x, t)| \le c\,\Lambda\epsilon^{-m}(|\xi|^2 + s_\epsilon^2)^{\frac{p-1}{2}}, \\ \qquad\qquad \langle D_\xi a_\epsilon(\xi, x, t)\zeta, \zeta\rangle \ge c\,\lambda(|\xi|^2 + s_\epsilon^2)^{\frac{p-2}{2}}|\zeta|^2, \end{cases}$$

for $s_\epsilon = (s^2 + \epsilon^2)^{\frac{1}{2}} > 0$. Here, the constants c are depending only on n and p. It will be proved in Lemma 2.13.

As usual, we denote p' as the Hölder conjugate of p and by p^* the Sobolev exponent of p. (Note that p^* can be any real number bigger than 1, provided that $p \ge n$.) We denote $d_H(X, Y)$ as the Hausdorff distance between two nonempty sets X and Y, namely,

$$d_H(X, Y) = \sup\{\operatorname{dist}(x, Y) : x \in X\} + \sup\{\operatorname{dist}(y, X) : y \in Y\}.$$

Remark 1.1. *As mentioned before, $a_k(\xi, x, t)$ is smooth with respect to ξ and x by Lemma 2.13. For Neumann boundary value problems, we need to consider extensions to compare weak solutions defined on different domains. In this paper, we consider Dirichlet boundary value problem with $\gamma \in W^{1,p}(\Omega)$ to obtain the main theorem without using extensions.*

We will only prove the parabolic case, because the elliptic case can be done in a similar way. To consider parabolic equations, we denote $\Omega_\tau = \Omega \times [0, \tau]$ and $\mathbb{R}^n_\tau = \mathbb{R}^n \times [0, \tau]$ for $\tau \in [0, T]$, where $T > 0$. We write $\langle\langle \cdot, \cdot \rangle\rangle_\Omega = \langle\langle \cdot, \cdot \rangle\rangle_{\langle W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)\rangle}$ as the pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$, where $W^{-1,p}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$. We carefully note that $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. We also note that for the consistency of the notation, we usually write $W_0^{1,p}(\mathbb{R}^n)$ instead of $W^{1,p}(\mathbb{R}^n)$. Here, we remark that $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$. For $\partial_t w$, we mean $\partial_t w \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ satisfying

$$\int_0^T \langle\langle \partial_t w, \varphi \rangle\rangle_\Omega \, dt = -\int_{\Omega_T} w\,\varphi_t \, dxdt \text{ for any } \varphi \in C_c^\infty(\Omega_T).$$

We consider a sequence of functions $\{u_k\}_{k=1}^\infty$ defined on the corresponding sequence of domains $\{\Omega^k\}_{k=1}^\infty$ in this paper. So to use convergence on $\{u_k\}_{k=1}^\infty$, we consider the zero extension as in the following definition. In this paper, '$\to$' means the strong convergence and '$\rightharpoonup$' means the weak convergence.

Definition 1.2. *For $1 < p < \infty$, we say $v_k \in L^{p'}(\Omega_T^k)$ ($k \in \mathbb{N}$) converges strongly-$*$ to $v_\infty \in L^{p'}(\Omega_T^\infty)$, which is denoted by $v_k \in L^{p'}(\Omega_T^k) \overset{*}{\to} v_\infty \in L^{p'}(\Omega_T^\infty)$, if*

$$\int_{\Omega_T^k} v_k \eta_k \, dxdt \to \int_{\Omega_T^\infty} v_\infty \eta_\infty \, dxdt,$$

for any $\eta_k \in L^p(\Omega_T^k)$ ($k \in \mathbb{N} \cup \{\infty\}$) satisfying

$$\bar\eta_k \rightharpoonup \bar\eta_\infty \text{ in } L^p(\mathbb{R}^n_T),$$

where $\bar\eta_k$ is the zero extension of η_k from Ω_T^k to $\mathbb{R}^n_T$.

Remark 1.3. *In Definition 1.2, if* $\Omega^k = \Omega^\infty$ *for any* $k \in \mathbb{N}$*, then* $v_k \to v_\infty$ *in* $L^{p'}(\Omega^\infty_T)$ *is equivalent to strong-∗ convergence, see Lemma 3.1.*

We use a similar definition for $W^{-1,p'}$. We remark that $W^{1,p}_0(\Omega)$ is reflexive when $1 < p < \infty$.

Definition 1.4. *For* $1 < p < \infty$*, we say that* $v_k \in W^{-1,p'}(\Omega^k)$ $(k \in \mathbb{N})$ *converges strongly-∗ to* $v_\infty \in W^{-1,p'}(\Omega^\infty)$*, which is denoted by* $v_k \in W^{-1,p'}(\Omega^k) \overset{*}{\to} v_\infty \in W^{-1,p'}(\Omega^\infty)$*, if*

$$\langle\!\langle v_k, \eta_k \rangle\!\rangle_{\Omega^k} \to \langle\!\langle v_\infty, \eta_\infty \rangle\!\rangle_{\Omega^\infty},$$

for any $\eta_k \in W^{1,p}_0(\Omega^k)$ $(k \in \mathbb{N} \cup \{\infty\})$ *satisfying*

$$(\bar{\eta}_k, D\bar{\eta}_k) \rightharpoonup (\bar{\eta}_\infty, D\bar{\eta}_\infty) \text{ in } L^p(\mathbb{R}^n, \mathbb{R}^{n+1})$$

where $\bar{\eta}_k$ *is the zero extension of* η_k *from* Ω^k *to* $\mathbb{R}^n$*.*

Definition 1.5. *For* $1 < p < \infty$*, we say that* $v_k \in L^{p'}(0, T; W^{-1,p'}(\Omega^k))$ $(k \in \mathbb{N})$ *converges strongly-∗ to* $v_\infty \in L^{p'}(0, T; W^{-1,p'}(\Omega^\infty))$*, denoted by* $v_k \in L^{p'}(0, T; W^{-1,p'}(\Omega^k)) \overset{*}{\to} v_\infty \in L^{p'}(0, T; W^{-1,p'}(\Omega^\infty))$*, if*

$$\int_0^T \langle\!\langle v_k, \eta_k \rangle\!\rangle_{\Omega^k} \, dt \to \int_0^T \langle\!\langle v_\infty, \eta_\infty \rangle\!\rangle_{\Omega^\infty} \, dt,$$

for any $\eta_k \in L^p(0, T; W^{1,p}_0(\Omega^k))$ $(k \in \mathbb{N} \cup \{\infty\})$ *satisfying*

$$(\bar{\eta}_k, D\bar{\eta}_k) \rightharpoonup (\bar{\eta}_\infty, D\bar{\eta}_\infty) \text{ in } L^p(\mathbb{R}^n_T, \mathbb{R}^{n+1})$$

where $\bar{\eta}_k \in L^p(0, T; W^{1,p}_0(\mathbb{R}^n))$ *is the zero extension of* η_k*.*

For $p > \frac{2n}{n+2}$ and an open bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, assume that

$$F \in L^p(\Omega_T, \mathbb{R}^n), \qquad f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$$

and

$$\gamma \in C([0, T]; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)) \quad \text{with} \quad \partial_t \gamma \in L^{p'}(0, T; W^{-1,p'}(\Omega)).$$

Let $u \in C([0, T]; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$ be the weak solution of

$$\begin{cases} \partial_t u - \operatorname{div} a(Du, x, t) &= f - \operatorname{div}(|F|^{p-2}F) & \text{in } \Omega_T, \\ u &= \gamma & \text{on } \partial_P \Omega_T. \end{cases} \tag{1.6}$$

Here, we say that $u \in \gamma + L^p(0, T; W^{1,p}_0(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ is the weak solution of (1.6), if

$$\int_0^T \langle\!\langle \partial_t u, \varphi \rangle\!\rangle_\Omega \, dt + \int_{\Omega_T} \langle a(Du, x, t), D\varphi \rangle \, dxdt = \int_{\Omega_T} \left[\langle |F|^{p-2}F, D\varphi \rangle + f\varphi \right] dxdt$$

holds for any $\varphi \in C^\infty_0(\Omega_T)$. Also for the initial condition, it means that

$$\lim_{h \searrow 0} \frac{1}{h} \int_0^h \int_\Omega |u(x, t) - \gamma(x, 0)|^2 \, dxdt = 0,$$

which is equivalent to $u(x, 0) = \gamma(x, 0)$ when $u \in C([0, T]; L^2(\Omega))$.

Now, we introduce the main result in this paper.

Theorem 1.6. *Let $\Omega^k \subset \mathbb{R}^n$ $(k \in \mathbb{N})$ be a sequence of open bounded domains with*

$$\lim_{k\to\infty} d_H(\partial\Omega^k, \partial\Omega) = 0. \tag{1.7}$$

For $k \in \mathbb{N}$, assume that $\epsilon_k > 0$, $F_k \in L^p(\Omega_T^k, \mathbb{R}^n)$, $f_k \in L^{p'}(0,T; W^{-1,p'}(\Omega^k))$ and

$$\gamma_k \in C([0,T]; L^2(\Omega^k)) \cap L^{p'}(0,T; W_0^{1,p}(\Omega^k)) \quad \textit{with} \quad \partial_t\gamma_k \in L^{p'}(0,T; W^{-1,p'}(\Omega^k))$$

satisfy that $\lim_{k\to\infty} \epsilon_k = 0$,

$$\begin{cases} f_k \in L^{p'}(0,T; W^{-1,p'}(\Omega^k)) & \overset{*}{\to} & f \in L^{p'}(0,T; W^{-1,p'}(\Omega)), \\ \partial_t\gamma_k \in L^{p'}(0,T; W^{-1,p'}(\Omega^k)) & \overset{*}{\to} & \partial_t\gamma \in L^{p'}(0,T; W^{-1,p'}(\Omega)), \end{cases} \tag{1.8}$$

and

$$\begin{cases} |F_k|^{p-2}F_k \in L^{p'}(\Omega_T^k, \mathbb{R}^n) & \overset{*}{\to} & |F|^{p-2}F \in L^{p'}(\Omega_T, \mathbb{R}^n), \\ \gamma_k \in L^p(\Omega_T^k) & \overset{*}{\to} & \gamma \in L^p(\Omega_T), \\ D\gamma_k \in L^p(\Omega_T^k, \mathbb{R}^n) & \overset{*}{\to} & D\gamma \in L^p(\Omega_T, \mathbb{R}^n). \end{cases} \tag{1.9}$$

Then for the weak solution $u_k \in C([0,T]; L^2(\Omega^k)) \cap L^p(0,T; W^{1,p}(\Omega^k))$ of

$$\begin{cases} \partial_t u_k - \operatorname{div} a_k(Du_k, x, t) & = & f_k - \operatorname{div}(|F_k|^{p-2}F_k) & \text{in } \Omega_T^k, \\ u_k & = & \gamma_k & \text{on } \partial_P\Omega_T^k. \end{cases} \tag{1.10}$$

where $a_k(\xi, x, t) = a_{\epsilon_k}(\xi, x, t)$, we have that

$$\lim_{k\to\infty} \left[\|Du_k - Du\|_{L^p(\Omega_T^k \cap \Omega_T)} + \|Du_k\|_{L^p(\Omega_T^k \setminus \Omega_T)} + \|Du\|_{L^p(\Omega_T \setminus \Omega_T^k)} \right] = 0, \tag{1.11}$$

where u is the weak solution of (1.6).

We refer to [13] for Calderón-Zygmund type estimates for a class of elliptic and parabolic systems with nonzero boundary data in rough domains such as Reifenberg flat domains.

Remark 1.7. *For the sake of convenience and simplicity, we employ the letters $c > 0$ throughout this paper to denote any constants which can be explicitly computed in terms of known quantities such as n, p, λ, Λ and the diameter of the domains. Thus the exact value denoted by c may change from line to line in a given computation.*

Remark 1.8. *We usually denote $\bar{g}$ as the natural zero extension of g for such space as $L^p(\Omega_T)$ and $L^{p'}(0,T; W^{-1,p'}(\Omega))$ which depends on the situations.*

1.2. Elliptic equations

We also have a result for elliptic equations which corresponds to Theorem 1.6. The proof is similar to that of Theorem 1.6, and we will only state the result.

Suppose that $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\begin{cases} a(\xi, x) \text{ is measurable in } x \text{ for every } \xi \in \mathbb{R}^n, \\ a(\xi, x) \text{ is } C^1\text{-regular in } \xi \text{ for every } x \in \mathbb{R}^n, \end{cases} \tag{1.12}$$

and the following ellipticity and growth conditions:

$$\begin{cases} |a(\xi,x)| + |D_\xi a(\xi,x)|(|\xi|^2+s^2)^{\frac{1}{2}} \le \Lambda(|\xi|^2+s^2)^{\frac{p-1}{2}}, \\ \qquad\qquad \langle D_\xi a(\xi,x)\zeta,\zeta\rangle \ge \lambda(|\xi|^2+s^2)^{\frac{p-2}{2}}|\zeta|^2, \end{cases} \tag{1.13}$$

for every $x,\xi,\zeta \in \mathbb{R}^n$ and for some constants $0<\lambda\le\Lambda$ and $s\ge 0$.

Under the assumptions (1.12) and (1.13), let $a_\epsilon(\xi,x)$ be a regularization of $a(\xi,x)$:

$$a_\epsilon(\xi,x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(\xi-\epsilon y, x-\epsilon z)\phi(y)\phi(z)\,dydz \qquad (0<\epsilon<1). \tag{1.14}$$

Then $a_\epsilon(\xi,x)$ satisfies the ellipticity and growth conditions, such as (1.2), and it is smooth enough, precisely,

$$\begin{cases} a_\epsilon(\xi,x) \text{ is } C^\infty\text{-regular in } \xi\in\mathbb{R}^n \text{ for every } x\in\mathbb{R}^n, \\ a_\epsilon(\xi,x) \text{ is } C^\infty\text{-regular in } x\in\mathbb{R}^n \text{ for every } \xi\in\mathbb{R}^n. \end{cases}$$

We have the following approximation results for elliptic problems.

Theorem 1.9. *For $1<p<\infty$ and an open bounded domain $\Omega\subset\mathbb{R}^n$ $(n\ge 2)$, assume that $F\in L^p(\Omega,\mathbb{R}^n)$, $f\in L^{(p^*)'}(\Omega)$ and $\gamma\in W^{1,p}(\Omega)$. Let $u\in\gamma+W_0^{1,p}(\Omega)$ be the weak solution of*

$$\begin{cases} -\mathrm{div}\, a(Du,x) &=& f-\mathrm{div}\,(|F|^{p-2}F) & \text{in } \Omega, \\ u &=& \gamma & \text{on } \partial\Omega. \end{cases}$$

Let $\Omega^k\subset\mathbb{R}^n$ $(k\in\mathbb{N})$ be a sequence of open bounded domains with

$$\lim_{k\to\infty} d_H(\partial\Omega^k,\partial\Omega) = 0.$$

For $k\in\mathbb{N}$, assume that $\epsilon_k>0$, $F_k\in L^p(\Omega^k,\mathbb{R}^n)$, $f_k\in L^{(p^*)'}(\Omega^k)$ and $\gamma\in W^{1,p}(\Omega^k)$ satisfy that

$$\lim_{k\to\infty}\left[\|F_k-F\|_{L^p(\Omega^k\cap\Omega)} + \|f_k-f\|_{L^{(p^*)'}(\Omega^k\cap\Omega)} + \|\gamma_k-\gamma\|_{W^{1,p}(\Omega^k\cap\Omega)}\right] = 0,$$

and

$$\lim_{k\to\infty}\left[\epsilon_k + \|F_k\|_{L^p(\Omega^k\setminus\Omega)} + \|f_k\|_{L^{(p^*)'}(\Omega^k\setminus\Omega)} + \|\gamma_k\|_{W^{1,p}(\Omega^k\setminus\Omega)}\right] = 0.$$

Then for the weak solution $u_k\in\gamma_k+W_0^{1,p}(\Omega^k)$ of

$$\begin{cases} \mathrm{div}\, a_k(Du_k,x) &=& -\mathrm{div}\,(|F_k|^{p-2}F_k)+f_k & \text{in } \Omega^k, \\ u_k &=& \gamma_k & \text{on } \partial\Omega^k. \end{cases}$$

where $a_k(\xi,x)=a_{\epsilon_k}(\xi,x)$, we have that

$$\lim_{k\to\infty}\left[\|Du_k-Du\|_{L^p(\Omega^k\cap\Omega)} + \|Du_k\|_{L^p(\Omega^k\setminus\Omega)} + \|Du\|_{L^p(\Omega\setminus\Omega^k)}\right] = 0.$$

2. Preliminaries

2.1. Basic results about functional analysis

We use the following results related to weak convergence and weak* convergence.

Proposition 2.1. *[12, Proposition 3.13 (iii)] Let* $\{f_i\}$ *be a sequence in* E^**. If* $f_i \overset{*}{\rightharpoonup} f$ *in* $\sigma(E^*, E)$ *then* $\{\|f_i\|\}$ *is bounded and* $\|f\| \le \liminf \|f_i\|$.

Proposition 2.2. *[12, Theorem 3.16 (Banach-Alaoglu-Bourbaki)] The closed unit ball* $B_{E^*} = \{f \in E^* : \|f\| \le 1\}$ *is compact in the weak-$*$ topology* $\sigma(E^*, E)$*.*

One can easily check that compactness in Proposition 2.2 implies sequential compactness for metric spaces.

Proposition 2.3. *If* E^* *is a metric space then any bounded sequence* $\{f_i\}$ *in* E^* *has a weakly-$*$ convergent subsequence.*

To apply Proposition 2.1 and Proposition 2.3 to Sobolev space, we use Proposition 2.4.

Proposition 2.4. *[12, Proposition 8.1]* $W^{1,p}$ *is a Banach space for* $1 \le p \le \infty$*.* $W^{1,p}$ *is reflexive for* $1 < p < \infty$ *and separable for* $1 \le p < \infty$*.*

To handle the dual space of $W_0^{1,p}(\Omega)$, we use [45, Corollary 10.49].

Proposition 2.5. *[45, Corollary 10.49] Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $1 \le p < \infty$*. Then* $h \in W^{-1,p'}(\Omega)$ *can be identified as*

$$\langle h, \varphi \rangle_\Omega = \int_\Omega \langle H, (\varphi, D\varphi) \rangle \, dx,$$

with

$$\|h\|_{W^{-1,p'}(\Omega)} = \left(\int_\Omega \sum_{i=0}^{n} |H_i|^{p'} \, dx \right)^{\frac{1}{p'}},$$

for some $H = (H_0, H_1, \cdots, H_n) \in L^{p'}(\Omega, \mathbb{R}^{n+1})$*.*

We have the following result from [51, Proposition III.1.2], [30, Lemma 2.1] and [50, Lemma 3.1].

Proposition 2.6. *[51, Proposition III.1.2] Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* $t_1 < t_2$ *and* $p > \frac{2n}{n+2}$*. Assume that* $v \in L^p(t_1, t_2; W_0^{1,p}(\Omega))$ *has a distributional derivative* $\partial_t v \in L^{p'}(t_1, t_2; W^{-1,p'}(\Omega))$*. Then there holds* $v \in C([t_1, t_2]; L^2(\Omega))$ *and moreover, the mapping* $t \mapsto \|v(\cdot, t)\|^2_{L^2(\Omega)}$ *is absolutely continuous on* $[t_1, t_2]$ *with*

$$\frac{d}{dt} \|v(\cdot, t)\|^2_{L^2(\Omega)} = 2 \langle\!\langle \partial_t v, v \rangle\!\rangle_\Omega \quad \text{a.e. on } [t_1, t_2],$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Omega$ *denotes the dual pairing between* $W^{-1,p'}(\Omega)$ *and* $W_0^{1,p}(\Omega)$*.*

2.2. Basic inequalities on elliptic condition

We use the following basic inequality in this paper.

Lemma 2.7. *[38, Lemma 3.2] For any $q > 1$ and $s \geq 0$, there exists $\kappa_1 = \kappa_1(n, q) \in (0, 1]$ such that*

$$|\xi - \zeta|^q \leq c\kappa^q(|\xi|^2 + s^2)^{\frac{q}{2}} + c\kappa^{q-2}(|\xi|^2 + |\zeta|^2 + s^2)^{\frac{q-2}{2}}|\xi - \zeta|^2,$$

for any $\kappa \in (0, \kappa_1]$.

We would like to emphasis that the inequalities in Lemmas 2.8 and 2.9 are obtained for $s \geq 0$ even when $1 < q < 2$. We remark that a different proof for $1 < q < 2$ was shown in [1, Lemma 2.1].

Lemma 2.8. *For any $q > 1$ and $s \geq 0$, we have that*

$$\int_0^1 (|\xi + \tau(\zeta - \xi)|^2 + s^2)^{\frac{q-2}{2}}\, d\tau = \int_0^1 (|\zeta + \tau(\xi - \zeta)|^2 + s^2)^{\frac{q-2}{2}}\, d\tau \leq c(|\xi|^2 + |\zeta|^2 + s^2)^{\frac{q-2}{2}},$$

for any $\xi, \zeta \in \mathbb{R}^n \setminus \{0\}$, where c depends only on q.

Proof. By changing variable, one can easily check that

$$\int_0^1 (|\xi + \tau(\zeta - \xi)|^2 + s^2)^{\frac{q-2}{2}}\, d\tau = \int_0^1 (|\zeta + \tau(\xi - \zeta)|^2 + s^2)^{\frac{q-2}{2}}\, d\tau,$$

and without loss of generality, we may assume $|\xi| \geq |\zeta|$.

If $q \geq 2$, then the lemma follows from the fact that

$$|\xi + \tau(\zeta - \xi)|^2 \leq 8(|\xi|^2 + |\zeta|^2) \qquad (\tau \in [0, 1]).$$

So it only remains to prove the lemma when $1 < q < 2$.

Next, suppose that $1 < q < 2$. We show the lemma by considering three cases:

(1). $2|\zeta - \xi| \leq |\xi|$,
(2). $|\xi| \leq 2|\zeta - \xi| \leq 2s$,
(3). $|\xi| \leq 2|\zeta - \xi|$ and $s < |\zeta - \xi|$.

(1). If $2|\zeta - \xi| \leq |\xi|$, then for any $\tau \in [0, 1]$ we have

$$|\xi + \tau(\zeta - \xi)| \geq |\xi| - |\tau(\zeta - \xi)| \geq \frac{|\xi|}{2} \geq \frac{|\xi| + |\zeta|}{4} \geq \frac{(|\xi|^2 + |\zeta|^2)^{\frac{1}{2}}}{4},$$

because we assumed that $|\xi| \geq |\zeta|$, which implies

$$\int_0^1 (|\xi + \tau(\zeta - \xi)|^2 + s^2)^{\frac{q-2}{2}}\, d\tau \leq c(q)\,(|\xi|^2 + |\zeta|^2 + s^2)^{\frac{q-2}{2}},$$

and the lemma is proved for the first case.

(2). If $|\xi| \leq 2|\zeta - \xi| \leq 2s$, then we obtain

$$|\xi|^2 + |\zeta|^2 + s^2 \leq |\xi|^2 + 2(|\xi|^2 + |\zeta - \xi|^2) + s^2 \leq 3(|\xi|^2 + |\zeta - \xi|^2 + s^2) \leq 18s^2,$$

which implies

$$\int_0^1 (|\xi + \tau(\zeta - \xi)|^2 + s^2)^{\frac{q-2}{2}}\, d\tau \le s^{q-2} \le c(q)\, (|\xi|^2 + |\zeta|^2 + s^2)^{\frac{q-2}{2}},$$

and the lemma is proved for the second case.

(3). Suppose that $|\xi| \le 2|\zeta - \xi|$ and $s < |\zeta - \xi|$. One can easily check that

$$\left\langle \xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}, \xi + \tau(\zeta - \xi) - \left(\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right)\right\rangle = 0,$$

which implies

$$|\xi + \tau(\zeta - \xi)|^2 = \left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right|^2 + \left(\tau + \frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}\right)^2 |\zeta - \xi|^2.$$

Then by changing variables, we obtain

$$\begin{aligned}
&\int_0^1 (|\xi + \tau(\zeta - \xi)|^2 + s^2)^{\frac{q-2}{2}}\, d\tau \\
&= \int_0^1 \left(\left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right|^2 + \left(\tau + \frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}\right)^2 |\zeta - \xi|^2 + s^2\right)^{\frac{q-2}{2}} d\tau \\
&= \int_{\frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}}^{1 + \frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}} \left(\left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right|^2 + \theta^2 |\zeta - \xi|^2 + s^2\right)^{\frac{q-2}{2}} d\theta \\
&\le c(q) \int_{\frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}}^{1 + \frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}} \left(\left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right| + |\theta||\zeta - \xi| + s\right)^{q-2} d\theta \\
&\le c(q)\, (I + II),
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
I &= \int_0^{\left|1 + \frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}\right|} \left(\left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right| + \theta|\zeta - \xi| + s\right)^{q-2} d\theta, \\
II &= \int_0^{\left|\frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}\right|} \left(\left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right| + \theta|\zeta - \xi| + s\right)^{q-2} d\theta.
\end{aligned}$$

By changing variables, we discover that

$$\begin{aligned}
I &= \frac{1}{|\zeta - \xi|} \int_{\left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right| + s}^{|\zeta - \xi|\left|1 + \frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}\right| + \left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right| + s} \kappa^{q-2}\, d\kappa, \\
&= \frac{\left[|\zeta - \xi|\left|1 + \frac{\langle \zeta - \xi, \xi\rangle}{|\zeta - \xi|^2}\right| + \left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right| + s\right]^{q-1} - \left[\left|\xi - \frac{\langle \zeta - \xi, \xi\rangle(\zeta - \xi)}{|\zeta - \xi|^2}\right| + s\right]^{q-1}}{(q-1)|\zeta - \xi|} \\
&\le \frac{c(q)\, (|\zeta - \xi| + |\xi| + s)^{q-1}}{(q-1)|\zeta - \xi|}.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
II &= \frac{1}{|\zeta-\xi|}\int_{\left|\xi-\frac{\langle\zeta-\xi,\xi\rangle(\zeta-\xi)}{|\zeta-\xi|^2}\right|+s}^{|\zeta-\xi|\left|\frac{\langle\zeta-\xi,\xi\rangle}{|\zeta-\xi|^2}\right|+\left|\xi-\frac{\langle\zeta-\xi,\xi\rangle(\zeta-\xi)}{|\zeta-\xi|^2}\right|+s}\kappa^{q-2}\,d\kappa,\\
&= \frac{\left[|\zeta-\xi|\left|\frac{\langle\zeta-\xi,\xi\rangle}{|\zeta-\xi|^2}\right|+\left|\xi-\frac{\langle\zeta-\xi,\xi\rangle(\zeta-\xi)}{|\zeta-\xi|^2}\right|+s\right]^{q-1}-\left[\left|\xi-\frac{\langle\zeta-\xi,\xi\rangle(\zeta-\xi)}{|\zeta-\xi|^2}\right|+s\right]^{q-1}}{(q-1)|\zeta-\xi|}\\
&\le \frac{c(q)\,(|\zeta-\xi|+|\xi|+s)^{q-1}}{(q-1)|\zeta-\xi|}.
\end{aligned}
$$

Since $|\zeta|\le|\xi|\le 2|\zeta-\xi|$ and $s<|\zeta-\xi|$, we have $|\xi|^2+|\zeta|^2+s^2\le 9|\zeta-\xi|^2$, and

$$
\frac{(|\zeta-\xi|+|\xi|+s)^{q-1}}{|\zeta-\xi|}\le\frac{c(q)\,|\zeta-\xi|^{q-1}}{|\zeta-\xi|}=c(q)\,|\zeta-\xi|^{q-2}\le c(q)\,(|\xi|^2+|\zeta|^2+s^2)^{\frac{q-2}{2}}.
$$

By the above three inequalities and (2.1), we find that the lemma holds when $|\xi|\le 2|\zeta-\xi|$ and $s<|\zeta-\xi|$. This completes the proof. □

Lemma 2.9. *For any $q>1$ and $s\ge 0$, we have that*

$$
\int_0^1(|\xi+\tau(\zeta-\xi)|^2+s^2)^{\frac{q-2}{2}}\,d\tau=\int_0^1(|\zeta+\tau(\xi-\zeta)|^2+s^2)^{\frac{q-2}{2}}\,d\tau\ge c(|\xi|^2+|\zeta|^2+s^2)^{\frac{q-2}{2}},
$$

for any $\xi,\zeta\in\mathbb{R}^n\setminus\{0\}$, where c depends only on q.

Proof. One can easily check that

$$
|\xi+t(\zeta-\xi)|^2+s^2\le c(q)(|\xi|^2+|\zeta|^2+s^2)\qquad(\tau\in[0,1]).
$$

If $1<q<2$, then

$$
\int_0^1(|\xi+\tau(\zeta-\xi)|^2+s^2)^{\frac{q-2}{2}}\,d\tau\ge c(q)\int_0^1(|\xi|^2+|\zeta|^2+s^2)^{\frac{q-2}{2}}\,d\tau\ge c(q)\,(|\xi|^2+|\zeta|^2+s^2)^{\frac{q-2}{2}},
$$

which prove the lemma for $1<q<2$.

To prove the lemma for the case $q\ge 2$, we assume that $|\xi|\ge|\zeta|$ without loss of generality. Then for $\tau\in[0,1/4]$, we have

$$
|\xi+\tau(\zeta-\xi)|\ge|\xi|-\tau|\zeta-\xi|\ge|\xi|-|\zeta-\xi|/4\ge|\xi|/2\ge c(q)\,(|\xi|^2+|\zeta|^2)^{\frac{1}{2}}.
$$

So we obtain

$$
\int_0^1(|\xi+\tau(\zeta-\xi)|^2+s^2)^{\frac{q-2}{2}}\,d\tau\ge c(q)\int_0^{\frac{1}{4}}(|\xi|^2+|\zeta|^2+s^2)^{\frac{q-2}{2}}\,d\tau\ge c(q)\,(|\xi|^2+|\zeta|^2+s^2)^{\frac{q-2}{2}},
$$

which prove the lemma for $q\ge 2$. This completes the proof. □

To compare $a(\xi,x,t)$ and $a(\zeta,x,t)$, we use the following lemma.

Lemma 2.10. *Under the assumptions* (1.1) *and* (1.2)*, we have*

$$|a(\xi,x,t)-a(\zeta,x,t)|^{\frac{p}{p-1}} \le c|\xi-\zeta|(|\xi|^2+|\zeta|^2+s^2)^{\frac{p-1}{2}},$$

for any $\xi,\zeta\in\mathbb{R}^n$.

Proof. We fix any $\xi,\zeta\in\mathbb{R}^n$. If $|\xi|=0$ or $|\zeta|=0$ then the lemma holds trivially from (1.1) and (1.2). So we assume that $\xi,\zeta\in\mathbb{R}^n\setminus\{0\}$. Since $|\xi-\zeta|^{\frac{1}{p-1}}\le c(|\xi|^2+|\zeta|^2+s^2)^{\frac{1}{2(p-1)}}$, we have from (1.2) and Lemma 2.8 that

$$\begin{aligned}
|a(\xi,x,t)-a(\zeta,x,t)|^{\frac{p}{p-1}} &= \left|\int_0^1 \frac{d}{d\tau}[a(\tau\xi+(1-\tau)\zeta,x,t)]\,d\tau\right|^{\frac{p}{p-1}}\\
&= \left|\int_0^1 D_\xi a(\tau\xi+(1-\tau)\zeta,x,t)(\xi-\zeta)\,d\tau\right|^{\frac{p}{p-1}}\\
&\le c|\xi-\zeta|^{\frac{p}{p-1}}\left(\int_0^1 (|\tau\xi+(1-\tau)\zeta|^2+s^2)^{\frac{p-2}{2}}\,d\tau\right)^{\frac{p}{p-1}}\\
&\le c|\xi-\zeta|^{\frac{p}{p-1}}(|\xi|^2+|\zeta|^2+s^2)^{\frac{p(p-2)}{2(p-1)}}\\
&\le c|\xi-\zeta|(|\xi|^2+|\zeta|^2+s^2)^{\frac{p-1}{2}}.
\end{aligned}$$

Since $\xi,\zeta\in\mathbb{R}^n$ were arbitrary chosen, the lemma follows. □

We show the following well-known inequality. We remark that a different proof for $0<q<2$ was shown in [1, Lemma 2.1] and [36, Lemma 2.1].

Lemma 2.11. *For any* $q>0$ *and* $s\ge 0$*, we have that*

$$\left|\left(|\xi|^2+s^2\right)^{\frac{q-2}{4}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{4}}\zeta\right|^2\le c\left(|\xi|^2+|\zeta|^2+s^2\right)^{\frac{q-2}{2}}|\xi-\zeta|^2,$$

and

$$\left\langle\left(|\xi|^2+s^2\right)^{\frac{q-2}{4}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{4}}\zeta,\xi-\zeta\right\rangle\ge c\left(|\xi|^2+|\zeta|^2+s^2\right)^{\frac{q-2}{4}}|\xi-\zeta|^2,$$

for any $\xi,\zeta\in\mathbb{R}^n$*, where c depends only on q.*

Proof. We fix any $\xi,\zeta\in\mathbb{R}^n$. If $|\xi|=0$ or $|\zeta|=0$ then the lemma holds trivially. So we assume that $\xi,\zeta\in\mathbb{R}^n\setminus\{0\}$. Then

$$\begin{aligned}
&\left(|\xi|^2+s^2\right)^{\frac{q-2}{4}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{4}}\zeta\\
&\quad=\int_0^1\frac{d}{d\tau}\left[\left(|\tau\xi+(1-\tau)\zeta|^2+s^2\right)^{\frac{q-2}{4}}(\tau\xi+(1-\tau)\zeta)\right]d\tau\\
&\quad=\int_0^1\frac{q-2}{2}\cdot\left(|\tau\xi+(1-\tau)\zeta|^2+s^2\right)^{\frac{q-6}{4}}\langle\tau\xi+(1-\tau)\zeta,\xi-\zeta\rangle(\tau\xi+(1-\tau)\zeta)\,d\tau\\
&\qquad+\int_0^1\left(|\tau\xi+(1-\tau)\zeta|^2+s^2\right)^{\frac{q-2}{4}}(\xi-\zeta)\,d\tau.
\end{aligned}$$

By taking $\frac{q}{2}+1 \in (1,\infty)$ instead for $q \in (1,\infty)$ in Lemma 2.8,

$$\left|\left(|\xi|^2+s^2\right)^{\frac{q-2}{4}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{4}}\zeta\right| \le c(q)|\xi-\zeta|\int_0^1\left(|\tau\xi+(1-\tau)\zeta|^2+s^2\right)^{\frac{q-2}{4}}\,d\tau$$
$$\le c(q)|\xi-\zeta|\left(|\xi|^2+|\zeta|^2+s^2\right)^{\frac{q-2}{4}}.$$

Also we get

$$\begin{aligned}&\left\langle\left(|\xi|^2+s^2\right)^{\frac{q-2}{4}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{4}}\zeta,\xi-\zeta\right\rangle\\ &=\int_0^1\frac{q-2}{2}\cdot\left(|\tau\xi+(1-\tau)\zeta|^2+s^2\right)^{\frac{q-6}{4}}|\langle\tau\xi+(1-\tau)\zeta,\xi-\zeta\rangle|^2\,d\tau\\ &\quad+\int_0^1\left(|\tau\xi+(1-\tau)\zeta|^2+s^2\right)^{\frac{q-2}{4}}|\xi-\zeta|^2\,d\tau.\end{aligned}$$

If $0<q\le 2$ then $1=\frac{2-q}{2}+\frac{q}{2}$ and $\frac{2-q}{2}\ge 0$. Also if $q>2$ then $\frac{q-2}{2}\ge 0$. Thus

$$\left\langle\left(|\xi|^2+s^2\right)^{\frac{q-2}{4}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{4}}\zeta,\xi-\zeta\right\rangle\ge\min\left\{\frac{q}{2},1\right\}\int_0^1\left(|\tau\xi+(1-\tau)\zeta|^2+s^2\right)^{\frac{q-2}{4}}|\xi-\zeta|^2\,d\tau.$$

By taking $\frac{q}{2}+1 \in (1,\infty)$ instead for $q \in (1,\infty)$ in Lemma 2.9,

$$\left\langle\left(|\xi|^2+s^2\right)^{\frac{q-2}{4}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{4}}\zeta,\xi-\zeta\right\rangle\ge c\left(|\xi|^2+|\zeta|^2+s^2\right)^{\frac{q-2}{4}}|\xi-\zeta|^2.$$

Since $\xi,\zeta\in\mathbb{R}^n$ were arbitrary chosen, the lemma follows. □

We will use the following lemma.

Lemma 2.12. *For any $q>1$ and $s\ge 0$, we have that*

$$\left|\left(|\xi|^2+s^2\right)^{\frac{q-2}{2}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{2}}\zeta\right|^{\frac{q}{q-1}}\le c\left(|\xi|^2+|\zeta|^2+s^2\right)^{\frac{q-1}{2}}|\xi-\zeta|,$$

for any $\xi,\zeta\in\mathbb{R}^n$, where c only depends on q.

Proof. Fix any $\xi,\zeta\in\mathbb{R}^n$. By taking $2q-2>0$ instead of $q\,(>0)$ in Lemma 2.11,

$$\left|\left(|\xi|^2+s^2\right)^{\frac{q-2}{2}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{2}}\zeta\right|^{\frac{q}{q-1}}\le c(q)\left(|\xi|^2+|\zeta|^2+s^2\right)^{\frac{q(q-2)}{2(q-1)}}|\xi-\zeta|^{\frac{q}{q-1}}.$$

By that $|\xi-\zeta|^{\frac{1}{q-1}}\le c\left(|\xi|^2+|\zeta|^2+s^2\right)^{\frac{1}{2(q-1)}}$,

$$\left|\left(|\xi|^2+s^2\right)^{\frac{q-2}{2}}\xi-\left(|\zeta|^2+s^2\right)^{\frac{q-2}{2}}\zeta\right|^{\frac{q}{q-1}}\le c(q)\left(|\xi|^2+|\zeta|^2+s^2\right)^{\frac{q-1}{2}}|\xi-\zeta|.$$

Since $\xi,\zeta\in\mathbb{R}^n$ were arbitrary chosen, the lemma follows. □

2.3. Regularization on the nonlinearities

To find the ellipticity and growth conditions of $a_\epsilon(\xi, x, t)$ in (1.5), we follow the approach in the proof of [31, Lemma 2] and [32, Lemma 3.1].

Lemma 2.13. *For* (1.5)*, we have*

$$\begin{cases} a_\epsilon(\xi, x, t) \text{ is } C^\infty\text{-regular in } \xi \in \mathbb{R}^n \text{ for every } (x, t) \in \mathbb{R}^{n+1}, \\ a_\epsilon(\xi, x, t) \text{ is } C^\infty\text{-regular in } x \in \mathbb{R}^n \text{ for every } \xi \in \mathbb{R}^n \text{ and } t \in \mathbb{R}, \end{cases} \tag{2.2}$$

and

$$\begin{cases} |a_\epsilon(\xi, x, t)| + |D_\xi a_\epsilon(\xi, x, t)|(|\xi|^2 + s_\epsilon^2)^{\frac{1}{2}} \le c\,\Lambda(|\xi|^2 + s_\epsilon^2)^{\frac{p-1}{2}}, \\ \qquad |D_x^m a_\epsilon(\xi, x, t)| + |D_\xi^m a_\epsilon(\xi, x, t)| \le c\,\Lambda\epsilon^{-m}(|\xi|^2 + s_\epsilon^2)^{\frac{p-1}{2}}, \\ \qquad\qquad \langle D_\xi a_\epsilon(\xi, x, t)\zeta, \zeta\rangle \ge c\,\lambda(|\xi|^2 + s_\epsilon^2)^{\frac{p-2}{2}}|\zeta|^2, \end{cases} \tag{2.3}$$

for $s_\epsilon = (s^2 + \epsilon^2)^{\frac{1}{2}}$. Here, the constants c are depending only on n and p.

Proof. Fix a vector $\xi \in \mathbb{R}^n$. Since $a(\xi, x, t)$ is C^1-regular in $\xi \in \mathbb{R}^n$ for every $x \in \mathbb{R}^n$, we find that $a_\epsilon(\xi, x, t)$ is C^1-regular in $\xi \in \mathbb{R}^n$ for every $x \in \mathbb{R}^n$. Also by changing variable, we have from (1.5) that

$$a_\epsilon(\xi, x, t) = \frac{1}{\epsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(\xi - \epsilon y, z, t)\phi(y)\phi\Big(\frac{x - z}{\epsilon}\Big)\,dydz,$$

which implies

$$D_x a_\epsilon(\xi, x, t) = \frac{1}{\epsilon^{n+1}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(\xi - \epsilon y, z, t)\phi(y)D\phi\Big(\frac{x - z}{\epsilon}\Big)\,dydz.$$

Moreover, from (1.2), the fact that $\operatorname{supp}\phi \subset \overline{B_1}$ and

$$\begin{aligned} D_x^m a_\epsilon(\xi, x, t) &= \frac{1}{\epsilon^{n+m}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(\xi - \epsilon y, z, t)\phi(y)D^m\phi\Big(\frac{x - z}{\epsilon}\Big)\,dydz \\ &= \frac{1}{\epsilon^m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(\xi - \epsilon y, x - \epsilon z, t)\phi(y)D^m\phi(z)\,dydz, \end{aligned}$$

for any $m \ge 0$, which implies that

$$\begin{aligned} |D_x^m a_\epsilon(\xi, x, t)| &\le \Lambda\epsilon^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (|\xi - \epsilon y|^2 + s^2)^{\frac{p-1}{2}}\phi(y)\,|D^m\phi(z)|\,dydz \\ &\le 2^{\frac{p-1}{2}}\Lambda\epsilon^{-m}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (|\xi|^2 + \epsilon^2 + s^2)^{\frac{p-1}{2}}\phi(y)\,|D^m\phi(z)|\,dydz \\ &\le 2^{\frac{p-1}{2}}\Lambda\epsilon^{-m}(|\xi|^2 + \epsilon^2 + s^2)^{\frac{p-1}{2}}\int_{\mathbb{R}^n} |D^m\phi(z)|\,dz, \end{aligned}$$

for any $m \ge 0$. Similarly, by changing variable, we have from (1.5) that

$$a_\epsilon(\xi, x, t) = \frac{1}{\epsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(y, x - \epsilon z, t)\phi\left(\frac{\xi - y}{\epsilon}\right)\phi(z)\,dydz,$$

and one can obtain that

$$|D_\xi^m a_\epsilon(\xi,x,t)| \le 2^{\frac{p-1}{2}}\Lambda\epsilon^{-m}(|\xi|^2+\epsilon^2+s^2)^{\frac{p-1}{2}}\int_{\mathbb{R}^n}|D^m\phi(y)|\,dz.$$

So $a_\epsilon(\xi,x,t)$ is C^∞-regular in $\xi\in\mathbb{R}^n$ for every $(x,t)\in\mathbb{R}^n$ and $a_\epsilon(\xi,x,t)$ is C^∞-regular in $x\in\mathbb{R}^n$ for every $\xi\in\mathbb{R}^n$ and $t\in\mathbb{R}$. Also the second inequality in (2.3) follows.

From (1.2), (1.5) and the fact that $\operatorname{supp}\phi\subset\overline{B_1}$, we have

$$\begin{aligned}\langle D_\xi a_\epsilon(\xi,x,t)\zeta,\zeta\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\langle D_\xi a(\xi-\epsilon y,x-\epsilon z,t)\zeta,\zeta\rangle\phi(y)\phi(z)\,dydz\\ &\ge \lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(|\xi-\epsilon y|^2+s^2)^{\frac{p-2}{2}}|\zeta|^2\phi(y)\phi(z)\,dydz\\ &\ge \lambda\int_{(B_1\setminus B_{\frac12})\cap\langle\xi,y\rangle\ge0}(|\xi|^2+|\epsilon y|^2+2\langle\xi,\epsilon y\rangle+s^2)^{\frac{p-2}{2}}|\zeta|^2\phi(y)\,dy\\ &\ge c(n,p)\,\lambda\Big(|\xi|^2+\frac{\epsilon^2}{4}+s^2\Big)^{\frac{p-2}{2}}|\zeta|^2\int_{(B_1\setminus B_{\frac12})\cap\langle\xi,y\rangle\ge0}\phi(y)\,dy\\ &\ge c(n,p)\,\lambda(|\xi|^2+s^2+\epsilon^2)^{\frac{p-2}{2}}|\zeta|^2,\end{aligned}$$

and the third inequality in (2.3) holds.

It only remains to prove the first inequality in (2.3). In view of (1.5), we have

$$\begin{aligned}|a_\epsilon(\xi,x,t)| &\le \Lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(|\xi-\epsilon y|^2+s^2)^{\frac{p-1}{2}}\phi(y)\phi(z)\,dydz\\ &\le 2^{\frac{p-1}{2}}\Lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(|\xi|^2+\epsilon^2+s^2)^{\frac{p-1}{2}}\phi(y)\phi(z)\,dydz\\ &= 2^{\frac{p-1}{2}}\Lambda(|\xi|^2+\epsilon^2+s^2)^{\frac{p-1}{2}}.\end{aligned}\tag{2.4}$$

If $16\epsilon^2\ge|\xi|^2+s^2$, then by changing variables and (1.5), we obtain

$$\begin{aligned}|D_\xi a_\epsilon(\xi,x,t)| &= \left|D_\xi\Big(\frac{1}{\epsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}a(y,x-\epsilon z,t)\phi\Big(\frac{\xi-y}{\epsilon}\Big)\phi(z)\,dydz\Big)\right|\\ &\le \frac{\Lambda}{\epsilon^{n+1}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(|y|^2+s^2)^{\frac{p-1}{2}}\left|D\phi\Big(\frac{\xi-y}{\epsilon}\Big)\right|\phi(z)\,dydz\\ &= \Lambda\epsilon^{-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(|\xi-\epsilon y|^2+s^2)^{\frac{p-1}{2}}|D\phi(y)|\phi(z)\,dydz\\ &\le 2^{\frac{p-1}{2}}\Lambda\epsilon^{-1}(|\xi|^2+\epsilon^2+s^2)^{\frac{p-1}{2}}\int_{\mathbb{R}^n}|D\phi(y)|\,dy.\end{aligned}$$

and from the fact that $16\epsilon^2\ge|\xi|^2+s^2$, we have $17\epsilon^2\ge|\xi|^2+\epsilon^2+s^2$ and

$$|D_\xi a_\epsilon(\xi,x,t)| \le 5\cdot2^{\frac{p-1}{2}}\Lambda(|\xi|^2+\epsilon^2+s^2)^{\frac{p-2}{2}}\int_{\mathbb{R}^n}|D\phi(y)|\,dy.\tag{2.5}$$

So we discover that the first inequality in (2.3) holds for the case $16\epsilon^2\ge|\xi|^2+s^2$.

On the other-hand, if $16\epsilon^2 \leq |\xi|^2 + s^2$, then we have

$$|\xi - \epsilon y|^2 + s^2 = |\xi|^2 - 2\epsilon\langle \xi, y\rangle + \epsilon^2|y|^2 + s^2 \geq \frac{|\xi|^2 + s^2 + \epsilon^2|y|^2}{2} \qquad (y \in \overline{B_1}),$$

and $\operatorname{supp}\phi \subset \overline{B_1}$ implies

$$\begin{aligned}|D_\xi a_\epsilon(\xi, x, t)| &\leq \left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_\xi a(\xi - \epsilon y, x - \epsilon z, t)\phi(y)\phi(z)\, dydz \right| \\ &\leq \Lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (|\xi - \epsilon y|^2 + s^2)^{\frac{p-2}{2}}\phi(y)\phi(z)\, dydz \\ &\leq 2\Lambda \int_{\mathbb{R}^n} (|\xi - \epsilon y|^2 + s^2)^{\frac{p}{2}}(|\xi|^2 + s^2 + \epsilon^2|y|^2)^{-1}\phi(y)\, dy,\end{aligned}$$

which implies that

$$|D_\xi a_\epsilon(\xi, x, t)| \leq c \int_{\mathbb{R}^n} (|\xi|^2 + s^2 + \epsilon^2|y|^2)^{\frac{p-2}{2}}\phi(y)\, dy. \tag{2.6}$$

We claim that if $16\epsilon^2 \leq |\xi|^2 + s^2$ and $|y| \leq 1$ then

$$(|\xi|^2 + s^2 + \epsilon^2|y|^2)^{\frac{p-2}{2}} \leq 2(|\xi|^2 + s^2 + \epsilon^2)^{\frac{p-2}{2}}. \tag{2.7}$$

If $p \geq 2$, then the claim (2.7) holds trivially. If $1 < p < 2$, then $16\epsilon^2 \leq |\xi|^2 + s^2$ implies

$$(|\xi|^2 + s^2 + \epsilon^2|y|^2)^{\frac{p-2}{2}} \leq (|\xi|^2 + s^2)^{\frac{p-2}{2}} \leq \Big(\frac{|\xi|^2 + s^2 + \epsilon^2}{2}\Big)^{\frac{p-2}{2}} \leq 2(|\xi|^2 + s^2 + \epsilon^2)^{\frac{p-2}{2}},$$

and we find that the claim (2.7) holds. Thus the claim (2.7) is proved. In light of (2.6) and (2.7), we have that if $16\epsilon^2 \leq |\xi|^2 + s^2$ then

$$|D_\xi a_\epsilon(\xi, x, t)| \leq c(|\xi|^2 + s^2 + \epsilon^2)^{\frac{p-2}{2}}. \tag{2.8}$$

Thus the first inequality in (2.3) follows from (2.4), (2.5) and (2.8). This completes the proof. □

Later, we will apply the gradient of the weak solution in Lemma 2.14 by considering a zero extension from Ω_T to $\mathbb{R}^n_T$.

Lemma 2.14. *For any $H \in L^p(\Omega_T, \mathbb{R}^n)$, we have that*

$$\lim_{\epsilon\searrow 0} \|a(H, \cdot) - a_\epsilon(H, \cdot)\|_{L^{\frac{p}{p-1}}(\Omega_T)} = 0.$$

Proof. Fix $\delta > 0$. From (1.5), we have

$$a(H(x,t), x, t) - a_\epsilon(H(x,t), x, t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [a(H(x,t), x, t) - a(H(x,t) - \epsilon y, x - \epsilon z, t)]\phi(y)\phi(z)\, dydz.$$

Let $\tilde{\Omega}_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$ and $\tilde{\Omega}_{\epsilon,T} = \tilde{\Omega}_\epsilon \times [0, T]$. Since $H \in L^p(\Omega_T, \mathbb{R}^n)$, there exists $\epsilon_\delta > 0$ such that if $\epsilon \in (0, \epsilon_\delta]$ then

$$\int_{\Omega_T\setminus\tilde{\Omega}_{\epsilon,T}} |H|^p\, dx < \delta,$$

which implies that

$$\|a(H,\cdot)-a_\epsilon(H,\cdot)\|_{L^{\frac{p}{p-1}}(\Omega_T\setminus\tilde{\Omega}_{\epsilon,T})} = \left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(\cdot),\cdot)-a(H(\cdot)-\epsilon y,\cdot-(\epsilon z,0))]\phi(y)\phi(z)\,dydz\right\|_{L^{\frac{p}{p-1}}(\Omega_T\setminus\tilde{\Omega}_{\epsilon,T})}$$
$$\leq c\left\|\left(|H(\cdot)|^2+s^2+\epsilon^2\right)^{\frac{p-1}{2}}\right\|_{L^{\frac{p}{p-1}}(\Omega_T\setminus\tilde{\Omega}_{\epsilon,T})}$$
$$\leq c\left[\delta+|\Omega_T\setminus\tilde{\Omega}_{\epsilon,T}|\,(s^p+\epsilon^p)\right]^{\frac{p-1}{p}},$$

for any $\epsilon\in(0,\epsilon_\delta]$. Thus

$$\limsup_{\epsilon\searrow 0}\|a(H,\cdot)-a_\epsilon(H,\cdot)\|_{L^{\frac{p}{p-1}}(\Omega_T\setminus\tilde{\Omega}_{\epsilon,T})} < c\delta^{\frac{p-1}{p}}.$$

Since $\delta>0$ was arbitrary chosen, we get

$$\lim_{\epsilon\searrow 0}\|a(H,\cdot)-a_\epsilon(H,\cdot)\|_{L^{\frac{p}{p-1}}(\Omega_T\setminus\tilde{\Omega}_{\epsilon,T})} = 0. \tag{2.9}$$

We now estimate $a(H,\cdot)-a_\epsilon(H,\cdot)$ on $\tilde{\Omega}_{\epsilon,T}$. By the triangle inequality,

$$\begin{aligned}&\|a(H,\cdot)-a_\epsilon(H,\cdot)\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}\\ &\quad=\left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(\cdot),\cdot)-a(H(\cdot)-\epsilon y,\cdot-(\epsilon z,0))]\phi(y)\phi(z)\,dydz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}\\ &\quad\leq I+II+III\end{aligned} \tag{2.10}$$

where

$$I=\left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(\cdot),\cdot)-a(H(\cdot-(\epsilon z,0)),\cdot-(\epsilon z,0))]\phi(y)\phi(z)\,dydz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})},$$
$$II=\left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(\cdot-(\epsilon z,0)),\cdot-(\epsilon z,0))-a(H(\cdot),\cdot-(\epsilon z,0))]\phi(y)\phi(z)\,dydz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})},$$
$$III=\left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(\cdot),\cdot-(\epsilon z,0))-a(H(\cdot)-\epsilon y,\cdot-(\epsilon z,0))]\phi(y)\phi(z)\,dydz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}.$$

We want to prove that the left-hand side of (2.10) goes to the zero as $\epsilon\searrow 0$.

To handle I, we use the standard approximation by mollifiers, see for instance [34, C. Theorem 6], to find that

$$\lim_{\epsilon\searrow 0}\left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(\cdot),\cdot)-a(H(\cdot-(\epsilon z,0)),\cdot-(\epsilon z,0))]\phi(y)\phi(z)\,dydz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}=0,$$

where we used that $a(H,\cdot)\in L^{\frac{p}{p-1}}(\Omega_T)$ and $\int_{\mathbb{R}^n}\phi(y)\,dy=1$, which implies that

$$\lim_{\epsilon\searrow 0} I = 0. \tag{2.11}$$

To handle II, we apply Hölder's inequality and Lemma 2.10 to find that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}[a(H(x-\epsilon z,t),x-\epsilon z,t)-a(H(x,t),x-\epsilon z,t)]\phi(z)dz\right|\\
&\quad\le\left|\int_{\mathbb{R}^n}|a(H(x-\epsilon z,t),x-\epsilon z,t)-a(H(x,t),x-\epsilon z,t)|^{\frac{p}{p-1}}\,\phi(z)\,dz\right|^{\frac{p-1}{p}}\left|\int_{\mathbb{R}^n}\phi(z)\,dz\right|^{\frac{1}{p}}\\
&\quad\le c\left|\int_{\mathbb{R}^n}|H(x-\epsilon z,t)-H(x,t)|(|H(x-\epsilon z,t)|^2+|H(x,t)|^2+s^2)^{\frac{p-1}{2}}\phi(z)\,dz\right|^{\frac{p-1}{p}}.
\end{aligned}$$

We apply Hölder's inequality to find that

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n}[a(H(\cdot-(\epsilon z,0)),\cdot-(\epsilon z,0))-a(H(\cdot),\cdot-(\epsilon z,0))]\phi(z)dz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}\\
&\le\left\|\int_{\mathbb{R}^n}|H(\cdot-(\epsilon z,0))-H(\cdot)|^p\phi(z)\,dz\right\|^{\frac{p-1}{p^2}}_{L^1(\tilde{\Omega}_{\epsilon,T})}\left\|\int_{\mathbb{R}^n}(|H(\cdot-(\epsilon z,0))|^2+|H(\cdot)|^2+s^2)^{\frac{p}{2}}\phi(z)\,dz\right\|^{\left(\frac{p-1}{p}\right)^2}_{L^1(\tilde{\Omega}_{\epsilon,T})},
\end{aligned}$$

and by using that $H\in L^p(\Omega_T,\mathbb{R}^n)$, we obtain that

$$\lim_{\epsilon\searrow 0}\left\|\int_{\mathbb{R}^n}[a(H(\cdot-(\epsilon z,0)),\cdot-(\epsilon z,0))-a(H(\cdot),\cdot-(\epsilon z,0))]\phi(z)\,dz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}=0,$$

which implies that

$$\lim_{\epsilon\searrow 0}II=0. \tag{2.12}$$

Last, to handle III, we find from Lemma 2.10 that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(x,t),x-\epsilon z,t)-a(H(x,t)-\epsilon y,x-\epsilon z,t)]\phi(y)\phi(z)\,dydz\\
&\quad\le c\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\epsilon y|(|H(x,t)|^2+|H(x,t)-\epsilon y|^2+s^2)^{\frac{p-1}{2}}\phi(y)\phi(z)\,dydz\\
&\quad\le c\epsilon\int_{\mathbb{R}^n}(|H(x,t)|^2+s^2+\epsilon^2)^{\frac{p-1}{2}}\phi(y)\,dy,
\end{aligned}$$

where we used that $\operatorname{supp}\phi\subset\overline{B_1}$ from (1.3). So by that $\int_{\mathbb{R}^n}\phi(y)\,dy=1$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(x,t),x-\epsilon z,t)-a(H(x,t)-\epsilon y,x-\epsilon z,t)]\phi(y)\phi(z)\,dydz\le c\epsilon(|H(x,t)|^2+s^2+\epsilon^2)^{\frac{p-1}{2}}.$$

So we again use Hölder's inequality to find that

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H(\cdot),\cdot-(\epsilon z,0))-a(H(\cdot)-\epsilon y,\cdot-(\epsilon z,0))]\phi(y)\phi(z)\,dz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}\\
&\quad\le c\epsilon\left\|(|H|^2+s^2+\epsilon^2)^{\frac{p-1}{2}}\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}.
\end{aligned}$$

By using $H\in L^p(\Omega_T,\mathbb{R}^n)$, we obtain that

$$\lim_{\epsilon\searrow 0}\left\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[a(H,\cdot-(\epsilon z,0))-a(H-\epsilon y,\cdot-(\epsilon z,0))]\phi(y)\phi(z)\,dz\right\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})}=0,$$

which implies that

$$\lim_{\epsilon\searrow 0} III = 0. \tag{2.13}$$

By combining (2.10), (2.11), (2.12) and (2.13), we find from that

$$\lim_{\epsilon\searrow 0} \|a(H,\cdot) - a_\epsilon(H,\cdot)\|_{L^{\frac{p}{p-1}}(\tilde{\Omega}_{\epsilon,T})} = 0,$$

and the lemma holds from (2.9). □

3. Regularization of nonlinear parabolic equations

This section is devoted to the proof of our main result, Theorem 1.6. We start with proving our main tools for convergence lemmas for the zero extensions, Lemmas 3.1–3.7. Then we apply these tools to obtain the convergence lemmas, Lemmas 3.8–3.10. To conclude our main result, we apply an indirect method. By negating the conclusion of Theorem 1.6, we show that (3.1) contradicts Lemma 3.9 and Lemma 3.10.

Let $\bar{u}_k \in L^p(0,T;W_0^{1,p}(\mathbb{R}^n))\cap L^\infty(0,T;L^2(\mathbb{R}^n))$ be the zero extension of $u_k-\gamma_k \in L^p(0,T;W_0^{1,p}(\Omega^k))\cap L^\infty(0,T;L^2(\Omega^k))$ in Theorem 1.6. Also we define $\bar{u} \in L^p(0,T;W_0^{1,p}(\mathbb{R}^n)) \cap L^\infty(0,T;L^2(\mathbb{R}^n))$ as the zero extension of $u-\gamma \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ in (1.6). To prove Theorem 1.6, we will assume that the conclusion of Theorem 1.6 does not hold. Then there exist $\delta_0 > 0$ and a subsequence, which will be still denoted as u_k ($k \in \mathbb{N}$), such that

$$\left[\|Du_k - Du\|_{L^p(\Omega_T^k\cap\Omega_T)} + \|Du_k\|_{L^p(\Omega_T^k\setminus\Omega_T)} + \|Du\|_{L^p(\Omega_T\setminus\Omega_T^k)}\right] > \delta_0.$$

So by (1.7) and (1.9), it follows that

$$\int_{\mathbb{R}^n_T} |D\bar{u}_k - D\bar{u}|^p \, dxdt > c\delta_0. \tag{3.1}$$

Later, we will show that a contradiction occurs due to (3.1).

To prove Theorem 1.6, we first derive the energy estimates for regularized parabolic problems in (1.10). We test (1.10) by $u_k - \gamma_k \in L^p(0,T;W_0^{1,p}(\Omega^k)) \cap C([0,T];L^2(\Omega^k))$ to find that

$$\begin{aligned}
&\int_0^\tau \langle\!\langle \partial_t u_k, u_k - \gamma_k\rangle\!\rangle_{\Omega^k} \, dt + \int_{\Omega^k_\tau} \langle a_k(Du_k,x,t), Du_k - D\gamma_k\rangle \, dxdt \\
&= \int_{\Omega^k_\tau} \langle |F_k|^{p-2}F_k, Du_k - D\gamma_k\rangle + f_k(u_k-\gamma_k) \, dxdt,
\end{aligned}$$

for any $\tau \in [0,T]$, which implies that

$$\begin{aligned}
&\int_0^\tau \langle\!\langle \partial_t(u_k-\gamma_k), u_k - \gamma_k\rangle\!\rangle_{\Omega^k} \, dt + \int_{\Omega^k_\tau} \langle a_k(Du_k,x,t) - a_k(D\gamma_k,x,t), Du_k - D\gamma_k\rangle \, dxdt \\
&= \int_{\Omega^k_\tau} \langle |F_k|^{p-2}F_k, Du_k - D\gamma_k\rangle + f_k(u_k-\gamma_k) \, dxdt \\
&\quad - \int_{\Omega^k_\tau} \langle a_k(D\gamma_k,x,t), Du_k - D\gamma_k\rangle \, dxdt - \int_0^\tau \langle\!\langle \partial_t\gamma_k, u_k - \gamma_k\rangle\!\rangle_{\Omega^k} \, dt,
\end{aligned}$$

for any $\tau \in [0, T]$. So by Poincaré's inequality and Lemma 2.7,

$$\sup_{0\le\tau\le T} \int_{\Omega^k} |(u_k - \gamma_k)(\cdot, \tau)|^2 \, dx + \int_{\Omega^k_T} |Du_k - D\gamma_k|^p \, dxdt$$
$$\le c \left[\|F_k\|_{L^p(\Omega^k_T)} + \|f_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)} + \|D\gamma_k\|_{L^p(\Omega^k_T)} + \|\partial_t \gamma_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)} \right].$$

Here, the constant $c > 0$ for Poincaré's inequality only depends on the size of the domain and $1 < p < \infty$, see [5, Theorem 6.30]. By taking $\bar{u}_k = u_k - \gamma_k \in L^p(0, T; W^{1,p}_0(\Omega^k)) \cap L^\infty(0, T; L^2(\Omega^k))$,

$$\sup_{0\le\tau\le T} \int_{\Omega^k} |\bar{u}_k(\cdot, \tau)|^2 \, dx + \int_{\Omega^k_T} |D\bar{u}_k|^p \, dxdt$$
$$\le c \left[\||F_k|^{p-2} F_k\|_{L^{p'}(\Omega^k_T)} + \|f_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)} + \|D\gamma_k\|_{L^p(\Omega^k_T)} + \|\partial_t \gamma_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)} \right]. \tag{3.2}$$

The domain Ω^k depends on the function $\bar{u}_k$ ($k \in \mathbb{N}$). To deal with the convergence of the functions, we need to consider the domain of the functions. It is the main reason why we adapted Definitions 1.2–1.5.

To use the compactness method, we need to show that the right-hand side of (3.2) is bounded uniformly. To do it, we use the zero extensions to $\mathbb{R}^n_T$, which makes the domain of the functions independent of $k \in \mathbb{N}$.

Let $\bar{v}_k \in L^p(0, T; W^{1,p}_0(\mathbb{R}^n))$ ($k \in \mathbb{N} \cup \{\infty\}$) be the zero extensions of $v_k \in L^p(0, T; W^{1,p}_0(\Omega^k))$ from Ω^k_T to $\mathbb{R}^n_T$. Also for $h_k \in W^{-1,p'}(\Omega^k)$ ($k \in \mathbb{N} \cup \{\infty\}$), we define $\bar{h}_k \in W^{-1,p'}(\mathbb{R}^n)$ which corresponds to the zero extension in Corollary 3.3. Under the assumption (1.7), we obtain the following results.

(1) [Lemma 3.1] If $v_k \in L^q(\Omega^k_T) \overset{*}{\to} v_\infty \in L^q(\Omega^\infty_T)$ ($1 < q < \infty$) then

$$\bar{v}_k \to \bar{v}_\infty \text{ in } L^q(\mathbb{R}^n_T).$$

(2) [Lemma 3.4] If $h_k \in W^{-1,p'}(\Omega^k) \overset{*}{\to} h_\infty \in W^{-1,p'}(\Omega^\infty)$ then

$$\bar{h}_k \overset{*}{\to} \bar{h}_\infty \text{ in } W^{-1,p'}(\mathbb{R}^n).$$

(3) [Lemma 3.5] If $h_k \in L^{p'}(0, T; W^{-1,p'}(\Omega^k)) \overset{*}{\to} h_\infty \in L^{p'}(0, T; W^{-1,p'}(\Omega^\infty))$ then

$$\bar{h}_k \overset{*}{\to} \bar{h}_\infty \text{ in } L^{p'}(0, T; W^{-1,p'}(\mathbb{R}^n)).$$

(4) [Lemma 3.6] If the sequence $\|v_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)}$ ($k \in \mathbb{N}$) is bounded then there exists $v_\infty \in L^{p'}(0, T; W^{-1,p'}(\Omega^\infty))$ with

$$\bar{v}_k \overset{*}{\rightharpoonup} \bar{v}_\infty \text{ in } L^{p'}(0, T; W^{-1,p'}(\mathbb{R}^n)).$$

(5) [Lemma 3.7] If the sequence $\|v_k\|_{L^\infty\left(0,T;L^2(\Omega^k)\right)}$ ($k \in \mathbb{N}$) is bounded then there exists $v_\infty \in L^\infty(0, T; L^2(\Omega^\infty))$ with

$$\bar{v}_k \overset{*}{\rightharpoonup} \bar{v}_\infty \text{ in } L^\infty(0, T; L^2(\mathbb{R}^n)).$$

We apply Lemmas 3.1–3.7 to (3.2) as follows. By using Lemma 3.1, we will show that the zero extensions of $|F_k|^{p-2}F_k$, γ_k and $D\gamma_k$ converge strongly-$*$. By using Lemma 3.5, we will show that the zero extensions of f_k and $\partial_t\gamma_k$ converge strongly-$*$. With Lemma 3.6, the existence of weakly-$*$ converging subsequence of $\partial_t\bar{u}_k$ in $L^{p'}(0,T;W^{-1,p'}(\mathbb{R}^n))$ will be obtained. Also with Lemma 3.7, the existence of weakly-$*$ converging subsequence of $\bar{u}_k$ in $L^\infty(0,T;L^2(\mathbb{R}^n))$ will be obtained.

We prove our main tools for convergence lemmas. From now on, we denote 1_E as the indicator function on the set E.

Lemma 3.1. *With the assumption* (1.7)*, suppose that* $1<q<\infty$ *and* $N\geq 1$. *If*

$$V_k \in L^{q'}(\Omega_T^k,\mathbb{R}^N) \ \overset{*}{\to}\ V_\infty \in L^{q'}(\Omega_T^\infty,\mathbb{R}^N),$$

then

$$\bar{V}_k \ \to\ \bar{V}_\infty \ \ in\ \ L^{q'}(\mathbb{R}^n_T,\mathbb{R}^N),$$

where $\bar{V}_k\in L^{q'}(\mathbb{R}^n_T,\mathbb{R}^N)$ *is the zero extension of* $V_k\in L^{q'}(\Omega_T^k,\mathbb{R}^N)$.

Proof. Suppose that $V_k \in L^{q'}(\Omega_T^k,\mathbb{R}^N) \ \overset{*}{\to}\ V_\infty \in L^{q'}(\Omega_T^\infty,\mathbb{R}^N)$. By (1.7),

$$\bar{\eta}\,1_{\Omega_T^k} \ \to\ \bar{\eta}\,1_{\Omega_T^\infty} \ \text{ in } \ L^q(\mathbb{R}^n_T,\mathbb{R}^N),$$

for any $\bar{\eta}\in L^q(\mathbb{R}^n_T,\mathbb{R}^N)$. So by Definition 1.2, we have that

$$\int_{\mathbb{R}^n_T}\langle\bar{V}_k,\bar{\eta}\rangle\,dxdt = \int_{\Omega_T^k}\langle V_k,\bar{\eta}\,1_{\Omega_T^k}\rangle\,dxdt \to \int_{\Omega_T^\infty}\langle V_\infty,\bar{\eta}\,1_{\Omega_T^\infty}\rangle\,dxdt = \int_{\mathbb{R}^n_T}\langle\bar{V}_\infty,\bar{\eta}\rangle\,dxdt,$$

which implies that

$$\bar{V}_k \rightharpoonup \bar{V}_\infty \ \text{ in } \ L^{q'}(\mathbb{R}^n_T,\mathbb{R}^N). \tag{3.3}$$

Suppose the lemma does not hold. Then there exist $\delta>0$ and a subsequence (which will be still denoted as $\{\bar{V}_k\}_{k=1}^\infty$) such that

$$\int_{\mathbb{R}^n_T}|\bar{V}_k-\bar{V}_\infty|^{q'}\,dxdt > \delta \qquad (k\in\mathbb{N}). \tag{3.4}$$

Choose $\bar{\eta}_k = |\bar{V}_k-\bar{V}_\infty|^{q'-2}(\bar{V}_k-\bar{V}_\infty)$ then

$$\|\bar{\eta}_k\|_{L^q(\mathbb{R}^n_T,\mathbb{R}^N)} = \|\bar{V}_k-\bar{V}_\infty\|^{\frac{1}{q-1}}_{L^{q'}(\mathbb{R}^n_T,\mathbb{R}^N)}. \qquad (k\in\mathbb{N}).$$

Since $(\bar{V}_k-\bar{V}_\infty)\rightharpoonup 0$ in $L^{q'}(\mathbb{R}^n_T,\mathbb{R}^N)$ and any weakly convergent sequence is bounded, we see that $\{\bar{\eta}_k\}_{k=1}^\infty$ is bounded in $L^q(\mathbb{R}^n_T,\mathbb{R}^N)$. So there exists a subsequence (which will be still denoted as $\{\bar{\eta}_k\}_{k=1}^\infty$) such that

$$\bar{\eta}_k \ \rightharpoonup\ \bar{\eta}_\infty \ \text{ in } \ L^q(\mathbb{R}^n_T,\mathbb{R}^N),$$

for some $\bar{\eta}_\infty\in L^q(\mathbb{R}^n_T,\mathbb{R}^N)$. By (1.7) and that $(\bar{V}_k-\bar{V}_\infty)\rightharpoonup 0$ in $L^{q'}(\mathbb{R}^n_T,\mathbb{R}^N)$,

$$\bar{\eta}_\infty = 0 \ \text{ in } \ \mathbb{R}^n_T\setminus\Omega_T^\infty.$$

Also we have that

$$\bar{\eta}_k \cdot 1_{\Omega_T^k} \rightharpoonup \bar{\eta}_\infty \cdot 1_{\Omega_T^\infty} \quad \text{in} \quad L^p(\mathbb{R}_T^n, \mathbb{R}^N), \tag{3.5}$$

because for any $\tilde{V} \in L^{q'}(\mathbb{R}_T^n, \mathbb{R}^N)$,

$$\int_{\mathbb{R}_T^n} \langle \tilde{V}, \bar{\eta}_k 1_{\Omega_T^k} \rangle \, dxdt = \int_{\mathbb{R}_T^n} \langle \tilde{V} \cdot 1_{\Omega_T^\infty}, \bar{\eta}_k \rangle \, dxdt + \int_{\mathbb{R}_T^n} \langle \tilde{V}(1_{\Omega_T^k} - 1_{\Omega_T^\infty}), \bar{\eta}_k \rangle \, dxdt \to \int_{\mathbb{R}_T^n} \langle \tilde{V}, \bar{\eta}_\infty 1_{\Omega_T^\infty} \rangle \, dxdt,$$

which holds from $|\Omega^k \setminus \Omega| \to 0$ and $|\Omega \setminus \Omega^k| \to 0$ by (1.7). From (3.5) and that $V_k \in L^{q'}(\Omega_T^k, \mathbb{R}^N) \overset{*}{\to} V_\infty \in L^{q'}(\Omega_T^\infty, \mathbb{R}^N)$, we use Definition 1.2 to find that

$$\int_{\mathbb{R}_T^n} \langle \bar{V}_k, \bar{\eta}_k \rangle \, dxdt = \int_{\Omega_T^k} \langle V_k, \bar{\eta}_k \cdot 1_{\Omega_T^k} \rangle \, dxdt \to \int_{\Omega_T^\infty} \langle V_\infty, \bar{\eta}_\infty \cdot 1_{\Omega_T^\infty} \rangle \, dxdt = \int_{\mathbb{R}_T^n} \langle \bar{V}_\infty, \bar{\eta}_\infty \rangle \, dxdt,$$

which implies that

$$\int_{\mathbb{R}_T^n} \langle \bar{V}_k - \bar{V}_\infty, \bar{\eta}_k \rangle \, dxdt = \int_{\mathbb{R}_T^n} \langle \bar{V}_k, \bar{\eta}_k \rangle \, dxdt - \int_{\mathbb{R}_T^n} \langle \bar{V}_\infty, \bar{\eta}_k \rangle \, dxdt \to 0. \tag{3.6}$$

On the other-hand, by (3.4), we find that

$$\int_{\mathbb{R}_T^n} \langle \bar{V}_k - \bar{V}_\infty, \bar{\eta}_k \rangle \, dxdt = \int_{\mathbb{R}_T^n} |\bar{V}_k - \bar{V}_\infty|^{q'} \, dxdt > \delta > 0 \qquad (k \in \mathbb{N}),$$

which contradicts (3.6). So the lemma follows. □

We have the following characterization for $h \in W^{-1,p'}(\Omega)$.

Lemma 3.2. *With the assumption* (1.7), *suppose that* $h \in W^{-1,p'}(\Omega)$ $(1 < p < \infty)$. *Then there exists* $v \in W_0^{1,p}(\Omega)$ *such that*

$$\int_\Omega \big\langle (|v|^{p-2}v, |Dv|^{p-2}Dv), (\varphi, D\varphi) \big\rangle \, dx = \langle\!\langle h, \varphi \rangle\!\rangle_{\langle W^{-1,p'}(\Omega), W_0^{1,p}(\Omega) \rangle},$$

for any $\varphi \in W_0^{1,p}(\Omega)$. *In addition, we have that* $\|h\|_{W^{-1,p'}(\Omega)} = \|v\|^{p-1}_{W_0^{1,p}(\Omega)}$.

Proof. Since $h \in W^{-1,p'}(\Omega)$, there exists $H = (H_0, H_1, \cdots, H_n) \in L^{p'}(\Omega, \mathbb{R}^{n+1})$ satisfying

$$\langle\!\langle h, \varphi \rangle\!\rangle_{\langle W^{-1,p'}(\Omega), W_0^{1,p}(\Omega) \rangle} = \int_\Omega \big\langle H, (\varphi, D\varphi) \big\rangle \, dx \ \text{ for any } \varphi \in W_0^{1,p}(\Omega),$$

by Proposition 2.5. Let $v \in W_0^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} |v|^{p-2}v - \operatorname{div} |Dv|^{p-2}Dv &= H_0 - \operatorname{div} [(H_1, \cdots, H_n)] \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then for any $\varphi \in W^{1,p}(\Omega)$, we get

$$\begin{aligned} \int_\Omega \big\langle (|v|^{p-2}v, |Dv|^{p-2}Dv), (\varphi, D\varphi) \big\rangle \, dx &= \int_\Omega \big\langle H, (\varphi, D\varphi) \big\rangle \, dx \\ &= \langle\!\langle h, \varphi \rangle\!\rangle_{\langle W^{-1,p'}(\Omega), W_0^{1,p}(\Omega) \rangle}. \end{aligned}$$

So by the definition of $\|\cdot\|_{W^{-1,p'}(\Omega)}$,

$$\|h\|_{W^{-1,p'}(\Omega)} = \sup_{\|\varphi\|_{W_0^{1,p}(\Omega)=1}} \langle\!\langle h, \varphi \rangle\!\rangle_{\langle W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)\rangle} \le \|v\|_{W_0^{1,p}(\Omega)}^{p-1}.$$

By taking $\varphi = \frac{v}{\|v\|_{W_0^{1,p}(\Omega)}} \in W_0^{1,p}(\Omega)$, we get

$$\|v\|_{W_0^{1,p}(\Omega)}^{p-1} \le \|h\|_{W^{-1,p'}(\Omega)}.$$

By combining the above two estimates, we get $\|h\|_{W^{-1,p'}(\Omega)} = \|v\|_{W_0^{1,p}(\Omega)}^{p-1}$. □

We extend $h \in L^{p'}(0,T; W^{-1,p'}(\Omega))$ to $\bar{h} \in L^{p'}(0,T; W^{-1,p'}(\mathbb{R}^n))$ in Corollary 3.3, which can be viewed as a natural zero extension because of (3.7).

Corollary 3.3. *With the assumption* (1.7)*, suppose that* $h \in W^{-1,p'}(\Omega)$ $(1 < p < \infty)$*. Then for* $v \in W_0^{1,p}(\Omega)$ *in Lemma 3.2, one can define* $\bar{h} \in W^{-1,p'}(\mathbb{R}^n)$ *as*

$$\left\langle\!\!\left\langle \bar{h}, \bar{\varphi} \right\rangle\!\!\right\rangle_{\langle W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)\rangle} = \int_{\mathbb{R}^n} \left\langle \left(|\bar{v}|^{p-2}\bar{v}, |D\bar{v}|^{p-2}D\bar{v} \right), (\bar{\varphi}, D\bar{\varphi}) \right\rangle dx, \tag{3.7}$$

for any $\bar{\varphi} \in W_0^{1,p}(\mathbb{R}^n)$*, where* $\bar{v} \in W_0^{1,p}(\mathbb{R}^n)$ *is the zero extension of* $v \in W_0^{1,p}(\Omega)$*. Moreover, we have that*

$$\left\langle\!\!\left\langle \bar{h}, \bar{\varphi} \right\rangle\!\!\right\rangle_{\langle W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)\rangle} = \langle h, \varphi \rangle_{\langle W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)\rangle} \tag{3.8}$$

for any $\varphi \in W_0^{1,p}(\Omega)$ *and the zero extension* $\bar{\varphi} \in W_0^{1,p}(\mathbb{R}^n)$ *of* $\varphi \in W_0^{1,p}(\Omega)$*. In addition,*

$$\|\bar{h}\|_{W^{-1,p'}(\mathbb{R}^n)} = \|\bar{v}\|_{W_0^{1,p}(\mathbb{R}^n)}^{p-1} = \|v\|_{W_0^{1,p}(\Omega)}^{p-1} = \|h\|_{W^{-1,p'}(\Omega)}.$$

In Definition 1.4, we defined a convergence for a sequence of the domains, say $h_k \in W^{-1,p'}(\Omega^k) \xrightarrow{*} h_\infty \in W^{-1,p'}(\Omega^\infty)$. But this convergence implies strong convergence by considering the zero extension in Corollary 3.3 as in the next lemmas.

Lemma 3.4. *Under the assumption* (1.7) *and* $1 < p < \infty$*, if* $h_k \in W^{-1,p'}(\Omega^k) \xrightarrow{*} h_\infty \in W^{-1,p'}(\Omega^\infty)$ *then*

$$\bar{h}_k \xrightarrow{*} \bar{h}_\infty \ \ in \ \ W^{-1,p'}(\mathbb{R}^n),$$

and

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} (|\bar{v}_k|^2 + |\bar{v}_\infty|^2)^{\frac{p-2}{2}} |\bar{v}_k - \bar{v}_\infty|^2 \, dx \to 0, \\ \displaystyle\int_{\mathbb{R}^n} \left(|D\bar{v}_k|^2 + |D\bar{v}_\infty|^2 \right)^{\frac{p-2}{2}} |D\bar{v}_k - D\bar{v}_\infty|^2 \, dx \to 0, \end{cases} \tag{3.9}$$

for $\bar{v}_k \in W_0^{1,p}(\mathbb{R}^n)$ *and* $\bar{h}_k \in W^{-1,p'}(\mathbb{R}^n)$ $(k \in \mathbb{N} \cup \{\infty\})$ *in Corollary 3.3.*

Proof. By using Corollary 3.3, define $\bar{h}_k \in W^{-1,p'}(\mathbb{R}^n)$ $(k \in \mathbb{N} \cup \{\infty\})$ as

$$\left\langle\!\left\langle \bar{h}_k, \bar{\varphi} \right\rangle\!\right\rangle_{\langle W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)\rangle} = \int_{\mathbb{R}^n} \left\langle \left(|\bar{v}_k|^{p-2}\bar{v}_k, |D\bar{v}_k|^{p-2}D\bar{v}_k\right), (\bar{\varphi}, D\bar{\varphi}) \right\rangle dx, \tag{3.10}$$

for any $\bar{\varphi} \in W_0^{1,p}(\mathbb{R}^n)$. Here, $v_k \in W_0^{1,p}(\Omega^k)$ $(k \in \mathbb{N} \cup \{\infty\})$ is defined in Lemma 3.2 and $\bar{v}_k \in W_0^{1,p}(\mathbb{R}^n)$ the zero extension of $v_k \in W_0^{1,p}(\Omega^k)$. Moreover,

$$\|\bar{h}_k\|_{W^{-1,p'}(\mathbb{R}^n)} = \|\bar{v}_k\|^{p-1}_{W_0^{1,p}(\mathbb{R}^n)} = \|v_k\|^{p-1}_{W_0^{1,p}(\Omega)} = \|h_k\|_{W^{-1,p'}(\Omega)} \qquad (k \in \mathbb{N} \cup \{\infty\}).$$

For $k \in \mathbb{N} \cup \{\infty\}$, let $V_k = \left(|v_k|^{p-2}v_k, |Dv_k|^{p-2}Dv_k\right) \in L^{p'}(\Omega^k, \mathbb{R}^{n+1})$ and $\bar{V}_k \in L^{p'}(\mathbb{R}^n, \mathbb{R}^{n+1})$ be the zero extension of V_k.

Suppose that (3.9) does not hold. Then there exist $\delta > 0$ and a subsequence, which will be still denoted as $\{\bar{v}_k\}_{k=1}^\infty$, such that

$$\int_{\mathbb{R}^n} (|\bar{v}_k|^2 + |\bar{v}_\infty|^2)^{\frac{p-2}{2}} |\bar{v}_k - \bar{v}_\infty|^2 \, dx + \int_{\mathbb{R}^n} \left(|D\bar{v}_k|^2 + |D\bar{v}_\infty|^2\right)^{\frac{p-2}{2}} |D\bar{v}_k - D\bar{v}_\infty|^2 \, dx > \delta \qquad (k \in \mathbb{N}). \tag{3.11}$$

Since $\bar{v}_k \, \|\bar{v}_k\|^{-1}_{W_0^{1,p}(\mathbb{R}^n)}$ is bounded in $W_0^{1,p}(\mathbb{R}^n)$, there exists a subsequence, which will be still denoted as $\bar{v}_k \, \|\bar{v}_k\|^{-1}_{W_0^{1,p}(\mathbb{R}^n)}$ $(k \in \mathbb{N})$, such that

$$\bar{v}_k \, \|\bar{v}_k\|^{-1}_{W_0^{1,p}(\mathbb{R}^n)} \rightharpoonup \tilde{v}_0 \ \text{ in } \ W_0^{1,p}(\mathbb{R}^n),$$

for some $v_0 \in W_0^{1,p}(\Omega^\infty)$ and the zero extension $\bar{v}_0 \in W_0^{1,p}(\mathbb{R}^n)$ of $v_0 \in W_0^{1,p}(\Omega^\infty)$. By taking $\bar{\varphi} = \bar{v}_k \, \|\bar{v}_k\|^{-1}_{W_0^{1,p}(\mathbb{R}^n)}$ in (3.10), we find from Definition 1.4 that

$$\begin{aligned}
\|\bar{v}_k\|^{p-1}_{W_0^{1,p}(\mathbb{R}^n)} &= \frac{1}{\|\bar{v}_k\|_{W_0^{1,p}(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \left\langle \left(|\bar{v}_k|^{p-2}\bar{v}_k, |D\bar{v}_k|^{p-2}D\bar{v}_k\right), (\bar{v}_k, D\bar{v}_k) \right\rangle dx \\
&= \left\langle\!\left\langle \bar{h}_k, \bar{v}_k \, \|\bar{v}_k\|^{-1}_{W_0^{1,p}(\mathbb{R}^n)} \right\rangle\!\right\rangle_{\langle W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)\rangle} \\
&= \left\langle\!\left\langle h_k, v_k \, \|\bar{v}_k\|^{-1}_{W_0^{1,p}(\mathbb{R}^n)} \right\rangle\!\right\rangle_{\langle W^{-1,p'}(\Omega^k), W_0^{1,p}(\Omega^k)\rangle} \\
&\xrightarrow{k\to\infty} \langle h_\infty, v_0 \rangle_{\langle W^{-1,p'}(\Omega^\infty), W_0^{1,p}(\Omega^\infty)\rangle}.
\end{aligned}$$

So $\bar{v}_k$ is bounded in $W_0^{1,p}(\mathbb{R}^n)$, and there exist $\bar{v}_0 \in W_0^{1,p}(\mathbb{R}^n)$, $\bar{V}_0 \in L^{p'}(\mathbb{R}^n, \mathbb{R}^{n+1})$ and a subsequence, which will be still denoted as $\{\bar{v}_k\}_{k=1}^\infty$, such that

$$\begin{cases} D\bar{v}_k \rightharpoonup D\bar{v}_0 & \text{in} \quad L^p(\mathbb{R}^n, \mathbb{R}^n), \\ \bar{v}_k \rightharpoonup \bar{v}_0 & \text{in} \quad L^p(\mathbb{R}^n), \\ \bar{V}_k \rightharpoonup \bar{V}_0 & \text{in} \quad L^{p'}(\mathbb{R}^n, \mathbb{R}^{n+1}). \end{cases} \tag{3.12}$$

Recall that $\bar{V}_k = \left(|\bar{v}_k|^{p-2}\bar{v}_k, |D\bar{v}_k|^{p-2}D\bar{v}_k\right) \in L^{p'}(\mathbb{R}^n, \mathbb{R}^{n+1})$ is the zero extension of $V_k = \left(|v_k|^{p-2}v_k, |Dv_k|^{p-2}Dv_k\right) \in L^{p'}(\Omega^k, \mathbb{R}^{n+1})$. Because of the assumption (1.7), one can also show that

$$\bar{v}_0 = 0 \ \text{ a.e. in } \ \mathbb{R}^n \setminus \Omega^\infty \quad \text{and} \quad \bar{V}_0 = 0 \ \text{ a.e. in } \ \mathbb{R}^n \setminus \Omega^\infty. \tag{3.13}$$

Also by (1.7),

$$\text{there exists } K \in \mathbb{N} \text{ such that } \ supp\,\varphi \subset\subset \Omega^k \,(k \geq K) \text{ for any } \varphi \in C_c^\infty(\Omega^\infty). \tag{3.14}$$

From (3.13), (3.14) and Definition 1.4, we obtain that

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_k, (\bar{\varphi}, D\bar{\varphi}) \right\rangle dx = \int_{\Omega^k} \langle V_k, (\varphi, D\varphi) \rangle \, dx \to \int_{\Omega^\infty} \langle V_\infty, (\varphi, D\varphi) \rangle \, dx,$$

for any $\varphi \in C_c^\infty(\Omega^\infty)$ and the zero extension $\bar{\varphi} \in C_c^\infty(\mathbb{R}^n)$ of $\varphi \in C_c^\infty(\Omega^\infty)$. Also from (3.12), (3.13) and (3.14), we obtain that

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_k, (\bar{\varphi}, D\bar{\varphi}) \right\rangle dx \to \int_{\mathbb{R}^n} \left\langle \bar{V}_0, (\bar{\varphi}, D\bar{\varphi}) \right\rangle dx = \int_{\Omega^\infty} \langle V_0, (\varphi, D\varphi) \rangle \, dx,$$

for any $\varphi \in C_c^\infty(\Omega^\infty)$ and the zero extension $\bar{\varphi} \in C_c^\infty(\mathbb{R}^n)$ of $\varphi \in C_c^\infty(\Omega^\infty)$. Thus

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_\infty - \bar{V}_0, (\varphi, D\varphi) \right\rangle dx = 0$$

for any $\varphi \in C_c^\infty(\Omega^\infty)$. For any $\varphi \in W_0^{1,p}(\Omega^\infty)$, there exists $\varphi_\epsilon \in C_c^\infty(\Omega^\infty)$ with $\|\varphi - \varphi_\epsilon\|_{W_0^{1,p}(\Omega^\infty)} < \epsilon$, which implies that

$$\left| \int_{\Omega^\infty} \left\langle \bar{V}_\infty - \bar{V}_0, (\varphi, D\varphi) \right\rangle dx \right| \leq \epsilon \left(\|\bar{V}_0\|_{L^{p'}(\Omega^\infty)} + \|\bar{V}_\infty\|_{L^{p'}(\Omega^\infty)} \right).$$

Since $\epsilon > 0$ was arbitrary chosen, we find that

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_\infty - \bar{V}_0, (\varphi, D\varphi) \right\rangle dx = \int_{\Omega^\infty} \left\langle \bar{V}_\infty - \bar{V}_0, (\varphi, D\varphi) \right\rangle dx = 0 \tag{3.15}$$

for any $\varphi \in W_0^{1,p}(\Omega^\infty)$.

Fix $\varphi \in C_c^\infty(\Omega^\infty)$. By (3.14), there exists $K \in \mathbb{N}$ with

$$\bar{v}_k - \bar{v}_\infty \varphi \in W_0^{1,p}(\Omega^k) \cap W_0^{1,p}(\mathbb{R}^n) \qquad (k \geq K).$$

By a direct calculation, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_k - \bar{v}_\infty, D[\bar{v}_k - \bar{v}_\infty]) \right\rangle dx \\ &= \int_{\mathbb{R}^n} \left\langle \bar{V}_k - \bar{V}_\infty, ((\bar{v}_k - \bar{v}_\infty \varphi), D[(\bar{v}_k - \bar{v}_\infty \varphi)]) \right\rangle dx \\ &\quad - \int_{\mathbb{R}^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_\infty (1-\varphi), D[\bar{v}_\infty (1-\varphi)]) \right\rangle dx. \end{aligned} \tag{3.16}$$

for any $k \geq K$. By (3.12) and (3.14), $(\bar{v}_k - \bar{v}_\infty \varphi) \rightharpoonup (\bar{v}_0 - \bar{v}_\infty \varphi)$ in $W_0^{1,p}(\mathbb{R}^n)$. So by Definition 1.4,

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_k, ((\bar{v}_k - \bar{v}_\infty \varphi), D(\bar{v}_k - \bar{v}_\infty \varphi)) \right\rangle dx \to \int_{\mathbb{R}^n} \left\langle \bar{V}_\infty, ((\bar{v}_0 - \bar{v}_\infty \varphi), D(\bar{v}_0 - \bar{v}_\infty \varphi)) \right\rangle dx,$$

and

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_\infty, ((\bar{v}_k - \bar{v}_\infty\varphi), D(\bar{v}_k - \bar{v}_\infty\varphi)) \right\rangle dx \to \int_{\mathbb{R}^n} \left\langle \bar{V}_\infty, ((\bar{v}_0 - \bar{v}_\infty)\varphi, D(\bar{v}_0 - \bar{v}_\infty\varphi)) \right\rangle dx,$$

which implies that

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_k - \bar{V}_\infty, ((\bar{v}_k - \bar{v}_\infty\varphi), D(\bar{v}_k - \bar{v}_\infty\varphi)) \right\rangle dx \to 0. \tag{3.17}$$

By (3.12),

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_\infty(1-\varphi), D[\bar{v}_\infty(1-\varphi)]) \right\rangle dx \to \int_{\mathbb{R}^n} \left\langle \bar{V}_0 - \bar{V}_\infty, (\bar{v}_\infty(1-\varphi), D[\bar{v}_\infty(1-\varphi)]) \right\rangle dx. \tag{3.18}$$

By combining (3.17) and (3.18), we use (3.15) to find that

$$\int_{\mathbb{R}^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_k - \bar{v}_\infty, D[\bar{v}_k - \bar{v}_\infty]) \right\rangle dx \to \int_{\mathbb{R}^n} \left\langle \bar{V}_0 - \bar{V}_\infty, (\bar{v}_\infty(1-\varphi), D[\bar{v}_\infty(1-\varphi)]) \right\rangle dx = 0, \tag{3.19}$$

because of that $\bar{v}_\infty(1-\varphi) \in W_0^{1,p}(\Omega^\infty)$. Then by Lemma 2.11,

$$\int_{\mathbb{R}^n} (|\bar{v}_k|^2 + |\bar{v}_\infty|^2)^{\frac{p-2}{2}} |\bar{v}_k - \bar{v}_\infty|^2 + \left(|D\bar{v}_k|^2 + |D\bar{v}_\infty|^2\right)^{\frac{p-2}{2}} |D\bar{v}_k - D\bar{v}_\infty|^2 \, dx \to 0,$$

but this contradicts (3.11) and we find that (3.9) holds. So by Lemma 2.12,

$$\begin{aligned}\int_{\mathbb{R}^n} |\bar{V}_k - \bar{V}_\infty|^{p'} \, dx \le{}& c\left[\int_{\mathbb{R}^n} \left(|D\bar{v}_k|^2 + |D\bar{v}_\infty|^2\right)^{\frac{p-2}{2}} |D\bar{v}_k - D\bar{v}_\infty|^2 \, dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} |D\bar{v}_k|^p + |D\bar{v}_\infty|^p \, dx\right]^{\frac{1}{2}} \\ &+ c\left[\int_{\mathbb{R}^n} \left(|\bar{v}_k|^2 + |\bar{v}_\infty|^2\right)^{\frac{p-2}{2}} |\bar{v}_k - \bar{v}_\infty|^2 \, dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} |\bar{v}_k|^p + |\bar{v}_\infty|^p \, dx\right]^{\frac{1}{2}} \\ \to{}& 0.\end{aligned}$$

This implies that

$$\|\bar{h}_k - \bar{h}_\infty\|_{W^{-1,p'}(\mathbb{R}^n)} = \sup_{\|\bar{\varphi}\|_{W_0^{1,p}(\mathbb{R}^n)}=1} \left\langle\!\left\langle \bar{h}_k - \bar{h}_\infty, \bar{\varphi} \right\rangle\!\right\rangle_{W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)} = \sup_{\|\bar{\varphi}\|_{W_0^{1,p}(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{\varphi}, D\bar{\varphi}) \right\rangle dx \to 0,$$

and the lemma follows. □

Lemma 3.5. *Under the assumption* (1.7) *and* $1 < p < \infty$, *suppose that* $h_k \in L^{p'}(0,T; W^{-1,p'}(\Omega^k)) \overset{*}{\to} h_\infty \in L^{p'}(0,T; W^{-1,p'}(\Omega^\infty))$. *Then*

$$\bar{h}_k \;\to\; \bar{h}_\infty \;\; in \;\; L^{p'}(0,T; W^{-1,p'}(\mathbb{R}^n)),$$

and

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n_T} (|\bar{v}_k|^2 + |\bar{v}_\infty|^2)^{\frac{p-2}{2}} |\bar{v}_k - \bar{v}_\infty|^2 \, dx \to 0, \\ \displaystyle\int_{\mathbb{R}^n_T} \left(|D\bar{v}_k|^2 + |D\bar{v}_\infty|^2\right)^{\frac{p-2}{2}} |D\bar{v}_k - D\bar{v}_\infty|^2 \, dx \to 0, \end{cases} \tag{3.20}$$

for $\bar{v}_k \in L^p(0,T; W_0^{1,p}(\mathbb{R}^n))$ *and* $\bar{h}_k \in L^{p'}(0,T; W^{-1,p'}(\mathbb{R}^n))$ $(k \in \mathbb{N} \cup \{\infty\})$ *in Corollary 3.3.*

Proof. For any $t \in [0,T]$, by using Corollary 3.3, define $\bar{h}_k(\cdot,t) \in W^{-1,p'}(\mathbb{R}^n)$ ($k \in \mathbb{N} \cup \{\infty\}$) as

$$\begin{aligned}&\left\langle\!\left\langle \bar{h}_k(\cdot,t), \bar{\varphi}(\cdot,t) \right\rangle\!\right\rangle_{\langle W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)\rangle} \\ &= \int_{\mathbb{R}^n} \left\langle \left(|\bar{v}_k(\cdot,t)|^{p-2}\bar{v}_k(\cdot,t), |D\bar{v}_k(\cdot,t)|^{p-2} D\bar{v}_k(\cdot,t) \right), (\bar{\varphi}(\cdot,t), D\bar{\varphi}(\cdot,t)) \right\rangle \, dx,\end{aligned} \tag{3.21}$$

for any $\bar{\varphi}(\cdot,t) \in W_0^{1,p}(\mathbb{R}^n)$. Here, $v_k(\cdot,t) \in W_0^{1,p}(\Omega^k)$ ($k \in \mathbb{N} \cup \{\infty\}$) is defined in Lemma 3.2 and $\bar{v}_k(\cdot,t) \in W_0^{1,p}(\mathbb{R}^n)$ is the zero extension of $v_k(\cdot,t) \in W_0^{1,p}(\Omega^k)$.

For any $t \in [0,T]$, let $\bar{V}_k(\cdot,t) \in L^{p'}(\mathbb{R}^n, \mathbb{R}^{n+1})$ ($k \in \mathbb{N} \cup \{\infty\}$) be the zero extension of

$$V_k(\cdot,t) := \left(|v_k(\cdot,t)|^{p-2} v_k(\cdot,t), |Dv_k(\cdot,t)|^{p-2} Dv_k(\cdot,t) \right) \in L^{p'}(\Omega^k, \mathbb{R}^{n+1}). \tag{3.22}$$

Suppose that (3.20) does not hold. Then there exist $\delta > 0$ and a subsequence, which will be still denoted as $\{\bar{v}_k\}_{k=1}^{\infty}$, such that

$$\int_{\mathbb{R}^n_T} \left(|\bar{v}_k|^2 + |\bar{v}_\infty|^2 \right)^{\frac{p-2}{2}} |\bar{v}_k - \bar{v}_\infty|^2 \, dxdt + \int_{\mathbb{R}^n_T} \left(|D\bar{v}_k|^2 + |D\bar{v}_\infty|^2 \right)^{\frac{p-2}{2}} |D\bar{v}_k - D\bar{v}_\infty|^2 \, dxdt > \delta \quad (k \in \mathbb{N}). \tag{3.23}$$

Since $\bar{v}_k \, \|\bar{v}_k\|^{-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))}$ ($k \in \mathbb{N}$) is bounded in $L^p(0,T;W_0^{1,p}(\mathbb{R}^n))$, there exist $v_0 \in L^p(0,T;W_0^{1,p}(\Omega^\infty))$ and a subsequence, which will be still denoted as $\bar{v}_k \, \|\bar{v}_k\|^{-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))}$ ($k \in \mathbb{N}$), such that

$$(\bar{v}_k, D\bar{v}_k) \, \|\bar{v}_k\|^{-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))} \rightharpoonup (\tilde{v}_0, D\tilde{v}_0) \ \text{ in } \ L^p(\mathbb{R}^n_T, \mathbb{R}^{n+1}),$$

where $\tilde{v}_0 \in L^p(0,T;W_0^{1,p}(\mathbb{R}^n))$ is the zero extension of $v_0 \in L^p(0,T;W_0^{1,p}(\Omega^\infty))$. By a direct calculation and Corollary 3.3,

$$\begin{aligned}\|\bar{v}_k\|^{p-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))} &= \frac{1}{\|\bar{v}_k\|_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n_T))}} \int_{\mathbb{R}^n_T} \left\langle \left(|\bar{v}_k|^{p-2}\bar{v}_k, |D\bar{v}_k|^{p-2} D\bar{v}_k \right), (\bar{v}_k, D\bar{v}_k) \right\rangle \, dxdt \\ &= \int_0^T \left\langle\!\!\left\langle \bar{h}_k(\cdot,t), \bar{v}_k(\cdot,t) \, \|\bar{v}_k\|^{-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))} \right\rangle\!\!\right\rangle_{\langle W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)\rangle} \, dt.\end{aligned}$$

Since $v_k(\cdot,t) \, \|\bar{v}_k\|^{-1}_{L^p(0,T;W_0^{1,p}(\Omega^k))} \in W_0^{1,p}(\Omega^k)$ ($k \in \mathbb{N}$), we find from (3.8) in Corollary 3.3 and Definition 1.5 that

$$\begin{aligned}&\int_0^T \left\langle\!\!\left\langle \bar{h}_k(\cdot,t), \bar{v}_k(\cdot,t) \, \|\bar{v}_k\|^{-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))} \right\rangle\!\!\right\rangle_{\langle W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)\rangle} \, dt \\ &= \int_0^T \left\langle\!\!\left\langle h_k(\cdot,t), v_k(\cdot,t) \, \|\bar{v}_k\|^{-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))} \right\rangle\!\!\right\rangle_{\langle W^{-1,p'}(\Omega^k), W_0^{1,p}(\Omega^k)\rangle} \, dt \\ &\to \int_0^T \langle\!\langle h_\infty(\cdot,t), v_0(\cdot,t) \rangle\!\rangle_{\langle W^{-1,p'}(\Omega^\infty), W_0^{1,p}(\Omega^\infty)\rangle} \, dt.\end{aligned}$$

By taking $\varphi = \bar{v}_k \, \|\bar{v}_k\|^{-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))}$ in (3.21), we combine the above equality and limit to find that

$$\|\bar{v}_k\|^{p-1}_{L^p(0,T;W_0^{1,p}(\mathbb{R}^n))} \to \int_0^T \langle\!\langle h_\infty(\cdot,t), v_0(\cdot,t) \rangle\!\rangle_{\langle W^{-1,p'}(\Omega^\infty), W_0^{1,p}(\Omega^\infty)\rangle} \, dt.$$

So $\bar{v}_k$ is bounded in $L^p(0,T;W_0^{1,p}(\mathbb{R}^n))$, and there exists a subsequence, which will be still denoted as $\{\bar{v}_k\}_{k=1}^{\infty}$, such that

$$\begin{cases} D\bar{v}_k \rightharpoonup D\bar{v}_0 & \text{in} \quad L^p(\mathbb{R}^n_T,\mathbb{R}^n), \\ \bar{v}_k \rightharpoonup \bar{v}_0 & \text{in} \quad L^p(\mathbb{R}^n_T), \\ \bar{V}_k \rightharpoonup \bar{V}_0 & \text{in} \quad L^{p'}(\mathbb{R}^n_T,\mathbb{R}^{n+1}), \end{cases} \tag{3.24}$$

where $\bar{v}_0 \in L^p(\mathbb{R}^n_T)$ is weakly differentiable in $\mathbb{R}^n_T$ with respect to x-variable. Because of the assumption (1.7), one can also show that

$$\bar{v}_0 = 0 \text{ a.e. in } \mathbb{R}^n_T \setminus \Omega^{\infty}_T \quad \text{and} \quad \bar{V}_0 = 0 \text{ a.e. in } \mathbb{R}^n_T \setminus \Omega^{\infty}_T. \tag{3.25}$$

Let $[w]_h(\cdot,t) = \frac{1}{h}\int_0^h w(\cdot,t+\tau)\,d\tau$ be Steklov average of w. In view of (1.7),

$$\text{there exists } K \in \mathbb{N} \text{ such that } supp\,\varphi \subset\subset \Omega^k \ (k \geq K) \text{ for any } \varphi \in C_c^{\infty}(\Omega^{\infty}). \tag{3.26}$$

By (3.21) and Definition 1.5, it follows that

$$\begin{aligned} \int_{\mathbb{R}^n} \Big\langle [\bar{V}_k]_h(x,t),(\bar{\varphi}(x,t),D\bar{\varphi}(x,t))\Big\rangle\,dx &= \frac{1}{h}\int_t^{t+h}\int_{\Omega^k} \langle V_k(x,\tau),(\varphi(x,t),D\varphi(x,t))\rangle\,dxd\tau \\ &\to \frac{1}{h}\int_t^{t+h}\int_{\Omega^{\infty}} \langle V_{\infty}(x,\tau),(\varphi(x,t),D\varphi(x,t))\rangle\,dxd\tau \\ &= \int_{\mathbb{R}^n} \Big\langle [\bar{V}_{\infty}]_h(x,t),(\bar{\varphi}(x,t),D\bar{\varphi}(x,t))\Big\rangle\,dx, \end{aligned}$$

for any $\varphi(\cdot,t) \in C_c^{\infty}(\Omega^{\infty})$. By (3.24) and (3.26),

$$\begin{aligned} \int_{\mathbb{R}^n} \Big\langle [\bar{V}_k]_h(x,t),(\bar{\varphi}(x,t),D\bar{\varphi}(x,t))\Big\rangle\,dx &= \frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^n} \Big\langle \bar{V}_k(x,\tau),(\varphi(x,t),D\varphi(x,t))\Big\rangle\,dxd\tau \\ &\to \frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^n} \Big\langle \bar{V}_0(x,\tau),(\varphi(x,t),D\varphi(x,t))\Big\rangle\,dxd\tau \\ &= \int_{\mathbb{R}^n} \Big\langle [\bar{V}_0]_h(x,t),(\bar{\varphi}(x,t),D\bar{\varphi}(x,t))\Big\rangle\,dx, \end{aligned}$$

for any $\varphi(\cdot,t) \in C_c^{\infty}(\Omega^{\infty})$. Thus

$$\int_{\mathbb{R}^n} \Big\langle [\bar{V}_{\infty} - \bar{V}_0]_h(x,t),(\bar{\varphi}(x,t),D\bar{\varphi}(x,t))\Big\rangle\,dx = 0$$

for any $\varphi(\cdot,t) \in C_c^{\infty}(\Omega^{\infty})$. For any $\varphi(\cdot,t) \in W_0^{1,p}(\Omega^{\infty})$, there exists $\varphi_{\epsilon}(\cdot,t) \in C_c^{\infty}(\Omega^{\infty})$ with $\|\varphi(\cdot,t) - \varphi_{\epsilon}(\cdot,t)\|_{W_0^{1,p}(\Omega^{\infty})} < \epsilon$. So we find that

$$\left|\int_{\mathbb{R}^n} \Big\langle [\bar{V}_{\infty} - \bar{V}_0]_h(x,t),(\bar{\varphi}(x,t),D\bar{\varphi}(x,t))\Big\rangle\,dx\right| \leq \epsilon\left[\|[\bar{V}_{\infty}]_h(\cdot,t)\|_{L^{p'}(\mathbb{R}^n)} + \|[\bar{V}_0]_h(\cdot,t)\|_{L^{p'}(\mathbb{R}^n)}\right],$$

for any $\varphi(\cdot,t) \in W_0^{1,p}(\Omega^{\infty})$ and the zero extension $\bar{\varphi}(\cdot,t) \in W_0^{1,p}(\mathbb{R}^n)$ of $\varphi(\cdot,t) \in W_0^{1,p}(\Omega^{\infty})$. Since $\epsilon > 0$ was arbitrary chosen, we find from (3.25) that

$$0 = \int_{\mathbb{R}^n} \Big\langle [\bar{V}_{\infty} - \bar{V}_0]_h(x,t),(\bar{\varphi}(x,t),D\bar{\varphi}(x,t))\Big\rangle\,dx = \int_{\Omega^{\infty}} \langle [V_{\infty} - V_0]_h(x,t),(\varphi(x,t),D\varphi(x,t))\rangle\,dx$$

for any $\varphi(\cdot,t) \in W_0^{1,p}(\Omega^\infty)$. We now integrate it with respect to time variable t to find that

$$0 = \int_\epsilon^{T-\epsilon} \int_{\Omega^\infty} \langle [V_\infty - V_0]_h(x,t), (\varphi(x,t), D\varphi(x,t)) \rangle \, dxdt$$

for any $0 < h < \epsilon < T$ and $\varphi \in L^p(0,T; W_0^{1,p}(\Omega))$. Since $V_\infty - V_0 \in L^{p'}(\Omega_T^\infty)$, we use [26, Lemma 3.2] to find that

$$0 = \int_\epsilon^{T-\epsilon} \int_{\Omega^\infty} \langle [V_\infty - V_0](x,t), (\varphi(x,t), D\varphi(x,t)) \rangle \, dxdt,$$

for any $0 < \epsilon < T$ and $\varphi \in L^p(0,T; W_0^{1,p}(\Omega^\infty))$. Thus

$$0 = \int_0^T \int_{\Omega^\infty} \langle [V_\infty - V_0](x,t), (\varphi(x,t), D\varphi(x,t)) \rangle \, dxdt, \tag{3.27}$$

for any $\varphi \in L^p(0,T; W_0^{1,p}(\Omega^\infty))$.

Fix $\varphi(\cdot,t) \in C_c^\infty(\Omega^\infty)$. By (3.26), there exists $K \in \mathbb{N}$ with

$$(\bar{v}_k - \bar{v}_\infty\varphi)(\cdot,t) \in W_0^{1,p}(\Omega^k) \cap W_0^{1,p}(\Omega^\infty) \qquad (k \geq K).$$

By a direct calculation,

$$\begin{aligned} &\int_{\mathbb{R}_T^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_k - \bar{v}_\infty, D[\bar{v}_k - \bar{v}_\infty]) \right\rangle dxdt \\ &\quad = \int_{\mathbb{R}_T^n} \left\langle \bar{V}_k - \bar{V}_\infty, ((\bar{v}_k - \bar{v}_\infty\varphi), D[\bar{v}_k - \bar{v}_\infty\varphi]) \right\rangle dxdt \\ &\qquad - \int_{\mathbb{R}_T^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_\infty(1-\varphi), D[\bar{v}_\infty(1-\varphi)]) \right\rangle dxdt. \end{aligned} \tag{3.28}$$

Also by (3.24), $(\bar{v}_k - \bar{v}_\infty\varphi, D[\bar{v}_k - \bar{v}_\infty\varphi]) \rightharpoonup (\bar{v}_0 - \bar{v}_\infty\varphi, D[\bar{v}_0 - \bar{v}_\infty\varphi])$ in $L^p(\mathbb{R}_T^n)$. So by Definition 1.5,

$$\int_{\mathbb{R}_T^n} \left\langle \bar{V}_k, (\bar{v}_k - \bar{v}_\infty\varphi, D[\bar{v}_k - \bar{v}_\infty\varphi]) \right\rangle dxdt \to \int_{\mathbb{R}_T^n} \left\langle \bar{V}_\infty, (\bar{v}_0 - \bar{v}_\infty\varphi, D[\bar{v}_0 - \bar{v}_\infty\varphi]) \right\rangle dxdt,$$

and

$$\int_{\mathbb{R}_T^n} \left\langle \bar{V}_\infty, (\bar{v}_k - \bar{v}_\infty\varphi, D[\bar{v}_k - \bar{v}_\infty\varphi]) \right\rangle dxdt \to \int_{\mathbb{R}_T^n} \left\langle \bar{V}_\infty, (\bar{v}_0 - \bar{v}_\infty\varphi, D[\bar{v}_0 - \bar{v}_\infty\varphi]) \right\rangle dxdt,$$

which implies that

$$\int_{\mathbb{R}_T^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_k - \bar{v}_\infty\varphi, D[\bar{v}_k - \bar{v}_\infty\varphi]) \right\rangle dxdt \to 0. \tag{3.29}$$

By (3.24),

$$\begin{aligned} &\int_{\mathbb{R}_T^n} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_\infty(1-\varphi), D[\bar{v}_\infty(1-\varphi)]) \right\rangle dxdt \\ &\quad \to \int_{\mathbb{R}_T^n} \left\langle \bar{V}_0 - \bar{V}_\infty, (\bar{v}_\infty(1-\varphi), D[\bar{v}_\infty(1-\varphi)]) \right\rangle dxdt. \end{aligned} \tag{3.30}$$

By combining (3.28), (3.29) and (3.30), we use (3.27) to find that

$$\begin{aligned}&\int_{\mathbb{R}^n_T} \left\langle \bar{V}_k - \bar{V}_\infty, (\bar{v}_k - \bar{v}_\infty, D[\bar{v}_k - \bar{v}_\infty]) \right\rangle dxdt \\ &\quad \to \int_{\mathbb{R}^n_T} \left\langle \bar{V}_0 - \bar{V}_\infty, (\bar{v}_\infty(1-\varphi), D[\bar{v}_\infty(1-\varphi)]) \right\rangle dxdt = 0,\end{aligned} \tag{3.31}$$

because of that $\bar{v}_\infty(1-\varphi) \in L^p(0,T; W_0^{1,p}(\Omega^\infty))$. So by Lemma 2.11 and (3.22),

$$\int_{\mathbb{R}^n_T} \left(|\bar{v}_k|^2 + |\bar{v}_\infty|^2\right)^{\frac{p-2}{2}} |\bar{v}_k - \bar{v}_\infty|^2 \, dxdt + \int_{\mathbb{R}^n_T} \left(|D\bar{v}_k|^2 + |D\bar{v}_\infty|^2\right)^{\frac{p-2}{2}} |D\bar{v}_k - D\bar{v}_\infty|^2 \, dxdt \to 0,$$

but this contradicts (3.23) and we find that (3.20) holds. Then by Lemma 2.12

$$\int_{\mathbb{R}^n_T} |\bar{V}_k - \bar{V}_\infty|^{p'} \, dxdt \to 0,$$

which implies that

$$\begin{aligned}\|\bar{h}_k - \bar{h}_\infty\|_{L^{p'}\left(0,T;W^{-1,p'}(\mathbb{R}^n)\right)} &= \int_0^T \sup_{\|\bar{\varphi}\|_{L^p\left(0,T;W_0^{1,p}(\mathbb{R}^n)\right)}=1} \left\langle\!\left\langle \bar{h}_k - \bar{h}_\infty, \bar{\varphi} \right\rangle\!\right\rangle_{\langle W^{-1,p'}(\mathbb{R}^n), W_0^{1,p}(\mathbb{R}^n)\rangle} dt \\ &= \int_0^T \sup_{\|\bar{\varphi}\|_{L^p\left(0,T;W_0^{1,p}(\mathbb{R}^n)\right)}=1} \int_{\mathbb{R}^n} \left\langle\!\left\langle [\bar{V}_k - \bar{V}_\infty], (\bar{\varphi}, D\bar{\varphi}) \right\rangle\!\right\rangle dxdt \\ &\to 0,\end{aligned}$$

and the lemma follows. □

To obtain a weak convergence for $\partial_t u_k \in L^{p'}(0,T; W^{-1,p'}(\Omega^k))$ $(k \in \mathbb{N})$, we consider the zero extension in Corollary 3.3. We remark that

$$\int_0^T \langle\!\langle h, \eta \rangle\!\rangle_\Omega \, dt = \int_0^T \left\langle\!\left\langle \bar{h}, \bar{\eta} \right\rangle\!\right\rangle_{\mathbb{R}^n} dt,$$

for any $\eta \in W_0^{1,p}(\Omega)$ and the zero extension $\bar{\eta} \in W_0^{1,p}(\mathbb{R}^n)$ of $\eta \in W_0^{1,p}(\Omega)$, where $\bar{h}$ is defined in Corollary 3.3.

Lemma 3.6. *Under the assumption* (1.7) *and* $1 < p < \infty$, *let* $\Omega^k \subset \mathbb{R}^n$ $(k \in \mathbb{N})$ *be a sequence of open bounded domains. If* $v_k \in L^{p'}(0,T; W^{-1,p'}(\Omega^k))$ $(k \in \mathbb{N})$ *satisfy*

$$\|v_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)} \le M \qquad (k \in \mathbb{N}),$$

for some $M > 0$, *then there exists* $v_\infty \in L^{p'}(0,T; W^{-1,p'}(\Omega^\infty))$ *such that*

$$\bar{v}_k \overset{*}{\rightharpoonup} \bar{v}_\infty \ \ in \ \ L^{p'}(0,T; W^{-1,p'}(\mathbb{R}^n)),$$

where $\bar{v}_k$ $(k \in \mathbb{N} \cup \{\infty\})$ *is defined in Corollary 3.3, which implies that*

$$\int_0^T \langle\!\langle \bar{v}_k(\cdot,t), \bar{\eta}(\cdot,t) \rangle\!\rangle_{\mathbb{R}^n} \, dt \to \int_0^T \langle\!\langle \bar{v}_\infty(\cdot,t), \bar{\eta}(\cdot,t) \rangle\!\rangle_{\mathbb{R}^n} \, dt$$

for any $\bar{\eta} \in L^p(0,T; W_0^{1,p}(\mathbb{R}^n))$.

Proof. Since $v_k \in L^{p'}(0,T;W_0^{-1,p'}(\Omega^k))$ $(k \in \mathbb{N})$, for each $t \in [0,T]$, there exists $V_k(\cdot,t) \in L^{p'}(\Omega^k, \mathbb{R}^{n+1})$ such that

$$\langle\langle v_k(\cdot,t), \varphi(\cdot)\rangle\rangle_{\Omega^k} = \int_{\Omega^k} \langle V_k(\cdot,t), (\varphi, D\varphi)(\cdot)\rangle\, dx \ \text{ for any } \varphi \in W_0^{1,p}(\Omega^k), \tag{3.32}$$

by Proposition 3.2. Moreover,

$$\|v_k(\cdot,t)\|_{W^{-1,p'}(\Omega^k)} = \inf\left\{\|V_k(\cdot,t)\|_{L^{p'}(\Omega^k,\mathbb{R}^{n+1})} : V_k(\cdot,t) \text{ satisfies (3.32)}\right\},$$

for any $t \in [0,T]$. So for $t \in [0,T]$, choose $V_k(\cdot,t) \in L^{p'}(\Omega^k, \mathbb{R}^{n+1})$ $(k \in \mathbb{N})$ so that

$$\|V_k(\cdot,t)\|_{L^{p'}(\Omega^k,\mathbb{R}^{n+1})} \le 2\|v_k(\cdot,t)\|_{W^{-1,p'}(\Omega^k)} \qquad (k \in \mathbb{N}),$$

which implies that

$$\|V_k\|_{L^{p'}(\Omega_T^k,\mathbb{R}^{n+1})} = \|V_k\|_{L^{p'}\left(0,T;L^{p'}(\Omega^k,\mathbb{R}^{n+1})\right)} \le 2\|v_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)} \le 2M.$$

for any $k \in \mathbb{N}$.

Let $\bar{V}_k$ be the zero extension of V_k from Ω_T^k to $\mathbb{R}_T^n$. Since $\|\bar{V}_k\|_{L^{p'}(\mathbb{R}_T^n,\mathbb{R}^{n+1})} \le 2M$ $(k \in \mathbb{N})$, by Proposition 2.3, there exists a weakly convergent subsequence, which will be still denoted by $\{\bar{V}_k\}_{k=1}^{\infty}$, which converges to $\bar{V}_\infty \in L^{p'}(\mathbb{R}_T^n, \mathbb{R}^{n+1})$, say

$$\bar{V}_k \rightharpoonup \bar{V}_\infty \ \text{ in } \ L^{p'}(\mathbb{R}_T^n, \mathbb{R}^{n+1}),$$

which implies that

$$\int_{\mathbb{R}_T^n} \langle \bar{V}_k, (\bar{\eta}, D\bar{\eta})\rangle\, dxdt \to \int_{\mathbb{R}_T^n} \langle \bar{V}_\infty, (\bar{\eta}, D\bar{\eta})\rangle\, dxdt, \tag{3.33}$$

for any $\bar{\eta} \in L^p(0,T;W_0^{1,p}(\mathbb{R}^n))$. Then one can check from (1.7) that $\bar{V}_\infty = 0$ a.e. in $\mathbb{R}_T^n \setminus \Omega_T^\infty$. So define $v_\infty \in L^{p'}(0,T;W^{-1,p'}(\Omega^\infty))$ as

$$\int_0^T \langle v_\infty(\cdot,t), \eta(\cdot,t)\rangle_{\Omega^\infty}\, dt = \int_{\Omega_T^\infty} \langle \bar{V}_\infty, (\eta, D\eta)\rangle\, dxdt,$$

for any $\eta \in L^p(0,T;W_0^{1,p}(\Omega^\infty))$. Then by Corollary 3.3,

$$\int_0^T \langle \bar{v}_\infty(\cdot,t), \bar{\eta}(\cdot,t)\rangle_{\mathbb{R}^n}\, dt = \int_{\mathbb{R}_T^n} \langle \bar{V}_\infty, (\bar{\eta}, D\bar{\eta})\rangle\, dxdt,$$

and

$$\int_0^T \langle \bar{v}_k(\cdot,t), \bar{\eta}(\cdot,t)\rangle_{\Omega^k}\, dt = \int_{\mathbb{R}_T^n} \langle \bar{V}_k, (\bar{\eta}, D\bar{\eta})\rangle\, dxdt,$$

for any $\bar{\eta} \in L^p(0,T;W_0^{1,p}(\mathbb{R}^n))$. So the lemma follows from (3.33). □

Lemma 3.7. *Under the assumption* (1.7) *and* $1 < p < \infty$, *let* $\Omega^k \subset \mathbb{R}^n$ $(k \in \mathbb{N})$ *be a sequence of open bounded domains. If* $v_k \in L^\infty(0,T;L^2(\Omega^k))$ $(k \in \mathbb{N})$ *satisfy*

$$\|v_k\|_{L^\infty\left(0,T;L^2(\Omega^k)\right)} \le M \qquad (k \in \mathbb{N}),$$

for some $M > 0$, then there exists $v_\infty \in L^\infty(0,T;L^2(\Omega^\infty))$ such that

$$\bar{v}_k \overset{*}{\rightharpoonup} \bar{v}_\infty \text{ in } L^\infty(0,T;L^2(\mathbb{R}^n))$$

where $\bar{v}_k$ is the zero extension of v_k to $L^\infty(0,T;L^2(\mathbb{R}^n))$ for $k \in \mathbb{N} \cup \{\infty\}$.

Proof. $L^\infty(0,T;L^2(\Omega^k))$ is dual of $L^1(0,T;L^2(\Omega^k))$ for $k \in \mathbb{N}\cup\{\infty\}$. We denote $\bar{v}_k$ as the zero extensions of v_k to $L^\infty(0,T;L^2(\mathbb{R}^n))$ for $k \in \mathbb{N} \cup \{\infty\}$. Since

$$\|\bar{v}_k\|_{L^\infty\left(0,T;L^2(\mathbb{R}^n)\right)} = \|v_k\|_{L^\infty\left(0,T;L^2(\Omega^k)\right)} \le M \qquad (k \in \mathbb{N}),$$

by Proposition 2.3 we find that there exists a weakly convergent subsequence, which will be still denoted as $\{\bar{v}_k\}_{k=1}^\infty$, which converges as

$$\bar{v}_k \overset{*}{\rightharpoonup} \bar{v}_\infty \text{ in } L^\infty(0,T;L^2(\mathbb{R}^n)).$$

We remark that weak-$*$ convergence was used instead of weak convergence, because $(L^\infty)^* \neq L^1$. One can easily check from (1.7) that $\bar{v}_\infty = 0$ a.e. in $\mathbb{R}^n_T \setminus \Omega^\infty_T$. So the lemma follows by taking $v_\infty = \bar{v}_\infty \cdot 1_{\Omega^\infty_T}$. □

Now recall the energy estimate (3.2).

$$\begin{aligned} &\sup_{0\le\tau\le T} \int_{\Omega^k} |\bar{u}_k(\cdot,\tau)|^2\, dx + \int_{\Omega^k_T} |D\bar{u}_k|^p\, dxdt \\ &\quad \le c\left[\||F_k|^{p-2}F_k\|_{L^{p'}(\Omega^k_T)} + \|f_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)} + \|D\gamma_k\|_{L^p(\Omega^k_T)} + \|\partial_t\gamma_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)}\right]. \end{aligned} \tag{3.34}$$

Let $\bar{F}_k, \bar{\gamma}_k, D\bar{\gamma}_k \in L^p(\mathbb{R}^n_T)$ be the zero extension of $F_k, \gamma_k, D\gamma_k \in L^p(\Omega^k_T)$, respectively. (We remark that $\bar{\gamma}_k$ might not be weakly differentiable in $\mathbb{R}^n_T$, but we abuse the notation for the simplicity of the computation.) We apply Lemma 3.1 to (1.9). Then

$$\begin{cases} |\bar{F}_k|^{p-2}\bar{F}_k \to |\bar{F}|^{p-2}\bar{F} & \text{in } L^{p'}(\mathbb{R}^n_T,\mathbb{R}^n), \\ \bar{\gamma}_k \to \bar{\gamma} & \text{in } L^p(\mathbb{R}^n_T), \\ D\bar{\gamma}_k \to D\bar{\gamma} & \text{in } L^p(\mathbb{R}^n_T,\mathbb{R}^n), \end{cases} \tag{3.35}$$

which implies that

$$\lim_{k\to\infty} \||F_k|^{p-2}F_k\|_{L^{p'}(\Omega^k_T)} = \lim_{k\to\infty} \||\bar{F}_k|^{p-2}\bar{F}_k\|_{L^{p'}(\mathbb{R}^n_T)} = \||\bar{F}|^{p-2}\bar{F}\|_{L^{p'}(\mathbb{R}^n_T)},$$

and

$$\lim_{k\to\infty} \|D\gamma_k\|_{L^p(\Omega^k_T)} = \lim_{k\to\infty} \|D\bar{\gamma}_k\|_{L^p(\mathbb{R}^n_T)} = \|D\bar{\gamma}\|_{L^p(\mathbb{R}^n_T)}.$$

Let $\bar{f}_k$, $\partial_t\bar{\gamma}_k$, $\bar{f}$ and $\partial_t\bar{\gamma}$ be the zero extension of $f_k, \partial_t\gamma_k \in L^{p'}(0,T;W^{-1,p'}(\Omega^k))$ and $f, \partial_t\gamma \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ in Corollary 3.3 respectively. By Corollary 3.3 and Lemma 3.5, we find from (1.8) that

$$\begin{cases} \bar{f}_k \overset{*}{\rightharpoonup} \bar{f} & \text{in } L^{p'}(0,T;W^{-1,p'}(\mathbb{R}^n)), \\ \partial_t\bar{\gamma}_k \overset{*}{\rightharpoonup} \partial_t\bar{\gamma} & \text{in } L^{p'}(0,T;W^{-1,p'}(\mathbb{R}^n)), \end{cases} \tag{3.36}$$

which implies that

$$\lim_{k\to\infty} \|f_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)} = \lim_{k\to\infty} \|\bar{f}_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\mathbb{R}^n)\right)} = \|\bar{f}\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right)},$$

and

$$\lim_{k\to\infty} \|\partial_t \gamma_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right)} = \lim_{k\to\infty} \|\partial_t \bar{\gamma}_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right)} = \|\partial_t \bar{\gamma}\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right)}.$$

So the right-hand side of (3.34) is bounded, and one can apply Aubin-Lions Lemma, Lemma 3.7 and the zero extension to find that there exists a subsequence of $\{\bar{u}_k\}_{k=1}^{\infty}$, which will be still denote by $\{\bar{u}_k\}_{k=1}^{\infty}$, and $\bar{u}_0 \in L^p(0,T;W_0^{1,p}(\mathbb{R}^n)) \cap L^\infty(0,T;L^2(\mathbb{R}^n))$ such that

$$\begin{cases} D\bar{u}_k \rightharpoonup D\bar{u}_0 & \text{in } L^p(\mathbb{R}^n_T, \mathbb{R}^n), \\ \bar{u}_k \to \bar{u}_0 & \text{in } L^p(\mathbb{R}^n_T), \\ \bar{u}_k \overset{*}{\rightharpoonup} \bar{u}_0 & \text{in } L^\infty(0,T;L^2(\mathbb{R}^n)). \end{cases} \tag{3.37}$$

Here, the compactness method is applied to some ball satisfying $B \supset \Omega^k$ $(k \in \mathbb{N})$ and $B \supset \Omega$ by using the zero extensions.

By (1.10),

$$\int_0^T \langle\!\langle \partial_t u_k, \varphi \rangle\!\rangle_{\Omega^k}\, dt = \int_{\Omega_T^k} \langle |F_k|^{p-2}F_k, D\varphi\rangle + f_k\varphi - \langle a_k(Du_k,x,t), D\varphi\rangle\, dxdt,$$

for any $\varphi \in L^p(0,T;W_0^{1,p}(\Omega^k))$. Then we see that $\|\partial_t u_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)}$ is bounded. We denote the zero extension of $\partial_t u_k \in L^{p'}(0,T;W^{-1,p'}(\Omega^k))$ in Corollary 3.3 as $\partial_t \bar{u}_k \in L^{p'}(0,T;W^{-1,p'}(\mathbb{R}^n))$. Then we find from Corollary 3.3 that

$$\|\partial_t \bar{u}_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\mathbb{R}^n)\right)} = \|\partial_t u_k\|_{L^{p'}\left(0,T;W^{-1,p'}(\Omega^k)\right)}\ (k \in \mathbb{N}) \text{ is bounded.} \tag{3.38}$$

So by Lemma 3.6, there exist $\partial_t u_0$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and a subsequence of $\{\bar{u}_k\}_{k=1}^{\infty}$, which will be still denoted by $\{\bar{u}_k\}_{k=1}^{\infty}$ such that

$$\partial_t \bar{u}_k \overset{*}{\rightharpoonup} \partial_t \bar{u}_0 \ \text{ in } \ L^{p'}(0,T;W^{-1,p'}(\mathbb{R}^n)). \tag{3.39}$$

Here, we denoted the zero extension of $\partial_t u_0 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ in Corollary 3.3 as $\partial_t \bar{u}_0 \in L^{p'}(0,T;W^{-1,p'}(\mathbb{R}^n))$. Define $u_0 = \bar{u}_0 + \gamma$ in Ω_T. Then we have that following lemma. We remark that a different proof is shown in Step 4 in the proof of [30, Lemma 5.1].

Lemma 3.8. *For $u_0 = \bar{u}_0 + \gamma$ in Ω_T, we have that*

$$\lim_{h\searrow 0} \frac{1}{h} \int_0^h \int_\Omega |u_0(x,t) - \gamma(x,0)|^2\, dxdt = 0.$$

Proof. Let $\hat{u}_k$ be the zero extension of $\bar{u}_k$ from $\mathbb{R}^n \times [0,T]$ to $\mathbb{R}^n \times [-T,T]$, which means that $\hat{u}_k = 0$ in $(\mathbb{R}^n \times [-T,T]) \setminus (\mathbb{R}^n \times [0,T])$. Also define $\partial_t \hat{u}_k$ as

$$\langle\!\langle \partial_t \hat{u}_k, \varphi \rangle\!\rangle_{\mathbb{R}^n} = \langle\!\langle \partial_t \bar{u}_k, \varphi \chi_{\Omega_T} \rangle\!\rangle_{\mathbb{R}^n} \quad \text{for any } \ \varphi \in L^p(-T,T;W^{1,p}(\mathbb{R}^n)).$$

Then we see that $\partial_t \hat{u}_k \in L^{p'}(-T, T; W^{-1,p'}(\mathbb{R}^n))$, because

$$\int_{-T}^{T} \langle\!\langle \partial_t \hat{u}_k, \varphi \rangle\!\rangle_{\mathbb{R}^n} \, dt = \int_0^T \langle\!\langle \partial_t \bar{u}_k, \varphi \rangle\!\rangle_{\mathbb{R}^n} \, dt = -\int_0^T \int_{\mathbb{R}^n} \bar{u}_k \, \varphi_t \, dxdt = -\int_{-T}^{T} \int_{\mathbb{R}^n} \hat{u}_k \, \varphi_t \, dxdt$$

for any $\varphi \in C_c^\infty(\mathbb{R}^n \times [-T, T])$. Here, we used that $\bar{u}_k = 0$ on $\mathbb{R}^n \times \{0\}$.

By (3.37) and (3.39), there exists a subsequence, which will be still denoted as $\hat{u}_k$ and $\partial_t \hat{u}_k$ ($k \in \mathbb{N}$), such that

$$\begin{cases} D\hat{u}_k \rightharpoonup D\hat{u}_0 & \text{in } L^p(\mathbb{R}^n \times (-T, T), \mathbb{R}^n), \\ \hat{u}_k \to \hat{u}_0 & \text{in } L^p(\mathbb{R}^n \times (-T, T)), \\ \hat{u}_k \overset{*}{\rightharpoonup} \hat{u}_0 & \text{in } L^\infty(-T, T; L^2(\mathbb{R}^n)). \end{cases} \tag{3.40}$$

and

$$\partial_t \hat{u}_k \overset{*}{\rightharpoonup} \partial_t \hat{u}_0 \ \text{ in } \ L^{p'}(-T, T; W^{-1,p'}(\mathbb{R}^n)),$$

for some $\hat{u}_0 \in L^p(-T, T; W_0^{1,p}(\mathbb{R}^n)) \cap L^\infty(-T, T; L^2(\mathbb{R}^n))$ and $\partial_t \hat{u}_0 \in L^{p'}(-T, T; W^{-1,p'}(\mathbb{R}^n))$. Then by Proposition 2.6, we have that $\hat{u}_0 \in C([-T, T]; L^2(\mathbb{R}^n))$, which implies that

$$0 = \lim_{h \nearrow 0} \frac{1}{h} \int_0^h \int_{\mathbb{R}^n} |\hat{u}_0|^2 \, dxdt = \lim_{h \searrow 0} \frac{1}{h} \int_0^h \int_{\mathbb{R}^n} |\hat{u}_0|^2 \, dxdt = \lim_{h \searrow 0} \frac{1}{h} \int_0^h \int_{\mathbb{R}^n} |\bar{u}_0|^2 \, dxdt,$$

where we used that $\hat{u}_0 = \bar{u}_0$ in $\mathbb{R}^n_T$, which holds from (3.37), (3.40) and that $\hat{u}_k$ is the zero extension of $\bar{u}_k$ from $\mathbb{R}^n_T$ to $\mathbb{R}^n \times [-T, T]$. Since $\bar{u}_0 = u_0 - \gamma$ in Ω, we get

$$\lim_{h \searrow 0} \frac{1}{h} \int_0^h \int_\Omega |u_0(x, t) - \gamma(x, t)|^2 \, dxdt = 0.$$

Since $\gamma \in C([0, T]; L^2(\Omega))$, we find that

$$\lim_{h \searrow 0} \frac{1}{h} \int_0^h \int_\Omega |\gamma(x, t) - \gamma(x, 0)|^2 \, dxdt = 0,$$

and the lemma follows. □

Lemma 3.9. *For the weak solutions $u \in \gamma + L^p(0, T; W_0^{1,p}(\Omega)) \cap C([0, T]; L^2(\Omega))$ of* (1.6) *and $u_k \in \gamma_k + L^p(0, T; W_0^{1,p}(\Omega^k)) \cap C([0, T]; L^2(\Omega^k))$ in* (1.10)*, we have that*

$$\lim_{k \to \infty} \int_{\mathbb{R}^n_T} |D\bar{u}_k - D\bar{u}|^p \varphi^p \, dxdt = 0 \ \textit{for any} \ \ \varphi \in C_c^\infty(\Omega) \textit{ with } 0 \le \varphi \le 1,$$

and

$$\lim_{k \to \infty} \int_{U_T} |D\bar{u}_k - D\bar{u}|^p \, dxdt = 0 \quad \textit{for any} \quad U \subset\subset \Omega. \tag{3.41}$$

Moreover, we have that

$$\begin{cases} D\bar{u}_k \rightharpoonup D\bar{u} & \textit{in } L^p(\mathbb{R}^n_T, \mathbb{R}^n), \\ \bar{u}_k \to \bar{u} & \textit{in } L^p(\mathbb{R}^n_T), \\ \bar{u}_k \overset{*}{\rightharpoonup} \bar{u} & \textit{in } L^\infty(0, T; L^2(\mathbb{R}^n)). \end{cases}$$

Proof. Recall from (1.7) that

$$\lim_{k\to\infty} d_H\left(\partial\Omega^k, \partial\Omega\right) = 0, \tag{3.42}$$

which implies that

$$\text{there exists } K \in \mathbb{N} \text{ such that } supp\,\varphi \subset\subset \Omega^k \,(k \geq K) \text{ for any } \varphi \in C_c^\infty(\Omega). \tag{3.43}$$

Fix $\varphi(x) \in C_c^\infty(\Omega)$ with $0 \leq \varphi \leq 1$, which is independent of t-variable. Choose $K \in \mathbb{N}$ in (3.43). Test (1.10) by $(\bar{u}_k - \bar{u}_0)\,\varphi^p$ to find that

$$\begin{aligned}\int_0^T \langle\!\langle \partial_t u_k, (\bar{u}_k - \bar{u}_0)\,\varphi^p \rangle\!\rangle_{\Omega^k}\, dt &+ \int_{\Omega_T^k} \left\langle a_k(Du_k, x, t), (D\bar{u}_k - D\bar{u}_0)\varphi^p + p(\bar{u}_k - \bar{u}_0)\varphi^{p-1}D\varphi \right\rangle\, dxdt \\ = \int_{\Omega_T^k} &\left\langle |F_k|^{p-2}F_k, (D\bar{u}_k - D\bar{u}_0)\varphi^p + p(\bar{u}_k - \bar{u}_0)\varphi^{p-1}D\varphi \right\rangle + f_k(\bar{u}_k - \bar{u}_0)\varphi^p\, dxdt,\end{aligned}$$

for any $k \geq K$. Recall that $\bar{u}_k = u_k - \gamma_k$, $\bar{u}_0 = u_0 - \gamma$ and $\varphi \in C_c^\infty(\Omega) \cap C_c^\infty(\Omega^k)$ for any $k \geq K$. For $(supp\,\varphi)_T = supp\,\varphi \times [0, T]$, we discover that

$$\begin{aligned}\int_0^T \langle\!\langle \partial_t (\bar{u}_k - \bar{u}_0), (\bar{u}_k - \bar{u}_0)\,\varphi^p \rangle\!\rangle_{\mathbb{R}^n}\, dt &+ \int_{\mathbb{R}_T^n \cap (supp\,\varphi)_T} \langle a_k(Du_k, x, t) - a_k(Du_0, x, t), (Du_k - Du_0)\varphi^p \rangle\, dxdt \\ = I_k + II_k + III_k + IV_k,&\end{aligned}$$

where

$$\begin{aligned} I_k &= \int_{\mathbb{R}_T^n \cap (supp\,\varphi)_T} \left\langle a_k(Du_k, x, t), (D\bar{\gamma}_k - D\bar{\gamma})\varphi^p - p(\bar{u}_k - \bar{u}_0)\varphi^{p-1}D\varphi \right\rangle\, dxdt, \\ II_k &= \int_{\mathbb{R}_T^n \cap (supp\,\varphi)_T} \left\langle |\bar{F}_k|^{p-2}\bar{F}_k, (D\bar{u}_k - D\bar{u}_0)\varphi^p \right\rangle\, dxdt \\ &\quad + \int_{\mathbb{R}_T^n \cap (supp\,\varphi)_T} \left\langle |\bar{F}_k|^{p-2}\bar{F}_k, p(\bar{u}_k - \bar{u}_0)\varphi^{p-1}D\varphi \right\rangle + \bar{f}_k(\bar{u}_k - \bar{u}_0)\varphi^p\, dxdt, \\ III_k &= -\int_{\mathbb{R}_T^n \cap (supp\,\varphi)_T} \langle a_k(Du_0, x, t), (Du_k - Du_0)\varphi^p \rangle\, dxdt, \\ IV_k &= -\int_0^T \langle \partial_t \bar{\gamma}_k + \partial_t \bar{u}_0, (\bar{u}_k - \bar{u}_0)\,\varphi^p \rangle_{\mathbb{R}^n}\, dt,\end{aligned}$$

for $k \geq K$. One can easily check from (3.35) and (3.37) that

$$\lim_{k\to\infty} I_k = 0. \tag{3.44}$$

By a direct calculation, we have

$$\begin{aligned} II_k &= \int_{\mathbb{R}_T^n \cap (supp\,\varphi)_T} \left\langle |\bar{F}|^{p-2}\bar{F}, (D\bar{u}_k - D\bar{u}_0)\varphi^p \right\rangle\, dxdt \\ &\quad + \int_{\mathbb{R}_T^n \cap (supp\,\varphi)_T} \left\langle |\bar{F}_k|^{p-2}\bar{F}_k - |\bar{F}|^{p-2}\bar{F}, (D\bar{u}_k - D\bar{u}_0)\varphi^p \right\rangle\, dxdt \\ &\quad + \int_{\mathbb{R}_T^n \cap (supp\,\varphi)_T} \left\langle |\bar{F}_k|^{p-2}\bar{F}_k, p(\bar{u}_k - \bar{u}_0)\varphi^{p-1}D\varphi \right\rangle + \bar{f}_k(\bar{u}_k - \bar{u}_0)\varphi^p\, dxdt.\end{aligned}$$

By (3.35)–(3.37),

$$\limsup_{k\to\infty} II_k = 0. \tag{3.45}$$

We handle III_k. By Lemma 2.14,

$$\lim_{k\to\infty} \|a_k(Du_0,\cdot)-a(Du_0,\cdot)\|_{L^{p'}\left(\mathbb{R}^n_T\cap(supp\,\varphi)_T\right)} \le \lim_{k\to\infty} \|a_k(Du_0,\cdot)-a(Du_0,\cdot)\|_{L^{p'}(\Omega_T)} = 0.$$

So by (3.37),

$$\limsup_{k\to\infty} III_k = 0. \tag{3.46}$$

By (3.36) and (3.37),

$$\limsup_{k\to\infty} IV_k = 0. \tag{3.47}$$

Since $\varphi=\varphi(x)$ and $0\le\varphi\le 1$, one can easily show that

$$\int_0^T \langle\!\langle \partial_t\left(\bar{u}_k-\bar{u}_0\right), \left(\bar{u}_k-\bar{u}_0\right)\varphi^p\rangle\!\rangle_{\mathbb{R}^n}\ dt = \int_{\mathbb{R}^n} \frac{\left|\left[\left(\bar{u}_k-\bar{u}_0\right)\varphi^{\frac{p}{2}}\right](x,T)\right|^2}{2}\,dx \ge 0.$$

because $\bar{u}_k = 0 = \bar{u}_0$ on $\mathbb{R}^n\times\{0\}$, which holds from Lemma 3.8. So by (3.44), (3.45), (3.46) and (3.47),

$$\int_{\mathbb{R}^n_T\cap(supp\,\varphi)_T} \langle a_k(Du_k,x,t)-a_k(Du_0,x,t),(Du_k-Du_0)\varphi^p\rangle\ dxdt \to 0,$$

because $\langle a_k(Du_k,x,t)-a_k(Du_0,x,t),(Du_k-Du_0)\varphi^p\rangle \ge 0$ in $\mathbb{R}^n_T\cap(supp\,\varphi)_T$, which implies that

$$\int_{\mathbb{R}^n_T\cap(supp\,\varphi)_T} (|Du_k|^2+|Du_0|^2+s^2)^{\frac{p-2}{2}}|Du_k-Du_0|^2\varphi^p dxdt \to 0.$$

For any $\kappa\in(0,\kappa_1]$, we have from Lemma 2.7 that

$$\begin{aligned}\int_{\mathbb{R}^n_T\cap(supp\,\varphi)_T} |Du_k-Du_0|^p\varphi^p\ dxdt &\le c\kappa^p \int_{\mathbb{R}^n_T\cap(supp\,\varphi)_T} (|Du_0|^2+s^2)^{\frac{p}{2}}\varphi^p\ dxdt \\ &\quad + c\kappa^{p-2}\int_{\mathbb{R}^n_T\cap(supp\,\varphi)_T} (|Du_k|^2+|Du_0|^2+s^2)^{\frac{p-2}{2}}|Du_k-Du_0|^2\varphi^p dxdt.\end{aligned}$$

So we find that

$$0 \le \limsup_{k\to\infty} \int_{\mathbb{R}^n_T\cap(supp\,\varphi)_T} |Du_k-Du_0|^p\varphi^p\ dxdt \le c\kappa^p \int_{\mathbb{R}^n_T\cap(supp\,\varphi)_T} (|Du_0|^2+s^2)^{\frac{p}{2}}\varphi^p\ dxdt.$$

Since $\kappa\in(0,\kappa_1]$ and $\varphi\in C_c^\infty(\Omega)$ were arbitrary chosen, we discover that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n_T\cap(supp\,\varphi)_T} |Du_k-Du_0|^p\varphi^p\ dxdt = 0 \ \text{ for any }\ \varphi\in C_c^\infty(\Omega) \text{ with } 0\le\varphi\le 1.$$

So by (3.35),

$$\lim_{k\to\infty} \int_{\mathbb{R}^n_T} |D\bar{u}_k-D\bar{u}_0|^p\varphi^p\ dxdt = 0 \ \text{ for any }\ \varphi\in C_c^\infty(\Omega) \text{ with } 0\le\varphi\le 1. \tag{3.48}$$

For any $U \subset\subset \Omega$, there exists a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ in Ω and $\eta = 1$ on U. Moreover, by (3.42), there exists $K \in \mathbb{N}$ such that

$$U \subset\subset \Omega^k \qquad (k \geq K). \tag{3.49}$$

So by (3.48),

$$\lim_{k\to\infty} \int_{U_T} |D\bar{u}_k - D\bar{u}_0|^p \, dxdt = 0 \quad \text{for any} \quad U \subset\subset \Omega. \tag{3.50}$$

By Corollary 3.3 and (3.39),

$$\int_0^T \langle\langle \partial_t \bar{u}_k, \bar{\varphi} \rangle\rangle_{\mathbb{R}^n} \, dt \overset{*}{\to} \int_0^T \langle\langle \partial_t \bar{u}_0, \bar{\varphi} \rangle\rangle_{\mathbb{R}^n} \, dt = \int_0^T \langle\langle \partial_t u_0, \bar{\varphi} \rangle\rangle_{\Omega} \, dt, \tag{3.51}$$

for any $\varphi \in C_0^\infty(\Omega_T)$.

Now, we show that u_0 is the weak solution of (1.6), which implies that $u = u_0$ by the uniqueness. Fix $\varphi \in C_0^\infty(\Omega_T)$ and choose $U \subset\subset \Omega$ with $\operatorname{supp}\varphi \subset \overline{U_T}$. By (3.42), there exists $K \in \mathbb{N}$ such that $U \subset\subset \Omega^k$ $(k \geq K)$. We have from (1.10) that

$$\int_0^T \langle\langle \partial_t u_k, \varphi \rangle\rangle_{\Omega^k} \, dt + \int_{\Omega_T^k} \langle a_k(Du_k, x, t), D\varphi \rangle \, dxdt = \int_{\Omega_T^k} \langle |F_k|^{p-2} F_k, D\varphi \rangle + f_k \varphi \, dxdt,$$

for any $k \geq K$. So by Lemma 2.10, Lemma 2.14, (3.35), (3.36), (3.50) and (3.51),

$$\int_0^T \langle\langle \partial_t u_0, \varphi \rangle\rangle_{\Omega} + \int_{\Omega_T} \langle a(Du_0, x, t), D\varphi \rangle \, dxdt = \int_{\Omega_T} \langle |F|^{p-2} F, D\varphi \rangle + f\varphi \, dxdt.$$

We find from Lemma 3.8 that $u_0 \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ is also the weak solution of (1.6). By uniqueness of the weak solution, we find that $u_0 = u$, and the lemma follows from (3.37), (3.48) and (3.50). □

We next estimate the concentration of $D\bar{u}_k$ near the boundary $\partial\Omega \times [0, T]$.

Lemma 3.10. *For any $\varphi \in C_c^\infty(\Omega)$ with $0 \leq \varphi \leq 1$, we have that*

$$\begin{aligned} &\limsup_{k\to\infty} \int_{\mathbb{R}_T^n} |D\bar{u}_k|^p \, (1 - \varphi^p) \, dxdt \\ &\quad \leq c \left[\int_{\Omega_T} (|Du|^2 + |D\gamma|^2 + s^2)^{\frac{p}{2}} \, (1 - \varphi^p) \, dxdt + \int_{\Omega} \frac{|[\bar{u}(1 - \varphi^p)^{\frac{1}{2}}](x, T)|^2}{2} \, dx \right]. \end{aligned}$$

Proof. Fix $\varphi \in C_c^\infty(\Omega)$ with $0 \leq \varphi \leq 1$. We have from (1.7) that

$$\text{there exists } K \in \mathbb{N} \text{ such that } \mathit{supp}\,\varphi \subset\subset \Omega^k \, (k \geq K) \text{ for any } \varphi \in C_c^\infty(\Omega). \tag{3.52}$$

We next take $\kappa = \kappa_1(n, p, \lambda, \Lambda)$ in Lemma 2.7 to find that

$$\begin{aligned} \int_{\Omega_T^k} |Du_k - D\gamma_k|^p \, (1 - \varphi^p) \, dxdt &\leq c \int_{\Omega_T^k} (|D\gamma_k|^2 + s^2)^{\frac{p}{2}} \, (1 - \varphi^p) \, dxdt \\ &\quad + c \int_{\Omega_T^k} (|Du_k|^2 + |D\gamma_k|^2 + s^2)^{\frac{p-2}{2}} |Du_k - D\gamma_k|^2 \, (1 - \varphi^p) \, dxdt, \end{aligned} \tag{3.53}$$

for any $k \geq K$. In view of (1.2), we discover that

$$\begin{aligned}&\int_{\Omega_T^k} (|Du_k|^2 + |D\gamma_k|^2 + s^2)^{\frac{p-2}{2}} |Du_k - D\gamma_k|^2 \,(1 - \varphi^p)\; dxdt \\ &\quad \leq c \int_{\Omega_T^k} \langle a(Du_k, x, t) - a(D\gamma_k, x, t), (Du_k - D\gamma_k)\rangle \,(1 - \varphi^p)\; dxdt,\end{aligned} \tag{3.54}$$

for any $k \geq K$.

We will estimate the limit superior of the right-hand side of (3.54). We test (1.10) by $(u_k - \gamma_k)\,(1 - \varphi^p)$ to find that

$$\int_{\Omega_T^k} \langle a_k(Du_k, x, t) - a_k(D\gamma_k, x, t), (Du_k - D\gamma_k)\,(1 - \varphi^p)\rangle \, dxdt = I_k + II_k + III_k + IV_k, \tag{3.55}$$

where

$$\begin{aligned}I_k &= \int_{\Omega_T^k} \langle a_k(Du_k, x, t), (u_k - \gamma_k)\, p\varphi^{p-1} D\varphi\rangle \; dxdt,\\ II_k &= -\int_{\Omega_T^k} \langle a_k(D\gamma_k, x, t), (Du_k - D\gamma_k)\,(1 - \varphi^p)\rangle \; dxdt,\\ III_k &= \int_{\Omega_T^k} \left\langle |F_k|^{p-2} F_k, D[(u_k - \gamma_k)\,(1 - \varphi^p)]\right\rangle + f_k(u_k - \gamma_k)\,(1 - \varphi^p)\; dxdt,\\ IV_k &= -\int_0^T \langle \partial_t u_k, (u_k - \gamma_k)\,(1 - \varphi^p)\rangle_{\Omega^k} \, dt,\end{aligned}$$

for any $k \geq K$.

We estimate the limit of the right-hand side as $k \to \infty$. Without loss of generality, assume that $k \geq K$. Then we have from (3.52) that

$$\varphi \in C_c^\infty(\Omega) \cap C_c^\infty(\Omega^k).$$

We first compute the limit of I_k. By the triangle inequality,

$$\begin{aligned}&\| |a_k(Du_k, x, t) - a(Du, x, t)|\, |D\varphi| \|_{L^{\frac{p}{p-1}}(\mathbb{R}_T^n \cap (supp\, \varphi)_T)} \\ &\leq \| |a_k(Du_k, x, t) - a_k(Du, x, t)|\, |D\varphi| \|_{L^{\frac{p}{p-1}}(\mathbb{R}_T^n \cap (supp\, \varphi)_T)} + \| |a_k(Du, x, t) - a(Du, x, t)|\, |D\varphi| \|_{L^{\frac{p}{p-1}}(\mathbb{R}_T^n \cap (supp\, \varphi)_T)}.\end{aligned}$$

Since $\varphi \in C_c^\infty(\Omega) \cap C_c^\infty(\Omega^k)$, we have from Lemma 2.10, Lemma 2.14 and (3.41) in Lemma 3.9 that

$$\lim_{k \to \infty} \| |a_k(Du_k, x, t) - a(Du, x, t)|\, |D\varphi| \|_{L^{\frac{p}{p-1}}(\mathbb{R}_T^n \cap (supp\, \varphi)_T)} = 0. \tag{3.56}$$

By Lemma 3.9, we have that $\bar{u}_k \to \bar{u}$ in $L^p(\mathbb{R}_T^n)$. Since $u_k - \gamma_k = \bar{u}_k$ in Ω_T^k and $u - \gamma = \bar{u}$ in Ω_T, we find from (3.50) that

$$I_k = \int_{\Omega_T^k} \langle a_k(Du_k, x, t), (u_k - \gamma_k) p\varphi^{p-1} D\varphi\rangle \, dxdt \to \int_{\Omega_T} \langle a(Du, x, t), (u - \gamma)\, p\varphi^{p-1} D\varphi\rangle \, dxdt. \tag{3.57}$$

Similarly, by the triangle inequality,

$$\begin{aligned}&\left\|a_k(D\gamma_k, x, t)\cdot 1_{\Omega_T^k} - a(D\gamma, x, t)\cdot 1_{\Omega_T}\right\|_{L^{p'}(\mathbb{R}^n_T)}\\ &\quad\le \left\|a_k(D\gamma_k, x, t)\cdot 1_{\Omega_T^k} - a_k(D\gamma, x, t)\cdot 1_{\Omega_T}\right\|_{L^{p'}(\mathbb{R}^n_T)} + \left\|a_k(D\gamma, x, t)\cdot 1_{\Omega_T} - a(D\gamma, x, t)\cdot 1_{\Omega_T}\right\|_{L^{p'}(\mathbb{R}^n_T)}.\end{aligned}$$

So we get from (3.35), Lemma 2.10 and Lemma 2.14 that

$$\lim_{k\to\infty}\left\|a_k(D\gamma_k, x, t)\cdot 1_{\Omega_T^k} - a(D\gamma, x, t)\cdot 1_{\Omega_T}\right\|_{L^{p'}(\mathbb{R}^n_T)} = 0,$$

and it follows from Lemma 3.9 that

$$\begin{aligned}II_k &= -\int_{\Omega_T^k}\langle a_k(D\gamma_k, x, t), (Du_k - D\gamma_k)(1-\varphi^p)\rangle\, dxdt\\ &= -\int_{\mathbb{R}^n_T}\langle a_k(D\gamma_k, x, t)\cdot 1_{\Omega_T^k}, D\bar{u}_k(1-\varphi^p)\rangle\, dxdt\\ &\to -\int_{\mathbb{R}^n_T}\langle a(D\gamma, x, t)\cdot 1_{\Omega_T}, D\bar{u}(1-\varphi^p)\rangle\, dxdt\\ &= -\int_{\Omega_T}\langle a(D\gamma, x, t), (Du - D\gamma)(1-\varphi^p)\rangle\, dxdt.\end{aligned}\tag{3.58}$$

Recall that

$$III_k = \int_{\Omega_T^k}\left\langle |F_k|^{p-2}F_k, D[(u_k-\gamma_k)(1-\varphi^p)]\right\rangle + f_k(u_k-\gamma_k)(1-\varphi^p)\ dxdt.$$

Then one can easily check from (3.35), (3.36) and Lemma 3.9 that

$$III_k \to \int_{\Omega_T}\left\langle |F|^{p-2}F, D[(u-\gamma)(1-\varphi^p)]\right\rangle + f(u-\gamma)(1-\varphi^p)\ dxdt.\tag{3.59}$$

Now, we estimate IV_k.

$$\begin{aligned}IV_k &= -\int_0^T \langle\langle \partial_t u_k, (u_k-\gamma_k)(1-\varphi^p)\rangle\rangle_{\Omega^k}\ dt\\ &= -\int_0^T \langle\langle \partial_t u_k - \partial_t\gamma_k, (u_k-\gamma_k)(1-\varphi^p)\rangle\rangle_{\Omega_T^k} - \langle\langle \partial_t\gamma_k, (u_k-\gamma_k)(1-\varphi^p)\rangle\rangle_{\Omega^k}\ dt.\end{aligned}$$

Since $\varphi = \varphi(x)$, $0\le\varphi\le 1$ and $u_k - \gamma_k = 0$ on $\Omega^k\times\{0\}$, we find that

$$\int_0^T \langle\langle \partial_t u_k - \partial_t\gamma_k, (u_k-\gamma_k)(1-\varphi^p)\rangle\rangle_{\Omega^k}\, dt = \int_{\Omega^k}\frac{|[(u_k-\gamma_k)(1-\varphi^p)^{\frac{1}{2}}](x,T)|^2}{2}\, dx \ge 0.$$

Since $u_k - \gamma_k = \bar{u}_k$ in Ω_T^k and $u - \gamma = \bar{u}$ in Ω_T, we find from (3.36) and Lemma 3.9 that

$$\int_0^T \langle\langle \partial_t\gamma_k, (u_k-\gamma_k)(1-\varphi^p)\rangle\rangle_{\Omega^k}\ dt \to \int_0^T \langle\langle \partial_t\gamma, (u-\gamma)(1-\varphi^p)\rangle\rangle_{\Omega}\ dt.$$

Thus

$$\limsup_{k\to\infty} IV_k \le -\int_0^T \langle\langle \partial_t\gamma, (u-\gamma)(1-\varphi^p)\rangle\rangle_\Omega \, dt. \tag{3.60}$$

In view of (3.55), we find from (3.57), (3.58), (3.59) and (3.60) that

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\Omega_T^k} \langle a_k(Du_k, x, t) - a_k(D\gamma_k, x, t), (Du_k - D\gamma_k)(1-\varphi^p)\rangle \, dxdt \\
&\quad\le \int_{\Omega_T} \langle a(Du, x, t), (u-\gamma)\, p\varphi^{p-1} D\varphi\rangle - \langle a(D\gamma, x, t), (Du - D\gamma)(1-\varphi^p)\rangle \, dxdt \\
&\qquad + \int_{\Omega_T} \left\langle |F|^{p-2}F, D[(u-\gamma)(1-\varphi^p)]\right\rangle + f(u-\gamma)(1-\varphi^p) \, dxdt \\
&\qquad - \int_0^T \langle\langle \partial_t\gamma, (u-\gamma)(1-\varphi^p)\rangle\rangle_\Omega \, dt.
\end{aligned}$$

By taking $(u-\gamma)(1-\varphi^p)$ in (1.6), we get that

$$\begin{aligned}
&\int_{\Omega_T} \langle a(Du, x, t), (u-\gamma)\, p\varphi^{p-1} D\gamma\rangle - \langle a(D\gamma, x, t), (Du - D\gamma)(1-\varphi^p)\rangle \, dxdt \\
&\quad + \int_{\Omega_T} \left\langle |F|^{p-2}F, D[(u-\gamma)(1-\varphi^p)]\right\rangle + g(u-\gamma)(1-\varphi^p) \, dxdt \\
&\quad = \int_{\Omega_T} \langle a(Du, x, t) - a(D\gamma, x, t), (Du - D\gamma)(1-\varphi^p)\rangle \, dxdt + \int_0^T \langle\langle \partial_t u, (u-\gamma)(1-\varphi^p)\rangle\rangle_\Omega \, dt.
\end{aligned}$$

Thus

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\Omega_T^k} \langle a_k(Du_k, x, t) - a_k(D\gamma_k, x, t), (Du_k - D\gamma_k)(1-\varphi^p)\rangle \, dxdt \\
&\quad\le \int_{\Omega_T} \langle a(Du, x, t) - a(D\gamma, x, t), (Du - D\gamma)(1-\varphi^p)\rangle \, dxdt + \int_0^T \langle\langle \partial_t u - \partial_t\gamma, (u-\gamma)(1-\varphi^p)\rangle\rangle_\Omega \, dt.
\end{aligned}$$

Since $\bar{u} = u - \gamma$, we find that

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\Omega_T^k} \langle a_k(Du_k, x, t) - a_k(D\gamma_k, x, t), (Du_k - D\gamma_k)(1-\varphi^p)\rangle \, dxdt \\
&\quad\le \int_{\Omega_T} \langle a(Du, x, t) - a(D\gamma, x, t), (Du - D\gamma)(1-\varphi^p)\rangle \, dxdt + \int_\Omega \frac{|[\bar{u}(1-\varphi^p)^{\frac{1}{2}}](x, T)|^2}{2} \, dx.
\end{aligned}$$

Since $\bar{u}_k = u_k - \gamma_k$, by (3.35), (3.53) and (3.54),

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\mathbb{R}^n_T} |D\bar{u}_k|^p (1-\varphi^p) \, dxdt \\
&\quad\le c\left[\int_{\Omega_T} (|Du|^2 + |D\gamma|^2 + s^2)^{\frac{p}{2}} (1-\varphi^p) \, dxdt + \int_\Omega \frac{|[\bar{u}(1-\varphi^p)^{\frac{1}{2}}](x, T)|^2}{2} \, dx\right],
\end{aligned}$$

and the lemma follows. □

3.1. Proof of Theorem 1.6

We are ready to prove Theorem 1.6.

Proof of Theorem 1.6. By Lemmas 3.9 and 3.10,

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{\mathbb{R}^n_T} |D\bar{u}_k - D\bar{u}|^p \, dxdt \\
&= \limsup_{k\to\infty} \left[\int_{\mathbb{R}^n_T} |D\bar{u}_k - D\bar{u}|^p \varphi^p \, dxdt + \int_{\mathbb{R}^n_T} |D\bar{u}_k - D\bar{u}|^p (1-\varphi^p) \, dxdt \right] \\
&\le c \left[\int_{\Omega_T} (|Du|^2 + |D\gamma|^2 + s^2)^{\frac{p}{2}} (1-\varphi^p) \, dxdt + \int_{\Omega} \frac{|[\bar{u}(1-\varphi^p)^{\frac{1}{2}}](x,T)|^2}{2} \, dx \right],
\end{aligned}$$

for any $\varphi \in C_c^\infty(\Omega)$ with $0 \le \varphi \le 1$. Since $\varphi \in C_c^\infty(\Omega)$ with $0 \le \varphi \le 1$ can be arbitrary chosen in the above estimates, one can choose a sequence of monotone increasing functions in $C_c^\infty(\Omega)$ which converges to 1 a.e. in Ω. Then by Lebesgue's dominated convergence theorem, we get

$$\limsup_{k\to\infty} \int_{\mathbb{R}^n_T} |D\bar{u}_k - D\bar{u}|^p \, dxdt \le 0.$$

This contradicts (3.1). So we find that (1.11) holds. □

Acknowledgments

Y. Kim was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (No. NRF-2020R1C1C1A01013363). S. Ryu was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1C1C1A01014310). P. Shin was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. NRF-2020R1I1A1A01066850). The authors would like to thank the referee for the careful reading of this manuscript and for offering valuable comments.

Conflict of interest

The authors declare no conflict of interest.

References

1. E. Acerbi, N. Fusco, Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$, *J. Math. Anal. Appl.*, **140** (1989), 115–135. https://doi.org/10.1016/0022-247X(89)90098-X
2. E. Acerbi, G. Mingione, Regularity results for a class of functionals with non-standard growth, *Arch. Rational Mech. Anal.*, **156** (2001), 121–140. https://doi.org/10.1007/s002050100117
3. E. Acerbi, G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew. Math.*, **584** (2005), 117–148. https://doi.org/10.1515/crll.2005.2005.584.117

4. E. Acerbi, G. Mingione, Gradient estimates for a class of parabolic systems, *Duke Math. J.*, **136** (2007), 285–320. https://doi.org/10.1215/S0012-7094-07-13623-8
5. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, Amsterdam: Elsevier/Academic Press, 2003.
6. P. Baroni, Lorentz estimates for degenerate and singular evolutionary systems, *J. Differ. Equations*, **255** (2013), 2927–2951. https://doi.org/10.1016/j.jde.2013.07.024
7. P. Baroni, Riesz potential estimates for a general class of quasilinear equations, *Calc. Var.*, **53** (2015), 803–846. https://doi.org/10.1007/s00526-014-0768-z
8. P. Baroni, V. Bögelein, Calderón-Zygmund estimates for parabolic $p(x,t)$-Laplacian systems, *Rev. Mat. Iberoam.*, **30** (2014), 1355–1386. https://doi.org/10.4171/RMI/817
9. P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, *Calc. Var.*, **57** (2018), 62. https://doi.org/10.1007/s00526-018-1332-z
10. L. Beck, G. Mingione, Lipschitz bounds and nonuniform ellipticity, *Commun. Pure Appl. Math.*, **73** (2020), 944–1034. https://doi.org/10.1002/cpa.21880
11. V. Bögelein, Global Calderón-Zygmund theory for nonlinear parabolic systems, *Calc. Var.*, **51** (2014), 555–596. https://doi.org/10.1007/s00526-013-0687-4
12. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, New York: Springer, 2011. https://doi.org/10.1007/978-0-387-70914-7
13. M. Bulíček, S.-S. Byun, P. Kaplický, J. Oh, S. Schwarzacher, On global L^q-estimates for systems with p-growth in rough domains, *Calc. Var.*, **58** (2019), 185. https://doi.org/10.1007/s00526-019-1621-1
14. S.-S. Byun, Y. Kim, Elliptic equations with measurable nonlinearities in nonsmooth domains, *Adv. Math.*, **288** (2016), 152–200. https://doi.org/10.1016/j.aim.2015.10.015
15. S.-S. Byun, J. Ok, S. Ryu, Global gradient estimates for general nonlinear parabolic equations in nonsmooth domains, *J. Differ. Equations*, **254** (2013), 4290–4326. https://doi.org/10.1016/j.jde.2013.03.004
16. S.-S. Byun, J. Ok, S. Ryu, Global gradient estimates for elliptic equations of $p(x)$-Laplacian type with BMO nonlinearity, *J. Reine Angew. Math.*, **715** (2016), 1–38. https://doi.org/10.1515/crelle-2014-0004
17. S.-S. Byun, L. Wang, Parabolic equations in time dependent Reifenberg domains, *Adv. Math.*, **212** (2007), 797–818. https://doi.org/10.1016/j.aim.2006.12.002
18. I. Chlebicka, A pocket guide to nonlinear differential equations in Musielak-Orlicz spaces, *Nonlinear Anal.*, **175** (2018), 1–27. https://doi.org/10.1016/j.na.2018.05.003
19. A. Cianchi, V. G. Maz'ya, Global Lipschitz regularity for a class of quasilinear elliptic equations, *Commun. Part. Diff. Eq.*, **36** (2011), 100–133. https://doi.org/10.1080/03605301003657843
20. A. Cianchi, V. G. Maz'ya, Global boundedness of the gradient for a class of nonlinear elliptic systems, *Arch. Rational Mech. Anal.*, **212** (2014), 129–177. https://doi.org/10.1007/s00205-013-0705-x
21. M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Rational Mech. Anal.*, **218** (2015), 219–273. https://doi.org/10.1007/s00205-015-0859-9

22. M. Colombo, G. Mingione, Regularity for double phase variational problems, *Arch. Rational Mech. Anal.*, **215** (2015), 443–496. https://doi.org/10.1007/s00205-014-0785-2

23. M. Colombo, G. Mingione, Calderón–Zygmund estimates and non-uniformly elliptic operators, *J. Funct. Anal.*, **270** (2016), 1416–1478. https://doi.org/10.1016/j.jfa.2015.06.022

24. C. De Filippis, G. Mingione, On the regularity of minima of non-autonomous functionals, *J. Geom. Anal.*, **30** (2020), 1584–1626. https://doi.org/10.1007/s12220-019-00225-z

25. C. De Filippis, G. Palatucci, Hölder regularity for nonlocal double phase equations, *J. Differ. Equations*, **267** (2019), 547–586. https://doi.org/10.1016/j.jde.2019.01.017

26. E. DiBenedetto, *Degenerate parabolic equations*, New York: Springer, 1993. https://doi.org/10.1007/978-1-4612-0895-2

27. L. Diening, B. Stroffolini, A. Verde, Everywhere regularity of functionals with φ-growth, *Manuscripta Math.*, **129** (2009), 449–481. https://doi.org/10.1007/s00229-009-0277-0

28. F. Duzaar, G. Mingione, Local Lipschitz regularity for degenerate elliptic systems, *Ann. Inst. H. Poincaré (C) Anal. Non Linéaire*, **27** (2010), 1361–1396. https://doi.org/10.1016/J.ANIHPC.2010.07.002

29. F. Duzaar, G. Mingione, Gradient estimates via non-linear potentials, *Amer. J. Math.*, **133** (2011), 1093–1149. https://doi.org/10.1353/ajm.2011.0023

30. A. H. Erhardt, Existence of solutions to parabolic problems with nonstandard growth and irregular obstacles, *Adv. Differential Equations*, **21** (2016), 463–504. https://doi.org/10.57262/ade/1457536498

31. L. Esposito, G. Mingione, Some remarks on the regularity of weak solutions of degenerate elliptic systems, *Rev. Mat. Complut.*, **11** (1998), 203–219. https://doi.org/10.5209/rev_REMA.1998.v11.n1.17325

32. L. Esposito, F. Leonetti, G. Mingione, Regularity results for minimizers of irregular integrals with (p, q) growth, *Forum Math.*, **14** (2002), 245–272. https://doi.org/10.1515/form.2002.011

33. L. Esposito, F. Leonetti, G. Mingione, Sharp regularity for functionals with (p, q) growth, *J. Differ. Equations*, **204** (2004), 5–55. https://doi.org/10.1016/j.jde.2003.11.007

34. L. Evans, *Partial differential equations*, 2 Eds., Providence, RI: American Mathematical Society, 2010.

35. M. Giaquinta, G. Modica, Remarks on the regularity of the minimizers of certain degenerate functionals, *Manuscripta Math.*, **57** (1986), 55–99. https://doi.org/10.1007/BF01172492

36. C. Hamburger, Regularity of differential forms minimizing degenerate elliptic functionals, *J. Reine Angew. Math.*, **431** (1992), 7–64. https://doi.org/10.1515/crll.1992.431.7

37. P. Hästö, J. Ok, Maximal regularity for local minimizers of non-autonomous functionals, *J. Eur. Math. Soc.*, **24** (2022), 1285–1334. https://doi.org/10.4171/JEMS/1118

38. Y. Kim, Gradient estimates for elliptic equations with measurable nonlinearities, *J. Math. Pure. Appl. (9)*, **114** (2018), 118–145. https://doi.org/10.1016/j.matpur.2017.11.003

39. T. Kuusi, G. Mingione, Universal potential estimates, *J. Funct. Anal.*, **262** (2012), 4205–4269. https://doi.org/10.1016/j.jfa.2012.02.018

40. T. Kuusi, G. Mingione, New perturbation methods for nonlinear parabolic problems, *J. Math. Pure Appl. (9)*, **98** (2012), 390–427. https://doi.org/10.1016/j.matpur.2012.02.004

41. T. Kuusi, G. Mingione, Linear potentials in nonlinear potential theory, *Arch. Rational Mech. Anal.*, **207** (2013), 215–246. https://doi.org/10.1007/s00205-012-0562-z

42. T. Kuusi, G. Mingione, Riesz potentials and nonlinear parabolic equations, *Arch. Rational Mech. Anal.*, **212** (2014), 727–780. https://doi.org/10.1007/s00205-013-0695-8

43. T. Kuusi, G. Mingione, Guide to nonlinear potential estimates, *Bull. Math. Sci.*, **4** (2014), 1–82. https://doi.org/10.1007/s13373-013-0048-9

44. T. Kuusi, G. Mingione, Vectorial nonlinear potential theory, *J. Eur. Math. Soc.*, **20** (2018), 929–1004. https://doi.org/10.4171/JEMS/780

45. G. Leoni, *A first course in Sobolev spaces*, Providence, RI: American Mathematical Society, 2009.

46. P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions, *Arch. Rational Mech. Anal.*, **105** (1989), 267–284. https://doi.org/10.1007/BF00251503

47. P. Marcellini, Regularity and existence of solutions of elliptic equations with p,q-growth conditions, *J. Differ. Equations*, **90** (1991), 1–30. https://doi.org/10.1016/0022-0396(91)90158-6

48. G. Mingione, The Calderón-Zygmund theory for elliptic problems with measure data, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **6** (2007), 195–261.

49. G. Mingione, Gradient potential estimates, *J. Eur. Math. Soc.*, **13** (2011), 459–486. https://doi.org/10.4171/JEMS/258

50. C. Scheven, Existence of localizable solutions to nonlinear parabolic problems with irregular obstacles, *Manuscripta Math.*, **146** (2015), 7–63. https://doi.org/10.1007/s00229-014-0684-8

51. R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Providence, RI: American Mathematical Society, 1997.

52. P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differ. Equations*, **51** (1984), 126–150. https://doi.org/10.1016/0022-0396(84)90105-0

53. K. Uhlenbeck, Regularity for a class of non-linear elliptic systems, *Acta Math.*, **138** (1977), 219–240. https://doi.org/10.1007/BF02392316

54. V. V. Zhikov, On some variational problems, *Russian J. Math. Phys.*, **5** (1997), 105–116.

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