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Research article

Lewy-Stampacchia inequality for noncoercive parabolic obstacle problems†

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Abstract: We investigate the obstacle problem for a class of nonlinear and noncoercive parabolic variational inequalities whose model is a Leray–Lions type operator having singularities in the coefficients of the lower order terms. We prove the existence of a solution to the obstacle problem satisfying a Lewy-Stampacchia type inequality.

Keywords: noncoercive evolution problems; obstacle problems; penalization; Lewy–Stampacchia inequality; Marcinkiewicz spaces

To Giuseppe Mingione, on the occasion of his 50th birthday, with regard and admiration.

1. Introduction

The aim of this paper is to study a nonlinear and noncoercive parabolic variational inequality with constraint and homogeneous Dirichlet boundary condition. The Lewy-Stampacchia inequality associated with it is addressed. After the first results of H. Lewy and G. Stampacchia [19] concerning inequalities in the context of superharmonic problems, there is by now a large literature concerning the theory of elliptic obstacle problems as well as of elliptic variational inequalities. We refer to [3, 16, 25] for a classical overview. For a more recent treatment related to nonlinear elliptic operators see also [23]. The obstacle problem for nonlocal and nonlinear operators has been cosidered in [17, 26]. An abstract and general version of the Lewy-Stampacchia inequality is given in [13]. Concerning the parabolic case, first existence results related to problems with time independent obstacles have been treated in [20] in the linear case and in [5] for the more general parabolic problems. The case of obstacles functions regular in time has been considered in [2, 5]. Existence and regularity theory for solutions of parabolic inequalities involving degenerate operators in divergence form have been established in [4, 18]. More recently in [15], the Authors prove Lewy-Stampacchia inequality for parabolic problems related to pseudomonotone type operators. In this paper we study a variational parabolic inequality for noncoercive operators that present singularities in the coeffcients of the lower order terms in the same spirit of [9, 12, 14].

Let us state the functional setting and the assumptions on the data.

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded open Lipschitz domain and let $\Omega_T := \Omega \times (0,T)$ be the parabolic cylinder over Ω of height $T > 0$. We shall denote by ∇v and $\partial_t v$ (or v_t) the spatial gradient and the time derivative of a function v respectively. We consider the class

$$W_p(0,T) := \left\{ v \in L^p(0,T,W_0^{1,p}(\Omega)) \colon v_t \in L^{p'}\left(0,T,W^{-1,p'}(\Omega)\right)\right\}, \tag{1.1}$$

where

$$\frac{2N}{N+2} < p < N. \tag{1.2}$$

and p' is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. In (1.1), $L^p(0,T,W_0^{1,p}(\Omega))$ and $L^{p'}\left(0,T,W^{-1,p'}(\Omega)\right)$ denote parabolic Banach spaces defined according to (2.7).

Given a measurable function $\psi \colon \Omega_T \cup \Omega \times \{0\} \to \mathbb{R}$, we are interested in finding functions $u \colon \Omega_T \to \mathbb{R}$ in the convex subset $\mathcal{K}_\psi(\Omega_T)$ of $W_p(0,T)$ defined as

$$\mathcal{K}_\psi(\Omega_T) := \left\{ v \in W_p\,(0,T) : \ v \geqslant \psi \quad \text{a.e. in } \Omega_T \right\}$$

and satisfying the following variational inequality

$$\int_0^T \langle u_t, v-u\rangle\, dt + \int_{\Omega_T} A(x,t,u,\nabla u)\cdot \nabla(v-u)\,dxdt \geqslant \int_0^T \langle f, v-u\rangle\,dt \quad \forall v \in \mathcal{K}_\psi(\Omega_T), \tag{1.3}$$

where

$$f \in L^{p'}(0,T,W^{-1,p'}(\Omega)) \tag{1.4}$$

and $\langle\cdot,\cdot\rangle$ denotes the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. The vector field

$$A = A(x,t,u,\xi) \colon \Omega_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is a Carathéodory function, i.e., A measurable w.r.t. $(x,t) \in \Omega_T$ for all $(u,\xi) \in \mathbb{R}\times\mathbb{R}^N$ and continuous w.r.t. $(u,\xi) \in \mathbb{R}\times\mathbb{R}^N$ for a.e. $(x,t) \in \Omega_T$, and such that for a.e. $(x,t) \in \Omega_T$ and for any $u \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^N$,

$$A(x,t,u,\xi)\cdot\xi \geqslant \alpha|\xi|^p - (b(x,t)|u|)^p - H(x,t) \tag{1.5}$$

$$[A(x,t,u,\xi) - A(x,t,u,\eta)]\cdot(\xi-\eta) > 0 \qquad \text{if } \xi \neq \eta \tag{1.6}$$

$$|A(x,t,u,\xi)| \leqslant \beta|\xi|^{p-1} + \left(\tilde{b}(x,t)|u|\right)^{p-1} + K(x,t) \tag{1.7}$$

hold true. Here α, β are positive constants, while H, K, b and $\tilde{b}$ are nonnegative measurable functions defined on Ω_T such that $H \in L^1(\Omega_T)$, $K \in L^{p'}(\Omega_T)$ and

$$b, \tilde{b} \in L^\infty\left(0,T,L^{N,\infty}(\Omega)\right), \tag{1.8}$$

where $L^{N,\infty}(\Omega)$ is the Marcinkiewicz space. For definitions of $L^{N,\infty}(\Omega)$ and $L^\infty\left(0,T,L^{N,\infty}(\Omega)\right)$ see Sections 2.2 and 2.3, respectively.

We assume that the obstacle function fulfills

$$\psi \in C^0\left([0,T],L^2(\Omega)\right) \cap L^p\left(0,T,W^{1,p}(\Omega)\right) \tag{1.9}$$

$$\psi \leqslant 0 \quad \text{a.e. in } \partial\Omega \times (0,T) \tag{1.10}$$

$$\psi_t \in L^{p'}(\Omega_T) \tag{1.11}$$

$$\psi(\cdot,0) \in W_0^{1,p}(\Omega). \tag{1.12}$$

For

$$u_0 \in L^2(\Omega) \tag{1.13}$$

we impose the following compatibility condition

$$u_0 \geqslant \psi(\cdot,0) \qquad \text{a.e. in } \Omega. \tag{1.14}$$

In the following, we will refer to a function $u \in \mathcal{K}_\psi(\Omega_T)$ satisfying (1.3) and such that $u(\cdot,0) = u_0$ as a *solution to the variational inequality* in the strong form with initial value u_0. Under previous assumptions the existence of a solution in the weak form can be proved, see [12]. However the existence of a solution in the sense stated above is not guaranteed even in simpler cases. Then we assume that the source term and the obstacle function are such that

$$g := f - \psi_t + \operatorname{div} A(x,t,\psi,\nabla\psi) = g^+ - g^- \quad \text{with} \quad g^+, g^- \in L^{p'}(0,T,W^{-1,p'}(\Omega))^+. \tag{1.15}$$

Here $L^{p'}(0,T,W^{-1,p'}(\Omega))^+$ denotes the non-negative elements of $L^{p'}(0,T,W^{-1,p'}(\Omega))$. Following the terminology of [7] or [15], (1.15) is equivalent to say that g is an element of the order dual $L^p(0,T,W_0^{1,p}(\Omega))^*$ defined as

$$L^p(0,T,W_0^{1,p}(\Omega))^* := \{g = g^+ - g^-,\ g^\pm \in L^{p'}(0,T,W^{-1,p'}(\Omega))^+\}.$$

Then, our main result reads as follows

Theorem 1.1. *Let* (1.2) *and* (1.4)–(1.15) *be in charge. Assume further that*

$$\mathscr{D}_b := \operatorname{dist}_{L^\infty(0,T,L^{N,\infty}(\Omega))}(b, L^\infty(\Omega_T)) < \frac{\alpha^{1/p}}{S_{N,p}}, \tag{1.16}$$

where $S_{N,p} = \omega_N^{-1/N}\frac{p}{N-p}$ *and* ω_N *denotes the measure of the unit ball of* $\mathbb{R}^N$*. Then, there exists at least a solution* $u \in \mathcal{K}_\psi(\Omega_T)$ *of the strong form of the variational inequality* (1.3) *satisfying* $u(\cdot,0) = u_0$*. Moreover, the following Lewy-Stampacchia inequality holds*

$$0 \leq \partial_t u - \operatorname{div} A(x,t,u,\nabla u) - f \leq g^- = (f - \partial_t\psi + \operatorname{div} A(x,t,\psi,\nabla\psi))^-. \tag{1.17}$$

In (1.16), $\mathscr{D}_b$ denotes the distance of b from $L^\infty(\Omega_T)$ in the space $L^\infty(0,T,L^{N,\infty}(\Omega))$ defined in (2.8) below.

Assumptions (1.8) on the coefficients of the lower order terms allow us to consider diffusion models in which the boundedness of the convective field with respect to the spatial variable is too restrictive (see [8]). The corresponding bounded case has been treated in [15].

We discuss condition (1.16) through an example. It's easy to verify that the operator

$$A(x,t,u,\xi) = |\xi|^{p-2}\xi + e^{-t}|u|^{p-2}u\left(\frac{\gamma}{|x|} + \frac{1}{\gamma}\arctan|x|\right)^{p-1}\frac{x}{|x|}$$

satisfies (1.5)–(1.8). According to (2.2) and (2.3) below, we get that

$$\mathscr{D}_b = \left(1 - \frac{1}{p}\right)^{1/p}\omega_N^{1/N}\gamma$$

and so (1.16) holds true whenever γ is small enough. On the other hand, we notice that (1.16) does not imply smallness of the norm of the coefficient b. Indeed

$$\|b\|_{L^\infty(0,T,L^{N,\infty}(\Omega))} \geqslant \frac{C}{\gamma}$$

for a constant C independent of γ.

Theorem 1.1 also applies in the case b and $\tilde{b}$ lie in a functional subspace of weak–L^N in which bounded functions are dense. For more details see also [10]. For other examples of operators satisfying conditions above we refer to [12].

We remark that for $f, \psi_t, \operatorname{div} A(x,t,\psi,\nabla\psi) \in L^{p'}(\Omega_T)$ condition (1.15) is satisfied. Then, Theorem 1.1 is comparable with the existence result of Lemma 3.1 in [4]. In order to prove our result, we consider a sequence of suitable penalization problems for which an existence result holds true (see [12]). Then we are able to construct a solution u to (1.3) as limit of solutions of such problems despite the presence of unbounded coefficients in the lower order terms.

2. Preliminary results

In this section we provide the notation and several preliminary results that will be fundamental in the sequel.

2.1. Notation

The symbol C (or $C_1, C_2, \dots$) will denote positive constant, possibly varying from line to line. For the dependence of C upon parameters, we will simply write $C = C(\cdot, \dots, \cdot)$. The positive and the negative part of a real number z will be denoted by z^+ and z^-, respectively, and are defined by $z^+ := \max\{z, 0\}$ and $z^- := -\min\{z, 0\}$. Given $z_1, z_2 \in \mathbb{R}$, we often use the notation $z_1 \wedge z_2$ and $z_1 \vee z_2$ in place of $\min\{z_1, z_2\}$ and $\max\{z_1, z_2\}$ respectively.

2.2. Lorentz spaces

Let Ω be a bounded domain in $\mathbb{R}^N$. For any $1 < p < \infty$ and $1 \leq q < \infty$, the Lorentz space $L^{p,q}(\Omega)$ is the set of real measurable functions f on Ω such that

$$\|f\|^q_{L^{p,q}} := p\int_0^\infty \left[\lambda_f(k)\right]^{\frac{q}{p}} k^{q-1}\,\mathrm{d}k < \infty.$$

Here $\lambda_f(k) := |\{x \in \Omega : |f(x)| > k\}|$ is the distribution function of f. When $p = q$, the Lorentz space $L^{p,p}(\Omega)$ coincides with the Lebesgue space $L^p(\Omega)$. When $q = \infty$, the space $L^{p,\infty}(\Omega)$ is the set of measurable functions f on Ω such that

$$\|f\|^p_{L^{p,\infty}} := \sup_{k>0} k^p \lambda_f(k) < \infty.$$

This set coincides with the Marcinkiewicz space weak-$L^p(\Omega)$. The expressions above do not define a norm in $L^{p,q}$ or $L^{p,\infty}$ respectively, in fact triangle inequality generally fails. Nevertheless, they are equivalent to a norm, which make $L^{p,q}(\Omega)$ and $L^{p,\infty}(\Omega)$ Banach spaces when endowed with them. An important role in the potential theory is played by these spaces as pointed out in [22].

For $1 \le q < p < r \le \infty$, the following inclusions hold

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega).$$

For $1 < p < \infty$, $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, if $f \in L^{p,q}(\Omega)$, $g \in L^{p',q'}(\Omega)$ we have the Hölder–type inequality

$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x \le \|f\|_{L^{p,q}}\,\|g\|_{L^{p',q'}}. \tag{2.1}$$

Since $L^\infty(\Omega)$ is not dense in $L^{p,\infty}(\Omega)$, for $f \in L^{p,\infty}(\Omega)$ in [6] the Authors stated the following

$$\mathrm{dist}_{L^{p,\infty}(\Omega)}(f, L^\infty(\Omega)) := \inf_{g\in L^\infty(\Omega)} \|f - g\|_{L^{p,\infty}(\Omega)}. \tag{2.2}$$

As already observed in [10, 11], we have

$$\mathrm{dist}_{L^{p,\infty}(\Omega)}(f, L^\infty(\Omega)) = \lim_{m\to+\infty} \|f\chi_{\{|f|>m\}}\|_{L^{p,\infty}} \tag{2.3}$$

and

$$\mathrm{dist}_{L^{p,\infty}(\Omega)}(f, L^\infty(\Omega)) = \lim_{m\to+\infty} \|f - \mathcal{T}_m f\|_{L^{p,\infty}},$$

where, for all $m > 0$, $\mathcal{T}_m$ is the truncation operator at levels $\pm m$, i.e.,

$$\mathcal{T}_m y := \min\{m, \max\{-m, y\}\} \qquad \text{for } y \in \mathbb{R}. \tag{2.4}$$

Another useful estimate is provided by the following sort of triangle inequality

$$\|f + \varepsilon g\|_{L^{p,\infty}} \leqslant (1 + \sqrt{\varepsilon})\,\|f\|_{L^{p,\infty}} + \sqrt{\varepsilon}(1 + \sqrt{\varepsilon})\,\|g\|_{L^{p,\infty}} \tag{2.5}$$

which holds true for $f, g \in L^{p,\infty}(\Omega)$ and $\varepsilon > 0$.

For $1 \le q < \infty$, any function in $L^{p,q}(\Omega)$ has zero distance to $L^\infty(\Omega)$. Indeed, $L^\infty(\Omega)$ is dense in $L^{p,q}(\Omega)$, the latter being continuously embedded into $L^{p,\infty}(\Omega)$.

Assuming that $0 \in \Omega$, $b(x) = \gamma/|x|$ belongs to $L^{N,\infty}(\Omega)$, $\gamma > 0$. For this function, we have

$$\mathrm{dist}_{L^{N,\infty}(\Omega)}(b, L^\infty(\Omega)) = \gamma\,\omega_N^{1/N}.$$

The Sobolev embedding theorem in Lorentz spaces [1, 24] reads as

Theorem 2.1. *Let us assume that $1 < p < N$, $1 \le q \le p$, then every function $u \in W_0^{1,1}(\Omega)$ verifying $|\nabla u| \in L^{p,q}(\Omega)$ actually belongs to $L^{p^*,q}(\Omega)$, where $p^* := \frac{Np}{N-p}$ is the Sobolev conjugate exponent of p and*

$$\|u\|_{L^{p^*,q}} \le S_{N,p}\,\|\nabla u\|_{L^{p,q}}, \tag{2.6}$$

where $S_{N,p}$ is the Sobolev constant given by $S_{N,p} = \omega_N^{-1/N}\frac{p}{N-p}$.

2.3. Parabolic spaces

Let $T > 0$ and X be a Banach space endowed with a norm $\|\cdot\|_X$. Then, the space $L^p(0,T,X)$ is defined as the class of all measurable functions $u\colon [0,T] \to X$ such that

$$\|u\|_{L^p(0,T,X)} := \left(\int_0^T \|u(t)\|_X^p \, \mathrm{d}t\right)^{1/p} < \infty \tag{2.7}$$

whenever $1 \le p < \infty$, and

$$\|u\|_{L^\infty(0,T,X)} := \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X < \infty$$

for $p = \infty$. The space $C^0([0,T],X)$ represents the class of all continuous functions $u\colon [0,T] \to X$ with the norm

$$\|u\|_{C^0([0,T],X)} := \max_{0\le t\le T} \|u(t)\|_X.$$

We essentially consider the case where X is either a Lorentz space or Sobolev space $W_0^{1,p}(\Omega)$. This space will be equipped with the norm $\|g\|_{W_0^{1,p}(\Omega)} := \|\nabla g\|_{L^p(\Omega)}$ for $g \in W_0^{1,p}(\Omega)$.

For $f \in L^\infty(0,T,L^{p,\infty}(\Omega))$ we define

$$\operatorname{dist}_{L^\infty(0,T,L^{p,\infty}(\Omega))}(f, L^\infty(\Omega_T)) = \inf_{g\in L^\infty(\Omega_T)} \|f - g\|_{L^\infty(0,T,L^{p,\infty}(\Omega))} \tag{2.8}$$

and as in (2.3) we find

$$\operatorname{dist}_{L^\infty(0,T,L^{p,\infty}(\Omega))}(f, L^\infty(\Omega_T)) = \lim_{m\to+\infty} \|f\chi_{\{|f|>m\}}\|_{L^\infty(0,T,L^{p,\infty}(\Omega))}. \tag{2.9}$$

In the class $W_p(0,T)$ defined in (1.1) and equipped with the norm

$$\|u\|_{W_p(0,T)} := \|u\|_{L^p(0,T,W^{1,p}(\Omega))} + \|u_t\|_{L^{p'}(0,T,W^{-1,p'}(\Omega))},$$

the following inclusion holds (see [27, Chapter III, page 106]).

Lemma 2.2. *Let $p > 2N/(N+2)$. Then $W_p(0,T)$ is contained into the space $C^0\left([0,T], L^2(\Omega)\right)$ and any function $u \in W_p(0,T)$ satisfies*

$$\|u\|_{C^0([0,T],L^2(\Omega))} \le C\|u\|_{W_p(0,T)}$$

for some constant $C > 0$.

Moreover, the function $t \in [0,T] \mapsto \|u(\cdot,t)\|^2_{L^2(\Omega)}$ is absolutely continuous and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|^2_{L^2(\Omega)} = \langle u_t(\cdot,t), u(\cdot,t)\rangle \qquad \text{for a.e. } t \in [0,T].$$

The compactness result due to Aubin–Lions reads as follows.

Lemma 2.3. *Let X_0, X, X_1 be Banach spaces with X_0 and X_1 reflexive. Assume that X_0 is compactly embedded into X and X is continuously embedded into X_1. For $1 < p, q < \infty$ let*

$$W := \{u \in L^p(0,T,X_0) \colon \partial_t u \in L^q(0,T,X_1)\}.$$

Then W is compactly embedded into $L^p(0,T,X)$.

As an example, we choose $q = p'$, $X_0 = W_0^{1,p}(\Omega)$, $X_1 = W^{-1,p'}(\Omega)$ and $X = L^p(\Omega)$ if $p \ge 2$ or $X = L^2(\Omega)$ for $\frac{2N}{N+2} < p < 2$. Therefore, we deduce

Lemma 2.4. *If $p > 2N/(N+2)$ then $W_p(0,T)$ is compactly embedded into $L^p(\Omega_T)$ and into $L^2(\Omega_T)$.*

3. A penalized problem

Let $\delta > 0$. We introduce the following initial–boundary value problem

$$\begin{cases} \partial_t u_\delta - \operatorname{div}\left[A(x,t,\max\{u_\delta,\psi\},\nabla u_\delta)\right] = \frac{1}{\delta}[(\psi-u_\delta)^+]^{q-1} + f \qquad \text{in } \Omega_T, \\ u_\delta = 0 \qquad \text{on } \partial\Omega\times(0,T), \\ u_\delta(\cdot,0) = u_0 \qquad \text{in } \Omega, \end{cases} \tag{3.1}$$

where

$$q := \min\{2,p\}.$$

Moreover, in this section we assume that

$$\psi \leqslant 0 \quad \text{a.e. in } \Omega_T. \tag{3.2}$$

We introduce the notation

$$\tilde{A}(x,t,w,\xi) := A(x,t,\max\{w,\psi\},\xi).$$

By the elementary inequality

$$|a \vee a'| \leqslant |a| \qquad \forall a\in\mathbb{R} \quad \forall a' \in (-\infty,0] \tag{3.3}$$

and recalling (1.5), (1.6) and (1.7), we easily deduce

$$\begin{aligned} &\tilde{A}(x,t,u,\xi)\cdot\xi \geqslant \alpha|\xi|^p - (b(x,t)|u|)^p - H(x,t) \\ &\left[\tilde{A}(x,t,u,\xi) - \tilde{A}(x,t,u,\eta)\right]\cdot(\xi-\eta) > 0 \qquad \text{if } \xi\neq\eta \\ &\left|\tilde{A}(x,t,u,\xi)\right| \leqslant \beta|\xi|^{p-1} + \left(\tilde{b}(x,t)|u|\right)^{p-1} + K(x,t) \end{aligned}$$

for a.e. $(x,t)\in\Omega_T$ and for any $u\in\mathbb{R}$ and $\xi,\eta\in\mathbb{R}^N$.

For $u_0\in L^2(\Omega)$ and $f\in L^{p'}(0,T,W^{-1,p'}(\Omega))$, a solution to problem (3.1) is a function

$$u_\delta \in C^0\left([0,T],L^2(\Omega)\right)\cap L^p(0,T,W_0^{1,p}(\Omega))$$

such that

$$\begin{aligned} -\int_{\Omega_T} u_\delta\varphi_t\,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_T}\tilde{A}(x,s,u_\delta,\nabla u_\delta)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{\delta}\int_{\Omega_T}[(\psi-u_\delta)^+]^{q-1}\varphi\,\mathrm{d}x\,\mathrm{d}s \\ + \int_\Omega u_0\varphi(x,0)\,\mathrm{d}x + \int_0^T\langle f,\varphi\rangle\,\mathrm{d}s \end{aligned}$$

for every $\varphi\in C^\infty(\bar{\Omega}_T)$ such that $\operatorname{supp}\varphi\subset[0,T)\times\Omega$.

By using the elementary inequality

$$(a+a')^\theta \leqslant a^\theta + a'^\theta \qquad \forall a,a'\in[0,+\infty) \qquad \forall\theta\in(0,1)$$

and Young inequality we see that

$$p<2 \quad \Longrightarrow \quad [(\psi-u)^+]^{p-1} \leqslant |\psi|^{p-1} + |u|^{p-1} \leqslant (p-1)\,(|u|+|\psi|) + 2(2-p).$$

Hence, by Theorem 4.2 and Remark 4.5 in [12] we get the following existence result.

Proposition 3.1. *Let* (1.2), (1.4)–(1.16) *and* (3.2) *be in charge. For every fixed* $\delta > 0$, *problem* (3.1) *admits a solution* $u_\delta \in C^0\Big([0,T], L^2(\Omega)\Big) \cap L^p(0,T,W_0^{1,p}(\Omega))$.

The arguments of [12] lead to some estimates for the sequence $\{u_\delta\}_{\delta>0}$. We propose here a proof that carefully keeps trace of the constants in the estimates.

Lemma 3.2. *Let* (1.2), (1.4)–(1.16) *and* (3.2) *be in charge. Any solution* $u_\delta \in C^0\Big([0,T], L^2(\Omega)\Big) \cap L^p(0,T,W_0^{1,p}(\Omega))$ *to problem* (3.1) *satisfies the following estimate*

$$\begin{aligned} \|u_\delta\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u_\delta\|^p_{L^p(\Omega_T)} \leqslant C(b,N,p,\alpha)\Big[\|u_0\|^2_{L^2(\Omega)} + \|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))} + \|H\|_{L^1(\Omega_T)} \\ + \Big(\|u_0\|^2_{L^2(\Omega)} + \|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))} + \|b\|^p_{L^p(\Omega_T)}\Big)^p \|b\|^p_{L^p(\Omega_T)}\Big]. \end{aligned} \tag{3.4}$$

Proof. We fix $t \in (0,T)$ and we set $\Omega_t := \Omega \times (0,t)$. We choose $\varphi := \mathcal{T}_1(u_\delta)\chi_{(0,t)}$ as a test function. If we let $\Phi(z) := \int_0^z \mathcal{T}_1(\zeta)\,\mathrm{d}\zeta$ for $z \in \mathbb{R}$, we have

$$\begin{aligned} &\int_\Omega \Phi(u_\delta(x,t))\,\mathrm{d}x + \int_{\Omega_t} \tilde{A}(x,s,u_\delta,\nabla u_\delta)\cdot\nabla\mathcal{T}_1(u_\delta)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad = \frac{1}{\delta}\int_{\Omega_t}[(\psi-u_\delta)^+]^{q-1}\mathcal{T}_1(u_\delta)\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_\Omega \Phi(u_0)\,\mathrm{d}x + \int_0^t \langle f,\mathcal{T}_1(u_\delta)\rangle\,\mathrm{d}s\,. \end{aligned}$$

Assumption (3.2) implies that $[(\psi-u_\delta)^+]^{q-1}\mathcal{T}_1(u_\delta) \leqslant 0$ a.e. in Ω_T, so we have

$$\begin{aligned} &\int_\Omega \Phi(u_\delta(x,t))\,\mathrm{d}x + \int_{\Omega_t\cap\{|u_\delta|\leqslant 1\}} \tilde{A}(x,s,u_\delta,\nabla u_\delta)\cdot\nabla u_\delta\,\mathrm{d}x\,\mathrm{d}s \\ &\quad \leqslant \int_\Omega \Phi(u_0(x,0))\,\mathrm{d}x + \int_0^t \langle f,\mathcal{T}_1(u_\delta)\rangle\,\mathrm{d}s\,. \end{aligned}$$

By (1.5) and (1.7) we deduce

$$\begin{aligned} &\int_\Omega \Phi(u_\delta(x,t))\,\mathrm{d}x + \alpha\int_{\Omega_t\cap\{|u_\delta|\leqslant 1\}} |\nabla u_\delta|^p\,\mathrm{d}x\,\mathrm{d}s \\ &\quad \leqslant \int_\Omega \Phi(u_0)\,\mathrm{d}x + \int_0^t \langle f,\mathcal{T}_1(u_\delta)\rangle\,\mathrm{d}s \\ &\qquad + \int_{\Omega_t\cap\{|u_\delta|\leqslant 1\}} (b|u_\delta \vee \psi|)^p\,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t\cap\{|u_\delta|\leqslant 1\}} H\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.5}$$

Now, as $0 \leqslant \Phi(z) \leqslant \frac{z^2}{2}$ for all $z \in \mathbb{R}$, we have

$$\int_\Omega \Phi(u_0)\,\mathrm{d}x \leqslant \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{3.6}$$

By Hölder and Young inequality we get

$$\begin{aligned}\int_0^t \langle f, \mathcal{T}_1(u_\delta)\rangle \,\mathrm{d}s &\leqslant \|f\|_{L^{p'}(0,T,W^{-1,p'}(\Omega))}\|\nabla \mathcal{T}_1(u_\delta)\|_{L^p(\Omega_t)}\\ &= \|f\|_{L^{p'}(0,T,W^{-1,p'}(\Omega))}\left(\int_{\Omega_t\cap\{|u_\delta|\leqslant 1\}} |\nabla(u_\delta)|^p \,\mathrm{d}x\,\mathrm{d}s\right)^{1/p}\\ &\leqslant \frac{\alpha}{2}\int_{\Omega_t\cap\{|u_\delta|\leqslant 1\}} |\nabla u_\delta|^p \,\mathrm{d}x\,\mathrm{d}s + C(\alpha,p)\|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))}.\end{aligned} \tag{3.7}$$

Finally, by (3.3)

$$\int_{\Omega_t\cap\{|u_\delta|\leqslant 1\}} (b|u_\delta \vee \psi|)^p \,\mathrm{d}x\,\mathrm{d}s \leqslant \int_{\Omega_t\cap\{|u_\delta|\leqslant 1\}} (b|u_\delta|)^p \,\mathrm{d}x\,\mathrm{d}s \leqslant \|b\|^p_{L^p(\Omega_T)}. \tag{3.8}$$

Gathering (3.6), (3.7), and (3.8) and using Hölder inequality, by (3.5) we have

$$\int_\Omega \Phi(u_\delta(x,t))\,\mathrm{d}x \leqslant M_0,$$

where

$$M_0 := C(N,p,\alpha)\left[\|u_0\|^2_{L^2(\Omega)} + \|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))} + \|b\|^p_{L^p(\Omega_T)}.\right] \tag{3.9}$$

It is easily seen that

$$\frac{|u|}{2} \leqslant \Phi(u) \quad \text{for } |u| \geqslant 1$$

and so

$$\sup_{0<t<T} |\{x\in\Omega:\ |u_\delta(x,t)| > k\}| \leqslant \frac{C(N,p,\alpha,\beta)M_0}{k} \qquad \forall k \geqslant 1. \tag{3.10}$$

We fix $t\in(0,T)$ and choose $\varphi := u_\delta\chi_{(0,t)}$ as a test function in (3.1). Again, assumption (3.2) implies that $[(\psi-u_\delta)^+]^{q-1}u_\delta \leqslant 0$ a.e. in Ω_T, then

$$\begin{aligned}&\frac{1}{2}\|u_\delta(\cdot,t)\|^2_{L^2(\Omega)} + \int_{\Omega_t} \tilde{A}(x,s,u_\delta,\nabla u_\delta)\cdot\nabla u_\delta \,\mathrm{d}x\,\mathrm{d}s\\ &\qquad\leqslant \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \int_0^t \langle f, u_\delta\rangle\,\mathrm{d}s.\end{aligned}$$

By Young inequality for $\varepsilon > 0$

$$\int_0^t \langle f, u_\delta\rangle\,\mathrm{d}s \leqslant \varepsilon\int_{\Omega_t} |\nabla u_\delta|^p \,\mathrm{d}x\,\mathrm{d}s + \frac{p-1}{p^p}\varepsilon^{1-p}\|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))}.$$

Then, by (1.5) we further have

$$\begin{aligned}&\|u_\delta(\cdot,t)\|^2_{L^2(\Omega)} + \alpha\int_{\Omega_t} |\nabla u_\delta|^p \,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \|u_0\|^2_{L^2(\Omega)} + \varepsilon\int_{\Omega_t} |\nabla u_\delta|^p\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + C(\varepsilon,p)\|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))} + \int_{\Omega_t} (b\,|u_\delta\vee\psi|)^p \,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t} H\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.11}$$

For $m > 0$ to be chosen later, we have from (3.3)

$$\int_{\Omega_t} (b\,|u_\delta \vee \psi|)^p \;\mathrm{d}x\,\mathrm{d}s \leqslant \int_{\Omega_t} (b\,|u_\delta|)^p \;\mathrm{d}x\,\mathrm{d}s = \int_{\Omega_t} (b\,\chi_{\{b\leqslant m\}}\,|u_\delta|)^p \;\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t} (b\,\chi_{\{b>m\}}\,|u_\delta|)^p \;\mathrm{d}x\,\mathrm{d}s. \tag{3.12}$$

We estimate separately the two terms in the right–hand side of (3.12). For $k > 1$ fixed, we obtain

$$\begin{aligned} &\int_{\Omega_t} (b\,\chi_{\{b\leqslant m\}}\,|u_\delta|)^p \;\mathrm{d}x\,\mathrm{d}s \\ &\qquad \leqslant m^p \int_0^t \mathrm{d}s \int_{\{|u_\delta(\cdot,s)|>k\}} |u_\delta|^p \,\mathrm{d}x + k^p \int_0^t \mathrm{d}s \int_\Omega b(x,s)^p \,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Now we apply Hölder inequality (2.1), estimates (2.6) and (3.10) to get

$$\begin{aligned} \int_0^t \mathrm{d}s \int_{\{|u_\delta(\cdot,s)|>k\}} |u_\delta|^p \,\mathrm{d}x &= \int_0^t \mathrm{d}s \int_\Omega |u_\delta \chi_{\{|u_\delta(\cdot,s)|>k\}}|^p \,\mathrm{d}x \\ &\leqslant \int_0^t \|\chi_{\{|u_\delta(\cdot,s)|>k\}}\|^p_{L^{N,\infty}(\Omega)} \|u_\delta\|^p_{L^{p^*,p}(\Omega)} \,\mathrm{d}s \\ &\leqslant \frac{S^p_{N,p} M_0^{p/N}}{k^{p/N}} \int_{\Omega_t} |\nabla u_\delta|^p \,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{3.14}$$

where M_0 is the constant in (3.9). On the other hand, using again Hölder inequality (2.1) and estimate (2.6) we have

$$\int_{\Omega_t} (b\,\chi_{\{b>m\}}\,|u_\delta|)^p \;\mathrm{d}x\,\mathrm{d}s \leqslant S^p_{N,p}\; \|b\,\chi_{\{b>m\}}\|^p_{L^\infty\left(0,T,L^{N,\infty}(\Omega)\right)} \int_{\Omega_t} |\nabla u_\delta|^p \,\mathrm{d}x\,\mathrm{d}s. \tag{3.15}$$

Inserting (3.13), (3.14) and (3.15) into (3.12) we obtain

$$\begin{aligned} &\int_{\Omega_t} (b\,|u_\delta \vee \psi|)^p \;\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \left[\frac{m^p S^p_{N,p} M_0^{p/N}}{k^{p/N}} + S^p_{N,p}\,\|b\,\chi_{\{b>m\}}\|^p_{L^\infty\left(0,T,L^{N,\infty}(\Omega)\right)} \right] \|\nabla u_\delta\|^p_{L^p(\Omega_t)} + k^p \int_0^t \mathrm{d}s \int_\Omega b(x,s)^p \,\mathrm{d}x. \end{aligned} \tag{3.16}$$

Observe that (3.11) and (3.16) imply

$$\begin{aligned} &\frac{1}{2}\,\|u_\delta(\cdot,t)\|^2_{L^2(\Omega)} + \alpha\,\|\nabla u_\delta\|^p_{L^p(\Omega_t)} \leqslant \\ &\quad \frac{1}{2}\,\|u_0\|^2_{L^2(\Omega)} + k^p \|b\|^p_{L^p(\Omega_T)} + \frac{p-1}{p^p} \varepsilon^{1-p} \|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))} + \|H\|_{L^1(\Omega_T)} \\ &\quad + \left[\varepsilon + \frac{m^p S^p_{N,p} M_0^{p/N}}{k^{p/N}} + S^p_{N,p}\,\|b\,\chi_{\{b>m\}}\|^p_{L^\infty\left(0,T,L^{N,\infty}(\Omega)\right)} \right] \|\nabla u_\delta\|^p_{L^p(\Omega_t)}. \end{aligned}$$

Now we choose $m > 0$ so large to guarantee

$$S^p_{N,p}\,\|b\,\chi_{\{b>m\}}\|^p_{L^\infty\left(0,T,L^{N,\infty}(\Omega)\right)} < \alpha.$$

The existence of such a value of m is a direct consequence of (1.16) and the characterization of distance in (2.9). It is also clear that m is a positive constant depending only on b, N, p and α. So we get

$$\begin{aligned}&\frac{1}{2}\,\|u_\delta(\cdot,t)\|^2_{L^2(\Omega)}+\alpha_1\,\|\nabla u_\delta\|^p_{L^p(\Omega_t)}\leqslant\\ &\frac{1}{2}\,\|u_0\|^2_{L^2(\Omega)}+k^p\|b\|^p_{L^p(\Omega_T)}+\frac{p-1}{p^p}\varepsilon^{1-p}\|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))}+\|H\|_{L^1(\Omega_T)}\\ &+\left[\varepsilon+\frac{m^pS^p_{N,p}M_0^{p/N}}{k^{p/N}}\right]\|\nabla u_\delta\|^p_{L^p(\Omega_t)}\end{aligned}$$

for some $\alpha_1=\alpha_1(b,N,p,\alpha)$. We may also choose $\varepsilon=\frac{\alpha_1}{2}$. Then the latter relation becomes

$$\begin{aligned}&\frac{1}{2}\,\|u_\delta(\cdot,t)\|^2_{L^2(\Omega)}+\frac{\alpha_1}{2}\,\|\nabla u_\delta\|^p_{L^p(\Omega_t)}\leqslant\\ &\frac{1}{2}\,\|u_0\|^2_{L^2(\Omega)}+k^p\|b\|^p_{L^p(\Omega_T)}+C_1(b,N,p,\alpha)\|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))}+\|H\|_{L^1(\Omega_T)}\\ &+C_2(b,N,p,\alpha)\left(\frac{M_0}{k}\right)^{p/N}\|\nabla u_\delta\|^p_{L^p(\Omega_t)}.\end{aligned}$$

We choose $k=M_0\left(\frac{\alpha_1}{4C_2}\right)^{N/p}$ so that $C_2\left(\frac{M_0}{k}\right)^{p/N}=\frac{\alpha_1}{4}$ and therefore

$$\begin{aligned}&\frac{1}{2}\,\|u_\delta(\cdot,t)\|^2_{L^2(\Omega)}+\frac{\alpha_1}{4}\,\|\nabla u_\delta\|^p_{L^p(\Omega_t)}\leqslant\\ &\frac{1}{2}\,\|u_0\|^2_{L^2(\Omega)}+C_3(b,N,p,\alpha)M_0^p\|b\|^p_{L^p(\Omega_T)}+C_1(b,N,p,\alpha)\|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))}+\|H\|_{L^1(\Omega_T)}.\end{aligned}$$

Taking into account the definition of M_0, the latter leads to the estimate (3.4). □

Lemma 3.3. *Let* (1.2)*,* (1.4)–(1.16) *and* (3.2) *be in charge. Assume further that g^- defined in* (1.15) *is such that*

$$g^-\in L^{q'}(\Omega_T). \tag{3.17}$$

Then, for every $\delta>0$, every solution u_δ of problem (3.1) *satisfies*

$$\|(u_\delta-\psi)^-\|^{q-1}_{L^q(\Omega_T)}\le\delta\|g^-\|_{L^{q'}(\Omega_T)}. \tag{3.18}$$

Moreover, there exists a positive constant C depending only on the data and independent on δ such that

$$\|\partial_t u_\delta\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\le C. \tag{3.19}$$

Proof. We use the function $\phi=(\psi-u_\delta)^+$ as a test function in the equation of Problem (3.1). Then, we get

$$\begin{aligned}\int_0^T\langle\partial_t u_\delta,(\psi-u_\delta)^+\rangle\,dt+\int_{\Omega_T}A(x,t,\max\{u_\delta,\psi\},\nabla u)\cdot\nabla(\psi-u_\delta)^+\,dxdt\\ =\frac{1}{\delta}\int_{\Omega_T}[(\psi-u_\delta)^+]^q dxdt+\int_0^T\langle f,(\psi-u_\delta)^+\rangle\,dt.\end{aligned}$$

Recalling (1.15), this implies

$$\begin{aligned}\frac{1}{\delta}\int_{\Omega_T}[(\psi-u_\delta)^+]^q dxdt = &\int_{\Omega_T} g^-(\psi-u_\delta)^+\,dxdt\\ &-\int_0^T \langle g^+,(\psi-u_\delta)^+\rangle\,dt\\ &-\int_0^T \langle \partial_t(\psi-u_\delta),(\psi-u_\delta)^+\rangle\,dt\\ &-\int_{\Omega_T\cap\{\psi>u_\delta\}}[A(x,t,\psi,\nabla\psi)-A(x,t,\psi,\nabla u_\delta)]\cdot\nabla(\psi-u_\delta)\,dxdt.\end{aligned}$$

By (1.14) we observe that

$$\int_0^T \langle \partial_t(\psi-u_\delta),(\psi-u_\delta)^+\rangle\,dt = \frac{1}{2}\|(u_\delta-\psi)^-(T)\|^2_{L^2(\Omega)}$$

hence, by (1.6) we get

$$\frac{1}{\delta}\int_{\Omega_T}[(\psi-u_\delta)^+]^q \le \int_{\Omega_T} g^-(\psi-u_\delta)^+\,dxdt.$$

Then, using Hölder inequality and dividing both sides of the inequality by $\|(\psi-u_\delta)^+)\|_{L^q((\Omega_T)}$ we obtain (3.18). To obtain (3.19) we fix $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))$ and then we observe that

$$\begin{aligned}\left|\int_0^T\langle\partial_t u_\delta,\varphi\rangle\,dt\right| \le &\left(\|A(\cdot,\cdot,\max\{u_\delta,\psi\},\nabla u_\delta)\|_{L^{p'}(\Omega_T)}+\|f\|_{L^{p'}(\Omega_T)}\right)\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))}\\ &+\frac{1}{\delta}\|(\psi-u_\delta)^+\|^{q-1}_{L^q(\Omega_T)}\|\varphi\|_{L^q(\Omega_T)}.\end{aligned}$$

At this point we observe that the definition of q and Holder inequality imply

$$\|\varphi\|_{L^q(\Omega_T)}\le C(p,|\Omega|,T)\|\varphi\|_{L^p(\Omega_T)}.$$

Finally, using (3.18) and Poncaré inequality slicewise, we conclude that

$$\left|\int_0^T\langle\partial_t u_\delta,\varphi\rangle\,dt\right|\le C(p,|\Omega|,T)\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))},$$

where C is a positive constant independent of δ. This immediately leads to (3.19). □

4. Proof of main result

We proceed step by step. We first prove the result under regularity assumptions on g and sign conditon (3.2) on the obstacle function ψ. Then we address the general case.

Proposition 4.1. *Let* (1.2), (1.4)–(1.16), (3.2) *and* (3.17) *be in charge. There exists at least solution* $u\in\mathcal{K}_\psi(\Omega_T)$ *to the variational inequality* (1.3) *such that* $u(\cdot,0)=u_0$ *in* Ω *and satisfying the following estimate*

$$\begin{aligned}\|u\|^2_{L^\infty(0,T,L^2(\Omega))}+\|\nabla u\|^p_{L^p(\Omega_T)}\leqslant C(b,N,p,\alpha)\Big[&\|u_0\|^2_{L^2(\Omega)}+\|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))}+\|H\|_{L^1(\Omega_T)}\\ &+\left(\|u_0\|^2_{L^2(\Omega)}+\|f\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))}+\|b\|^p_{L^p(\Omega_T)}\right)^p\|b\|^p_{L^p(\Omega_T)}\Big].\end{aligned}\tag{4.1}$$

Proof. By Proposition 3.1, for every $\delta > 0$ there exists a solution $u_\delta \in C^0\left([0,T], L^2(\Omega)\right) \cap L^p(0,T,W_0^{1,p}(\Omega))$ to problem (3.1) satisfying (3.4). Hence we have that, by Lemma 3.3 and Lemma 2.2, there exists $u \in C^0\left([0,T], L^2(\Omega)\right) \cap L^p(0,T,W_0^{1,p}(\Omega))$ such that

$$u_\delta \to u \quad \text{strongly in } L^p(\Omega_T) \tag{4.2}$$

$$\nabla u_\delta \rightharpoonup \nabla u \quad \text{weakly in } L^p\left(\Omega_T, \mathbb{R}^N\right) \tag{4.3}$$

$$u_\delta \overset{*}{\rightharpoonup} u \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega))$$

$$\partial_t u_\delta \rightharpoonup \partial_t u \quad \text{weakly in } L^{p'}(0,T,W^{-1,p'}(\Omega))$$

as $\delta \to 0^+$. By semicontinuity, (3.4) implies (4.1)

We claim that the limit function u solves the variational inequality (1.3) in the strong form.

It is immediate to check that

$$u(\cdot,0) = u_0 \qquad \text{a.e. in } \Omega, \tag{4.4}$$

$$u \geq \psi \text{ a.e. in } \quad \Omega_T. \tag{4.5}$$

Indeed, (4.4) holds since $u_\delta(\cdot,0) = u_0$ a.e. in Ω for every $\delta > 0$. On the other hand, if we pass to the limit as $\delta \to 0^+$ in (3.18) and take into account (4.2) we have $\|(u-\psi)^-\|_{L^{2\wedge p}(\Omega_T)} = 0$ which clearly implies (4.5).

Our next goal is to prove that

$$\nabla u_\delta \to \nabla u \qquad \text{a.e. in } \Omega_T \tag{4.6}$$

as $\delta \to 0^+$. We test the penalized equation by $\mathcal{T}_1(u_\delta - u)$ and since condition (4.5) implies

$$\int_{\Omega_T} [(\psi - u_\delta)^+]^{q-1} \mathcal{T}_1(u_\delta - u)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0$$

we get the following inequality

$$\int_0^T \langle \partial_t u_\delta, \mathcal{T}_1(u_\delta - u)\rangle\, dt + \int_{\Omega_T} A(x,t,u_\delta \vee \psi, \nabla u_\delta) \cdot \nabla \mathcal{T}_1(u_\delta - u)\, dz \leqslant \int_0^T \langle f, \mathcal{T}_1(u_\delta - u)\rangle\, dt. \tag{4.7}$$

If we set $\Phi(z) := \int_0^z \mathcal{T}_1(\zeta)\,\mathrm{d}\zeta$, by (4.4) we obtain

$$\int_0^T \langle \partial_t u_\delta, \mathcal{T}_1(u_\delta - u)\rangle\, dt = \int_\Omega \Phi(u_\delta - u)(x,T)\,\mathrm{d}x + \int_0^T \langle \partial_t u, \mathcal{T}_1(u_\delta - u)\rangle\, dt.$$

Because of (4.3), the latter term in the last inequality vanishes in the limit as $\delta \to 0$. So, as Φ is nonnegative, we get

$$\limsup_{\delta\to 0} \int_0^T \langle \partial_t u_\delta, \mathcal{T}_1(u_\delta - u)\rangle\, dt \geqslant 0.$$

Again by (4.3), the right hand side of (4.7) vanishes in the limit as $\delta \to 0$, and so (4.7) implies

$$\limsup_{\delta\to 0} \int_{\Omega_T \cap \{|u_\delta - u| \leqslant 1\}} A(x,t,u_\delta \vee \psi, \nabla u_\delta) \cdot \nabla(u_\delta - u)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0. \tag{4.8}$$

By (1.7), (3.2) and (3.3) we have

$$\begin{aligned}|A(x,t,u_\delta\vee\psi,\nabla u)|\chi_{\{|u_\delta-u|\leqslant 1\}} &\leqslant \beta|\nabla u|^{p-1}+(\tilde{b}|u_\delta|)^{p-1}\chi_{\{|u_\delta-u|\leqslant 1\}}+K\\ &\leqslant \beta|\nabla u|^{p-1}+C(p)\tilde{b}^{p-1}+C(p)(\tilde{b}|u|)^{p-1}+K\end{aligned}$$

therefore, by the dominated convergence theorem and by (4.2), we get

$$\lim_{\delta\to 0}\int_{\Omega_T\cap\{|u_\delta-u|\leqslant 1\}} A(x,t,u_\delta\vee\psi,\nabla u)\cdot\nabla(u_\delta-u)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{4.9}$$

Combining (4.8) and (4.9) and by (1.6) we get

$$\lim_{\delta\to 0}\int_{\Omega_T}\left[A(x,t,u_\delta\vee\psi,\nabla u_\delta)-A(x,t,u_\delta\vee\psi,\nabla u)\right]\cdot\nabla\mathcal{T}_1(u_\delta-u)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{4.10}$$

Using again (1.6), relation (4.10) gives

$$\left[A(x,t,u_\delta\vee\psi,\nabla u_\delta)-A(x,t,u_\delta\vee\psi,\nabla u)\right]\cdot\nabla(u_\delta-u)\chi_{\{|u_\delta-u|\leqslant 1\}}\to 0 \qquad \text{a.e. in } \Omega_T$$

and so by (4.2) we get

$$\left[A(x,t,u_\delta\vee\psi,\nabla u_\delta)-A(x,t,u_\delta\vee\psi,\nabla u)\right]\cdot\nabla(u_\delta-u)\to 0 \qquad \text{a.e. in } \Omega_T$$

as $\delta\to 0$. By Lemma 3.1 in [21] we deduce that (4.6) holds.

We let $v\in\mathcal{K}_\psi(\Omega_T)$. It is clear that $[(\psi-u_\delta)^+]^{q-1}\,\mathcal{T}_\lambda(u_\delta-v)\leqslant 0$ a.e. in Ω_T and for every $\lambda>0$. For this reason, if we use $\mathcal{T}_\lambda(u_\delta-v)$ as a test function in (3.1) we deduce

$$\begin{aligned}&\int_0^T\langle\partial_t u_\delta,\mathcal{T}_\lambda(u_\delta-v)\rangle\,\mathrm{d}t+\int_{\Omega_T}\left[A(x,t,u_\delta\vee\psi,\nabla u_\delta)-A(x,t,u_\delta\vee\psi,\nabla v)\right]\cdot\nabla\mathcal{T}_\lambda(u_\delta-v)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqslant\int_0^T\langle f,\mathcal{T}_\lambda(u_\delta-v)\rangle\,\mathrm{d}t-\int_{\Omega_T}A(x,t,u_\delta\vee\psi,\nabla v)\cdot\nabla\mathcal{T}_\lambda(u_\delta-v)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.11}$$

We set $\Phi_\lambda(z):=\int_0^z\mathcal{T}_\lambda(\zeta)\,\mathrm{d}\zeta$ and we have

$$\begin{aligned}\int_0^T\langle\partial_t u_\delta,\mathcal{T}_\lambda(u_\delta-v)\rangle\,\mathrm{d}t&=\int_0^T\langle\partial_t v,\mathcal{T}_\lambda(u_\delta-v)\rangle\,\mathrm{d}t+\int_0^T\langle\partial_t u_\delta-\partial_t v,\mathcal{T}_\lambda(u_\delta-v)\rangle\,\mathrm{d}t\\ &=\int_0^T\langle\partial_t v,\mathcal{T}_\lambda(u_\delta-v)\rangle\,\mathrm{d}t+\int_\Omega\Phi_\lambda(u_\delta-v)(x,T)\,\mathrm{d}x-\int_\Omega\Phi_\lambda(u_0-v(x,0))\,\mathrm{d}x.\end{aligned} \tag{4.12}$$

We observe that Lemma 2.2 applies because of (3.4) and (3.19), so

$$u_\delta(\cdot,t)\rightharpoonup u(\cdot,t) \quad \text{weakly in } L^2(\Omega) \text{ for all } t\in[0,T].$$

This convergence and the Lipschitz continuity of Φ_λ gives $\Phi_\lambda(u_\delta-v)(\cdot,T)\rightharpoonup\Phi_\lambda(u-v)(\cdot,T)$ weakly in $L^2(\Omega)$, then

$$\lim_{\delta\to 0}\int_\Omega\Phi_\lambda(u_\delta-v)(x,T)\,\mathrm{d}x=\int_\Omega\Phi_\lambda(u-v)(x,T)\,\mathrm{d}x. \tag{4.13}$$

On the other hand, by Fatou lemma, we are able to pass to the limit as $\delta \to 0$ in the third term on the left–hand side of (4.11). Indeed, for this term we know by the monotonicity condition (1.6) that the integrand is nonnegative and we have already observed that u_δ and ∇u_δ converge a.e. according to (4.2) and (4.6) respectively. We only need to handle the term

$$\int_{\Omega_T} A(x,t,u_\delta \vee \psi, \nabla v)\cdot \nabla \mathcal{T}_\lambda(u_\delta - v)\,\mathrm{d}x\,\mathrm{d}t.$$

This can be done arguing similarly as for the case $\lambda = 1$. By (1.7) we have

$$|A(x,t,u_\delta \vee \psi, \nabla v)|\chi_{\{|u_\delta - v|\leqslant \lambda\}} \leqslant \beta|\nabla v|^{p-1} + K + C(p)\lambda^{p-1}\left(\tilde{b}^{p-1} + (\tilde{b}|v|)^{p-1}\right).$$

By (4.2) and (4.5) we obtain $A(x,t,u_\delta \vee \psi, \nabla v) \to A(x,t,u,\nabla v)$ a.e. in Ω_T, Therefore, by the dominated convergence theorem, $A(x,t,u_\delta \vee \psi, \nabla v) \to A(x,t,u,\nabla v)$ strongly in $L^{p'}(\Omega_T, \mathbb{R}^N)$, and this yields

$$\lim_{\delta\to 0}\int_{\Omega_T} A(x,t,u_\delta,\nabla v)\cdot\nabla\mathcal{T}_\lambda(u_\delta - v)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} A(x,t,u,\nabla v)\cdot\nabla\mathcal{T}_\lambda(u-v)\,\mathrm{d}x\,\mathrm{d}t.$$

Taking into account the latter relation and also (4.12) and (4.13), we can now pass to the limit as $\delta \to 0$ in (4.11) and obtain

$$\int_0^T \langle \partial_t v, \mathcal{T}_\lambda(u-v)\rangle\,\mathrm{d}t + \int_\Omega \Phi_\lambda(u-v)(x,T)\,\mathrm{d}x - \int_\Omega \Phi_\lambda(u_0 - v(x,0))\,\mathrm{d}x$$
$$+ \int_{\Omega_T} A(x,t,u,\nabla u)\cdot\nabla\mathcal{T}_\lambda(u-v)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T \langle f, \mathcal{T}_\lambda(u-v)\rangle\,\mathrm{d}t.$$

Since

$$\begin{aligned}
\mathcal{T}_\lambda(u-v) &\to u - v && \text{strongly in } L^p(0,T,W_0^{1,p}(\Omega)) \text{ as } \lambda\to\infty,\\
\Phi_\lambda(u-v)(\cdot,T) &\to \frac{1}{2}|u_0 - v(\cdot,0)|^2 && \text{strongly in } L^1(\Omega) \text{ as } \lambda\to\infty\\
\Phi_\lambda(u_0 - v(\cdot,0)) &\to \frac{1}{2}|u(\cdot,0) - v(\cdot,0)|^2 && \text{strongly in } L^1(\Omega) \text{ as } \lambda\to\infty
\end{aligned}$$

and also observing that

$$\int_0^T \langle \partial_t v, u - v\rangle\,\mathrm{d}t = \int_0^T \langle \partial_t u, u-v\rangle\,\mathrm{d}t + \frac{1}{2}\int_\Omega |u_0 - v(\cdot,0)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u(\cdot,T) - v(\cdot,T)|^2\,\mathrm{d}x$$

we conclude that (1.3) holds. □

Next result shows that a Lewy–Stampacchia inequality can be derived under some suitable assuption, that we are going to remove later.

Proposition 4.2. *Let* (1.2)*,* (1.4)–(1.16)*,* (3.2) *and* (3.17) *be in charge. If we also assume that*

$$\begin{aligned}
&g^- \in L^{p'}(\Omega_T)\cap L^p(0,T,W_0^{1,p}(\Omega))\\
&g^- \geqslant 0 \qquad \text{a.e. in } \Omega_T\\
&\partial_t g^- \in L^{q'}(\Omega_T)
\end{aligned}$$

the solution u of the obstacle problem constructed in Proposition 4.1 satisfies the Lewy–Stampacchia inequality (1.17).

Proof. We define

$$z_\delta := g^- - \frac{1}{\delta}\left[(\psi - u_\delta)^+\right]^{q-1}.$$

For $k \geqslant 1$ we also define

$$\begin{aligned}\eta_k(y) &:= (q-1)\int_0^{y^+} \min\{k, s^{q-2}\}\,\mathrm{d}s\\ \Psi_k(x,t,\lambda) &:= -\left(g^- - \frac{1}{\delta}\eta_k(\lambda^-)\right)^-\\ \Lambda_k(x,t,\lambda) &:= \int_0^\lambda \Psi_k(x,t,\sigma)\,\mathrm{d}\sigma.\end{aligned}$$

Thanks to Lemma 4.3 in [15] we are able to test (3.1) by $\Psi_k(x,s,u_\delta-\psi)\chi_{(0,t)}$ for $t \in (0,T)$, obtaining

$$\begin{aligned}&-\int_{\Omega_t}\partial_t\Lambda_k(x,s,u_\delta-\psi)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \Lambda_k(x,t,(u_\delta-\psi)(x,t))\,\mathrm{d}x - \int_\Omega \Lambda_k(x,0,(u_\delta-\psi)(x,0))\,\mathrm{d}x\\ &\qquad -\int_{\Omega_t}\left[A(x,s,u_\delta\vee\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right]\cdot\nabla\left(g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-)\right)^-\mathrm{d}x\,\mathrm{d}s\\ &\qquad -\int_{\Omega_t} z_\delta\left(g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-)\right)^-\mathrm{d}x\,\mathrm{d}s\\ &\quad = -\int_0^t\left\langle g^+, \left(g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-)\right)^-\right\rangle\mathrm{d}s \leqslant 0.\end{aligned} \tag{4.14}$$

By (1.14) we have

$$\int_\Omega \Lambda_k(x,0,(u_\delta-\psi)(x,0))\,\mathrm{d}x = 0.$$

We also have

$$\begin{aligned}-\int_{\Omega_t}\partial_t\Lambda_k(x,s,u_\delta-\psi)\,\mathrm{d}x\,\mathrm{d}s &= -\int_{\Omega_t}\partial_t g^-\int_0^{u_\delta-\psi}\chi_{\left\{g^- - \frac{1}{\delta}\eta_k(\tau^-)<0\right\}}\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}s\\ &= -\int_{\Omega_t}\partial_t g^-\int_0^{-(u_\delta-\psi)^-}\chi_{\left\{g^- - \frac{1}{\delta}\eta_k(\tau^-)<0\right\}}\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}s\\ &\geqslant -\int_{\Omega_t}|\partial_t g^-||(u_\delta-\psi)^-|\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

So, taking into account (4.14), we have

$$\begin{aligned}&-\int_{\Omega_t}|\partial_t g^-||(u_\delta-\psi)^-|\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega\Lambda_k(x,t,(u_\delta-\psi)(x,t))\,\mathrm{d}x - \int_{\Omega_t} z_\delta\left(g^- - \frac{1}{\delta}\eta_k\left((u_\delta-\psi)^-\right)\right)^-\mathrm{d}x\,\mathrm{d}s\\ &\quad -\int_{\Omega_t}\left[A(x,s,u_\delta\vee\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right]\cdot\nabla\left(g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-)\right)^-\mathrm{d}x\,\mathrm{d}s \leqslant 0.\end{aligned} \tag{4.15}$$

We remark that

$$-\int_{\Omega_t} z_\delta \left(g^- - \frac{1}{\delta}\eta_k\left((u_\delta-\psi)^-\right)\right)^- \mathrm{d}x\,\mathrm{d}s$$
$$= -\int_{\Omega_t}\left(g^- - \frac{1}{\delta}\left[(\psi-u_\delta)^+\right]^{q-1}\right)\left(g^- - \frac{1}{\delta}\eta_k\left((u_\delta-\psi)^-\right)\right)^- \mathrm{d}x\,\mathrm{d}s.$$

Since we have $\left\{g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-) < 0\right\} \subset \{u_\delta < \psi\}$ then

$$-\int_{\Omega_t}\left[A(x,s,u_\delta\vee\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right]\cdot\nabla\left(g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-)\right)^- \mathrm{d}x\,\mathrm{d}s$$
$$= \int_{\Omega_t}\chi_{\left\{g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-)<0\right\}}\left[A(x,s,\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right]\cdot\nabla\left(g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-))\right)\mathrm{d}x\,\mathrm{d}s.$$

By (1.6) it follows that

$$\left[A(x,s,\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right]\cdot\nabla\left(g^- - \frac{1}{\delta}\eta_k((u_\delta-\psi)^-))\right)$$
$$\geqslant \frac{1}{\delta}\eta_k'((u_\delta-\psi)^-)\left[A(x,s,\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right]\cdot\nabla(u_\delta-\psi)$$
$$- \left|\left[A(x,s,\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right]\right|\left|\nabla g^-\right|$$
$$\geqslant -\left|A(x,s,\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right|\left|\nabla g^-\right|.$$

Hence, we deduce from (4.15)

$$-\int_{\Omega_t}|\partial_t g^-||(u_\delta-\psi)^-|\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \Lambda_k(x,t,(u_\delta-\psi)(x,t))\,\mathrm{d}x$$
$$-\int_{\Omega_t}\left(g^- - \frac{1}{\delta}\left[(\psi-u_\delta)^+\right]^{q-1}\right)\left(g^- - \frac{1}{\delta}\eta_k\left((u_\delta-\psi)^-\right)\right)^- \mathrm{d}x\,\mathrm{d}s$$
$$-\int_{\Omega_t}\left|A(x,s,\psi,\nabla u_\delta) - A(x,s,\psi,\nabla\psi)\right|\left|\nabla g^-\right|\mathrm{d}x\,\mathrm{d}s \leqslant 0.$$

Now, we pass to the limit as $k\to\infty$. In particular, by using the monotone convergence theorem, we have

$$\lim_{k\to\infty}\int_\Omega \Lambda_k(x,t,(u_\delta-\psi)(x,t))\,\mathrm{d}x = -\int_\Omega \mathrm{d}x\int_0^{(u_\delta-\psi)(x,t)}\left(g^- - \frac{1}{\delta}\left[\sigma^-\right]^{q-1}\right)^- \mathrm{d}\sigma \geqslant 0$$

and also

$$-\lim_{k\to\infty}\int_{\Omega_t}\left(g^- - \frac{1}{\delta}\left[(\psi-u_\delta)^+\right]^{q-1}\right)\left(g^- - \frac{1}{\delta}\eta_k\left((u_\delta-\psi)^-\right)\right)^- \mathrm{d}x\,\mathrm{d}s = \|z_\delta^-\|^2_{L^2(\Omega_t)}$$

We gather the previous relations, and (since $t\in(0,T)$ is arbitrary) we get

$$-\int_{\Omega_T}|\partial_t g^-||(u_\delta-\psi)^-|\,\mathrm{d}x\,\mathrm{d}s + \|z_\delta^-\|^2_{L^2(\Omega_T)} \leqslant \int_{\Omega_T}\chi_{\{\psi>u_\delta\}}\left|A(x,t,\psi,\nabla u_\delta) - A(x,t,\psi,\nabla\psi)\right|\left|\nabla g^-\right|\mathrm{d}x\,\mathrm{d}s.$$

Since it is clear that

$$\lim_{\delta\to0}\int_{\Omega_T}|\partial_t g^-||(u_\delta-\psi)^-|\,\mathrm{d}x\,\mathrm{d}s = 0$$

we obtain

$$\limsup_{\delta\to 0} \|z_\delta^-\|^2_{L^2(\Omega_T)} \leqslant \limsup_{\delta\to 0} \int_{\Omega_t} \chi_{\{\psi>u_\delta\}} \, |A(x,t,\psi,\nabla u_\delta) - A(x,t,\psi,\nabla\psi)| \, |\nabla g^-| \, \mathrm{d}x \, \mathrm{d}s. \tag{4.16}$$

Observing that (4.2), (4.5) and (4.6) hold, then

$$F_\delta := \chi_{\{\psi>u_\delta\}} \, |A(x,t,\psi,\nabla u_\delta) - A(x,t,\psi,\nabla\psi)| \to 0 \qquad \text{a.e. in } \Omega_T$$

as $\delta \to 0$. By (1.7), (3.2) and (3.4), F_δ is also bounded in $L^{p'}(\Omega_T)$, hence $F_\delta \rightharpoonup 0$ in $L^{p'}(\Omega_T)$. We deduce

$$\lim_{\delta\to 0} \int_{\Omega_T} \chi_{\{\psi>u_\delta\}} \, |A(x,t,\psi,\nabla u_\delta) - A(x,t,\psi,\nabla\psi)| \, |\nabla g^-| \, \mathrm{d}x \, \mathrm{d}s = 0.$$

By (4.16) we obtain

$$\lim_{\delta\to 0} \|z_\delta^-\|^2_{L^2(\Omega_T)} = 0.$$

Hence we have

$$0 \leqslant \frac{1}{\delta} \, [(u_\delta - \psi)^-]^{q-1} = \partial_t u_\delta - \mathrm{div} A(\cdot,\cdot,u_\delta \vee \psi, \nabla u_\delta) - f$$

and so

$$0 \leqslant \partial_t u - \mathrm{div} A(\cdot,\cdot,u,\nabla u) - f.$$

Similarly, rewriting (3.1) as follows

$$z_\delta^+ + \partial_t u_\delta - \mathrm{div} A(\cdot,\cdot,u_\delta \vee \psi, \nabla u_\delta) - f = g^- + z_\delta^-$$

then

$$\partial_t u - \mathrm{div} A(\cdot,\cdot,u,\nabla u) - f \leqslant g^-$$

and the proof is completed. □

Next result provides the one of Theorem 1.1 under the assumption (3.2) but removing condition (3.17).

Proposition 4.3. *Let* (1.2), (1.4)–(1.16) *and* (3.2) *be in charge. There exists at least solution* $u \in \mathcal{K}_\psi(\Omega_T)$ *to the variational inequality* (1.3) *satisfying* $u(\cdot,0) = u_0$ *in* Ω*, the estimate* (4.1) *and the Lewy–Stampacchia inequality* (1.17).

Proof. We know that

$$g := f - \psi_t + \mathrm{div}\, A(x,t,\psi,\nabla\psi) = g^+ - g^-,$$

where $g^\pm$ are nonnegative elements of $L^{p'}(0,T,W^{-1,p'}(\Omega))$. By using a regularization procedure, due to [7] Lemma p. 593, and Lemma 4.1 in [15], we find a sequence $\{g_n^-\}_{n\in\mathbb{N}}$ of nonnegative functions such that

$$g_n^- \in L^{p'}(\Omega_T) \cap L^p(0,T,W_0^{1,p}(\Omega))$$
$$g_n \geqslant 0 \quad \text{a.e. in } \Omega_T$$
$$\partial_t g_n^- \in L^{q'}(\Omega_T)$$

and

$$g_n^- \to g^- \quad \text{in } L^{p'}(0,T,W^{-1,p'}(\Omega)) \text{ as } n \to \infty.$$

We define

$$f_n = \psi_t - \operatorname{div} A(x,t,\psi,\nabla\psi) + g^+ - g_n^-.$$

It is clear that

$$f_n \to f \quad \text{in } L^{p'}(0,T,W^{-1,p'}(\Omega))$$

as $n \to \infty$. Due to the regularity assumptions on g_n^-, we get the existence of $u_n \in \mathcal{K}_\psi(\Omega_T)$ with $u_n(\cdot,0) = u_0$ in Ω such that for every $v \in \mathcal{K}_\psi(\Omega_T)$ we have

$$\int_0^T \langle \partial_t u_n, v - u_n\rangle\, dt + \int_{\Omega_T} A(x,t,u_n,\nabla u_n)\cdot\nabla(v-u_n)\, dxdt \geqslant \int_0^T \langle f_n, v-u_n\rangle\, dt. \tag{4.17}$$

Moreover, the subsequent estimate holds

$$\begin{aligned}\|u_n(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla u_n\|^p_{L^p(\Omega_t)} \leqslant C(b,N,p,\alpha)\Big[&\|u_0\|^2_{L^2(\Omega)} + \|f_n\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))} + \|H\|_{L^1(\Omega_T)}\\ &+ \left(\|u_0\|^2_{L^2(\Omega)} + \|f_n\|^{p'}_{L^{p'}(0,T,W^{-1,p'}(\Omega))} + \|b\|^p_{L^p(\Omega_T)}\right)^p \|b\|^p_{L^p(\Omega_T)}\Big]\end{aligned}$$

and the following Lewy-Stampacchia inequality holds

$$0 \le \partial_t u_n - \operatorname{div} A(x,t,u_n,\nabla u_n) - f_n \le g_n^-. \tag{4.18}$$

Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ is strongly converging (and hence bounded) in $L^{p'}(0,T,W^{-1,p'}(\Omega))$, we obtain

$$\sup_{0<t<T}\int_\Omega |u_n(\cdot,t)|^2\, dx + \int_{\Omega_T} |\nabla u_n|^p\, dxdt \leqslant C$$

for some positive constant C independent of n. Moreover, the Lewy–Stampacchia inequality (4.18) implies a uniform bound of this kind

$$\|\partial_t u_n\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C$$

again for some positive constant C independent of n. Therefore, there exists $u \in C^0\left([0,T],L^2(\Omega)\right) \cap L^p(0,T,W^{1,p}_0(\Omega))$ with $u(\cdot,0) = u_0$ in Ω such that

$$\begin{aligned} u_n &\to u \quad \text{strongly in } L^p(\Omega_T) \\ \nabla u_n &\rightharpoonup \nabla u \quad \text{weakly in } L^p\left(\Omega_T,\mathbb{R}^N\right) \\ u_n &\overset{*}{\rightharpoonup} u \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)) \\ \partial_t u_n &\rightharpoonup \partial_t u \quad \text{weakly in } L^{p'}(0,T,W^{-1,p'}(\Omega)) \end{aligned} \tag{4.19}$$

as $n \to \infty$. Obviously (4.19) implies $u \geqslant \psi$ a.e. in Ω_T. If we summarize, we have $u \in \mathcal{K}_\psi(\Omega_T)$ and then $v_n := u_n - \mathcal{T}_1(u_n - u) \in \mathcal{K}_\psi(\Omega_T)$. Hence, we use v_n as a test function in (4.17) and, arguing as in the proof of Proposition 4.1, we obtain

$$\nabla u_n \to \nabla u \qquad \text{a.e. in } \Omega_T$$

as $n \to \infty$. For fixed $\lambda > 0$ and $v \in \mathcal{K}_\psi(\Omega_T)$ we also have $v_{n,\lambda} := u_n - \mathcal{T}_\lambda(u_n - v) \in \mathcal{K}_\psi(\Omega_T)$. Arguing again as in the proof of Proposition 4.1, we get (1.3) passing to the limit (first as $n \to \infty$ and then as $\lambda \to \infty$) in the inequality obtained by testing (4.17) by $v_{n,\lambda}$. □

Finally, we remove condition (3.2), i.e., we are able to prove Theorem 1.1.

Proof of Theorem 1.1. The convex set $\mathcal{K}_\psi(\Omega_T)$ is nonempty and one can find $w \in \mathcal{K}_\psi(\Omega_T)$ such that $w(\cdot, 0) = \psi(\cdot, 0)$ in Ω (see for details Remark 2.1 in [15]). Let us define

$$\begin{aligned} \hat{A}(x, t, u, \eta) &:= A(x, t, u + w, \eta + \nabla w) \\ \hat{f} &:= f - \partial_t w \\ \hat{\psi} &:= \psi - w \\ \hat{u}_0 &:= u_0 - w(\cdot, 0). \end{aligned}$$

Hence $\hat{f} \in L^{p'}(0, T, W^{-1,p'}(\Omega))$ and $\hat{\psi}$ and ψ share the same trace on $\partial\Omega \times (0, T)$. Therefore, one can conclude

$$\begin{aligned} \hat{\psi} &\leqslant 0 \quad \text{a.e. in } \Omega_T \\ \hat{\psi}(\cdot, 0) &= 0 \quad \text{a.e. in } \Omega. \end{aligned}$$

Moreover, the vector field $\hat{A}$ enjoys similar properties as A. This is trivial for conditions (1.6) and (1.7). As in [12], properties of A and Young inequality, we have for $\varepsilon > 0$

$$\hat{A}(x, t, u, \xi) \cdot \xi \geqslant (\alpha - \beta\, \varepsilon^p)\, |\xi + \nabla w|^p - \left(b^p + \varepsilon^p\, \tilde{b}^p\right) |u + w|^p - H_1$$

with a suitable $H_1 \in L^1(\Omega_T)$. Moreover, as an elementary consequence of the convexity of $|\ |^p$, for $0 < \vartheta < 1$ we find a constant $C = C(\vartheta, p) > 0$ such that

$$|\xi + \nabla w|^p \geqslant \vartheta^p\, |\xi|^p - C\, |\nabla w|^p\,, \qquad |u + w|^p \leqslant \vartheta^{-p}\, |u|^p + C\, |w|^p.$$

Hence, we get coercivity condition for $\hat{A}$:

$$\hat{A}(x, u, \xi) \cdot \xi \geqslant \hat{\alpha}\, |\xi|^p - (\hat{b}\, |u|)^p - \hat{H},$$

where we set

$$\hat{\alpha} = (\alpha - \beta\, \varepsilon^p)\, \vartheta^p\,, \qquad \hat{b} = \frac{b + \varepsilon\, \tilde{b}}{\vartheta}$$

and denoted by $\hat{H}$ a suitable nonnegative function in $L^1(\Omega_T)$. Obviously, we can make $\hat{\alpha}$ arbitrarily close to α, by choosing ε close to 0 and ϑ close to 1. Using inequality (2.5) for b and $\tilde{b}$ in place of f and g, respectively, we can easily show that also $\mathscr{D}_{\hat{b}}$ is arbitrarily close to $\mathscr{D}_b$, again by choosing ε close to 0 and ϑ close to 1. Indeed, we have

$$\begin{aligned} &\operatorname{dist}_{L^\infty(0,T,L^{N,\infty}(\Omega))}(\hat{b}, L^\infty(\Omega_T)) \\ &\quad \leqslant \frac{1 + \sqrt{\varepsilon}}{\vartheta} \operatorname{dist}_{L^\infty(0,T,L^{N,\infty}(\Omega))}(b, L^\infty(\Omega_T)) + \frac{\sqrt{\varepsilon}(1 + \sqrt{\varepsilon})}{\vartheta}\, ||\tilde{b}||_{L^\infty(0,T,L^{N,\infty}(\Omega))}. \end{aligned}$$

By (1.16) we can also have

$$\mathscr{D}_{\hat{b}} < \frac{\hat{\alpha}^{1/p}}{S_{N,p}}.$$

We observe that

$$\hat{f} - \hat{\psi}_t + \operatorname{div} A(x, t, \hat{\psi}, \nabla\hat{\psi}) = f - \psi_t + \operatorname{div} \hat{A}(x, t, \psi, \nabla\psi).$$

We can apply Proposition 4.3 for the operator $\hat{A}$. Therefore, we obtain the existence of a function $\hat{u} \in \mathcal{K}_{\hat{\psi}}(\Omega_T)$ such that

$$\hat{u}(\cdot, 0) = \hat{u}_0 \qquad \text{in } \Omega \tag{4.20}$$

and the following parabolic variational inequality

$$\int_0^T \langle \hat{u}_t, \hat{v} - \hat{u}\rangle \,\mathrm{d}t + \int_{\Omega_T} \hat{A}(x, t, \hat{u}, \nabla\hat{u}) \cdot \nabla(\hat{v} - \hat{u}) \,\mathrm{d}x \,\mathrm{d}t \geqslant \int_0^T \langle \hat{f}, \hat{v} - \hat{u}\rangle \,\mathrm{d}t$$

holds true for every admissible function $\hat{v} \in \mathcal{K}_{\hat{\psi}}(\Omega_T)$. Since any $v \in \mathcal{K}_{\psi}(\Omega_T)$ can be rewritten as $v = \hat{v} + w$ for some $\hat{v} \in \mathcal{K}_{\hat{\psi}}(\Omega_T)$, by (4.20), by the definitions of $\hat{A}$, $\hat{f}$ and $\hat{\psi}$, we see that the variational inequality (1.3) holds true with $u := \hat{u} + w$ and for any admissible function $v \in \mathcal{K}_{\psi}(\Omega_T)$. The fact that $u \in \mathcal{K}_{\psi}(\Omega_T)$ and $u(\cdot, 0) = u_0$ in Ω is obvious, and this concludes the proof. □

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Conflict of interest

The authors declare no conflict of interest.

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