http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(3): 1–9.
DOI:10.3934/mine.2023050
Received: 17 August 2022
Revised: 18 August 2022
Accepted: 18 August 2022
Published: 19 August 2022

Editorial

Partial differential equations from theory to applications: Dedicated to Alberto Farina, on the occasion of his 50th birthday†

Serena Dipierro[1,*] **and Luca Lombardini**[2]

[1] Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, WA6009 Crawley, Australia

[2] Institute for Analysis and Scientific Computing, Technische Universität Wien, Wiedner Hauptstraße 8–10, 1040 Vienna, Austria

† **This contribution is part of the Special Issue:** Partial Differential Equations from theory to applications—Dedicated to Alberto Farina, on the occasion of his 50th birthday
Guest Editors: Serena Dipierro; Luca Lombardini
Link: www.aimspress.com/mine/article/5752/special-articles

* **Correspondence:** Email: serena.dipierro@uwa.edu.au.

Abstract: Partial differential equations are a classical and very active field of research. One of its salient features is to break the rigid distinction between the evolution of the theory and the applications to real world phenomena, since the two are intimately intertwined in the harmonious development of such a fascinating and multifaceted topic of investigation.

Keywords: partial differential equations; rigidity, symmetry and classification results; regularity theory; nonlocal equations; applications

"Un giocatore lo vedi dal coraggio, dall'altruismo e dalla fantasia"
(Francesco De Gregori, La leva calcistica della classe '68)

This special issue at a glance

Besides being a champion of chess and of *des chiffres et des lettres*, during the first fifty years of his life Alberto Farina has also left a permanent trace in partial differential equations. We are all familiar with his depth of thought and recognizable style and it would be virtually impossible to recall and comment on all his fundamental contributions in different areas.

Alberto at work.

To cut a long story short, we just mention that the articles by Alberto for which MathSciNet lists more than fifty citations (at least one per year of Alberto's life) cover in depth a broad spectrum of partial differential equations, including:

- one-dimensional symmetry of entire solutions, inspired by a celebrated conjecture by Ennio De Giorgi [24, 28, 31],
- overdetermined boundary value problems [23, 32],
- Liouville-type theorems for solutions of the Lane-Emden equation [12, 25] and of general semilinear equations [21],
- non-existence results for Liouville-type equations [14, 26].

We also know how proud Alberto is of his papers written with James Serrin [29, 30], in which they have creatively looked at the class of quasilinear elliptic equations with the property that any entire solution must necessarily be constant (of course, the simple Laplace equation does not belong to this class, but the family of equations considered includes the p-Laplacian and the mean curvature equations with appropriate weights).

Among all the problems studied by Alberto, one of his favorites is probably De Giorgi's Conjecture [15]. This is the problem, in the original statement by De Giorgi:

Let us consider a solution $u \in C^2(\mathbb{R}^n)$ of

$$\Delta u = u^3 - u$$

such that $|u| \leq 1$, $\frac{\partial u}{\partial x_n} > 0$ in the whole $\mathbb{R}^n$. Is it true that, for every $\lambda \in \mathbb{R}$, the sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$? (This problem is related to [40, 41]).

In this editorial, we do not dive into the full story of De Giorgi's Conjecture (for this, see [31] and the references therein). Rather, we recall some of the very many results obtained by Alberto on this topic, also in relation with p-Laplace operators

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \tag{1}$$

and the mean curvature operator

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{2}$$

For the sake of brevity, we will say that "u is one-dimensional" when the claim in De Giorgi's Conjecture holds true (notice indeed that this claim is equivalent to the possibility of writing u as a function of only one variable).

Several ideas stemmed out from Alberto's "Habilitation à diriger des recherches", especially in connection with the notion of "stable" solutions: namely a solution of an equation with a variational structure is called stable if the second derivative of the corresponding energy functional is nonnegative definite. Not only is this a natural condition to consider from the perspective of the calculus of variations (for instance, local minima satisfy this condition). It is also strictly related to the geometry of the solution, as indeed solutions satisfying the monotonicity assumption in the statement of De Giorgi's Conjecture can be proved to be stable.

In this setting, a special case of [28, Theorems 1.1 and 1.5] gives that:

Let $\mathcal{N}u$ be either the p-Laplace operator in (1) *for some $p \geq 1$ or the mean curvature operator in* (2)*.*

Let $u \in C^1(\mathbb{R}^2)\cap C^2(\{\nabla u \neq 0\})$, with $\nabla u \in L^\infty(\mathbb{R}^2)\cap W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, be a stable weak solution of $\mathcal{N}u = f(u)$ in $\mathbb{R}^2$, where f is a locally Lipschitz function.

If $\mathcal{N}u$ is the p-Laplace operator with $p \in [1,2)$, assume additionally that $\{\nabla u = 0\} = \varnothing$.

Then, u is one-dimensional.

Notice that this is a more general case than De Giorgi's Conjecture in dimension 2.

Moreover, a special case of [28, Theorem 1.2] gives that:

Let $\mathcal{N}u$ be either the p-Laplace operator in (1) *for some $p > 1$ or the mean curvature operator in* (2)*.*

Let $u \in C^1(\mathbb{R}^3)\cap W^{1,\infty}(\mathbb{R}^3)$ be a stable weak solution of $\mathcal{N}u = f(u)$ in $\mathbb{R}^3$, where f is a locally Lipschitz function.

Assume additionally that $\{\nabla u = 0\} = \varnothing$ and $\frac{\partial u}{\partial x_3} \geq 0$.

Then, u is one-dimensional.

Notice that this is a more general case than De Giorgi's Conjecture in dimension 3.

One typical feature of Alberto's work is also to relate the symmetry properties of a solution to the geometry of the solution's profiles at infinity. In particular, the monotonicity assumption in the statement of De Giorgi's Conjecture allows one to define the two profiles

$$\overline{u}(x') := \lim_{x_n\to+\infty} u(x',x_n) \qquad \text{and} \qquad \underline{u}(x') := \lim_{x_n\to-\infty} u(x',x_n),$$

where the notation $(x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$ has been used.

In this setting, it follows from a particular case of [33, Theorem 1.1] that one can deduce the symmetry of the solution from the symmetry of the profiles. For this, one says that a function is "two-dimensional" when it can be written as a function of at most two variables, and we have that:

Let $p \in (1, +\infty)$ *and* $u \in W^{1,p}_{\mathrm{loc}}(R^n)$ *be a weak solution of*

$$\Delta_p u = -u(1 - u^2)^{p-1}$$

such that $|u| \le 1$, $\frac{\partial u}{\partial x_n} > 0$ *in the whole* $\mathbb{R}^n$.

Suppose that both $\overline{u}$ *and* $\underline{u}$ *are two-dimensional.*

Then, $\overline{u}$ *is identically* $+1$, $\underline{u}$ *is identically* -1 *and* u *is a local minimum of the energy functional.*

Also, if $n \le 8$, *then* u *is one-dimensional.*

We observe that this statement when $p = 2$ proves De Giorgi's Conjecture, under the additional assumption that the limit profiles are two-dimensional.

Another very recognizable line of investigation consists in deducing symmetry results from the knowledge of the shape/geometry of a level set of the solution. As an example, one can obtain a symmetry result under the assumption that "one level set is a complete graph", meaning that there exist $\lambda \in \mathbb{R}$ and $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\{u = \lambda\} = \Big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n = \Gamma(x')\Big\}.$$

We remark that this graphicality condition is compatible with the monotonicity assumption in De Giorgi's Conjecture, but it is not ensured by it. In this setting, from [33, Theorem 1.3] one obtains the symmetry of the solution if one level set is a complete graph:

Let us consider a solution $u \in C^2(\mathbb{R}^n)$ *of*

$$\Delta u = u^3 - u$$

such that $|u| \le 1$, $\frac{\partial u}{\partial x_n} > 0$ *in the whole* $\mathbb{R}^n$.

Assume that one level set of u *is a complete graph.*

Then, $\overline{u}$ *is identically* $+1$, $\underline{u}$ *is identically* -1 *and* u *is a local minimum of the energy functional.*

Also, if $n \le 8$, *then* u *is one-dimensional.*

See also [35] for related results based on the limit interface of the solution.

A problem strictly related to De Giorgi's Conjecture embraces the case in which the solution attains the limits $+1$ and -1 at infinity in a uniform way. In this case, the problem is often named Gibbons' Conjecture, after theoretical physicist Gary William Gibbons. In his seminal paper [24], Alberto gave a positive answer to Gibbons' Conjecture in every dimension (the same result was obtained independently and with different methods in [4, 5]):

Let $u \in C^2(\mathbb{R}^n)$ *be a bounded solution of*

$$\Delta u = u^3 - u$$

in $\mathbb{R}^n$ *such that*

$$\lim_{x_n \to +\infty} \sup_{x' \in \mathbb{R}^{n-1}} \Big| u(x', x_n) - 1 \Big| = 0 \qquad \textit{and} \qquad \lim_{x_n \to -\infty} \sup_{x' \in \mathbb{R}^{n-1}} \Big| u(x', x_n) + 1 \Big| = 0.$$

Then, u *is one-dimensional.*

See also [34] for related results in more general settings.

De Giorgi's Conjecture is also closely related to Liouville-type results. To showcase this link, we recall for instance [21, Theorem 1.1]:

Assume that $f \in C^1(\mathbb{R})$*,* $f \le 0$ *and* $n \le 4$*. Let* $u \in C^2(\mathbb{R}^n)$ *be a bounded and stable solution of*

$$\Delta u = f(u).$$

Then, u is constant.

This result has been recently sharpened in [22, Theorem 1] for dimension $n \le 10$ (addressing also the case of solutions that are assumed to be only bounded from below). See [25, 26] also for classification results with power-type nonlinearities, sign-changing and unbounded solutions, etc. The case of exponential nonlinearities is also addressed in [14].

Recently, a great attention has been devoted also to a nonlocal variant of De Giorgi's Conjecture in which the Laplace operator is replaced by the fractional Laplacian, defined, for $s \in (0, 1)$, up to normalizing constants, as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy,$$

with the integral computed in Cauchy's Principal Value sense.

This type of problems is interesting also in connection with long-range phase transitions and, for $s \in \left(0, \frac{1}{2}\right)$, which is often referred to as the "genuinely nonlocal regime", is related to the theory of nonlocal minimal surfaces (see [20] for further details).

One of the works by Alberto also settles this problem in dimension 3 for the genuinely nonlocal regime. Indeed, from [19, Theorem 1.1] we have that:

Let $n \le 3$ *and* $s \in \left(0, \frac{1}{2}\right)$*. Let us consider a solution* $u \in C^2(\mathbb{R}^n)$ *of*

$$-(-\Delta)^s u = u^3 - u$$

such that $|u| \le 1$*,* $\frac{\partial u}{\partial x_n} > 0$ *in the whole* $\mathbb{R}^n$*.*

Then, u is one-dimensional.

We also observe that in the original formulation by De Giorgi, the problem was explicitly related to the study of minimal surfaces via the reference to [40, 41]. This intimate connection is reflected into many of Alberto's works, in which the symmetry properties of solutions of partial differential equations are linked to Bernstein-type results classifying entire graphical solutions of geometric equations.

In this respect, many of the works by Alberto are connected with the Bernstein problem and, as an example, we recall here [27, Theorem 1.1]:

Assume that $n \ge 8$ *and let* $u \in C^2(\mathbb{R}^n)$ *be a solution of the minimal surface equation*

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

If $n - 7$ *partial derivatives of u are bounded on one side (not necessarily the same), then u is an affine function.*

This result is sharp, since one can also construct examples of non-affine solutions in dimension $n \ge 8$ with $n - 8$ partial derivatives that are bounded on one side.

Remembering a Champion. Ireneo (with Francesco Totti and Enrico Valdinoci) on the day of his (and Totti's) retirement.

We also mention that a version of this result dealing with nonlocal minimal surfaces has been obtained in [11].

We hope that this special issue reflects Alberto's tastes and scientific passions, also covering a variety of research directions. Though it is simplistic to try to frame these directions into stiff categories, the articles collected in this volume do, in our opinion, overlap with many of Alberto's main interests and include fundamental topics, such as:

- rigidity, symmetry and classification results [3, 6, 7, 13, 16, 36, 39],
- regularity theory [9, 17],
- nonlocal equations [1, 2, 8, 10],
- topical applications [18, 37, 38].

On a sad note, in the period of time in which this special issue was launched and completed, the great figure of Ireneo Peral has left us. Tragically, his contribution to this special issue [2] will remain as one of his last works. We will miss Ireneo and we will always remember his brilliance, enthusiasm and generosity.

But on a positive note, Ireneo has not left us alone and many of his disciples, friends and collaborators are putting together a new special issue commemorating Ireneo's prominence. In a sense, we are happy to pass the baton to this incoming special issue, which we see as a prolongation of Ireneo's marvelous and unique scientific matrice.

Acknowledgments

We thank all the authors and all the referees for their invaluable contributions.

SD has been supported by the Australian Research Council DECRA DE180100957 *PDEs, free boundaries and applications*.

Conflict of interest

The authors declare no conflict of interest.

References

1. N. Abatangelo, S. Jarohs, A. Saldaña, Fractional Laplacians on ellipsoids, *Mathematics in Engineering*, **3** (2021), 1–34. http://doi.org/10.3934/mine.2021038
2. B. Abdellaoui, P. Ochoa, I. Peral, A note on quasilinear equations with fractional diffusion, *Mathematics in Engineering*, **3** (2021), 1–28. http://doi.org/10.3934/mine.2021018
3. F. G. Alessio, P. Montecchiari, Gradient Lagrangian systems and semilinear PDE, *Mathematics in Engineering*, **3** (2021), 1–28. http://doi.org/10.3934/mine.2021044
4. M. T. Barlow, R. F. Bass, C. Gui, The Liouville property and a conjecture of De Giorgi, *Commun. Pure Appl. Math.*, **53** (2000), 1007–1038. http://doi.org/10.1002/1097-0312(200008)53:8<1007::AID-CPA3>3.0.CO;2-U
5. H. Berestycki, F. Hamel, R. Monneau, One-dimensional symmetry of bounded entire solutions of some elliptic equations, *Duke Math. J.*, **103** (2000), 375–396. http://doi.org/10.1215/S0012-7094-00-10331-6
6. B. Bianchini, G. Colombo, M. Magliaro, L. Mari, P. Pucci, M. Rigoli, Recent rigidity results for graphs with prescribed mean curvature, *Mathematics in Engineering*, **3** (2021), 1–48. http://doi.org/10.3934/mine.2021039
7. D. Castorina, G. Catino, C. Mantegazza, A triviality result for semilinear parabolic equations, *Mathematics in Engineering*, **4** (2022), 1–15. http://doi.org/10.3934/mine.2022002
8. A. Cesaroni, M. Novaga, Second-order asymptotics of the fractional perimeter as $s \to 1$, *Mathematics in Engineering*, **2** (2020), 512–526. http://doi.org/10.3934/mine.2020023
9. M. Cirant, K. R. Payne, Comparison principles for viscosity solutions of elliptic branches of fully nonlinear equations independent of the gradient, *Mathematics in Engineering*, **3** (2021), 1–45. http://doi.org/10.3934/mine.2021030
10. M. Conti, F. Dell'Oro, V. Pata, Exponential decay of a first order linear Volterra equation, *Mathematics in Engineering*, **2** (2020), 459–471. http://doi.org/10.3934/mine.2020021
11. M. Cozzi, A. Farina, L. Lombardini, Bernstein-Moser-type results for nonlocal minimal graphs, *Commun. Anal. Geom.*, **29** (2021), 761–777. http://doi.org/10.4310/CAG.2021.v29.n4.a1
12. L. Damascelli, A. Farina, B. Sciunzi, E. Valdinoci, Liouville results for m-Laplace equations of Lane-Emden-Fowler type, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **26** (2009), 1099–1119. http://doi.org/10.1016/j.anihpc.2008.06.001
13. L. D'Ambrosio, M. Gallo, A. Pugliese, A note on the Kuramoto-Sivashinsky equation with discontinuity, *Mathematics in Engineering*, **3** (2021), 1–29. http://doi.org/10.3934/mine.2021041
14. E. N. Dancer, A. Farina, On the classification of solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$: stability outside a compact set and applications, *Proc. Amer. Math. Soc.*, **137** (2009), 1333–1338. http://doi.org/10.1090/S0002-9939-08-09772-4

15. E. De Giorgi, Convergence problems for functionals and operators, In: *Proceedings of the international meeting on recent methods in nonlinear analysis (Rome, 1978)*, Bologna: Pitagora, 1979, 131–188.

16. A. De Luca, V. Felli, Unique continuation from the edge of a crack, *Mathematics in Engineering*, **3** (2021), 1–40. http://doi.org/10.3934/mine.2021023

17. D. De Silva, O. Savin, On the boundary Harnack principle in Hölder domains, *Mathematics in Engineering*, **4** (2022), 1–12. http://doi.org/10.3934/mine.2022004

18. G. Delvoye, O. Goubet, F. Paccaut, Comparison principles and applications to mathematical modelling of vegetal meta-communities, *Mathematics in Engineering*, **4** (2022), 1–17. http://doi.org/10.3934/mine.2022035

19. S. Dipierro, A. Farina, E. Valdinoci, A three-dimensional symmetry result for a phase transition equation in the genuinely nonlocal regime, *Calc. Var.*, **57** (2018), 15. http://doi.org/10.1007/s00526-017-1295-5

20. S. Dipierro, E. Valdinoci, Long-range phase coexistence models: recent progress on the fractional Allen-Cahn equation, In: *Topics in applied analysis and optimisation*, Cham: Springer, 2019, 121–138. http://doi.org/10.1007/978-3-030-33116-0_5

21. L. Dupaigne, A. Farina, Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$, *J. Eur. Math. Soc.*, **12** (2010), 855–882. http://doi.org/10.4171/JEMS/217

22. L. Dupaigne, A. Farina, Classification and Liouville-type theorems for semilinear elliptic equations in unbounded domains, *Anal. PDE*, **15** (2022), 551–566. http://doi.org/10.2140/apde.2022.15.551

23. A. Farina, B. Kawohl, Remarks on an overdetermined boundary value problem, *Calc. Var.*, **31** (2008), 351–357. http://doi.org/10.1007/s00526-007-0115-8

24. A. Farina, Symmetry for solutions of semilinear elliptic equations in $\mathbf{R}^N$ and related conjectures, (Italian), *Ricerche Mat.*, **48** (1999), 129–154.

25. A. Farina, On the classification of solutions of the Lane-Emden equation on unbounded domains of $\mathbb{R}^N$, *J. Math. Pures Appl. (9)*, **87** (2007), 537–561. http://doi.org/10.1016/j.matpur.2007.03.001

26. A. Farina, Stable solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$, *C. R. Math.*, **345** (2007), 63–66. http://doi.org/10.1016/j.crma.2007.05.021

27. A. Farina, A sharp Bernstein-type theorem for entire minimal graphs, *Calc. Var.*, **57** (2018), 123. http://doi.org/10.1007/s00526-018-1392-0

28. A. Farina, B. Sciunzi, E. Valdinoci, Bernstein and De Giorgi type problems: new results via a geometric approach, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, **7** (2008), 741–791. http://doi.org/10.2422/2036-2145.2008.4.06

29. A. Farina, J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations, *J. Differ. Equations*, **250** (2011), 4367–4408. http://doi.org/10.1016/j.jde.2011.02.007

30. A. Farina, J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations, II, *J. Differ. Equations*, **250** (2011), 4409–4436. http://doi.org/10.1016/j.jde.2011.02.016

31. A. Farina, E. Valdinoci, The state of the art for a conjecture of De Giorgi and related problems, In: *Recent progress on reaction-diffusion systems and viscosity solutions*, Hackensack, NJ: World Sci. Publ., 2009, 74–96. http://doi.org/10.1142/9789812834744_0004

32. A. Farina, E. Valdinoci, Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems, *Arch. Rational Mech. Anal.*, **195** (2010), 1025–1058. http://doi.org/10.1007/s00205-009-0227-8
33. A. Farina, E. Valdinoci, 1D symmetry for solutions of semilinear and quasilinear elliptic equations, *Trans. Amer. Math. Soc.*, **363** (2011), 579–609. http://doi.org/10.1090/S0002-9947-2010-05021-4
34. A. Farina, E. Valdinoci, Rigidity results for elliptic PDEs with uniform limits: an abstract framework with applications, *Indiana Univ. Math. J.*, **60** (2011), 121–141. http://doi.org/10.1512/iumj.2011.60.4433
35. A. Farina, E. Valdinoci, 1D symmetry for semilinear PDEs from the limit interface of the solution, *Commun. Part. Diff. Eq.*, **41** (2016), 665–682. http://doi.org/10.1080/03605302.2015.1135165
36. M. Fogagnolo, A. Pinamonti, Strict starshapedness of solutions to the horizontal p-Laplacian in the Heisenberg group, *Mathematics in Engineering*, **3** (2021), 1–15. http://doi.org/10.3934/mine.2021046
37. F. Gazzola, E. M. Marchini, The moon lander optimal control problem revisited, *Mathematics in Engineering*, **3** (2021), 1–14. http://doi.org/10.3934/mine.2021040
38. A. Jüngel, U. Stefanelli, L. Trussardi, A minimizing-movements approach to GENERIC systems, *Mathematics in Engineering*, **4** (2022), 1–18. http://doi.org/10.3934/mine.2022005
39. R. Magnanini, G. Poggesi, Interpolating estimates with applications to some quantitative symmetry results, *Mathematics in Engineering*, **5** (2023), 1–21. http://doi.org/10.3934/mine.2023002
40. L. Modica, S. Mortola, Il limite nella Γ-convergenza di una famiglia di funzionali ellittici, (Italian), *Boll. Un. Mat. Ital. A (5)*, **14** (1977), 526–529.
41. L. Modica, S. Mortola, Un esempio di Γ^--convergenza, (Italian), *Boll. Un. Mat. Ital. B (5)*, **14** (1977), 285–299.

AIMS Press