http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(3): 1–19.
DOI:10.3934/mine.2023049
Received: 22 March 2022
Revised: 01 August 2022
Accepted: 01 August 2022
Published: 16 August 2022

Research article

Flow by Gauss curvature to the L_p dual Minkowski problem†

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† **This contribution is part of the Special Issue:** Nonlinear PDEs and geometric analysis
Guest Editors: Julie Clutterbuck; Jiakun Liu
Link: www.aimspress.com/mine/article/6186/special-articles

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Abstract: In the paper [20], the authors introduced a Gauss curvature flow to study the Aleksandrov problem and the dual Minkowski problem. The paper [20] treated the cases when one can establish the uniform estimate for the Gauss curvature flow. In this paper, we study the L_p dual Minkowski problem, an extension of the dual Minkowski problem. We deal with some cases in which there is no uniform estimate for the Gauss curvature flow. We adopt the topological method from [13] to find a special initial condition such that the Gauss curvature flow converges to a solution of the L_p dual Minkowski problem.

Keywords: Gauss curvature flow; Monge-Ampere equation; Minkowski problem; variational method; a priori estimates

Dedicated to Professor Neil Trudinger on the occasion of his 80th birthday.

1. Introduction

Let $\mathcal{M}_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$ enclosing the origin. In [20], the authors studied the following anisotropic Gauss curvature flow:

$$\begin{cases} \partial_t X(x,t) = -f(\nu) r^\alpha K(x,t)\nu, \\ X(x,0) = X_0(x), \end{cases} \tag{1.1}$$

where $K(\cdot,t)$ is the Gauss curvature of the hypersurface $\mathcal{M}_t$, parametrised by $X(\cdot,t): \mathbb{S}^n \to \mathbb{R}^{n+1}$, $\nu(\cdot,t)$ is the unit outer normal at $X(\cdot,t)$, f is a given positive smooth function on $\mathbb{S}^n$, and $r = |X(x,t)|$ is the distance from the origin to the point $X(x,t)$.

The Gauss curvature flow (1.1) was introduced to study the existence of solutions to the dual Minkowski problem proposed in [16]. It can be formulated as solving the following Monge-Ampère equation on the unit sphere $\mathbb{S}^n$,

$$\det(\nabla^2 u + uI)(x) = \frac{f(x)}{u}(|\nabla u|^2 + u^2)^{\frac{\alpha}{2}}, \quad x \in \mathbb{S}^n, \tag{1.2}$$

where u denotes the support function of a hypersurface solution $\mathcal{M}$. By establishing the a priori estimates and studying the convergence of the normalized flows of (1.1), the following results were proved in [20].

Theorem 1.1. ([20]) Let f be a smooth and positive function on the sphere $\mathbb{S}^n$.

(i) If $\alpha > n+1$, there is a unique smooth, uniformly convex solution to (1.2).

(ii) If $\alpha = n+1$, assume that f satisfies the condition (1.3) below, then there is a smooth, uniformly convex solution to (1.2). The solution is unique up to dilation.

(iii) If $\alpha < n+1$ and f is even, there is an origin-symmetric, smooth and uniformly convex solution to (1.2).

(iv) If $f \equiv 1$, then the solution must be a sphere when $\alpha \geq n+1$, and the origin-symmetric solution must be a sphere when $0 \leq \alpha < n+1$.

When $\alpha = n+1$, Eq (1.2) is the Aleksandrov problem. It is known that there is a necessary and sufficient condition for the existence of solutions, namely

$$\begin{aligned} &\int_{\mathbb{S}^n} f = |\mathbb{S}^n|, \\ &\int_{\omega} f < |\mathbb{S}^n| - |\omega^*|, \ \ \forall \text{ convex domain } \omega \subset \mathbb{S}^n, \end{aligned} \tag{1.3}$$

where ω^* is the dual set of ω.

In this paper, as in [5, 10, 20] we employ a Gauss curvature flow to study the existence of solutions to the L_p dual Minkowski problem introduced in [23]

which extends the dual Minkowski problem (1.2). Let f and g be positive functions on $\mathbb{S}^n$, and $p, q \in \mathbb{R}$. We study the existence of solutions to the following equation,

$$\det(\nabla^2 u + uI)(x) = \frac{f(x)u^{p-1}(|\nabla u|^2 + u^2)^{\frac{n+1-q}{2}}}{g\left(\frac{\nabla u(x) + ux}{\sqrt{|\nabla u|^2 + u^2}}\right)}, \quad x \in \mathbb{S}^n. \tag{1.4}$$

Equation (1.4) contains the L_p dual Minkowski problem as a special case (namely when $g \equiv 1$). It extends the L_p-Minkowski problem (when $q = n+1$ and $g \equiv 1$) and the dual Minkowski problem (when $p = 0$ and $g \equiv 1$).

In particular, when $g \equiv 1$, $p = 1$, $q = n+1$, Eq (1.4) is the classical Minkowski problem, which asks for the existence of closed convex hypersurfaces with prescribed surface area measure. It is a major impetus for the development of fully nonlinear PDEs. The L_p-Minkowski problem, introduced in [22], concerns the existence of closed convex hypersurfaces with prescribed p-area measures. It extends the classical Minkowski problem and includes the logarithmic Minkowski problem ($p = 0$), and the centro-affine Minkowski problem ($p = -n - 1$) as special cases [2, 11]. In the last two decades, great progress has been made in the study of the L_p-Minkowski problem. There is a rich phenomena on the existence and multiplicity of solutions, depending on the range of the exponent p (see e.g., [1, 9, 12, 15, 18, 19]).

For general exponents p and q, Eq (1.4) has been studied in [4, 6–8, 17]. Suppose that $g \equiv 1$. The existence of smooth solutions to (1.4) was proved in [17] for $p > q$, and in [8] for $p = q$. When $p < q$, the solution may not be smooth in general. Weak solutions were obtained in [4] when $p > 1$ and $q > 0$, and later on in [8] for all $p > 0$ and $q \in \mathbb{R}$. If only origin-symmetric solutions are concerned, Eq (1.4) was solved in [6] when $pq \geq 0$, and in [7] when $q > 0$ and $-q^* < p < 0$ where q^* is defined as

$$q^* = \begin{cases} \dfrac{q}{q-n} & \text{if } q \geq n+1, \\ \dfrac{nq}{q-1} & \text{if } 1 < q < n+1, \\ +\infty & \text{if } 0 < q \leq 1. \end{cases}$$

Suppose now $g \not\equiv 1$. Equation (1.4) with $p = q = 0$ characterises the Gauss image problem proposed by [3], which extends the classical Aleksandrov problem. It was also considered in [21] from the optimal transportation viewpoint.

The main result of this paper is the following.

Theorem 1.2. Let $f, g \in C^{1,1}(\mathbb{S}^n)$ be positive functions satisfying $c_0^{-1} \leq f, g \leq c_0$ for some constant $c_0 > 1$. Suppose that $q > n$ and

$$p < \begin{cases} -\dfrac{nq}{q-1}, & \text{if } q \geq n+1, \\ -\dfrac{q}{q-n}, & \text{if } n < q < n+1. \end{cases} \tag{1.5}$$

Then there is a uniformly convex and $C^{3,\alpha}$-smooth positive solution to (1.4), where $\alpha \in (0, 1)$.

When $q = n + 1$ and $g \equiv 1$, Theorem (1.2) recovers the main result in [13]. The range of p and q in Theorem 1.2 has no overlap with that in [7], and to the best of our knowledge, has not been studied in other papers.

To prove Theorem 1.2, we will employ the following Gauss curvature flow,

$$\frac{\partial X}{\partial t}(x, t) = -\eta(t) \frac{f(\nu)}{g(\xi)} \langle X, \nu \rangle^p |X|^{n+1-q} K(x, t) \nu + X(x, t), \tag{1.6}$$

where $X(\cdot, t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a parametrisation of the evolving convex hypersurface $\mathcal{M}_t$, $\xi = X/|X|$, ν and K are respectively the unit outward normal and the Gauss curvature of $\mathcal{M}_t$. The multiplier $\eta(t)$ is given by

$$\eta(t) = \Big[\int_{\mathbb{S}^n} f(x) u^p(x, t) dx \Big]^{\frac{1}{-p} - 1} \Big[\int_{\mathbb{S}^n} g(\xi) r^q(\xi, t) d\xi \Big]^{1 - \frac{1}{q}}, \tag{1.7}$$

and as before, $u(x, t)$ and $r(\xi, t)$ are the support and radial functions of $\mathcal{M}_t$.

Denote by $\mathcal{K}_o$ the set of convex bodies $\Omega \subset \mathbb{R}^{n+1}$ containing the origin in its interior. We will show that (1.6) is a gradient flow to the following functional:

$$\mathcal{J}_{p,q}(\Omega) = \Big[\int_{\mathbb{S}^n} g(\xi) r^q(\xi) d\xi\Big]^{\frac{1}{q}} + \Big[\int_{\mathbb{S}^n} u^p(x) f(x) dx\Big]^{\frac{1}{-p}}, \tag{1.8}$$

where $\Omega \in \mathcal{K}_o$, u and r are the support function and the radial function of Ω, respectively. If $\Omega \in \mathcal{K}_o$ is a critical point of the functional (1.8), then the support function of Ω satisfies Eq (1.4).

For the Gauss curvature flow (1.6), a main issue is the lack of uniform estimates, namely the control of the eccentricity of Ω (i.e., the eccentricity of the minimum ellipsoid of Ω). Our strategy is to use a topological method to find a special initial condition such that the evolving hypersurfaces $\mathcal{M}_t = \partial\Omega_t$ satisfy

$$B_r(0) \subset \Omega_t \subset B_R(0), \tag{1.9}$$

for some positive constants $R \geq r > 0$ independent of t. Once the solution satisfies such a C^0-estimates, one can establish higher order a priori estimates for the flow (1.6). Hence by the monotonicity of the functional (1.8), the flow converges to a solution of (1.4).

To find the special initial hypersurface, we assume that $q > n$ and (1.5) such that the functional $\mathcal{J}_{p,q}(\Omega)$ will become very large if either the volume of Ω is sufficiently large or small, or the eccentricity of Ω is sufficiently large. This property enables us to find the special initial hypersurface by using the topological method and the variational structure of the equation as in [13], where we proved the existence of solutions to the L_p-Minkowski problem in the super-critical case (namely when $q = n + 1$ and $p < -n - 1$). Although the approach is similar to that in [13], equation (1.4) and the associated flow (1.6) are more complicated than the corresponding ones in [13]. Therefore, we need to present sufficient details of the argument for the approach.

We will consider in Section 2 the a priori estimates for the flow (1.6). In Section 3, we combine the a priori estimates and the topological method to find a special initial condition such that the flow converges to a solution of (1.4), and thus prove Theorem 1.2. Section 4 contains further remarks on some variants of Theorem 1.2.

2. A priori estimates for the Gauss curvature flow (1.6)

For a closed convex hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$, the support function of $\mathcal{M}$ is given by

$$u(x) = \langle x, \nu_{\mathcal{M}}^{-1}(x)\rangle, \quad \forall\, x \in \mathbb{S}^n,$$

where $\nu_{\mathcal{M}} : \mathcal{M} \to \mathbb{S}^n$ is the Gauss map and $\nu_{\mathcal{M}}^{-1}$ is its inverse. It is well known that $\nu_{\mathcal{M}}^{-1}(x) = u(x)x + \nabla u(x)$, and the Gauss curvature of $\mathcal{M}$ at $\nu_{\mathcal{M}}^{-1}(x)$ is given by

$$K = 1/\det(u_{ij} + u\delta_{ij}), \tag{2.1}$$

where $u_{ij} := \nabla^2_{ij} u$. Assume that $\mathrm{Cl}(\mathcal{M}) \in \mathcal{K}_o$, where $\mathrm{Cl}(\mathcal{M})$ denotes the convex body enclosed by $\mathcal{M}$. Recall that the radial function of $\mathcal{M}$, denoted by r, is given by

$$r(\xi) = \max\{\lambda : \lambda\xi \in \mathrm{Cl}(\mathcal{M})\} \;\; \forall\, \xi \in \mathbb{S}^n. \tag{2.2}$$

Denote $\vec{r}(\xi) = r(\xi)\xi$. We also define the radial Gauss mapping by

$$\mathscr{A}_{\mathcal{M}}(\xi) = \nu_{\mathcal{M}}(\vec{r}(\xi)) \ \ \forall\, \xi \in \mathbb{S}^n.$$

It is easy to verify that

$$r \circ \mathscr{A}_{\mathcal{M}}^{-1}(x) = |\nu_{\mathcal{M}}^{-1}(x)| = (u^2(x) + |\nabla u(x)|^2)^{\frac{1}{2}}.$$

For any convex body $\Omega \in \mathcal{K}_o$, its polar dual Ω^* is given by

$$\Omega^* = \{x \in \mathbb{R}^{n+1} : x \cdot y \leq 1, \forall\, y \in \Omega\}.$$

Let $\mathcal{M}_t$ be a solution to the flow (1.6) and $X(\cdot, t)$ be its parametrisation. Consider the new parametrisation

$$\overline{X}(x, t) = X(\nu_{\mathcal{M}_t}^{-1}(x), t).$$

It is straightforward to compute

$$\frac{\partial \overline{X}}{\partial t} = \sum_i \frac{\partial X}{\partial z^i} \frac{\partial (\nu_{\mathcal{M}_t}^{-1})_i}{\partial t} + \frac{\partial X}{\partial t}.$$

Since the first term on the right hand side is tangential, taking inner product with the unit outer normal of $\mathcal{M}_t$ gives that

$$\partial_t u(x, t) = \langle x, \partial_t \overline{X}(x, t)\rangle = \langle x, \partial_t X(\nu_{\mathcal{M}_t}^{-1}(x), t)\rangle.$$

Hence by (2.1), the flow (1.6) can be expressed as

$$\partial_t u(x, t) = -\frac{\eta(t) f(x)}{g(\frac{\nabla u + ux}{\sqrt{u^2 + |\nabla u|^2}})} \frac{u^p (u^2 + |\nabla u|^2)^{\frac{n+1-q}{2}}}{\det(\nabla^2 u + uI)} + u. \tag{2.3}$$

Theorem 2.1. Suppose that both f and g are positive and $C^{1,1}$-smooth. Let $u(\cdot, t)$ be a positive, smooth and uniformly convex solution to (2.3), $t \in [0, T)$. Assume that

$$1/C_0 \leq u(x, t) \leq C_0 \tag{2.4}$$

for all $(x, t) \in \mathbb{S}^n \times [0, T)$. Then

$$C^{-1} I \leq (\nabla^2 u + uI)(x, t) \leq CI \ \ \forall\, (x, t) \in \mathbb{S}^n \times [0, T), \tag{2.5}$$

where C is a positive constant depending only on n, p, q, C_0, $\min_{\mathbb{S}^n} f$, $\min_{\mathbb{S}^n} g$, $\|f\|_{C^{1,1}(\mathbb{S}^n)}$, $\|g\|_{C^{1,1}(\mathbb{S}^n)}$, and the initial condition $u(\cdot, 0)$, but is independent of T.

Proof. We first observe that, by the convexity

$$|\nabla u(x, t)| \leq \max_{\mathbb{S}^n} u(\cdot, t) \leq C_0, \ \ \forall\, (x, t) \in \mathbb{S}^n \times [0, T). \tag{2.6}$$

It also yields the bound of $\eta(t)$ defined by (1.7):

$$1/C_1 \le \eta(t) \le C_1, \quad \text{for all } t \in [0, T), \tag{2.7}$$

where C_1 depends only on $n, p, q, \min_{\mathbb{S}^n} f, \min_{\mathbb{S}^n} g, \max_{\mathbb{S}^n} f, \max_{\mathbb{S}^n} g$ and C_0.

Recently in [14], we studied the centro-affine Minkowski problem. We established in [14] the a priori estimates for a more general equation of the form

$$\partial_t u(x,t) = -\eta(t)\Phi(x, u, \nabla u)[\det(\nabla^2 u + uI)]^{-1} + u(x,t), \tag{2.8}$$

so that when

$$\Phi(x, u, \nabla u) = \frac{f(x)}{g(\frac{\nabla u + ux}{\sqrt{u^2+|\nabla u|^2}})} u^p (u^2 + |\nabla u|^2)^{\frac{n+1-q}{2}},$$

the Eq (2.8) becomes (2.3).

By virtue of Lemma 6.1 and Lemma 6.2 in [14], and using (2.6), (2.7) and (2.8), we conclude (2.5) as desired and hence complete the proof. □

By the second derivative estimates (2.5), Eq (2.3) becomes uniformly parabolic. Hence, by Krylov's regularity theory, we have the following $C^{3,\alpha}$ estimate,

$$\|u(\cdot, t)\|_{C^{3,\alpha}(\mathbb{S}^n)} \le C \;\; \forall\, (x,t) \in \mathbb{S}^n \times [0, T), \tag{2.9}$$

for any given $\alpha \in (0, 1)$, where the constant C depends only on n, p, q, C_0, $\min_{\mathbb{S}^n} f$, $\min_{\mathbb{S}^n} g$, $\|f\|_{C^{1,1}(\mathbb{S}^n)}$, $\|g\|_{C^{1,1}(\mathbb{S}^n)}$, and the initial condition $u(\cdot, 0)$. By the a priori estimates (2.9), we have the longtime existence of solutions to the flow (1.6), provided that u satisfies (2.4).

Theorem 2.2. Assume the conditions in Theorem 2.1. Let $T_{\max}$ be the maximal time such that the solution $u(\cdot, t)$ exists on $[0, T_{\max})$. Then $T_{\max} = \infty$ and u satisfies the estimates (2.5) and (2.9).

3. Proof of Theorem 1.2

In this section, we use a topological method to select an initial hypersurface $\mathcal{N}_0$, such that the flow (1.6) deforms $\mathcal{N}_0$ to a solution of (1.4).

3.1. Monotonicity of the functional (1.8)

We first prove the monotonicity of $\mathcal{J} := \mathcal{J}_{p,q}$ under the flow (1.6). Recall that

$$\mathcal{J}(\Omega) = \Big[\int_{\mathbb{S}^n} g(\xi) r^q(\xi) d\xi\Big]^{\frac{1}{q}} + \Big[\int_{\mathbb{S}^n} u^p(x) f(x) dx\Big]^{\frac{1}{-p}}.$$

Lemma 3.1. Suppose $\mathcal{M}_t$, $t \in [0, T)$, is a solution to the flow (1.6) in $\mathcal{K}_o$. Then

$$\frac{d}{dt}\mathcal{J}(\Omega_t) \ge 0,$$

where $\Omega_t = \mathrm{Cl}(\mathcal{M}_t)$. Equality holds if and only if the support function of $\mathcal{M}_t$ satisfies

$$\det(\nabla^2 u + uI) = \frac{f u^{p-1}(|\nabla u|^2 + u^2)^{\frac{n+1-q}{2}}}{g(\frac{\nabla u + ux}{\sqrt{u^2+|\nabla u|^2}})}\eta(t), \tag{3.1}$$

where η is given in (1.7).

Proof. The following formulas can be found in [20]:

$$\begin{aligned} \frac{\partial_t r}{r}(\xi, t) &= \frac{\partial_t u}{u}(\mathscr{A}_{\mathcal{M}_t}(\xi), t), \\ |\mathrm{Jac}\mathscr{A}|(\xi) &= \frac{r^{n+1}K(\vec{r}(\xi,t))}{u(\mathscr{A}_{\mathcal{M}_t}(\xi))}, \end{aligned} \tag{3.2}$$

where $\mathrm{Jac}\mathscr{A}$ is the Jacobian of the radial Gauss mapping.

By virtue of (2.1), (2.3) and (3.2), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{J}(\Omega_t) &= \Big[\int_{\mathbb{S}^n} g r^q d\xi\Big]^{\frac{1}{q}-1}\int_{\mathbb{S}^n} g r^q \frac{\partial_t r}{r} d\xi - \Big[\int_{\mathbb{S}^n} u^p f dx\Big]^{\frac{1}{-p}-1}\int_{\mathbb{S}^n} f u^p \frac{\partial_t u}{u} dx \\ &= \Big[\int_{\mathbb{S}^n} g r^q d\xi\Big]^{\frac{1}{q}-1}\int_{\mathbb{S}^n} g \frac{r^{q-n-1}}{K}\partial_t u dx - \Big[\int_{\mathbb{S}^n} u^p f dx\Big]^{\frac{1}{-p}-1}\int_{\mathbb{S}^n} f u^{p-1}\partial_t u dx \\ &= \Big[\int_{\mathbb{S}^n} g r^q d\xi\Big]^{\frac{1}{q}-1}\int_{\mathbb{S}^n} g \frac{r^{q-n-1}}{uK}\Big(u - \eta(t)\frac{f}{g} r^{n+1-q} u^p K\Big)\partial_t u dx \geq 0. \end{aligned}$$

Clearly, equality $\frac{d}{dt}\mathcal{J}(\Omega_t) = 0$ holds if and only if $u(\cdot, t)$ satisfies (3.1). □

The proof of Lemma 3.1 also verifies that (1.4) is the Euler-Lagrange equation of the functional (1.8) up to a constant. Note that once we have a solution to (3.1), by a proper rescaling, we can obtain a solution to (1.4).

In the following, we aim to find an initial condition $u(\cdot, 0)$ such that (2.4) is satisfied.

3.2. *An estimate for the functional* (1.8)

For any convex body Ω in $\mathbb{R}^{n+1}$, let $E(\Omega)$ denote John's minimum ellipsoid of Ω. We have

$$\frac{1}{n+1}E(\Omega) \subset \Omega \subset E(\Omega).$$

Let $a_1(\Omega) \leq a_2(\Omega) \leq \cdots \leq a_{n+1}(\Omega)$ be the lengths of semi-axes of $E(\Omega)$. Denote $e_{\mathcal{M}} = e_\Omega = \frac{a_{n+1}(\Omega)}{a_1(\Omega)}$ the eccentricity of $\mathcal{M} := \partial\Omega$ (or the eccentricity of Ω).

In this subsection, we will show Proposition 3.2 by assuming that Lemma 3.3 and Lemma 3.4 hold. The proofs of Lemma 3.3 and Lemma 3.4 will be presented after the proof of Proposition 3.2.

Proposition 3.2. Suppose that $q > n$ and p satisfies (1.5). Suppose that f, g satisfy $c_0^{-1} \leq f, g \leq c_0$ for some constant $c_0 > 1$. For any given constant $A > \mathcal{J}(B_1(0))$, if one of the quantities e_Ω, $\mathrm{Vol}(\Omega)$, $[\mathrm{Vol}(\Omega)]^{-1}$, and $[\mathrm{dist}(O, \partial\Omega)]^{-1}$ is sufficiently large, then $\mathcal{J}(\Omega) \geq A$.

Proof. We divide the proof into three steps.
Step 1: If $e_\Omega \geq e$ for a large constant $e > 1$, we have $\mathcal{J}(\Omega) > A$.

By Lemmas 3.3 and 3.4, we have

$$-\frac{1}{p}\int_{\mathbb{S}^n} u^p dx \geq C\frac{d^{n-|p|}}{\prod_{j=2}^{n+1} a_j}, \tag{3.3}$$

$$\frac{1}{q}\int_{\mathbb{S}^n} r^q d\xi \geq C a_{n+1}^{q-n}\prod_{j=1}^{n} a_j. \tag{3.4}$$

From (3.3) and (3.4), there exists a constant $C > 0$ depending only on n, p, q, c_0 such that

$$\begin{aligned}
[\mathcal{J}(\Omega)]^2 &\geq C\Big[\int_{\mathbb{S}^n} u^p dx\Big]^{-\frac{1}{p}}\Big[\int_{\mathbb{S}^n} r^q d\xi\Big]^{\frac{1}{q}} \\
&\geq C d^{\frac{n}{|p|}-1} a_{n+1}^{1-\frac{n}{q}}\Big[\prod_{j=2}^{n+1} a_j^{-\frac{1}{|p|}}\Big]\Big[\prod_{j=1}^{n} a_j^{\frac{1}{q}}\Big] \\
&= C\Big[\frac{a_1}{d}\Big]^{1-\frac{n}{|p|}}\Big[\frac{a_2}{a_1}\Big]^{1-\frac{n}{|p|}-\frac{1}{q}}\Big[\frac{a_3}{a_2}\Big]^{1-\frac{n-1}{|p|}-\frac{2}{q}}\cdots\Big[\frac{a_{n+1}}{a_n}\Big]^{1-\frac{1}{|p|}-\frac{n}{q}}.
\end{aligned}$$

That is

$$[\mathcal{J}(\Omega)]^2 \geq C\Big[\frac{a_1}{d}\Big]^{1-\frac{n}{|p|}}\prod_{j=1}^{n}\Big[\frac{a_{j+1}}{a_j}\Big]^{1-\frac{n+1-j}{|p|}-\frac{j}{q}}. \tag{3.5}$$

Condition (1.5) yields that

$$1-\frac{n}{|p|} > 0, \quad \text{and} \quad 1-\frac{n+1-j}{|p|}-\frac{j}{q} > 0,\ \forall\ j = 1,\cdots,n.$$

To see this, if $q \geq n+1$, then $|p| > \frac{nq}{q-1}$ and

$$\begin{aligned}
1-\frac{n+1-j}{|p|}-\frac{j}{q} &> 1-\frac{(n+1-j)(q-1)}{nq}-\frac{j}{q} \\
&= \frac{(q-n-1)(j-1)}{nq} \geq 0.
\end{aligned}$$

While if $n < q < n+1$, then $|p| > \frac{q}{q-n}$ and

$$\begin{aligned}
1-\frac{n+1-j}{|p|}-\frac{j}{q} &> 1-\frac{(n+1-j)(q-n)+j}{q} \\
&= \frac{(n-j)(n+1-q)}{q} \geq 0.
\end{aligned}$$

Note that $a_1 \geq d$. If e_Ω is large, there is a j such that a_{j+1}/a_j is large. We see from (3.5) that $\mathcal{J}(\Omega) \geq A$.
Step 2: If either $\mathrm{Vol}(\Omega) \leq v_0$ or $\mathrm{Vol}(\Omega) \geq v_0^{-1}$, for a small constant $v_0 > 0$, then $\mathcal{J}(\Omega) > A$.

We have

$$\mathcal{J}(\Omega) \geq \Big[\int_{\mathbb{S}^n} r^q g d\xi\Big]^{\frac{1}{q}} \geq Cd \geq \frac{Cd}{a_{n+1}}[\mathrm{Vol}(\Omega)]^{\frac{1}{n+1}}, \tag{3.6}$$

and

$$\mathcal{J}(\Omega) \geq \Big[\int_{\mathbb{S}^n} u^p f dx \Big]^{\frac{1}{|p|}} \geq \frac{C}{a_{n+1}} \geq \frac{a_1}{a_{n+1}} \frac{C}{[\mathrm{Vol}(\Omega)]^{\frac{1}{n+1}}}. \tag{3.7}$$

If a_1/d is large, then (3.5) implies that $\mathcal{J}(\Omega) \geq A$. Therefore we may assume $a_1 \leq Cd$ for some C, and so $a_1/a_{n+1} \leq Cd/a_{n+1}$. Hence if either d/a_{n+1} or a_1/a_{n+1} is sufficiently close to 0, then (3.5) again shows that $\mathcal{J}(\Omega) \geq A$. Hence we assume that d/a_{n+1} or a_1/a_{n+1} are away from 0. By (3.6) and (3.7), if either $\mathrm{Vol}(\Omega)$ or $[\mathrm{Vol}(\Omega)]^{-1}$ is large, then $\mathcal{J}(\Omega) \geq A$.

Step 3: If $\mathrm{dist}(O, \partial\Omega) \leq d_0$ for a sufficiently small $d_0 > 0$, then $\mathcal{J}(\Omega) > A$.

In this case, we may assume that $a_j \leq Cd$ for all $1 \leq j \leq n+1$ for some $C \geq 1$. As discussed in Step 2, $a_1 \leq Cd$, otherwise we are done. If a_j/d is sufficiently large for some j, then e_Ω is also huge as $e_\Omega \geq a_j/a_1 \geq a_j/(Cd)$. Step 1 shows that $\mathcal{J}(\Omega) \geq A$.

Under the above assumption, if d is sufficiently small, then $\mathrm{Vol}(\Omega)$ becomes very small. By Step 2, we have $\mathcal{J}(\Omega) > A$. □

Lemma 3.3. Let $\Omega \in \mathcal{K}_o$. Suppose $q > 0$. There exists a constant $C > 0$ depending only on n and q such that

$$\frac{1}{q} \int_{\mathbb{S}^n} r^q(\xi) d\xi \geq C a_{n+1}^{q-n} \prod_{i=1}^{n} a_i.$$

Here, r is the radial function of Ω, and $a_1 \leq \cdots \leq a_{n+1}$ are the lengths of semi-axes of $E(\Omega)$.

Proof. By a proper rotation of coordinates, we assume that $E = E(\Omega)$ is given by

$$E - \zeta_E = \Big\{ z \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{z_i^2}{a_i^2} \leq 1 \Big\},$$

where $\zeta_E = (\zeta_1, \cdots, \zeta_{n+1})$ is the center of E. We can further assume that $\zeta_{n+1} \geq 0$.

Since $\frac{1}{n+1}E \subset \Omega$, we have

$$u(\mathbf{e}_{n+1}) \geq \zeta_{n+1} + \frac{1}{n+1} a_{n+1}. \tag{3.8}$$

Hence, there exists a point $p_0 \in \Omega$ such that

$$p_0 \cdot \mathbf{e}_{n+1} = u(\mathbf{e}_{n+1}) \geq \zeta_{n+1} + \frac{1}{n+1} a_{n+1}.$$

Consider the hyperplane L which is orthogonal to $\mathbf{e}_{n+1}$ and passes through ζ_E:

$$L = \{ z \in \mathbb{R}^{n+1} : (z - \zeta_E) \cdot \mathbf{e}_{n+1} = 0 \}.$$

Let $P = L \cap \frac{1}{n+1}E$ be the intersection of L with the ellipsoid $\frac{1}{n+1}E$, and V be the cone in $\mathbb{R}^{n+1}$ with base P and the vertex p_0. Clearly $V \subset \Omega$.

Case 1: $q > n+1$. Let us consider the following subset of V:

$$V' = \Big\{ z \in V : z_{n+1} - \zeta_{n+1} \geq \frac{1}{2}(p_0 \cdot \mathbf{e}_{n+1} - \zeta_{n+1}) \Big\}.$$

This together with (3.8) implies that

$$|z| \geq \frac{a_{n+1}}{2(n+1)}, \quad \forall\, z \in V'. \tag{3.9}$$

Using $V' \subset \Omega$, $q > n+1$ and (3.9), we have

$$\frac{1}{q}\int_{\mathbb{S}^n} r^q d\xi = \int_{\Omega} |z|^{q-n-1} dz \geq \int_{V'} |z|^{q-n-1} dz \geq C a_{n+1}^{q-n-1}\,\mathrm{Vol}(V'), \tag{3.10}$$

where C is a constant depending only on n and q.

It is easy to see that

$$\mathrm{Vol}(V') \geq c_n a_{n+1} \prod_{i=1}^{n} a_i. \tag{3.11}$$

Combining (3.10) and (3.11), we conclude that

$$\int_{\mathbb{S}^n} r^q d\xi \geq C a_{n+1}^{q-n} \prod_{i=1}^{n} a_i.$$

Case 2: $0 < q \leq n+1$. Since Ω contains the origin and $\Omega \subset E$, we have

$$|z| \leq c_n a_{n+1}, \ \forall\, z \in \Omega. \tag{3.12}$$

Using $\frac{1}{n+1}E \subset \Omega$, $n < q \leq n+1$ and (3.12), we derive that

$$\frac{1}{q}\int_{\mathbb{S}^n} r^q d\xi = \int_{\Omega} |z|^{q-n-1} dz \geq C a_{n+1}^{q-n-1}\,\mathrm{Vol}(\Omega) \geq C a_{n+1}^{q-n} \prod_{i=1}^{n} a_i.$$

This completes the proof. □

Lemma 3.4. Let $\Omega \in \mathcal{K}_o$. Suppose $p < 0$. There exists a constant $C > 0$ depending only on n and p such that

$$-\frac{1}{p}\int_{\mathbb{S}^n} u^p dx \geq C \frac{d^{n-|p|}}{\prod_{j=2}^{n+1} a_j}.$$

Here, u is the support function of Ω, $d = \mathrm{dist}(O, \partial\Omega)$ and $a_1 \leq \cdots \leq a_{n+1}$ are the lengths of semi-axes of $E(\Omega)$.

Proof. By a proper rotation of coordinates, we assume that $E = E(\Omega)$ is given by

$$E - \zeta_E = \left\{ z \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{z_i^2}{a_i^2} \leq 1 \right\},$$

where $\zeta_E = (\zeta_1, \cdots, \zeta_{n+1})$ is the center of E. Let $x_0 \in \mathbb{S}^n$ be a point such that $u(x_0) = \min_{\mathbb{S}^n} u = d$, where u is the support function of Ω and $d = \mathrm{dist}(O, \partial\Omega)$. We choose $i_\#$ and switch $\mathbf{e}_{i_\#}$ and $-\mathbf{e}_{i_\#}$ if necessary such that

$$x_0 \cdot \mathbf{e}_{i_\#} = \max\{|x_0 \cdot \mathbf{e}_i| : 1 \leq i \leq n+1\}.$$

This implies that $x_0 \cdot \mathbf{e}_{i_\#} \geq c_n$. We use c_n to denote a constant which depends only on n but may change from line to line.

Let $w(x) = u(x) + u(-x)$, $x \in \mathbb{S}^n$, be the width of Ω in x. Since $\frac{1}{n+1}E(\Omega) \subset \Omega \subset E(\Omega)$, we have

$$d \leq \min_{\mathbb{S}^n} w \leq c_n a_{i_\#}$$

and

$$\frac{2a_i}{n+1} \leq w(\mathbf{e}_i) \leq 2a_i, \quad \forall\, i = 1, \cdots, n+1.$$

By switching $\mathbf{e}_i$ and $-\mathbf{e}_i$ if necessary, we assume that $u(\mathbf{e}_i) \leq c_n a_i$ for all $i = 1, \cdots, n+1$.

Consider the cone $\mathcal{V}$ in $\mathbb{R}^{n+1}$ with the vertex $p_0 = r^*(x_0)x_0$ and the base

$$C := \text{convex hull of } \{O, r^*(\mathbf{e}_k)\mathbf{e}_k\}_{k \neq i_\#}.$$

Here, r^* is the radial function of the polar dual Ω^* of Ω:

$$\Omega^* = \{y \in \mathbb{R}^{n+1} : y \cdot z \leq 1\ \forall\, z \in \Omega\}.$$

Let $\mathcal{V}'$ be a subset of $\mathcal{V}$:

$$\mathcal{V}' = \Big\{z = (z_1, \cdots, z_{n+1}) \in \mathcal{V} : z_{i_\#} \geq \frac{r^*(x_0)}{2} x_0 \cdot \mathbf{e}_{i_\#}\Big\}.$$

Recall that $r^* = 1/u$. So $r^*(x_0) = \frac{1}{d}$. Since $x_0 \cdot \mathbf{e}_{i_\#} \geq c_n$, we see that

$$\frac{1}{c_n d} \leq |z| \leq \frac{1}{d} \quad \text{for all } z \in \mathcal{V}'.$$

The second inequality above follows by $\mathcal{V}' \subset \Omega^* \subset B_{1/d}(0)$ (as $B_d(0) \subset \Omega$). Therefore

$$\begin{aligned} -\frac{1}{p}\int_{\mathbb{S}^n} u^p dx &= \frac{1}{|p|}\int_{\mathbb{S}^n} (r^*)^{|p|} \\ &= \int_{\Omega^*} |z|^{|p|-n-1} dz \\ &\geq \int_{\mathcal{V}'} |z|^{|p|-n-1} dz \\ &\geq C d^{n+1-|p|}\,\mathrm{Vol}(\mathcal{V}'). \end{aligned} \tag{3.13}$$

Since $r^*(\mathbf{e}_k) = \frac{1}{u(\mathbf{e}_k)} \geq \frac{c_n}{a_k}$ for all $k \geq 1$, we obtain

$$\mathrm{Vol}(\mathcal{V}') \geq \frac{c_n}{d} a_{i_\#} \prod_{j=1}^{n+1} a_j^{-1}.$$

Plugging this into (3.13), we obtain

$$-\frac{1}{p}\int_{\mathbb{S}^n} u^p dx \geq C d^{n-|p|} a_{i_\#} \prod_{j=1}^{n+1} a_j^{-1} \geq C \frac{d^{n-|p|}}{\prod_{j=2}^{n+1} a_j}.$$

□

Remark 3.5. Let $\mathcal{M}_t$, $t \in [0, T_{\max})$, be a solution to (1.6). By Proposition 3.2, if $\mathcal{J}(\mathcal{M}_t) < A$ for a constant A independent of t, then there exist positive constants e_0, v_0, d_0 depending on A, but independent of t, such that

$$e_{\mathcal{M}_t} \le e_0, \quad v_0 \le \mathrm{Vol}(\mathcal{M}_t) \le v_0^{-1}, \text{ and } B_{d_0}(0) \subset \Omega_t, \tag{3.14}$$

where Ω_t is the convex body enclosed by $\mathcal{M}_t$. Note that (3.14) implies (2.4). Hence, the a priori estimates (2.5) and (2.9) hold, and one has the long-time existence of solution (Theorem 2.2). Therefore, all we need is to establish the condition $\mathcal{J}(\mathcal{M}_t) < A$ for some constant A.

3.3. A modified flow of (1.6)

We introduce a modified flow (as in [13]) such that for any initial condition, the solution exists for all time $t \ge 0$. It is more convenient to work with a flow which exists for all $t \ge 0$.

Let us fix a constant

$$A_0 = 10\|g\|_{L^1(\mathbb{S}^n)} + 10(n+1)\|f\|_{L^1(\mathbb{S}^n)}. \tag{3.15}$$

If the minimum ellipsoid of Ω is $B_1(0)$, then $\frac{1}{n+1} B_1(0) \subset \Omega \subset B_1(0)$ and hence

$$\mathcal{J}(\Omega) \le \frac{1}{2} A_0. \tag{3.16}$$

For a closed, smooth and uniformly convex hypersurface $\mathcal{N}$ such that $\Omega_0 = \mathrm{Cl}(\mathcal{N}) \in \mathcal{K}_o$, we define $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ as follows:

a): If $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t)) < A_0$ for all time $t \ge 0$, let $\bar{\mathcal{M}}_{\mathcal{N}}(t) = \mathcal{M}_{\mathcal{N}}(t)$ for all $t \ge 0$, where $\mathcal{M}_{\mathcal{N}}(t)$ is the solution to (1.6).

b): If $\mathcal{J}(\mathcal{N}) < A_0$, and $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t))$ reaches A_0 at the first time $t_0 > 0$, we define

$$\bar{\mathcal{M}}_{\mathcal{N}}(t) = \begin{cases} \mathcal{M}_{\mathcal{N}}(t), & \text{if } 0 \le t < t_0, \\ \mathcal{M}_{\mathcal{N}}(t_0), & \text{if } t \ge t_0. \end{cases}$$

c): If $\mathcal{J}(\mathcal{N}) \ge A_0$, we let $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$ for all $t \ge 0$.

For convenience, we call $\bar{\mathcal{M}}_{\mathcal{N}}$ a modified flow of (1.6). By the a priori estimates in Section 2, $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ is smooth for any fixed time t, and Lipschitz continuous in time t. Moreover, we have the following properties.

i) $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ is defined for all time $t \ge 0$, and by Lemma 3.1, $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t))$ is non-decreasing. In particular, we have $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) \le \max\{A_0, \mathcal{J}(\mathcal{N})\}\ \forall\, t \ge 0$.

ii) If either $\mathrm{dist}(O, \mathcal{N})$ is very small, or $\mathrm{Vol}(\Omega_0)$ is sufficiently large or small, or e_{Ω_0} is sufficiently large, by Proposition 3.2, we have $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}\ \forall\, t \ge 0$.

3.4. Homology of a class of ellipsoids

Here we recall the homology of a class of ellipsoids $\mathcal{A}_I$ introduced in [13], such that an ellipsoid E with $\mathcal{J}(E) < A_0$ is contained in $\mathcal{A}_I$. By Proposition 3.2, we have

Corollary 3.6. For the constant A_0 given by (3.15), there exist sufficiently small constants $\bar{d}$ and $\bar{v}$, and sufficiently large constant $\bar{e}$, such that for any $\Omega \in \mathcal{K}_o$,

(i) if $\mathrm{dist}(O, \partial\Omega) \leq \bar{d}$, then $\mathcal{J}(\Omega) > A_0$;

(ii) if $e_\Omega \geq \bar{e}$, then $\mathcal{J}(\Omega) > A_0$;

(iii) if $\mathrm{Vol}(\Omega) \leq \bar{v}$ or $\mathrm{Vol}(\Omega) \geq 1/(n+1)^{n+1}\bar{v}^{-1}$, then $\mathcal{J}(\Omega) > A_0$.

Let $\mathcal{K}$ be the metric space consisting of non-empty, compact and convex sets in $\mathbb{R}^{n+1}$, equipped with the Hausdorff distance. Denote by $\bar{\mathcal{K}}_o$ the closure of $\mathcal{K}_o$ in $\mathcal{K}$.

Fix the constants $\bar{d}, \bar{v}, \bar{e}$ in Corollary 3.6. Let $\mathcal{A}_I$ be the set of ellipsoids $E \in \bar{\mathcal{K}}_o$ such that $\bar{v} \leq \mathrm{Vol}(E) \leq 1/\bar{v}$, and $e_E \leq \bar{e}$. Denote by $\mathcal{A}$ the following subset of $\mathcal{A}_I$

$$\mathcal{A} = \{E \in \mathcal{A}_I : \mathrm{Vol}(E) = \omega_n, \text{ and either } e_E = \bar{e} \text{ or } \mathrm{dist}(O, \partial E) = 0\}.$$

Here, $\omega_n = |B_1(0)|$ is the volume of $B_1(0)$, and e_E is the eccentricity of E.

We also denote by $\mathcal{E}_I$ the set of ellipsoids in $\mathcal{A}_I$ centred at the origin, and by $\mathcal{E}$ the set of ellipsoids in $\mathcal{A}$ centred at the origin. These sets are all metric spaces by equipping the Hausdorff distance.

It was proved in [13] that $\mathcal{E}_I$ is contractible and so the homology $H_k(\mathcal{E}_I) = 0$ for all $k \geq 1$. Moreover, $\mathcal{A}_I$ is homeomorphic to $\mathcal{E}_I \times B_1(0)$. Hence, $\mathcal{A}_I$ is contractible and the homology

$$H_k(\mathcal{A}_I) = 0 \ \text{ for all } k \geq 1. \tag{3.17}$$

Denote

$$\mathcal{P} = \{E \in \mathcal{A}_I : \text{either } \mathrm{Vol}(E) = \bar{v}, \text{ or } \mathrm{Vol}(E) = 1/\bar{v}, \text{ or } e_E = \bar{e}, \text{ or } O \in \partial E\}. \tag{3.18}$$

It is the boundary of $\mathcal{A}_I$ if we regard $\mathcal{A}_I$ as a set in the topological space of all ellipsoids. Moreover, there is a retraction Ψ from $\mathcal{A}_I \setminus \{B_1\}$ to $\mathcal{P}$. Namely, $\Psi : \mathcal{A}_I \setminus \{B_1\} \to \mathcal{P}$ is continuous and $\Psi|_{\mathcal{P}} = id$. The following two theorems were also proved in [13].

Proposition 3.7. We have the following results.

(i) $H_{k+1}(\mathcal{P}) = H_k(\mathcal{A})$ for all $k \geq 1$.

(ii) There is a long exact sequence

$$\cdots \to H_{k+1}(\mathcal{A}) \to H_k(\mathcal{E} \times \mathbb{S}^n) \to H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \to H_k(\mathcal{A}) \to \cdots.$$

Proposition 3.8. Let $n_* = \frac{n(n+1)}{2}$. The homology group $H_{n^*+n-1}(\mathcal{E}) = \mathbb{Z}$.

3.5. Selection of a good initial condition

In this subsection, we use Propositions 3.7 and 3.8 to select a special initial condition in $\mathcal{A}_I$ such that the solution to the Gauss curvature flow (1.6) satisfies the uniform estimate. The idea is similar to that in [13].

For any ellipsoid $\mathcal{N}$ such that $\mathrm{Cl}(\mathcal{N}) \in \mathcal{A}_I$, let $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ be the solution to the modified flow. We have the following properties:

1) If $\mathrm{Cl}(\mathcal{N})$ is close to $\mathcal{P}$ in Hausdorff distance or in $\mathcal{P}$, we have $\mathcal{J}(\mathcal{N}) \geq A_0$ and so $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$ for all t (see Corollary 3.6).
2) If $\mathrm{Cl}(\mathcal{N})$ is close to $B_1(0)$ in Hausdorff distance, then $\mathcal{J}(\mathcal{N}) < A_0$.
3) By our definition of the modified flow, $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) < \max\{A_0, \mathcal{J}(\mathcal{N})\}$ for all t. Hence by Remark 3.5, if $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ is not identical to $\bar{\mathcal{M}}_{\mathcal{N}}(0) = \mathcal{N}$, then

$$e_{\bar{\mathcal{M}}_{\mathcal{N}}(t)} \leq \bar{e}, \quad \bar{v} \leq \mathrm{Vol}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) \leq 1/\bar{v}, \text{ and } B_{\bar{d}}(0) \subset \mathrm{Cl}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) \ \forall\, t \geq 0. \tag{3.19}$$

With these properties, we can prove the following key lemma.

Lemma 3.9. For every $t > 0$, there exists $\mathcal{N} = \mathcal{N}_t$ with $\mathrm{Cl}(\mathcal{N}) \in \mathcal{A}_I$, such that the minimum ellipsoid of $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ is the unit ball $B_1(0)$.

Proof. Suppose by contradiction that there exists $t' > 0$ such that, for any $\Omega \in \mathcal{A}_I$, $E_{\mathcal{N}}(t') \neq B_1(0)$, where $\mathcal{N} = \partial\Omega$ and $E_{\mathcal{N}}(t')$ is the minimum ellipsoid of $\Omega_{\mathcal{N}}(t') := \mathrm{Cl}(\bar{\mathcal{M}}_{\mathcal{N}}(t'))$.

By Corollary 3.6, $E_{\mathcal{N}}(t') \in \mathcal{A}_I$. Hence we can define a continuous map $T : \mathcal{A}_I \to \mathcal{P}$ by

$$\Omega \in \mathcal{A}_I \mapsto E_{\mathcal{N}}(t') \in \mathcal{A}_I \setminus \{B_1\} \mapsto \Psi(E_{\mathcal{N}}(t')) \in \mathcal{P},$$

where Ψ is the retraction after (3.18), and $B_1 = B_1(0)$ for short. Note that when $\Omega \in \mathcal{P}$, we have $\mathcal{J}(\Omega) \geq A_0$ and thus $E_{\mathcal{N}}(t') = E_{\mathcal{N}}(0) = \Omega$. This implies that $T|_{\mathcal{P}} = id_{\mathcal{P}}$. Hence, T is a retraction from $\mathcal{A}_I$ to $\mathcal{P}$, and so there is an injection from $H_*(\mathcal{P})$ to $H_*(\mathcal{A}_I)$. By (3.17), we then have

$$H_k(\mathcal{P}) = 0 \text{ for all } k \geq 1.$$

It follows from Proposition 3.7 (ii) that

$$H_k(\mathcal{E} \times \mathbb{S}^n) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \text{ for all } k \geq 1.$$

Computing the left-hand side by the Künneth formula and using the fact $H_k(\mathbb{S}^n) = \mathbb{Z}$ if $k = 0$ or $k = n$, and $H_k(\mathbb{S}^n) = 0$ otherwise, we further obtain

$$H_k(\mathcal{E}) \oplus H_{k-n}(\mathcal{E}) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n).$$

However, this contradicts Proposition 3.8 by taking $k = n^* + 2n - 1$ in the above. □

In the following we prove the convergence of the flow (1.6) with a specially chosen initial condition. Take a sequence $t_k \to \infty$ and let $\mathcal{N}_k = \mathcal{N}_{t_k}$ be the initial data from Lemma 3.9. By our choice of A_0 (see (3.15) and (3.16)), Lemma 3.9 implies that

$$\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_k}(t_k)) \leq \frac{1}{2} A_0. \tag{3.20}$$

Hence, by the monotonicity of the functional $\mathcal{J}$, we have

$$\bar{\mathcal{M}}_{\mathcal{N}_k}(t) = \mathcal{M}_{\mathcal{N}_k}(t) \ \ \forall\, t \le t_k.$$

Since $\mathrm{Cl}(\mathcal{N}_k) \in \mathcal{A}_I$ and $B_{\bar{d}}(0) \subset \mathrm{Cl}(\mathcal{N}_k)$, by Blaschke's selection theorem, there is a subsequence of $\mathcal{N}_k$ which converges in Hausdorff distance to a limit $\mathcal{N}_*$ such that $\mathrm{Cl}(\mathcal{N}_*) \in \mathcal{A}_I$ and $B_{\bar{d}} \subset \mathrm{Cl}(\mathcal{N}_*)$.

Next, we show that the flow (1.6) starting from $\mathcal{N}_*$ satisfying $\mathcal{J}(\mathcal{M}_{\mathcal{N}_*}(t)) < A_0$ for all t.

Lemma 3.10. For any $t \ge 0$, we have

$$\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) \le \frac{3}{4}A_0.$$

Hence

$$\bar{\mathcal{M}}_{\mathcal{N}_*}(t) = \mathcal{M}_{\mathcal{N}_*}(t) \ \ \forall\, t > 0.$$

Proof. For any given $t > 0$, since $\mathcal{N}_k \to \mathcal{N}_*$ and $t_k \to \infty$, when k is sufficiently large such that $t_k > t$, we have

$$\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) - \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) \le \frac{1}{4}A_0.$$

By the monotonicity of the functional $\mathcal{J}$,

$$\mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) \le \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t_k)).$$

Combining above two inequalities with (3.20), we obtain that

$$\begin{aligned}
\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) &= \mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) - \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) + \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) \\
&\le \mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) - \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) + \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t_k)) \\
&\le \frac{1}{4}A_0 + \frac{1}{2}A_0 = \frac{3}{4}A_0.
\end{aligned}$$

This completes the proof. □

3.6. Convergence of the flow and existence of solutions to (1.4)

Let $\Omega_{\mathcal{N}_*}(t) = \mathrm{Cl}(\mathcal{M}_{\mathcal{N}_*}(t))$ and $u(\cdot, t)$ be its support function. By Lemma 3.10, $\mathcal{M}_{\mathcal{N}_*}(t)$ satisfies (3.19). Hence,

$$\bar{d} \le u(x,t) \le C, \ \ \forall\, (x,t) \in \mathbb{S}^n \times [0,\infty),$$

where $C = (n+1)/(\bar{v}\omega_{n-1}\bar{d}^n)$. Hence, condition (2.4) holds, and we obtain the existence of solutions to (1.4) as follows.

Proof of Theorem 1.2. Denote $\mathcal{M}(t) = \mathcal{M}_{\mathcal{N}_*}(t)$ and $\mathcal{J}(t) = \mathcal{J}(\mathcal{M}(t))$. By Lemma 3.1 and Lemma 3.10,

$$\mathcal{J}(t) < A_0 \text{ and } \mathcal{J}'(t) \ge 0 \ \ \forall\, t \ge 0.$$

Therefore,

$$\int_0^\infty \mathcal{J}'(t)dt \le \limsup_{T\to\infty} \mathcal{J}(T) - \mathcal{J}(0) \le A_0.$$

This implies that there exists a sequence $t_i \to \infty$ such that

$$\mathcal{J}'(t_i) = \Big[\int_{\mathbb{S}^n} g r^q d\xi\Big]^{\frac{1}{q}-1} \int_{\mathbb{S}^n} g \frac{r^{q-n-1}}{uK}\Big(u - \eta(t)\frac{f}{g} r^{n+1-q} u^p K\Big)\partial_t u dx\Big|_{t=t_i} \to 0.$$

Passing to a subsequence, we obtain by the a priori estimates (2.9) that $u(\cdot, t_i) \to u_\infty$ in $C^{3,\alpha}(\mathbb{S}^n)$-topology and u_∞ satisfies

$$\det(\nabla^2 u + uI) = \lambda \frac{f u^{p-1}(|\nabla u|^2 + u^2)^{\frac{n+1-q}{2}}}{g(\frac{\nabla u + ux}{\sqrt{u^2+|\nabla u|^2}})},$$

where $\lambda = \lim_{t_i\to\infty} \eta(t_i)$. As $p \neq q$, it can be seen that $u_\infty^\lambda = \lambda^{\frac{1}{p-q}} u_\infty$ satisfies (1.4). □

4. Further remarks

This section is devoted to some variants of Theorem 1.2. We show that when g is a positive function defined on $\mathbb{R}^{n+1}$ instead of $\mathbb{S}^n$, our argument still works. We first show Eq (4.1) below admits a solution u up to some multiplier $\lambda > 0$ when p and q are in the same range as in Theorem 1.2. Since now g is a function on $\mathbb{R}^{n+1}$, it does not imply that the equation with $\lambda = 1$ has a solution by a scaling argument as mentioned in Section 3.6.

Theorem 4.1. Let $f \in C^{1,1}(\mathbb{S}^n)$ and $g \in C^{1,1}(\mathbb{R}^{n+1})$ be two positive functions satisfying $c_0^{-1} \le f, g \le c_0$ for some $c_0 > 1$. Suppose $q > n$ and p satisfies (1.5). Then there is a constant $\lambda > 0$, and a uniformly convex and $C^{3,\alpha}$ function u, $\alpha \in (0, 1)$, such that

$$\det(\nabla^2 u + uI)(x) = \lambda \frac{f(x) u^{p-1}(|\nabla u|^2 + u^2)^{\frac{n+1-q}{2}}}{g(\nabla u(x) + ux)}, \quad x \in \mathbb{S}^n. \tag{4.1}$$

Proof. Similarly to (1.6), we study the flow

$$\frac{\partial X}{\partial t}(x, t) = -\eta(t)\frac{f(\nu)}{g(X)}\langle X, \nu\rangle^p |X|^{n+1-q} K(x, t)\nu + X(x, t), \tag{4.2}$$

where

$$\eta(t) = \Big[\int_{\mathbb{S}^n} f(x) u^p(x, t) dx\Big]^{\frac{1}{-p}-1}\Big[q\int_{\mathbb{S}^n}\int_0^{r(\xi,t)} g(\tau\xi)\tau^{q-1} d\tau d\xi\Big]^{1-\frac{1}{q}}.$$

The same computation as in Lemma 3.1 implies that the functional

$$\mathcal{J}^{(1)}(\Omega) = \Big[q\int_{\mathbb{S}^n}\int_0^{r(\xi)} g(\tau\xi)\tau^{q-1} d\tau d\xi\Big]^{\frac{1}{q}} + \Big[\int_{\mathbb{S}^n} u^p(x) f(x) dx\Big]^{\frac{1}{-p}}$$

is non-decreasing along (4.2).

Since g is bounded and positive, we have

$$c_0^{-1}\Big[\int_{\mathbb{S}^n} r^q d\xi\Big]^{\frac{1}{q}} \le \Big[q\int_{\mathbb{S}^n}\int_0^{r(\xi)} g(\tau\xi)\tau^{q-1} d\xi d\tau\Big]^{\frac{1}{q}} \le c_0\Big[\int_{\mathbb{S}^n} r^q d\xi\Big]^{\frac{1}{q}}.$$

Therefore, Proposition 3.2 is also valid for the functional $\mathcal{J}^{(1)}$.

As a result, we can define the modified flow for (4.2) as in Section 3.3 with $A_0 = 10c_0|\mathbb{S}^n| + 10(n+1)\|f\|_{L^1(\mathbb{S}^n)}$. By using the topological argument as in Section 3.5, we can show the existence of an initial hypersurface $\mathcal{M}_0$ with $\mathrm{Cl}(\mathcal{M}_0) \in \mathcal{K}_o$ such that the evolving hypersurface $\mathcal{M}_t = \partial\Omega_t$ of the flow (4.2) satisfies

$$B_r(0) \subset \Omega_t \subset B_R(0)$$

for some constants $R > r > 0$ which are independent of time t. Equivalently, the support function of $\mathcal{M}_t$, which satisfies the parabolic equation

$$\partial_t u(x,t) = -\frac{\eta(t) f(x)}{g(\nabla u + ux)} \frac{u^p (u^2 + |\nabla u|^2)^{\frac{n+1-q}{2}}}{\det(\nabla^2 u + uI)} + u, \tag{4.3}$$

enjoys the C^0-estimates:

$$C_0^{-1} \leq u(\cdot, t) \leq C_0.$$

Since (4.3) is of the form (2.8), Theorem 2.1 is valid for (4.3). Therefore, $u(\cdot, t)$ exists for all time $t \geq 0$ and is of $C^{3,\alpha}$-smooth.

The same argument as in Section 3.6 implies that $u(\cdot, t_i)$ converges to a solution of (4.1) with $\lambda = \lim_{t_i \to \infty} \eta(t_i)$. □

When $q \geq n+1$ and $p < -q$, we can show the existence of solutions to (4.1) with $\lambda = 1$.

Theorem 4.2. Let $f \in C^{1,1}(\mathbb{S}^n)$ and $g \in C^{1,1}(\mathbb{R}^{n+1})$ be two positive functions satisfying $c_0^{-1} \leq f, g \leq c_0$ for some $c_0 > 1$. Suppose $q \geq n+1$ and $p < -q$. Then there is a uniformly convex and $C^{3,\alpha}$ function to (4.1) with $\lambda = 1$, where $\alpha \in (0,1)$.

Proof. Consider the functional

$$\mathcal{J}^{(2)}(\Omega) = -\frac{1}{p}\int_{\mathbb{S}^n} u^p(x) f(x) dx + \int_{\mathbb{S}^n}\int_0^{r(\xi)} g(\tau\xi)\tau^{q-1} d\tau\, d\xi,$$

and the flow

$$\frac{\partial X}{\partial t}(x,t) = -\frac{f(\nu)K(x,t)\langle X, \nu\rangle^p}{g(X)} |X|^{n+1-q}\nu + X(x,t). \tag{4.4}$$

Similar calculations as in Lemma 3.1 show that $\mathcal{J}^{(2)}$ is non-decreasing under (4.4).

One can verify that Proposition 3.2 holds for the functional $\mathcal{J}^{(2)}$.

Step 1: if $d = \mathrm{dist}(O, \partial\Omega)$ is sufficiently small, then $\mathcal{J}^{(2)}(\Omega)$ is sufficiently large. We adopt the same notations as in Proposition 3.2. By virtue of (3.3) and (3.4), we obtain

$$[\mathcal{J}^{(2)}(\Omega)]^2 \geq C\Big(\int_{\mathbb{S}^n} u^p dx\Big)\Big(\int_{\mathbb{S}^n} r^q d\xi\Big) \geq C a_1 a_{n+1}^{q-n-1} d^{n-|p|},$$

where $C > 0$ depends only on n, p, q and c_0. As $q \geq n+1$ and $a_1 \geq d$, we have

$$[\mathcal{J}^{(2)}(\Omega)]^2 \geq C d^{q-|p|}.$$

This shows that if d is tiny, then $\mathcal{J}^{(2)}(\Omega)$ is huge.
Step 2: if $\mathrm{Vol}(\Omega)$ or $[\mathrm{Vol}(\Omega)]^{-1}$ is sufficiently small, then $\mathcal{J}^{(2)}(\Omega)$ is sufficiently large. Since $\mathrm{Vol}(\Omega)$ being small implies that d is tiny, we are done by Step 1 in this case. Suppose that $\mathrm{Vol}(\Omega)$ is huge. The conclusion then follows from

$$\mathcal{J}^{(2)}(\Omega) \geq C \int_{\mathbb{S}^n} r^q d\xi \geq C[\mathrm{Vol}(\Omega)]^{\frac{q}{n+1}}. \tag{4.5}$$

Step 3: if e_Ω is sufficiently large, then so is $\mathcal{J}^{(2)}(\Omega)$. By Step 1, we assume without loss of generality that d is bounded from below by a constant $C > 0$ depending on n, p, q and c_0. Using (4.5),

$$[\mathcal{J}^{(2)}(\Omega)]^{\frac{n+1}{q}} \geq C\,\mathrm{Vol}(\Omega) \geq C e_\Omega d^{n+1} \geq C e_\Omega.$$

This proves Step 3.

To complete the proof, we introduce a modified flow of (4.4) and use the topological argument as in Section 3 to find the needed initial hypersurface $\mathcal{N}$, such that the evolving hypersurfaces $\mathcal{M}_t$ with $\mathcal{M}_0 = \mathcal{N}$ satisfy $B_r(0) \subset \Omega_t \subset B_R(0)$ for some constants $R > r > 0$. Since the support function $u(\cdot, t)$ of (4.4) satisfies a parabolic equation of the form (2.8), the higher order derivative estimates follow by Theorem 2.1. The remaining proof follows exactly as that of Theorem 1.2. Since (4.4) does not contain an integral term like $\eta(t)$ in (4.2), along a sequence of times $\{t_i\}_{i=1}^{\infty}$, $u(\cdot, t_i)$ converges to a solution of (4.1) with $\lambda = 1$. □

Acknowledgments

The first author was supported by ARC DE210100535. The second author was supported by NSFC12031017 and the Fundamental Research Funds for the Central Universities (No. 226-2022-00157). The third author was supported by ARC DP200101084.

Conflict of interest

The authors declare no conflict of interest.

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