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Research article

Linear stability analysis of overdetermined problems with non-constant data†

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Abstract: We study an overdetermined problem that arises as the Euler-Lagrange equation of a weighted variational problem in elasticity. Based on a detailed linear analysis by spherical harmonics, we prove the existence and local uniqueness as well as an optimal stability estimate for the shape of a domain allowing the solvability of the overdetermined problem. Our linear analysis reveals that the solution structure is strongly related to the choice of parameters in the problem. In particular, the global uniqueness holds for the pair of the parameters lying in a triangular region.

Keywords: overdetermined problem; stability estimate; symmetry; spherical harmonics; implicit function theorem

1. Introduction

The main purpose of the present paper is to derive a sharp quantitative stability estimate for the rigidity of the spherical configuration of Ω under non-radial perturbations of the boundary data g, where a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ admits a solution u to the overdetermined problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, ν is the unit outer normal vector to $\partial\Omega$, and f, g are prescribed functions of the form

$$f(x) = (n+\alpha)|x|^{\alpha}, \quad g(x) = g_0(\xi)\,|x|^{\beta}, \quad \xi = \frac{x}{|x|}, \tag{1.2}$$

where $\alpha,\beta \in \mathbb{R}$, g_0 is a function defined on the unit sphere $\mathbb{S}$ and the coefficient $n+\alpha$ is only for the normalization. In order to clarify the sense in which (1.1) is to be understood, we shall additionally assume $\alpha > -n$, so that

$$u(x) = \frac{1-|x|^{\alpha+2}}{\alpha+2} \tag{1.3}$$

is a solution to (1.1) in the sense of distributions when Ω is the unit ball $\mathbb{B}$, $g_0 = 1$ and $\beta \in \mathbb{R}$. Equation (1.1) arises as the Euler-Lagrange equation of a variational problem for a weighted torsional rigidity (see Section 2). The particular case $\alpha = \beta = 0$ has been extensively studied in the literature and is sometimes referred to as Serrin's overdetermined problem.

In the case where f, g are positive constants (i.e., $\alpha = \beta = 0$ and $g_0 > 0$ a constant), it is well-known that Ω must be a ball if (1.1) has a solution $u \in C^2(\overline{\Omega})$. This rigidity result was proved in a seminal paper [34] by Serrin, with an innovative argument called the method of moving planes based on Alexandrov's reflection principle and a refined boundary point lemma for corners, which in fact applies to nonlinear equations. Weinberger [37] provided an alternative proof based on the observation that, if u satisfies (1.1), the Cauchy-Schwarz deficit

$$d(u) := |D^2u|^2 - \frac{(\Delta u)^2}{n} \geq 0$$

becomes identically zero and thus u is a quadratic function as (1.3) (see [22, 23, 31] for refined arguments). Another interesting proof was introduced by Brandolini, Nitsch, Salani and Trombetti [7] using an integral quantity related to Newton's inequalities involving elementary symmetric functions of the eigenvalues of the Hessian matrix D^2u.

There have been numerous studies on the stability of the spherical configuration of Ω when $f = n$ (i.e., $\alpha = 0$) and g is slightly perturbed from a constant. Here we recall a few relevant results (without mentioning technical assumptions) from a methodological point of view, but not in chronological order. In order to describe the results in a unified manner, let Ω_ρ be the bounded star-shaped domain enclosed by

$$\partial\Omega_\rho := \{(1+\rho(\xi))\xi \mid \xi \in \mathbb{S}\} \tag{1.4}$$

for $\rho \in C^{2+\gamma}(\mathbb{S})$ with $-1 < \rho(\xi) < \infty$ and $0 < \gamma < 1$, and let u_ρ denote a unique solution to the Dirichlet problem consisting in the first two equations in (1.1) for $\Omega = \Omega_\rho$. Aftalion, Busca and Reichel [3] initiated the stability analysis of (1.1) by developing a quantitative version of the method of moving planes and proved that, up to translation,

$$\|\rho\|_{L^\infty(\mathbb{S})} \leq C \left| \log \left\| \frac{\partial u_\rho}{\partial \nu} + 1 \right\|_{C^1(\partial\Omega_\rho)} \right|^{-1/n} \tag{1.5}$$

holds if the quantity $\|\partial_\nu u_\rho + 1\|_{C^1(\partial\Omega_\rho)}$ is sufficiently small. This inequality shows that the deviation ρ of domain Ω_ρ from the unit ball $\Omega_0 = \mathbb{B}$ can be controlled by that of the Neumann data g from the constant $c = 1$. This method was further developed by Ciraolo, Magnanini and Vespri [13] using a quantitative

Harnack's inequality, and the logarithmic estimate (1.5) was sharpened to a power-type estimate. In fact, these two results apply to general nonlinear equations. For the particular case $f = n$, in a series of papers [22–25], Magnanini and Poggesi improved (1.5) by establishing an integral identity that relates $d(u)$ to the deviation $\partial_\nu u + 1$ and estimating both sides of the identity. The resulting estimate is

$$\|\rho\|_{L^\infty(\mathbb{S})} \leq C \left\| \frac{\partial u_\rho}{\partial \nu} + 1 \right\|_{L^2(\partial\Omega_\rho)}^{\tau_n},$$

where $\tau_2 = 1$, $\tau_3 = 1 - \varepsilon$ for any $\varepsilon > 0$, and $\tau_n = 4/(n+1)$ for $n \geq 4$ (see [25] for a sharper estimate in the case $n = 3$). In particular, this estimate is optimal for $n = 2$, and almost optimal for $n = 3$, as one can confirm by choosing Ω_ρ as ellipsoids that linear estimates (i.e., $\tau_n = 1$) are sharpest. Optimal linear estimates for any spatial dimensions $n \geq 2$ have been established either when the norm of the left hand side is weakened or when the norm of the right hand side is strengthened. Indeed, Feldman [16] proved

$$\|\rho\|_{L^1(\mathbb{S})} \leq C \left\| \frac{\partial u_\rho}{\partial \nu} + 1 \right\|_{L^2(\partial\Omega_\rho)}$$

by refining an argument of Brandolini, Nitsch, Salani and Trombetti [8], in which a power-type stability estimate was obtained for the first time by exploiting their own proof of the symmetry in [7]. Another optimal estimate obtained by Gilsbach and the present author [18] states that

$$\|\rho\|_{C^{2+\gamma}(\mathbb{S})} \leq C \left\| \frac{\partial u_\rho}{\partial \nu} + 1 \right\|_{C^{2+\gamma}(\partial\Omega_\rho)}. \tag{1.6}$$

This estimate is a consequence of the detailed linear analysis of (1.1) for $\alpha = \beta = 0$ based on a new implicit function theorem for triplets of Banach spaces, and it also provides the existence and local uniqueness of Ω_ρ for small perturbations of g from $c = 1$.

For non-constant f, g, the overdetermined problem (1.1) or other variants were treated in [1, 2, 4, 5, 9, 19, 20, 28–30, 35]. Bianchini, Henrot and Salani [6] studied the existence, uniqueness and geometric properties of Ω for (1.1) with $\alpha = 0$, $\beta > 0$ and $\beta \neq 1$ by a variational method and the maximum principle. In particular, for $\alpha = 0$ and $\beta > 1$, they proved the stability estimate

$$\|\rho\|_{L^\infty(\mathbb{S})} \leq C \left\| \frac{\partial u_\rho}{\partial \nu} + |x|^\beta \right\|_{L^\infty(\partial\Omega_\rho)}, \tag{1.7}$$

or equivalently

$$\|\rho\|_{L^\infty(\mathbb{S})} \leq C \, \|g_0 - 1\|_{L^\infty(\mathbb{S})}.$$

The restriction $\beta > 1$ hinges on the availability of the comparison principle for domains Ω with different values of g, and indeed (1.7) was proved by a comparison of Ω_ρ with radial domains. Note that a priori estimates for $\alpha = \beta = 0$ such as (1.6) cannot yield the corresponding estimates for general α, β by a direct use of the triangle inequality.

Our purpose in this paper is to prove an optimal quantitative stability estimate of the radial configuration of Ω for non-radial perturbations of g in any spatial dimensions $n \geq 2$ by linearization approach as in [18]. Since this approach relies only on the non-degeneracy of the linearized problem, we can treat general α, β unless

$$\alpha - \beta + 1 \in \mathbb{N} \cup \{0\}.$$

In fact, our linear analysis suggests that a symmetry-breaking bifurcation should occur at these exceptional values of α, β. We refer to [10–12, 14, 15, 21, 26, 27, 33, 36] for the linearization approach to bifurcation phenomena in overdetermined problems. We also emphasize that our approach yields the existence and local uniqueness of Ω for given perturbations g_0 even if $\beta \leq 0$ for which the variational method in general fails (see [6]).

To state our main result, for $k \in \mathbb{N}$ and $0 < \gamma < 1$, we set $h^{k+\gamma}(\overline{\Omega})$, called the little Hölder space, to be a closed proper subspace of $C^{k+\gamma}(\overline{\Omega})$ defined as the closure of $C^{\infty}(\overline{\Omega})$ in $C^{k+\gamma}(\overline{\Omega})$, and similarly we define $h^{k+\gamma}(\Gamma)$ for a hypersurface Γ. The little Hölder space $h^{k+\gamma}(\mathbb{S})$ is suitable for our linearization approach, since the set of spherical harmonics spans a dense subspace of $h^{k+\gamma}(\mathbb{S})$.

Theorem 1.1. *Let $\alpha > -n$, $\beta \in \mathbb{R}$ and $0 < \gamma < 1$ satisfy*

$$\alpha - \beta + 1 \notin \mathbb{N} \cup \{0\}.$$

Then, there are $\delta, \varepsilon > 0$ such that, for any $g_0 \in h^{2+\gamma}(\mathbb{S})$ with $\|g_0 - 1\|_{h^{2+\gamma}(\mathbb{S})} < \delta$, there exists a unique $\rho \in h^{3+\gamma}(\mathbb{S})$ with $\|\rho\|_{h^{3+\gamma}(\mathbb{S})} < \varepsilon$ such that (1.1) *is solvable in $\Omega = \Omega_\rho$ with f, g defined by* (1.2). *Moreover, $\rho = \rho(g_0)$ satisfies the following:*

(i) If $g_0 \to 1$ in $h^{2+\gamma}(\mathbb{S})$, then $\rho \to 0$ in $h^{3+\gamma}(\mathbb{S})$.

(ii) There is a constant $C > 0$ such that

$$\|\rho\|_{h^{2+\gamma}(\mathbb{S})} \leq C\|g_0 - 1\|_{h^{2+\gamma}(\mathbb{S})} \tag{1.8}$$

holds for any g_0 with $\|g_0 - 1\|_{h^{2+\gamma}(\mathbb{S})} < \delta$.

In the case where $\alpha - \beta + 1 < 0$, the uniqueness in fact holds among all bounded domains Ω having C^1-boundary with $0 \in \Omega$.

Remark 1.2. Theorem 1.1 contains, as a special case, the radial symmetry of Ω for $g_0 = 1$. In fact, the global rigidity/symmetry of Ω for $\alpha - \beta + 1 < 0$ has its counterpart for the endpoint case $\alpha - \beta + 1 = 0$ (see Proposition 2.2). Moreover, we prove a global uniqueness result in this special case $\alpha - \beta + 1 = 0$, where the solvability of (1.1) is invariant under rescaling of Ω (see Proposition 5.2).

Remark 1.3. *The stability estimate* (1.8) *still holds for $h^{k+\gamma}(\mathbb{S})$ with arbitrary $k \geq 2$ in both sides. This can be easily verified, as all the succeeding arguments equally proceed with $h^{k-1+\gamma}, h^{k+\gamma}, h^{k+1+\gamma}$ instead of $h^{1+\gamma}, h^{2+\gamma}, h^{3+\gamma}$.*

The structure of the present paper is as follows. In Section 2, we shall discuss the radial symmetry of Ω in the case $g_0 = 1$ with various different techniques. In Section 3, we introduce a functional analytic formulation of (1.1) and derive the linearized problem. A detailed linear analysis is carried out by spherical harmonics. In Section 4, we derive the stability estimate (1.8) as well as the existence and local uniqueness of Ω by an implicit function theorem in [18]. Lastly, in Section 5, we study the global uniqueness of Ω when $\alpha - \beta + 1 \leq 0$.

2. Radial symmetry of domains

This section concerns the radial symmetry of Ω when it admits a solution u to (1.1) for radial data f, g. Although some of the symmetry results presented in this section are well-known or easily deduced from existing methods, we briefly discuss them so as to compare the well-known arguments with ours.

Let us begin with a variational structure of (1.1). Indeed, (1.1) is derived as the Euler-Lagrange equation of the minimization problem of the generalized torsion functional

$$J(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{(\int_\Omega uf\,dx)^2}$$

among all sets Ω of equal weighted volume

$$V(\Omega) = \int_\Omega g^2\,dx.$$

In the case where g_0 is a positive constant and

$$-\frac{n+2}{2} < \alpha \le 0 \le \beta,$$

we can show that $J(\Omega)$ is minimized when Ω is a ball centered at the origin by a rearrangement inequality as in Pólya [32]. Indeed, Sobolev's inequality implies that $J(\Omega)$ is attained by a nonnegative function $u_\Omega \in H_0^1(\Omega)$ for $\alpha > -(n+2)/2$ and thus

$$J(\Omega) = \frac{\int_\Omega |\nabla u_\Omega|^2\,dx}{\left(\int_\Omega u_\Omega f\,dx\right)^2}.$$

If we denote by Ω^* the ball centered at the origin having the same volume as Ω, and by u_Ω^* the symmetric decreasing rearrangement of u_Ω, we see that

$$J(\Omega^*) \le \frac{\int_{\Omega^*} |\nabla u_\Omega^*|^2\,dx}{\left(\int_{\Omega^*} u_\Omega^* f\,dx\right)^2} \le J(\Omega), \quad V(\Omega^*) \le V(\Omega)$$

Thus, choosing a larger ball $B \supset \Omega^*$ with $V(B) = V(\Omega)$, we have

$$J(B) \le J(\Omega^*) \le J(\Omega),$$

with equality only if Ω is a ball. We emphasize that this symmetry result only holds for the minimizer Ω, but not for every critical point Ω that admits a solution u to (1.1).

The method of moving planes can be used to deduce the radial symmetry of any bounded domain Ω having C^2-boundary in which (1.1) has a solution u with

$$-n < \alpha \le 0 \le \beta. \tag{2.1}$$

Indeed, the method is based on the comparison between the solution u and its reflection $\tilde{u}(x) := u(x')$ in the hyperplane $x_1 = \lambda$ in a maximal cap

$$\Omega_\lambda := \{x = (x_1, x_2, \ldots, x_n) \in \Omega \mid x_1 > \lambda\},$$

where $\lambda \ge 0$ is chosen to be the smallest number so that the reflected caps

$$\Omega'_\mu := \{x' = (2\mu - x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x \in \Omega_\mu\} \quad (\mu \ge \lambda)$$

are all contained in Ω. Since f is non-increasing and g is non-decreasing in the radial direction if (2.1) holds, the difference $u - \tilde{u}$ is subharmonic in Ω_λ, and Hopf's boundary lemma or its refined version by Serrin [34] derives a contradiction at a boundary point $x \in \partial\Omega_\lambda$ unless $\lambda = 0$ or u is symmetric with respect to the hyperplane $x_1 = \lambda$. The same argument with $x_1 = \lambda$ moved from the opposite side, i.e., from $\lambda = -\infty$ toward $\lambda = 0$, deduces the symmetry of Ω with respect to x_1. Hence, choosing the moving plane in every direction, we can obtain the radial symmetry of Ω.

The aforementioned arguments by Weinberger [37] and Brandolini, Nitsch, Salani and Trombetti [7] using integral quantities and algebraic inequalities apparently work only for $\alpha = \beta = 0$.

Our approach here is based on the existence of a spherical foliation of $\mathbb{R}^n \setminus \{0\}$ consisting of the boundaries of parametrized solutions $\Omega(t)$ (see [29, 30], where a similar argument was used for a different overdetermined problem). This argument only relies on the structure of spherical solutions and is irrelevant to the monotonicity of f, g; and thus our result applies even to the case $\alpha > 0$ or $\beta < 0$. Moreover, the result holds under a minimal regularity assumption on $\partial\Omega$. However, we point out that the result does not fully cover the case (2.1).

Proposition 2.1. *Let $\alpha > -n$, $\beta \in \mathbb{R}$ satisfy*

$$\alpha - \beta + 1 < 0,$$

and let Ω be a bounded domain having C^1-boundary and $0 \in \Omega$. If (1.1) has a solution $u \in C^1(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega \setminus \{0\})$ for f, g defined by (1.2) with $g_0 = 1$, then Ω must be the unit ball $\mathbb{B}$.

Proof. For $0 < t < \infty$, let us consider the parametrized overdetermined problem

$$\begin{cases} -\Delta u = (n+\alpha)|x|^\alpha & \text{in } \Omega(t), \\ u = 0 & \text{on } \partial\Omega(t), \\ -\dfrac{\partial u}{\partial \nu} = t^{\alpha-\beta+1}|x|^\beta & \text{on } \partial\Omega(t). \end{cases} \tag{2.2}$$

It is easy to check that $\Omega(t) = \mathbb{B}_t$, the ball centered at the origin with radius t, has a solution $u = u_t$ to (2.2) given by

$$u_t := \frac{t^{\alpha+2} - |x|^{\alpha+2}}{\alpha+2}.$$

Now let us suppose that there is a bounded domain Ω admitting a solution $u \in C^2(\overline{\Omega} \setminus \{0\})$ to (1.1) with $g_0 = 1$ and $0 \in \Omega$. We choose the largest number $t_* > 0$ and the smallest number $t^* > 0$ such that

$$\mathbb{B}_{t_*} \subset \Omega \subset \mathbb{B}_{t^*}. \tag{2.3}$$

We will show by contradiction that $t_* \geq 1$ and $t^* \leq 1$; and thus $\Omega = \mathbb{B}$. Let us suppose $t_* < 1$ and take a point $x_0 \in \partial\Omega \cap \partial\mathbb{B}_{t_*}$. Then, $w := u - u_{t_*}$ satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{B}_{t_*}, \\ w = u \geq 0 & \text{on } \partial\mathbb{B}_{t_*}, \\ w(x_0) = 0 = \min\limits_{x \in \overline{\mathbb{B}_{t_*}}} w(x). \end{cases} \tag{2.4}$$

Hence we arrive at a contradiction as

$$0 \geq \frac{\partial w}{\partial \nu}(x_0) = \frac{\partial u}{\partial \nu}(x_0) - \frac{\partial u_{t_*}}{\partial \nu}(x_0) = -(t_*)^\beta + (t_*)^{\alpha+1} > 0.$$

Similarly, $t^* > 1$ leads to a contradiction by considering $w := u - u_{t^*}$ in Ω. □

We also obtain a symmetry result in the endpoint case $\alpha - \beta + 1 = 0$. In this particular case, $\mathbb{B}_t$ for arbitrary radius $t > 0$ allows the solvability of (1.1) for $g_0 = 1$. Indeed, u_t solves (2.2) in $\Omega(t) = \mathbb{B}_t$ with $t^{\alpha-\beta+1} = 1$ for any $t > 0$.

Proposition 2.2. *Let $\alpha > -n$, $\beta \in \mathbb{R}$ satisfy*

$$\alpha - \beta + 1 = 0,$$

and let Ω be a bounded domain having C^1-boundary and $0 \in \Omega$. If (1.1) *has a solution $u \in C^1(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega \setminus \{0\})$ for f, g defined by* (1.2) *with $g_0 = 1$, then Ω must be a ball centered at the origin.*

Proof. The proof proceeds similarly as before, except that in the inclusion (2.3) we will only prove that $t_* = t^*$. If this is not true, then $\mathbb{B}_{t_*} \subsetneq \Omega$ and $w := u - u_{t_*}$ satisfies (2.4) and $w > 0$ in $\mathbb{B}_{t_*}$ by the strong maximum principle. Hence by Hopf's lemma (used in $\mathbb{B}_{t_*}$) we arrive at a contradiction as

$$0 > \frac{\partial w}{\partial \nu}(x_0) = \frac{\partial u}{\partial \nu}(x_0) - \frac{\partial u_{t_*}}{\partial \nu}(x_0) = -(t_*)^\beta + (t_*)^{\alpha+1} = 0.$$

Thus $B_{t_*} = \Omega$ as desired. □

In the proofs above, the crucial step for the radial symmetry of Ω is the construction of a foliated family of domains $\Omega(t)$. This technique will be used in Section 5 to prove the uniqueness of Ω with a different foliation.

3. Linearized problem

In this section, we will first derive a functional analytic formulation of the problem of finding Ω for a prescribed g such that (1.1) is solvable, as presented in [18]. For this purpose, throughout this section we fix $\alpha > -n, \beta \in \mathbb{R}$ and $0 < \gamma < 1$, and set

$$\mathcal{U}_\delta^{k+\gamma} = \left\{\rho \in h^{k+\gamma}(\mathbb{S}) \mid \|\rho\|_{h^{k+\gamma}(\mathbb{S})} < \delta\right\}$$

for $\delta > 0$ and $k \in \mathbb{N}$. If $\delta > 0$ is sufficiently small, for $\rho \in \mathcal{U}_\delta^{2+\gamma}$, we may define the domain Ω_ρ by (1.4) and a diffeomorphism $\theta_\rho \in h^{2+\alpha}(\overline{\mathbb{B}}, \overline{\Omega_\rho})$ by

$$\theta_\rho(x) = \begin{cases} x + \eta\left(|x| - 1\right)\rho\left(\dfrac{x}{|x|}\right)\dfrac{x}{|x|} & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$

where $\eta \in C^\infty(\mathbb{R})$ is a cut-off function satisfying $0 \leq \eta \leq 1$, $|\eta'| \leq 4$, $\eta(r) = 1$ for $|r| \leq 1/4$, and $\eta(r) = 0$ for $|r| \geq 3/4$. This induces the pullback and pushforward isomorphisms $\theta_\rho^* \in \mathrm{Isom}(h^{k+\gamma}(\overline{\Omega_\rho}), h^{k+\gamma}(\overline{\mathbb{B}}))$,

$\theta_*^\rho \in \mathrm{Isom}(h^{k+\gamma}(\overline{\mathbb{B}}), h^{k+\gamma}(\overline{\Omega_\rho}))$ for $0 \le k \le 2$, as well as the corresponding boundary isomorphisms, defined by

$$\theta_\rho^* u = u \circ \theta_\rho, \quad \theta_*^\rho u = u \circ \theta_\rho^{-1}.$$

For each $\rho \in \mathcal{U}_\delta^{2+\gamma}$, the Dirichlet problem consisting in the first two equations in (1.1) has a unique solution $u_\rho \in h^{2+\gamma}(\overline{\Omega_\rho})$ by the Schauder theory. Consequently, we can define the mapping $F \in C(\mathcal{U}_\delta^{2+\gamma} \times h^{1+\gamma}(\mathbb{S}), h^{1+\gamma}(\mathbb{S}))$ by

$$F(\rho, g_0) = \theta_\rho^* \left[\frac{\partial u_\rho}{\partial \nu_\rho} + g_0 |x|^\beta \right], \tag{3.1}$$

where $\nu_\rho \in h^{1+\gamma}(\partial\Omega_\rho, \mathbb{R}^n)$ is the unit outer normal vector field on $\partial\Omega_\rho$. Thus, for a given $g_0 \in h^{1+\gamma}(\mathbb{S})$, our problem reduces to finding a solution $\rho \in \mathcal{U}_\delta^{2+\gamma}$ to

$$F(\rho, g_0) = 0.$$

Indeed, for such a ρ, u_ρ additionally satisfies the Neumann boundary condition in (1.1). In terms of these notations, the spherical solution $\Omega = \mathbb{B}$ for $g_0 = 1$ corresponds to $F(0, 1) = 0$.

In order to construct a solution ρ for $g_0 \neq 1$ by an implicit function theorem, we will differentiate (3.1) with respect to ρ. At this point, we encounter a regularity issue, that is, we need to impose the higher regularity assumption $\rho \in h^{3+\gamma}(\mathbb{S})$ for the differentiability of F as stated in the following lemma. Note that $u_\rho \in h^{3+\gamma}(\overline{\Omega_\rho})$ under this assumption. Here, we shall use the notation

$$N_\rho(x) = |x| - 1 - \rho\left(\frac{x}{|x|}\right) \quad (x \neq 0), \tag{3.2}$$

by which ν_ρ and the normal and tangential components $\mu_\rho, \tau_\rho \in h^{2+\gamma}(\partial\Omega_\rho, \mathbb{R}^n)$ of the vector field $\theta_*^\rho \nu_0$ on $\partial\Omega_\rho$ are represented by

$$\nu_\rho = \frac{\nabla N_\rho}{|\nabla N_\rho|}, \quad \mu_\rho = \frac{\nu_\rho}{|\nabla N_\rho|}, \quad \tau_\rho = \frac{x}{|x|} - \frac{\nu_\rho}{|\nabla N_\rho|}, \tag{3.3}$$

where we have used $\theta_*^\rho \nu_0(x) = \nu_0(\xi) = \xi = x/|x|$ for $x = \xi + \rho(\xi)\xi \in \partial\Omega_\rho$.

Lemma 3.1. *For sufficiently small $\delta > 0$, we have*

$$F \in C^1(\mathcal{U}_\delta^{3+\gamma} \times h^{1+\gamma}(\mathbb{S}), h^{1+\gamma}(\mathbb{S})) \cap C(\mathcal{U}_\delta^{2+\gamma} \times h^{1+\gamma}(\mathbb{S}), h^{1+\gamma}(\mathbb{S})),$$

and the following hold:

(i) The Fréchet derivative of F with respect to ρ is given by, for $\tilde\rho \in h^{3+\gamma}(\mathbb{S})$,

$$\partial_\rho F(\rho, g_0)\,[\tilde\rho] = \theta_\rho^* \left[H_{\partial\Omega_\rho} p + \frac{\partial p}{\partial \nu_\rho} - \frac{f \theta_*^\rho \tilde\rho}{|\nabla N_\rho|} + \frac{\partial^2 u_\rho}{\partial \tau_\rho \partial \nu_\rho} \theta_*^\rho \tilde\rho \right] + \beta(1+\rho)^{\beta-1} g_0 \tilde\rho \tag{3.4}$$

where $p \in h^{2+\gamma}(\overline{\Omega_\rho})$ is the unique solution to

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega_\rho, \\ p = -\dfrac{\partial u_\rho}{\partial \nu_\rho} \dfrac{\theta_*^\rho \tilde\rho}{|\nabla N_\rho|} & \text{on } \partial\Omega_\rho, \end{cases} \tag{3.5}$$

and $H_{\partial\Omega_\rho} \in h^{1+\gamma}(\partial\Omega_\rho)$ is the mean curvature of $\partial\Omega_\rho$ normalized in such a way that $H_{\partial\mathbb{B}} = n - 1$.

(ii) $\partial_\rho F(\rho, g_0)$ *has a continuous extension in* $\mathcal{L}(h^{2+\gamma}(\mathbb{S}), h^{1+\gamma}(\mathbb{S}))$ *and*

$$\partial_\rho F \in C^\omega(\mathcal{U}_\delta^{3+\gamma} \times h^{1+\gamma}(\mathbb{S}), \mathcal{L}(h^{2+\gamma}(\mathbb{S}), h^{1+\gamma}(\mathbb{S}))).$$

Proof. We give a sketch of proof and refer to [18, Lemma 2.1] for details. Let us first derive the formula (3.4) in the simplest case where $\rho = 0$. To this end, for $\tilde{\rho} \in h^{3+\alpha}(\mathbb{S})$ and small $\varepsilon \in \mathbb{R}$, we substitute $\rho = \varepsilon\tilde{\rho}$ and the formal expansion $u_\rho = u_0 + \varepsilon p + o(\varepsilon)$ into

$$\begin{cases} -\Delta u_\rho = f & \text{in } \Omega_\rho, \\ \quad u_\rho = 0 & \text{on } \partial\Omega_\rho \end{cases}$$

and use the corresponding equation for u_0 to obtain

$$\begin{aligned} f(x) = -\Delta u_\rho(x) &= f(x) - \varepsilon\Delta p(x) + o(\varepsilon) && \text{for } x \in \Omega_0 = \mathbb{B}, \\ 0 = u_\rho(x + \varepsilon\tilde{\rho}\nu_0) &= \varepsilon\frac{\partial u_0}{\partial \nu_0}(x)\tilde{\rho} + \varepsilon p(x) + o(\varepsilon) && \text{for } x \in \partial\Omega_0 = \mathbb{S}. \end{aligned}$$

Thus, letting $\varepsilon \to 0$, we see that p satisfies (3.5) for $\rho = 0$. Moreover, for $x \in \mathbb{S}$,

$$\begin{aligned} F(\varepsilon\tilde{\rho}, g_0)(x) &= \frac{\partial u_\rho}{\partial \nu_\rho}(x + \varepsilon\tilde{\rho}\nu_0) + g_0(x)(1 + \varepsilon\tilde{\rho})^\beta \\ &= \frac{\partial u_\rho}{\partial \nu_0}(x + \varepsilon\tilde{\rho}\nu_0) + \varepsilon\frac{\partial u_\rho}{\partial \tau}(x + \varepsilon\tilde{\rho}\nu_0) + g_0(x)\,(1 + \varepsilon\beta\tilde{\rho}) + o(\varepsilon) \\ &= F(0, g_0)(x) + \varepsilon\frac{\partial^2 u_0}{\partial {\nu_0}^2}(x)\tilde{\rho} + \varepsilon\frac{\partial p}{\partial \nu_0}(x) + \varepsilon\frac{\partial u_0}{\partial \tau}(x) + \varepsilon\beta g_0(x)\tilde{\rho} + o(\varepsilon) \\ &= F(0, g_0)(x) - \varepsilon f(x)\tilde{\rho} + \varepsilon H_{\partial\Omega_0} p(x) + \varepsilon\frac{\partial p}{\partial \nu_0}(x) + \varepsilon\beta g_0(x)\tilde{\rho} + o(\varepsilon), \end{aligned}$$

where we used the fact that ν_ρ and $\partial^2 u_0/\partial\nu_0^2$ can be represented by a tangent vector τ to $\mathbb{S}$ and the Laplace-Beltrami operator $\Delta_{\mathbb{S}}$ on $\mathbb{S}$ as

$$\begin{aligned} \nu_\rho &= \nu_0 + \varepsilon\tau + o(\varepsilon), \\ \frac{\partial^2 u_0}{\partial {\nu_0}^2}(x) &= \Delta u_0(x) - \Delta_{\mathbb{S}} u_0(x) - H_{\mathbb{S}}\frac{\partial u_0}{\partial \nu_0}(x) = -f(x) - H_{\mathbb{S}}\frac{\partial u_0}{\partial \nu_0}(x). \end{aligned}$$

This shows (i) for $\rho = 0$ by letting $\varepsilon \to 0$. For general $\rho \neq 0$, we use the same argument as above with the reference domain Ω_ρ instead of $\Omega_0 = \mathbb{B}$. In particular, every occurrence of $\tilde{\rho}$ must now be replaced by $\theta_*^\rho\tilde{\rho}/|\nabla N_\rho|$ and the extra term $\varepsilon(\theta_*^\rho\tilde{\rho})\partial_{\tau_\rho}\partial_{\nu_\rho}u_\rho$ appears in the expansion of $F(\rho + \varepsilon\tilde{\rho}\nu_0, g)$, since for $x \in \mathbb{S}$

$$x + (\rho(x) + \varepsilon\tilde{\rho}(x))\nu_0 = \theta_\rho(x) + \varepsilon\tilde{\rho}(x)\left\{\mu_\rho(\theta_\rho(x)) + \tau_\rho(\theta_\rho(x))\right\}.$$

For (ii), we observe that the formula (3.4) with (3.5) still makes sense for $\rho \in \mathcal{U}_\delta^{3+\gamma}$ and $\tilde{\rho} \in h^{2+\gamma}(\mathbb{S})$; and the extension in (ii) is thus defined. Finally, the analyticity of $\partial_\rho F$ follows from that of $\mathcal{U}_\delta^{3+\gamma} \ni \rho \mapsto \theta_\rho^* H_{\partial\Omega_\rho} \in h^{1+\gamma}(\mathbb{S})$, $\theta_\rho^* u_\rho \in h^{3+\gamma}(\overline{\mathbb{B}})$ and $\theta_\rho^* p \in h^{2+\gamma}(\overline{\mathbb{B}})$. □

Remark 3.2. The required higher regularity $\rho \in \mathcal{U}_\delta^{3+\gamma}$ is adequate in view of the formula in (i), since, if $\rho \in h^{k+\gamma}(\mathbb{S})$, then at most $H_{\partial\Omega_\rho} \in h^{k-2+\gamma}(\partial\Omega_\rho)$, $u_\rho \in h^{k+\gamma}(\overline{\Omega_\rho})$ and $p \in h^{k-1+\gamma}(\overline{\Omega_\rho})$.

As stated in the next lemma, the representation formula (3.4) of the Fréchet derivative of F in Lemma 3.1 yields a characterization of the invertibility of the extended operator $\partial_\rho F(\rho, g_0)$ in terms of the elliptic boundary value problem

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega_\rho, \\ \left(H_{\partial\Omega_\rho} - \dfrac{f}{g} + \dfrac{1}{g}\dfrac{\partial g}{\partial \nu_\rho}\right) p + \dfrac{\partial p}{\partial \nu_\rho} = \varphi & \text{on } \partial\Omega_\rho. \end{cases} \tag{3.6}$$

Note that $F(\rho, g_0) = 0$ implies that $g = -\partial_{\nu_\rho} u_\rho \in h^{2+\gamma}(\partial\Omega_\rho)$ and $g > 0$ on $\partial\Omega_\rho$, where the latter follows from the maximum principle.

Lemma 3.3. *Suppose that $\rho \in \mathcal{U}_\delta^{3+\gamma}$ and $g_0 \in h^{2+\gamma}(\mathbb{S})$ satisfy $F(\rho, g_0) = 0$. Then, the inverse*

$$\partial_\rho F(\rho, g_0)^{-1} \in \mathcal{L}(h^{1+\gamma}(\mathbb{S}), h^{2+\gamma}(\mathbb{S}))$$

exists if and only if (3.6) *has a unique solution $p \in h^{2+\gamma}(\overline{\Omega_\rho})$ for $\varphi \in h^{1+\gamma}(\partial\Omega_\rho)$. Furthermore, the inverse is then given by*

$$\partial_\rho F(\rho, g_0)^{-1}\left[\theta_\rho^* \varphi\right] = \theta_\rho^*\left[\frac{p|\nabla N_\rho|}{g}\right]. \tag{3.7}$$

Proof. By assumption, $-\partial_{\nu_\rho} u_\rho = g$ on $\partial\Omega_\rho$. Moreover, in view of (1.2) and (3.3),

$$\theta_\rho^* \frac{\partial^2 u_\rho}{\partial \tau_\rho \partial \nu_\rho} = -\theta_\rho^* \frac{\partial g}{\partial \tau_\rho} = -\beta(1+\rho)^{\beta-1} g_0 + \theta_\rho^*\left[\frac{1}{|\nabla N_\rho|}\frac{\partial g}{\partial \nu_\rho}\right] \quad \text{on } \mathbb{S}.$$

Hence, the boundary condition in (3.5) becomes

$$\theta_*^\rho \tilde{\rho} = \frac{p|\nabla N_\rho|}{g} \quad \text{on } \partial\Omega_\rho, \tag{3.8}$$

and the remaining condition in (3.5) and (3.4) are

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega_\rho, \\ \left(H_{\partial\Omega_\rho} - \dfrac{f}{g} + \dfrac{1}{g}\dfrac{\partial g}{\partial \nu_\rho}\right) p + \dfrac{\partial p}{\partial \nu_\rho} = \theta_*^\rho \partial_\rho F(\rho, g_0)\,[\tilde{\rho}] & \text{on } \partial\Omega_\rho. \end{cases}$$

Since $\theta_\rho^*, \theta_*^\rho$ are isomorphisms and (3.8) yields a one-to-one correspondence between $\tilde{\rho}$ and p, the invertibility of $\partial_\rho F(\rho, g_0) \in \mathcal{L}(h^{2+\gamma}(\mathbb{S}), h^{1+\gamma}(\mathbb{S}))$ is equivalent to the unique existence of a solution $p \in h^{2+\gamma}(\overline{\Omega_\rho})$ to (3.6) for any given boundary data $\varphi \in h^{1+\gamma}(\partial\Omega_\rho)$. The formula (3.7) follows from the above equations. □

A well-known sufficient condition (see Gilbarg and Trudinger [17, Theorem 6.31]) for the unique solvability of (3.6) is

$$H_{\partial\Omega_\rho} - \frac{f}{g} + \frac{1}{g}\frac{\partial g}{\partial \nu_\rho} > 0 \quad \text{on } \partial\Omega_\rho.$$

In particular, for $\rho = 0$ and $g_0 = 1$, this positivity condition is nothing but

$$\alpha - \beta + 1 < 0.$$

In fact, we can classify all the values of α, β for which (3.6) is uniquely solvable by virtue of spherical harmonics. To this end, we recall some basic properties of spherical harmonics. Let us denote by H_l the vector space of all homogeneous harmonic polynomials of degree $l \in \mathbb{N} \cup \{0\}$ on $\mathbb{R}^n$. The dimension $d_l^{(n)}$ of H_l is given by $d_0^{(n)} = 1$, $d_1^{(n)} = n$ and

$$d_l^{(n)} = \binom{l+n-1}{l} + \binom{l+n-3}{l-2} \qquad (l \geq 2).$$

If we regard H_l as a subspace of $L^2(\mathbb{S})$ and choose an orthonormal basis

$$\{h_{l,1}, h_{l,2}, \dots, h_{l,d_l^{(n)}}\} \subset H_l,$$

then it is known that

$$\mathcal{B} = \bigcup_{l=0}^{\infty} \{h_{l,1}, h_{l,2}, \dots, h_{l,d_l^{(n)}}\}$$

forms a complete orthonormal system of $L^2(\mathbb{S})$. In particular, $u \in h^{k+\gamma}(\mathbb{S})$ can be expressed by its Fourier series in $L^2(\mathbb{S})$ as

$$u = \sum_{l=0}^{\infty} \sum_{m=1}^{d_l^{(n)}} \hat{u}_{l,m} h_{l,m}.$$

Moreover, if $u \in C^\infty(\mathbb{S})$, the coefficients $\hat{u}_{l,m}$ are rapidly decaying so that the series on the right hand side converges in the norm in $C^{k+\gamma}(\mathbb{S})$. Thus, the linear subspace spanned by $\mathcal{B}$ is dense in $h^{k+\gamma}(\mathbb{S})$. We also note that $h_{l,m}$ satisfies

$$\frac{\partial h_{l,m}}{\partial \nu} = x \cdot \nabla h_{l,m} = l h_{l,m} \quad \text{on } \mathbb{S}.$$

In other words, the Dirichlet-to-Neumann operator $\mathcal{N} \in \mathcal{L}(h^{k+\gamma}(\mathbb{S}), h^{k-1+\gamma}(\mathbb{S}))$, for $k \geq 2$, defined by

$$\mathcal{N}\varphi = \frac{\partial v}{\partial \nu},$$

where $v \in h^{k+\gamma}(\overline{\mathbb{B}})$ is the unique solution to the Dirichlet problem

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{B}, \\ \quad v = \varphi & \text{on } \mathbb{S}, \end{cases}$$

satisfies

$$\mathcal{N} h_{l,m} = l h_{l,m}.$$

Lemma 3.4. *Let $\rho = 0$, $g_0 = 1$, $\alpha > -n$ and $\beta \in \mathbb{R}$. The boundary value problem* (3.6) *has a unique solution $p \in h^{2+\gamma}(\overline{\mathbb{B}})$ for any $\varphi \in h^{1+\gamma}(\mathbb{S})$ if and only if*

$$\alpha - \beta + 1 \notin \mathbb{N} \cup \{0\}. \tag{3.9}$$

Proof. The boundary condition in (3.6) can be written as

$$(-1 - \alpha + \beta)\, p + \frac{\partial p}{\partial \nu} = \varphi.$$

Hence, (3.6) is uniquely solvable if and only if

$$(-1-\alpha+\beta)I+\mathcal{N}\in\mathcal{L}(h^{2+\gamma}(\mathbb{S}),h^{1+\gamma}(\mathbb{S})) \tag{3.10}$$

is invertible. As remarked earlier, the latter holds if $-1-\alpha+\beta>0$; and in particular $(I+\mathcal{N})^{-1}\in\mathcal{L}(h^{1+\gamma}(\mathbb{S}),h^{2+\gamma}(\mathbb{S}))$ exists. Thus, by the Fredholm theory (see e.g., [17, Theorem 5.3]) applied to

$$((-1-\alpha+\beta)I+\mathcal{N})\,(I+\mathcal{N})^{-1}=I+(-2-\alpha+\beta)(I+\mathcal{N})^{-1},$$

where $(I+\mathcal{N})^{-1}$ is compact as a mapping from $h^{1+\gamma}(\mathbb{S})$ to itself, the range of (3.10) is closed and the invertibility follows from the surjectivity of (3.10). Now, since

$$(-1-\alpha+\beta)h_{l,m}+\mathcal{N}h_{l,m}=(l-1-\alpha+\beta)h_{l,m},$$

the condition (3.9) implies that the range of (3.10) contains the linear span of $\mathcal{B}$ and hence its closure $h^{1+\gamma}(\mathbb{S})$. On the other hand, if there is an $l\in\mathbb{N}\cup\{0\}$ such that $\alpha-\beta+1=l$, then obviously (3.10) is not injective from the above computation. □

4. Quantitative stability estimates

Our goal in this section is to derive the existence, uniqueness and regularity of a solution ρ to the nonlinear equation $F(\rho,g_0)=0$, based on the linear analysis in the previous section. Lemmas 3.3 and 3.4 show that the linearized operator $\partial_\rho F(0,1)$ has the inverse

$$\partial_\rho F(0,1)^{-1}\in\mathcal{L}(h^{1+\gamma}(\mathbb{S}),h^{2+\gamma}(\mathbb{S})) \tag{4.1}$$

as long as (3.9) holds. This indicates that $F(\rho,g_0)=0$ can be locally solved and the solution map $g_0\mapsto\rho$ is differentiable.

However, a classical perturbation method generally fails for our nonlinear equation defined by (3.1). Indeed, in contrast to $F(\cdot,g_0)\in C^1(\mathcal{U}_\delta^{3+\gamma},h^{1+\gamma}(\mathbb{S}))$, the inverse (4.1) only recovers a partial regularity that is not sufficient for a successive approximation of the form

$$\rho_{j+1}=\rho_j-\partial_\rho F(0,1)^{-1}F(\rho_j,g_0),\quad \rho_0=0$$

to converge, since $\rho_j\in h^{3+\gamma}(\mathbb{S})$ only results in $\rho_{j+1}\in h^{2+\gamma}(\mathbb{S})$. This regularity deficit called the loss of derivatives can be circumvented by the following implicit function theorem introduced by Gilsbach and the author [18, Theorem 4.2]. It requires some additional regularity assumptions on F at the single point $(\rho,g_0)=(0,1)$, but provides the existence, local uniqueness and differentiability of $g_0\mapsto\rho$.

Proposition 4.1. *Let $X_2\subset X_1\subset X_0$, $Z_2\subset Z_1\subset Z_0$ and Y be Banach spaces with inclusions being continuous embeddings, and let D_j be a neighborhood of a point $(x_0,y_0)\in X_2\times Y$ in $X_j\times Y$ with $D_2\subset D_1\subset D_0$ for $j=0,1,2$. Suppose that*

$$F\in C^1(D_2,Z_1)\cap C(D_1,Z_1)\cap C^1(D_1,Z_0)\cap C(D_0,Z_0) \tag{4.2}$$

satisfies the following conditions:

(a) $F(x_0, y_0) = 0$;

(b) $\partial_x F(x, y) \in \mathcal{L}(X_2, Z_1) \cap \mathcal{L}(X_1, Z_0)$ *has an extension with*

$$\partial_x F \in C(D_2, \mathcal{L}(X_1, Z_1)) \cap C(D_1, \mathcal{L}(X_0, X_0));$$

(c) F *is differentiable at* (x_0, y_0) *as a mapping from* D_j *to* Z_j *for* $j = 1, 2$;

(d) $\partial_x F(x_0, y_0)^{-1} \in \mathcal{L}(Z_j, X_j)$ *exists for* $j = 0, 1, 2$.

Then, there are a neighborhood U_1 *of* $x_0 \in X_1$, *a neighborhood* V *of* $y_0 \in Y$ *and a mapping* $\upsilon : V \to U_1$ *satisfying*

(i) $F(\upsilon(y), y) = 0$ *for all* $y \in V$;

(ii) $\upsilon(y_0) = x_0$ *and* $\upsilon(y) \to x_0$ *in* X_1 *as* $y \to y_0$ *in* Y;

(iii) $F(x, y) = 0$, $x \in U_1$ *and* $y \in V$ *imply that* $x = \upsilon(y)$;

(iv) $\upsilon \in C^1(V, X_0)$ *and*

$$\upsilon'(y) = -\partial_x F(\upsilon(y), y)^{-1} \partial_y F(\upsilon(y), y) \in \mathcal{L}(Y, X_0).$$

Remark 4.2. The continuity in (ii) is not explicitly stated in [18, Theorem 4.2]. But it is clear from the proofs of [18, Theorems 4.1 and 4.2]: choose an arbitrarily small neighborhood $U_1' \subset U_1$ of x_0 and then take a small $V' \subset V$ accordingly and use the contraction mapping principle to get $\upsilon(y) \in U_1'$ for $y \in V'$.

This proposition enables us to handle nonlinear problems having a particular type of loss of derivatives specified in the conditions (b) and (d). The assumption (c) can be regarded as no loss of derivatives occurring at the single point (x_0, y_0).

In order to apply Proposition 4.1 to our problem with F defined by (3.1), we set, for $j = 0, 1, 2$,

$$X_j = h^{j+2+\gamma}(\mathbb{S}), \quad Z_j = h^{j+1+\gamma}(\mathbb{S}), \quad Y = h^{2+\gamma}(\mathbb{S}), \quad D_j = \mathcal{U}_\delta^{j+2+\gamma} \times Y.$$

As in Lemma 3.1, we have

$$\begin{aligned} F \in & C^1(\mathcal{U}_\delta^{4+\gamma} \times h^{2+\gamma}(\mathbb{S}), h^{2+\gamma}(\mathbb{S})) \cap C(\mathcal{U}_\delta^{3+\gamma} \times h^{2+\gamma}(\mathbb{S}), h^{2+\gamma}(\mathbb{S})) \\ & \cap C^1(\mathcal{U}_\delta^{3+\gamma} \times h^{1+\gamma}(\mathbb{S}), h^{1+\gamma}(\mathbb{S})) \cap C(\mathcal{U}_\delta^{2+\gamma} \times h^{1+\gamma}(\mathbb{S}), h^{1+\gamma}(\mathbb{S})). \end{aligned}$$

This implies that F meets the regularity assumption (4.2). Similarly, (a), (b) are easily confirmed with $(x_0, y_0) = (0, 1)$. Now we use Lemma 3.4 and its variant in higher regular spaces to conclude that the non-degeneracy condition (d) is satisfied if (3.9) holds. For the remaining condition (c), we recall that the loss of derivatives is caused by the regularity of the mean curvature $H_{\partial\Omega_\rho}$ (see Remark 3.2) and by several other terms in (3.4). However, at $(\rho, g_0) = (0, 1)$ where $\partial\Omega_0 = \mathbb{S}$, non-smooth terms vanish and we have

$$\partial_\rho F(0, 1)[\tilde{\rho}] = (n-1)p + \frac{\partial p}{\partial \nu} - (n+\alpha)\tilde{\rho} + \beta\tilde{\rho}$$

and p is as smooth as $\tilde{\rho}$ by (3.5). Thus one can check that $F \in C(D_j, Z_j)$, with the image space having the stronger topology, is still differentiable at $(\rho, g_0) = (0, 1)$ for $j = 1, 2$. As a conclusion, we obtain open sets $U_1 \subset h^{3+\gamma}(\mathbb{S})$ and $V \subset h^{2+\gamma}(\mathbb{S})$ with $(0, 1) \in U_1 \times V$ and a solution map $\rho : V \to U_1$ such that

(i) $F(\rho(g_0), g_0) = 0$ for all $g_0 \in h^{2+\gamma}(\mathbb{S})$;

(ii) $\rho(1) = 0$ and $\rho(g_0) \to 0$ in $h^{3+\gamma}(\mathbb{S})$ as $g_0 \to 1$ in $h^{2+\gamma}(\mathbb{S})$;

(iii) $F(\tilde{\rho}, g_0) = 0$, $\tilde{\rho} \in U_1$ and $g_0 \in V$ imply that $\tilde{\rho} = \rho(g_0)$.

Moreover, $\rho \in C^1(V, h^{2+\gamma}(\mathbb{S}))$ and hence the linear stability estimate

$$\|\rho(g_0)\|_{h^{2+\gamma}(\mathbb{S})} \leq C\|g_0 - 1\|_{h^{2+\gamma}(\mathbb{S})}$$

holds for any $g_0 \in V$ in a small neighborhood of $1 \in h^{2+\gamma}(\mathbb{S})$. This concludes the proof of Theorem 1.1, except the last assertion on the global uniqueness of Ω.

5. Global uniqueness by foliations

Our remaining task is to prove that Ω_ρ constructed in the previous section is the only possible bounded domain having a solution u to (1.1) in the case where

$$\alpha - \beta + 1 < 0.$$

The technique we employ is based on the construction of a foliation of $\mathbb{R}^n \setminus \{0\}$ by the boundaries of particular solutions $\Omega(t)$ to (1.1), as in the proof of Propositions 2.1 and 2.2. However, for non-constant g_0, the spherical foliation is no longer suitable and we need to construct a non-spherical one. We rely on an argument used by Bianchini, Henrot and Salani [6, Theorem 3.4], where the uniqueness is proved for $\alpha = 0$ by constructing a non-spherical foliation as the boundaries of the rescaled family

$$t\Omega := \{tx \in \mathbb{R}^n \mid x \in \Omega\}.$$

The following proposition is a generalization of [6, Theorem 3.4] to the case of arbitrary $\alpha > -n$, which completes the proof of Theorem 1.1.

Proposition 5.1. *Let $\alpha > -n$, $\beta \in \mathbb{R}$ and $g_0 \in C(\mathbb{S})$ satisfy*

$$\alpha - \beta + 1 < 0, \quad g_0(\xi) > 0,$$

and let Ω and $\tilde{\Omega}$ be bounded domains having C^1-boundaries and $0 \in \Omega \cap \tilde{\Omega}$. Suppose that (1.1) *in Ω and $\tilde{\Omega}$ respectively admit solutions*

$$u \in C^1(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega \setminus \{0\}),$$
$$\tilde{u} \in C^1(\overline{\tilde{\Omega}} \setminus \{0\}) \cap C^2(\tilde{\Omega} \setminus \{0\})$$

with f, g defined by (1.2)*. Then $\Omega = \tilde{\Omega}$.*

Proof. The proof is similar to that of Proposition 2.1 with $\Omega(t) = t\Omega$ for $0 < t < \infty$. It is easy to see that the parametrized overdetermined problem

$$\begin{cases} -\Delta u = (n + \alpha)|x|^\alpha & \text{in } \Omega(t), \\ \quad u = 0 & \text{on } \partial\Omega(t), \\ -\dfrac{\partial u}{\partial \nu} = t^{\alpha-\beta+1} g_0(\xi)|x|^\beta & \text{on } \partial\Omega(t). \end{cases}$$

has a solution

$$u_t(x) := t^{\alpha+2} u\left(\frac{x}{t}\right) \quad (x \in \Omega(t)).$$

As in the proof of Proposition 2.1, we can choose the largest number $t_* > 0$ and the smallest number $t^* > 0$ such that

$$\Omega(t_*) \subset \tilde{\Omega} \subset \Omega(t^*),$$

and prove that $t_* \geq 1$ and $t^* \leq 1$ and hence $\tilde{\Omega} = \Omega(1) = \Omega$ by contradiction. Indeed, if $t_* < 1$, we take a point $x_0 \in \partial\tilde{\Omega} \cap \partial\Omega(t_*)$ and observe that $w := \tilde{u} - u_{t_*}$ satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega(t_*), \\ \quad w = \tilde{u} \geq 0 & \text{on } \partial\Omega(t_*), \\ w(x_0) = 0 = \min\limits_{x \in \overline{\Omega(t_*)}} w(x). \end{cases} \tag{5.1}$$

Hence we arrive at a contradiction as

$$\begin{aligned} 0 \geq \frac{\partial w}{\partial \nu}(x_0) &= \frac{\partial \tilde{u}}{\partial \nu}(x_0) - \frac{\partial u_{t_*}}{\partial \nu}(x_0) \\ &= \left\{-1 + (t_*)^{\alpha-\beta+1}\right\} g_0\left(\frac{x_0}{|x_0|}\right)|x_0|^\beta > 0. \end{aligned}$$

Similarly, $t^* > 1$ leads to a contradiction by considering $w := \tilde{u} - u_{t^*}$ in $\tilde{\Omega}$. □

In the endpoint case $\alpha - \beta + 1 = 0$, the existence of Ω for $g_0 \neq 1$ is not guaranteed due to the degeneracy of $\partial_\rho F(0, 1)$. However, we can prove the uniqueness of Ω up to dilation as in Proposition 2.2.

We say that Ω satisfies the interior sphere condition if for any point $x \in \partial\Omega$ there is a ball $B \subset \Omega$ such that $x \in \partial B$. In particular, this condition is fulfilled if $\partial\Omega$ is of class C^2. The interior sphere condition allows us to use Hopf's lemma.

Proposition 5.2. *Let $\alpha > -n$, $\beta \in \mathbb{R}$ and $g_0 \in C(\mathbb{S})$ satisfy*

$$\alpha - \beta + 1 = 0, \quad g_0(\xi) > 0,$$

and let Ω and $\tilde{\Omega}$ be bounded domains having C^1-boundaries and $0 \in \Omega \cap \tilde{\Omega}$, and moreover suppose that Ω or $\tilde{\Omega}$ satisfies the interior sphere condition. If (1.1) *in Ω and $\tilde{\Omega}$ respectively admit solutions*

$$u \in C^1(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega \setminus \{0\}),$$
$$\tilde{u} \in C^1(\overline{\tilde{\Omega}} \setminus \{0\}) \cap C^2(\tilde{\Omega} \setminus \{0\})$$

with f, g defined by (1.2)*, then $\tilde{\Omega} = t\Omega$ for some $t > 0$.*

Proof. We may suppose that Ω satisfies the interior sphere condition. The same argument as in the proof of Proposition 5.1 yields $\Omega(t_*) \subset \tilde{\Omega}$ with a common boundary point $x_0 \in \partial\tilde{\Omega} \cap \partial\Omega(t_*)$. If $\Omega(t_*) \neq \tilde{\Omega}$, then $w := \tilde{u} - u_{t_*}$ satisfies (5.1) and $w > 0$ in $\Omega(t_*)$ by the strong maximum principle. Hence Hopf's lemma derives a contradiction as

$$0 > \frac{\partial w}{\partial \nu}(x_0) = \frac{\partial \tilde{u}}{\partial \nu}(x_0) - \frac{\partial u_{t_*}}{\partial \nu}(x_0) = 0.$$

Thus $\Omega(t_*) = \tilde{\Omega}$ as desired. □

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Conflict of interest

The author declares no conflict of interest.

References

1. A. Acker, Interior free boundary problems for the Laplace equation, *Arch. Rational Mech. Anal.*, **75** (1981), 157–168. https://doi.org/10.1007/BF00250477
2. A. Acker, Uniqueness and monotonicity of solutions for the interior Bernoulli free boundary problem in the convex, *n*-dimensional case, *Nonlinear Anal. Theor.*, **13** (1989), 1409–1425. https://doi.org/10.1016/0362-546X(89)90102-8
3. A. Aftalion, J. Busca, W. Reichel, Approximate radial symmetry for overdetermined boundary value problems, *Adv. Differential Equations*, **4** (1999), 907–932.
4. V. Agostiniani, R. Magnanini, Symmetries in an overdetermined problem for the Green's function, *Discrete Cont. Dyn. Syst. S*, **4** (2011), 791–800. https://doi.org/10.3934/dcdss.2011.4.791
5. C. Bianchini, A Bernoulli problem with non-constant gradient boundary constraint, *Appl. Anal.*, **91** (2012), 517–527. https://doi.org/10.1080/00036811.2010.549479
6. C. Bianchini, A. Henrot, P. Salani, An overdetermined problem with non-constant boundary condition, *Interfaces Free Bound.*, **16** (2014), 215–241. https://doi.org/10.4171/IFB/318
7. B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, Serrin-type overdetermined problems: allternative proof, *Arch. Rational Mech. Anal.*, **190** (2008), 267–280. https://doi.org/10.1007/s00205-008-0119-3
8. B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, On the stability of the Serrin problem, *J. Differ. Equations*, **245** (2008), 1566–1583. https://doi.org/10.1016/j.jde.2008.06.010
9. L. A. Caffarelli, H. W. Alt, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.*, **325** (1981), 105–144. https://doi.org/10.1515/crll.1981.325.105
10. L. Cavallina, T. Yachimura, On a two-phase Serrin-type problem and its numerical computation, *ESAIM: COCV*, **26** (2020), 65. https://doi.org/10.1051/cocv/2019048
11. L. Cavallina, Local analysis of a two phase free boundary problem concerning mean curvature, *Indiana Univ. Math. J.*, in press.
12. L. Cavallina, T. Yachimura, Symmetry breaking solutions for a two-phase overdetermined problem of Serrin-type, In: *Current trends in analysis, its applications and computation (Proceedings of the 12th ISAAC Congress, Aveiro, Portugal, 2019)*, Birkhäuser, 2022, in press.
13. G. Ciraolo, R. Magnanini, V. Vespri, Hölder stability for Serrin's overdetermined problem, *Annali di Matematica*, **195** (2016), 1333–1345. https://doi.org/10.1007/s10231-015-0518-7

14. M. Del Pino, F. Pacard, J. Wei, Serrin's overdetermined problem and constant mean curvature surfaces, *Duke Math. J.*, **164** (2015), 2643–2722. https://doi.org/10.1215/00127094-3146710

15. M. M. Fall, I. A. Minlend, T. Weth, Unbounded periodic solutions to Serrin's overdetermined boundary value problem, *Arch. Rational Mech. Anal.*, **223** (2017), 737–759. https://doi.org/10.1007/s00205-016-1044-5

16. W. M. Feldman, Stability of Serrin's problem and dynamic stability of a model for contact angle motion, *SIAM J. Math. Anal.*, **50** (2018), 3303–3326. https://doi.org/10.1137/17M1143009

17. D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Berlin, Heidelberg: Springer, 2001. https://doi.org/10.1007/978-3-642-61798-0

18. A. Gilsbach, M. Onodera, Linear stability estimates for Serrin's problem via a modified implicit function theorem, *Calc. Var.*, **60** (2021), 241. https://doi.org/10.1007/s00526-021-02107-1

19. A. Henrot, M. Onodera, Hyperbolic solutions to Bernoulli's free boundary problem, *Arch. Rational Mech. Anal.*, **240** (2021), 761–784. https://doi.org/10.1007/s00205-021-01620-z

20. A. Henrot, H. Shahgholian, The one phase free boundary problem for the p-Laplacian with non-constant Bernoulli boundary condition, *Trans. Amer. Math. Soc.*, **354** (2002), 2399–2416. https://doi.org/10.1090/S0002-9947-02-02892-1

21. N. Kamburov, L. Sciaraffia, Nontrivial solutions to Serrin's problem in annular domains, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **38** (2021), 1–22. https://doi.org/10.1016/j.anihpc.2020.05.001

22. R. Magnanini, G. Poggesi, On the stability for Alexandrov's soap bubble theorem, *J. Anal. Math.*, **139** (2019), 179–205. https://doi.org/10.1007/s11854-019-0058-y

23. R. Magnanini, G. Poggesi, Serrin's problem and Alexandrov's soap bubble theorem: enhanced stability via integral identities, *Indiana Univ. Math. J.*, **69** (2020), 1181–1205. https://doi.org/10.1512/iumj.2020.69.7925

24. R. Magnanini, G. Poggesi, Nearly optimal stability for Serrin's problem and the Soap Bubble theorem, *Calc. Var.*, **59** (2020), 35. https://doi.org/10.1007/s00526-019-1689-7

25. R. Magnanini, G. Poggesi, Interpolating estimates with applications to some quantitative symmetry results, *Mathematics in Engineering*, **5** (2023), 1–21. https://doi.org/10.3934/mine.2023002

26. F. Morabito, P. Sicbaldi, Delaunay type domains for an overdetermined elliptic problem in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, *ESAIM: COCV*, **22** (2016), 1–28. https://doi.org/10.1051/cocv/2014064

27. F. Morabito, Symmetry breaking bifurcations for two overdetermined boundary value problems with non-constant Neumann condition on exterior domains in $\mathbb{R}^3$, *Commun. Part. Diff. Eq.*, **46** (2021), 1137–1161. https://doi.org/10.1080/03605302.2020.1871363

28. M. Onodera, Geometric flows for quadrature identities, *Math. Ann.*, **361** (2015), 77–106. https://doi.org/10.1007/s00208-014-1062-2

29. M. Onodera, On the symmetry in a heterogeneous overdetermined problem, *Bull. Lond. Math. Soc.*, **47** (2015), 95–100. https://doi.org/10.1112/blms/bdu098

30. M. Onodera, Dynamical approach to an overdetermined problem in potential theory, *J. Math. Pure. Appl.*, **106** (2016), 768–796. https://doi.org/10.1016/j.matpur.2016.03.011

31. L. E. Payne, P. W. Schaefer, Duality theorems in some overdetermined boundary value problems, *Math. Method. Appl. Sci.*, **11** (1989), 805–819. https://doi.org/10.1002/mma.1670110606

32. G. Pólya, Torsional rigidity, principal frequency, electrostatic capacity and symmetrization, *Quart. Appl. Math.*, **6** (1948), 267–277. https://doi.org/10.1090/QAM/26817

33. F. Schlenk, P. Sicbaldi, Bifurcating extremal domains for the first eigenvalue of the Laplacian, *Adv. Math.*, **229** (2012), 602–632. https://doi.org/10.1016/j.aim.2011.10.001

34. J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.*, **43** (1971), 304–318. https://doi.org/10.1007/BF00250468

35. H. Shahgholian, Diversifications of Serrin's and related symmetry problems, *Complex Var. Elliptic Equ.*, **57** (2012), 653–665. https://doi.org/10.1080/17476933.2010.504848

36. P. Sicbaldi, New extremal domains for the first eigenvalue of the Laplacian in flat tori, *Calc. Var.*, **37** (2010), 329–344. https://doi.org/10.1007/s00526-009-0264-z

37. H. F. Weinberger, Remark on the preceding paper of Serrin, *Arch. Rational Mech. Anal.*, **43** (1971), 319–320. https://doi.org/10.1007/BF00250469