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Research article

A note on construction of nonnegative initial data inducing unbounded solutions to some two-dimensional Keller–Segel systems[†]

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Abstract: It was shown that unbounded solutions of the Neumann initial-boundary value problem to the two-dimensional Keller–Segel system can be induced by initial data having large negative energy if the total mass $\Lambda \in (4\pi, \infty) \setminus 4\pi \cdot \mathbb{N}$ and an example of such an initial datum was given for some transformed system and its associated energy in Horstmann–Wang (2001). In this work, we provide an alternative construction of nonnegative nonradially symmetric initial data enforcing unbounded solutions to the original Keller–Segel model.

Keywords: chemotaxis; Keller–Segel system; local sensing; infinite-time blow-up; unbounded solution; Lyapunov functional

1. Introduction

The main purpose of this note is to provide an alternative construction of nonnegative and nonradially symmetric initial data for some Keller–Segel-type models which will enforce finite or infinite blowup. Consider the following functional:

$$\mathcal{F}(u,v) := \int_\Omega \left(u \log u - uv + \frac{1}{2}|\nabla v|^2 + \frac{1}{2}v^2 \right) dx,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary $\partial\Omega$ and a pair of nonnegative smooth functions (u, v). The main result of this note is stated as follows.

Theorem 1.1. *For any $M > 0$ and $\Lambda \in (4\pi, \infty)$ there exists a pair of nonnegative functions $(u_0, v_0) \in (C^\infty(\overline{\Omega}))^2$ satisfying*

$$\begin{cases} \|u_0\|_{L^1(\Omega)} = \Lambda, \\ \mathcal{F}(u_0, v_0) < -M. \end{cases}$$

The above functional $\mathcal{F}(u, v)$ appears in the study of the minimal Keller–Segel system:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = 0 & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

and also one of the following chemotaxis model featuring a signal-dependent motility function of the negative exponential type:

$$\begin{cases} u_t = \Delta(e^{-v} u) & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = 0 & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Classical positive solutions of (1.1) satisfy the following energy-dissipation identity ([4, 9]):

$$\frac{d}{dt}\mathcal{F}(u, v)(t) + \int_\Omega u \left|\nabla \log u - \nabla v\right|^2 \, dx + \|v_t\|^2_{L^2(\Omega)} = 0,$$

while for the classical solutions to (1.2), there holds ([2]):

$$\frac{d}{dt}\mathcal{F}(u, v)(t) + \int_\Omega u e^{-v} \left|\nabla \log u - \nabla v\right|^2 \, dx + \|v_t\|^2_{L^2(\Omega)} = 0.$$

In both cases, the above energy identities will immediately give rise to the a priori upper bound for $\mathcal{F}(u, v)(t)$. On the other hand, for any given initial data of small total mass such that $\|u_0\|_{L^1(\Omega)} < 4\pi$, one could derive a lower bound for the energy functional and then the classical solutions of both systems (1.1) and (1.2) exist globally in time and remain bounded uniformly in the two-dimensional setting (see [2, 4, 7, 9]). For large data, unbounded solutions of the above problems could be constructed based on observations of the variational structure of the stationary problem and by taking an advantage of the subtle connection between its associated functional with the energy $\mathcal{F}$. In [5] the authors introduced a transformation problem of the original system (1.1) with the unknowns being the cell density and the relative signal concentration. Then they constructed unbounded solutions for the transformed problem, which in turn implied blowup of the original one.

In this note we would rather to construct an unbounded solution to the original system (1.1) or (1.2) in a more direct way. To this aim, let us sketch the main idea of the construction of an unbounded solution following [11] (see also [5]). First, the corresponding stationary solutions (u_s, v_s) to (1.1) or (1.2) satisfy the following problem:

$$\begin{cases} v_s - \Delta v_s = \dfrac{\Lambda}{\int_\Omega e^{v_s}\,dx} e^{v_s} & \text{in } \Omega, \\ u_s = \dfrac{\Lambda}{\int_\Omega e^{v_s}\,dx} e^{v_s} & \text{in } \Omega, \\ \dfrac{\partial v_s}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for some $\Lambda > 0$. Denote

$$\mathcal{S}(\Lambda) := \left\{ (u_s, v_s) \in C^2(\overline{\Omega}) : (u_s, v_s) \text{ is a solution to (1.3)} \right\}$$

for $\Lambda > 0$. By [5, Lemma 3.5] and [10, Theorem 1], for $\Lambda \notin 4\pi\mathbb{N}$ there exists some $C > 0$ such that

$$\sup\{\|(u_s, v_s)\|_{L^\infty(\Omega)} : (u_s, v_s) \in \mathcal{S}(\Lambda)\} \le C$$

and

$$\mathcal{F}_*(\Lambda) := \inf\{\mathcal{F}(u_s, v_s) : (u_s, v_s) \in \mathcal{S}(\Lambda)\} \ge -C.$$

On the other hand, let (u, v) be the classical positive solution to (1.1) or (1.2) in $\Omega \times (0, \infty)$. If the solution is uniform-in-time bounded, by the compactness method (cf. [13, Lemma 3.1]), there exist a sequence of time $\{t_k\} \subset (0, \infty)$ and a solution (u_s, v_s) to (1.3) with $\Lambda = \|u_0\|_{L^1(\Omega)}$ such that $\lim_{k\to\infty} t_k = \infty$ and that

$$\lim_{k\to\infty} (u(t_k), v(t_k)) = (u_s, v_s) \quad \text{in } C^2(\overline{\Omega}),$$

as well as

$$\mathcal{F}(u_s, v_s) \le \mathcal{F}(u_0, v_0).$$

Thus taking account of the above discussion, for a pair of nonnegative functions (u_0, v_0) satisfying

$$\begin{cases} \|u_0\|_{L^1(\Omega)} = \Lambda \notin 4\pi\mathbb{N}, \\ \mathcal{F}(u_0, v_0) < \mathcal{F}_*(\Lambda), \end{cases} \tag{1.4}$$

the corresponding solution must be unbounded or blow up in finite time.

Recently in [2], we constructed nonnegative initial data satisfying (1.4) when $\Lambda \in (8\pi, \infty)$ in the radially symmetric case, which differs from those given in [5]. However, it was left open whether our idea for a construction of adequate initial data can be extended to the nonradial symmetric case if $\Lambda \in (4\pi, 8\pi)$. Theorem 1.1 of the present work gives an affirmative answer to this question and as a consequence, we have an alternative proof of the following corollaries ([5]).

Corollary 1.2. *For any* $\Lambda \in (4\pi, \infty)\backslash 4\pi\mathbb{N}$ *there exists a nonnegative initial datum* (u_0, v_0) *satisfying* (1.4) *such that the corresponding classical solution of* (1.1) *satisfies either:*

- *exists globally in time and* $\limsup_{t\to\infty}(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}) = \infty$*;*
- *blows up in finite time.*

Remark 1.3. *Finite time blowup solutions of the corresponding parabolic-elliptic system are constructed if* $\Lambda > 4\pi$ *in [8].*

As to the system (1.2), global existence of classical solutions with any nonnegative initial data was guaranteed in [2], which excluded the possibility of finite-time blowup. Hence, we arrive at the following:

Corollary 1.4. *For any* $\Lambda \in (4\pi, \infty)\backslash 4\pi\mathbb{N}$ *there exists a nonnegative initial datum* (u_0, v_0) *satisfying* (1.4) *such that the corresponding global classical solution of* (1.2) *blows up at time infinity.*

In previous works [3, 6, 12, 13], nonnegative initial data with large negative energy were constructed in several modified situations, e.g., the higher dimensional setting, the nonlinear diffusion case, the nonlinear sensitivity case and the indirect signal case. In those works, the initial datum has a concentration at an interior point of Ω. Similarly, in our precedent work [2], we constructed an initial datum which concentrates at the origin based on certain perturbation of the rescaled explicit solutions to the elliptic system

$$\begin{cases} -\Delta V = U & x \in \mathbb{R}^2, \\ e^V = U & x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} U = 8\pi, \end{cases}$$

provided that the total mass $\Lambda > 8\pi$. However, without the radially symmetric requirement and when $4\pi < \Lambda < 8\pi$, we need to construct an initial datum that concentrates at a boundary point. To this aim, some cut-off and folding-up techniques are introduced. Besides, a lemma of analysis (Lemma 2.2) plays a crucial role in estimating the value of each individual integral in the energy functional and in order to get vanishing estimations of the error terms, the radius of the cut-off function used in our case needs to depend on the rescaled parameter as well, which in contrast was fixed in the radially symmetric case in [2].

2. Proof of Theorem 1.1

A straightforward calculation leads us to the following lemma.

Lemma 2.1. *For any* $\lambda \geq 1$ *and* $r \in (0, 1)$, *the functions*

$$u_\lambda(x) := \frac{8\lambda^2}{(1 + \lambda^2|x|^2)^2}, \quad v_\lambda(x) := 2\log\frac{1 + \lambda^2}{1 + \lambda^2|x|^2} + \log 8 \qquad \textit{for all } x \in \mathbb{R}^2,$$

satisfy

$$\int_{\mathbb{R}^2} u_\lambda\, dx = 8\pi, \quad u_\lambda(x) \leq 8\lambda^2, \quad v_\lambda(x) > \log 8 > 0 \quad \textit{in } B_r(0) := \{x \in \mathbb{R}^2 \,|\, |x| < r\}.$$

Since $\partial\Omega$ is C^2 class, for any boundary point $P \in \partial\Omega$ there exist some $R' = R'_P \in (0, 1)$ and some C^2 function $\gamma_P : \mathbb{R} \to \mathbb{R}$ such that

$$\Omega \cap B_{R'}(0) = \{(x_1, x_2) \in B_{R'}(0) \,|\, x_2 > \gamma_P(x_1)\}$$

(cf. [1, Appendix C.1]). Moreover since Ω is a bounded domain, we can find some point $P_0 = (P_1, P_2) \in \partial\Omega$ satisfying that there exists $R \in (0, R')$ such that

$$(\gamma_{P_0})''(x_1) \geq 0 \qquad \text{for all } |P_1 - x_1| < R. \tag{2.1}$$

By translation, we may assume $P_0 = (0, 0)$. Hereafter we fix the above $R \in (0, 1)$ and $\gamma = \gamma_{P_0}$. In this setting, we have the following lemma:

Lemma 2.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a radially symmetric, nonnegative and continuous function. For any $r \in (0, R)$ it follows that*

$$\frac{1}{2}\int_{B_r(0)} f(x)\,dx - K(R)\left(\sup_{x\in B_r(0)} f(x)\right)\cdot r^3 \leq \int_{B_r(0)\cap\Omega} f(x)\,dx \leq \frac{1}{2}\int_{B_r(0)} f(x)\,dx,$$

where

$$K(R) := \max_{|\xi|\leq R} \gamma''(\xi) > 0. \tag{2.2}$$

Proof. We first note that for any $r \in (0, R)$,

$$\Omega \cap B_r(0) = \{(x_1, x_2) \in B_r(0) \,|\, x_2 > \gamma(x_1)\}.$$

Since $\gamma(0) = 0$ and the assumption (2.1), it follows by Taylor's theorem that for all $x_1 \in (-R, R)$ we have

$$\gamma'(0)x_1 \leq \gamma(x_1) \leq \gamma'(0)x_1 + \frac{1}{2}K(R)\cdot x_1^2,$$

where $K(R) := \max_{|\xi|\leq R}\gamma''(\xi) > 0$. Thus we can deduce that

$$A_{+\varepsilon} \subset (\Omega \cap B_r(0)) \subset A,$$

where

$$\begin{aligned} A_{+\varepsilon} &:= \{(x_1, x_2) \in B_r(0) \,|\, x_2 > \gamma'(0)x_1 + \frac{1}{2}K(R)\cdot r^2\}, \\ A &:= \{(x_1, x_2) \in B_r(0) \,|\, x_2 > \gamma'(0)x_1\}. \end{aligned}$$

By denoting

$$B_{+\varepsilon} := \{(x_1, x_2) \in B_r(0) \,|\, \gamma'(0)x_1 + \frac{1}{2}K(R)\cdot r^2 \geq x_2 > \gamma'(0)x_1\},$$

we confirm that

$$A_{+\varepsilon} = A \setminus B_{+\varepsilon}.$$

Since the radial symmetry of f implies

$$\int_A f(x)\,dx = \frac{1}{2}\int_{B_r(0)} f(x)\,dx,$$

we have

$$\frac{1}{2}\int_{B_r(0)} f(x)\,dx - \int_{B_{+\varepsilon}} f(x)\,dx \leq \int_{\Omega\cap B_r(0)} f(x)\,dx \leq \frac{1}{2}\int_{B_r(0)} f(x)\,dx.$$

Since

$$|B_{+\varepsilon}| \leq \frac{1}{2}K(R)r^2 \cdot 2r = K(R)r^3,$$

we have that

$$\frac{1}{2}\int_{B_r(0)} f(x)\,dx - \left(\sup_{x\in B_r(0)} f(x)\right)\cdot K(R)\cdot r^3 \leq \int_{\Omega\cap B_r(0)} f(x)\,dx \leq \frac{1}{2}\int_{B_r(0)} f(x)\,dx,$$

which concludes the proof. □

For any $0 < \eta_1 < \eta_2$ we can construct a radially symmetric function $\phi_{\eta_2,\eta_1} \in C^\infty(\mathbb{R}^2)$ satisfying

$$\phi_{\eta_2,\eta_1}(B(0,\eta_1)) = \{1\},\ 0 \leq \phi_{\eta_2,\eta_1} \leq 1,\ \phi_{\eta_2,\eta_1}(\mathbb{R}^2 \setminus B(0,\eta_2)) = \{0\},\ x\cdot\nabla\phi_{\eta_2,\eta_1}(x) \leq 0.$$

For any $\lambda > \max\{1, (\frac{4}{R})^{\frac{6}{5}}\}$, we fix

$$r := \lambda^{-\frac{5}{6}}, \qquad r_1 := \frac{r}{2},$$

and then $0 < r_1 < r < \min\{1, \frac{R}{4}\}$. Noting that

$$f(\lambda) := 1 - \frac{1}{1+(\lambda r_1)^2} = 1 - \frac{4}{4+\lambda^{\frac{1}{3}}} \nearrow 1 \quad \text{as } \lambda \to \infty,$$

and by the increasing property of f, we can find $\lambda_* > \max\{1, (\frac{4}{R})^{\frac{6}{5}}\}$ such that

$$4\pi\cdot f(\lambda_*) - 8K(R)\lambda_*^{-\frac{1}{2}} > 2\pi,$$

where $K(R)$ is defined in (2.2). Here we confirm that for any $\lambda > \lambda_*$,

$$4\pi\cdot f(\lambda) - 8K(R)\lambda^{-\frac{1}{2}} > 2\pi.$$

Now we define the pair

$$(u_0, v_0) := (au_\lambda\phi_{r,r_1}\chi_\Omega, av_\lambda\phi_{\frac{R}{2},\frac{R}{4}}\chi_\Omega)$$

with some $a > 0$. Here we remark that u_0 and v_0 are nonnegative functions belonging to $C^\infty(\overline{\Omega})$.

Lemma 2.3. *Let $\Lambda \in (4\pi, \infty)$. For $\lambda > \lambda_*$ there exists*

$$a = a(\lambda) \in \left(\frac{\Lambda}{4\pi}, \frac{\Lambda}{2\pi}\right) \tag{2.3}$$

such that

$$\int_\Omega u_0\,dx = \Lambda. \tag{2.4}$$

Proof. Firstly by changing variables, we see that

$$\begin{aligned}\int_{B(0,\ell)} u_\lambda\,dx &= 8\int_{B(0,\lambda\ell)} \frac{dy}{(1+|y|^2)^2}\\ &= 8\pi\int_0^{(\lambda\ell)^2} \frac{d\tau}{(1+\tau)^2}\\ &= 8\pi\cdot\left(1-\frac{1}{1+(\lambda\ell)^2}\right) \quad \text{for } \ell>0,\end{aligned} \tag{2.5}$$

and that

$$8\pi\cdot\left(1-\frac{1}{1+(\lambda r_1)^2}\right) < \int_{B_r(0)} u_\lambda\phi_{r,r_1}\,dx < 8\pi\cdot\left(1-\frac{1}{1+(\lambda r)^2}\right).$$

Here in light of the radial symmetry of $u_\lambda\phi_{r,r_1}$, we can invoke Lemma 2.2 to have

$$4\pi\cdot\left(1-\frac{1}{1+(\lambda r_1)^2}\right) - K(R)8\lambda^2 r^3 < \int_\Omega u_\lambda\phi_{r,r_1}\chi_\Omega\,dx < 4\pi\cdot\left(1-\frac{1}{1+(\lambda r)^2}\right),$$

where we used

$$\max_{x\in B_r(0)} u_\lambda\phi_{r,r_1}(x) = 8\lambda^2 \quad \text{and} \quad \int_\Omega u_\lambda\phi_{r,r_1}\chi_\Omega\,dx = \int_{B_r(0)\cap\Omega} u_\lambda\phi_{r,r_1}\,dx.$$

By the choice of $r>0$, we have

$$4\pi\cdot f(\lambda) - 8K(R)\lambda^{-\frac{1}{2}} < \int_\Omega u_\lambda\phi_{r,r_1}\chi_\Omega\,dx.$$

Therefore for any $\lambda>\lambda_*$ we find some $a=a(\lambda)$ satisfying

$$\frac{\Lambda}{4\pi} < a < \frac{\Lambda}{2\pi}$$

and (2.4). We conclude the proof. □

Lemma 2.4. *There exists $C>0$ such that for all $\lambda>\lambda_*$,*

$$\int_\Omega u_0\log u_0\,dx \le 8\pi a\log\lambda + C, \tag{2.6}$$

where $a=a(\lambda)$ is defined in Lemma 2.3.

Proof. Since $s\log s\le t\log t+\frac{1}{e}$ for $s\le t$ and $u_0\le au_\lambda\chi_{B_r(0)\cap\Omega}$, it follows

$$\begin{aligned}\int_\Omega u_0\log u_0\,dx &\le \int_\Omega (au_\lambda\chi_{B_r(0)\cap\Omega})\log(au_\lambda\chi_{B_r(0)\cap\Omega})\,dx + \frac{|\Omega|}{e}\\ &\le a\int_\Omega u_\lambda\chi_{B_r(0)\cap\Omega}\log u_\lambda\,dx + (a\log a + e^{-1})\int_\Omega u_\lambda\,dx + \frac{|\Omega|}{e}.\end{aligned}$$

Since $\log u_\lambda \le \log(8\lambda^2) = 2\log\lambda + \log 8$ and $\int_\Omega u_\lambda \le 8\pi$, we have

$$\int_\Omega u_0\log u_0\,dx \le 2a\log\lambda\int_\Omega u_\lambda\chi_{B_r(0)\cap\Omega}\,dx + 8\pi(a\log 8 + a\log a + e^{-1}) + \frac{|\Omega|}{e}.$$

By Lemma 2.2 we obtain

$$\int_\Omega u_\lambda \chi_{B_r(0)\cap\Omega} \le \frac{1}{2}\int_{B_r(0)} u_\lambda \le \frac{1}{2}\int_{\mathbb{R}^2} u_\lambda = 4\pi.$$

Therefore

$$\int_\Omega u_0 \log u_0 \, dx \quad \le \quad 8\pi a \log \lambda + C,$$

where we remark that the constant C is independent of a and λ in view of (2.3). We conclude the proof. □

Lemma 2.5. *There exists $C > 0$ such that for all $\lambda > \lambda_*$,*

$$\int_\Omega u_0 v_0 \, dx \ge 16\pi a^2 \log \lambda - \frac{64\pi a^2 \log \lambda}{4 + \lambda^{\frac{1}{3}}} - K(R)\lambda^{-\frac{1}{2}}(2\log(1+\lambda^2) + \log 8) - C, \tag{2.7}$$

where $a = a(\lambda)$ is defined in Lemma 2.3.

Proof. Using $v_\lambda > 0$ in $B(0, r)$, $u_0 = 0$ on $B(0, r)^c$ and $r_1 < \frac{R}{4}$, we see that

$$\int_\Omega u_0 v_0 \, dx \quad \ge \quad a^2 \int_{B(0,r_1)} u_\lambda v_\lambda \chi_{B_{r_1}(0)\cap\Omega} \, dx.$$

Since $u_\lambda v_\lambda$ is radially symmetric and

$$\max_{x\in B_{r_1}(0)} u_\lambda v_\lambda(x) = 8\lambda^2(2\log(1+\lambda^2) + \log 8),$$

we apply Lemma 2.2 and recall $r_1 = 2^{-1}\lambda^{-\frac{5}{6}}$ to deduce that

$$\begin{aligned}\int_\Omega u_0 v_0 \, dx \quad &\ge \quad \frac{1}{2}a^2 \int_{B(0,r_1)} u_\lambda v_\lambda \, dx - K(R) 8\lambda^2(2\log(1+\lambda^2) + \log 8) \cdot r_1^3 \\ &= \quad \frac{1}{2}a^2 \int_{B(0,r_1)} u_\lambda v_\lambda \, dx - K(R)\lambda^{-\frac{1}{2}}(2\log(1+\lambda^2) + \log 8).\end{aligned}$$

Since

$$v_\lambda(x) > 2\log \frac{1+\lambda^2}{1+\lambda^2|x|^2} \qquad \text{for } x \in B(0, r_1),$$

we have that

$$\begin{aligned}\frac{1}{2}a^2 \int_{B(0,r_1)} u_\lambda v_\lambda \, dx \quad &\ge \quad \frac{1}{2}a^2 \int_{B(0,r_1)} u_\lambda \cdot 2\log \frac{1+\lambda^2}{1+\lambda^2|x|^2} \, dx \\ &> \quad 2a^2 \log \lambda \int_{B(0,r_1)} u_\lambda \, dx - a^2 \int_{B(0,r_1)} u_\lambda \log(1+\lambda^2|x|^2) \, dx.\end{aligned}$$

By (2.5), it follows

$$2a^2 \log \lambda \int_{B(0,r_1)} u_\lambda \, dx \quad \ge \quad 2a^2 \log \lambda \cdot 8\pi \left(1 - \frac{1}{1+(\lambda r_1)^2}\right)$$

$$= \quad 16\pi a^2 \log\lambda - \frac{64\pi a^2 \log\lambda}{4+\lambda^{\frac{1}{3}}}.$$

On the other hand, by (2.3) and direct calculations we see

$$\begin{aligned} a^2 \int_{B(0,r_1)} u_\lambda \log(1+\lambda^2|x|^2)\,dx &= 8a^2 \int_{B(0,r_1)} \frac{\lambda^2 \log(1+\lambda^2|x|^2)}{(1+\lambda^2|x|^2)^2}\,dx \\ &= 16\pi a^2 \int_0^{\lambda r_1} \frac{s\log(1+s^2)}{(1+s^2)^2}\,ds \\ &< 8\pi a^2 \int_0^\infty \frac{\log(1+\xi)}{(1+\xi)^2}\,d\xi < \infty. \end{aligned}$$

Combining above estimates, we obtain that

$$\int_\Omega u_0 v_0\,dx \geq 16\pi a^2 \log\lambda - \frac{64\pi a^2\log\lambda}{4+\lambda^{\frac{1}{3}}} - K(R)\lambda^{-\frac{1}{2}}(2\log(1+\lambda^2)+\log 8) - C$$

for $\lambda > \lambda_*$ with some positive constant C, which is independent of a and λ due to (2.3). □

Lemma 2.6. *For any $\varepsilon_1 > 0$ there exists $C(\varepsilon_1) > 0$ such that for all $\lambda > \lambda_*$,*

$$\frac{1}{2}\int_\Omega \left(v_0^2 + |\nabla v_0|^2\right)dx \leq 8\pi(1+\varepsilon_1)a^2\log\lambda + C(\varepsilon_1), \tag{2.8}$$

where $a = a(\lambda)$ is defined in Lemma 2.3.

Proof. Since

$$\frac{1+\lambda^2}{1+\lambda^2|x|^2} \leq \frac{1+\lambda^2}{\lambda^2|x|^2} \leq \left(\frac{2}{|x|}\right)^2 \qquad \text{for } \lambda > 1,$$

we see that for $\lambda > 1$

$$|v_\lambda(x)| \leq 4\log\frac{2}{|x|} + \log 8 \ \text{ in } B_1(0).$$

Hence it follows from straightforward calculations that there is a positive constant C satisfying

$$\begin{aligned} \int_\Omega v_0^2\,dx &\leq a^2 \int_{B_1(0)} \left(4\log\frac{2}{|x|} + \log 8\right)^2 dx \\ &\leq C, \end{aligned} \tag{2.9}$$

where the constant C is independent of a and λ due to (2.3).

Moreover by Young's inequality, for any $\varepsilon_1 > 0$ there exists $C'(\varepsilon_1) > 0$ such that

$$\begin{aligned} |\nabla v_0|^2 =& a^2|\phi_{\frac{R}{2},\frac{R}{4}}\nabla v_\lambda + \nabla\phi_{\frac{R}{2},\frac{R}{4}} v_\lambda|^2 \chi_{B_{\frac{R}{2}}(0)\cap\Omega} \\ \leq& a^2(1+\varepsilon_1)\phi^2_{\frac{R}{2},\frac{R}{4}}|\nabla v_\lambda|^2\chi_{B_{\frac{R}{2}}(0)\cap\Omega} + C'(\varepsilon_1)a^2|\nabla\phi_{\frac{R}{2},\frac{R}{4}}|^2 v_\lambda^2 \chi_{B_{\frac{R}{2}}(0)\cap\Omega}. \end{aligned}$$

Since by (2.9) we have some $C > 0$ such that

$$a^2\int_\Omega |\nabla\phi_{\frac{R}{2},\frac{R}{4}}|^2 v_\lambda^2 \chi_{B_{\frac{R}{2}}(0)\cap\Omega}\,dx \leq C$$

and by the direct calculations, we have

$$|\nabla v_\lambda(x)| = \frac{4\lambda^2|x|}{1+\lambda^2|x|^2},$$

and then we infer that

$$\begin{aligned}\int_\Omega |\nabla v_0|^2\,dx &\le a^2(1+\varepsilon_1)\int_\Omega \phi^2_{\frac{R}{2},\frac{R}{4}}|\nabla v_\lambda|^2\chi_{B_{\frac{R}{2}}(0)\cap\Omega}\,dx + C'(\varepsilon_1)a^2\int_\Omega |\nabla\phi_{\frac{R}{2},\frac{R}{4}}|^2 v_\lambda^2 dx\\ &\le 16a^2(1+\varepsilon_1)\int_{B_{\frac{R}{2}}(0)\cap\Omega}\frac{\lambda^4|x|^2}{(1+\lambda^2|x|^2)^2}\,dx + C''(\varepsilon_1)\end{aligned}$$

with some $C''(\varepsilon_1) > 0$. Since $\frac{\lambda^4|x|^2}{(1+\lambda^2|x|^2)^2}$ is radially symmetric, we can invoke Lemma 2.2 to see

$$\int_\Omega |\nabla v_0|^2\,dx \le 8a^2(1+\varepsilon_1)\int_{B_{\frac{R}{2}}(0)}\frac{\lambda^4|x|^2}{(1+\lambda^2|x|^2)^2}\,dx + C''(\varepsilon_1),$$

thus

$$\frac{1}{2}\int_\Omega |\nabla v_0|^2\,dx \le 4a^2(1+\varepsilon_1)\int_{B_1(0)}\frac{\lambda^4|x|^2}{(1+\lambda^2|x|^2)^2}\,dx + \frac{C''(\varepsilon_1)}{2}.$$

On the other hand,

$$\begin{aligned}\int_{B_1(0)}\frac{\lambda^4|x|^2}{(1+\lambda^2|x|^2)^2}\,dx &= \pi\int_0^{\lambda^2}\frac{\tau}{(1+\tau)^2}\,d\tau\\ &\le \pi\int_0^{\lambda^2}\frac{1}{1+\tau}\,d\tau\\ &= \pi\log(1+\lambda^2).\end{aligned}$$

Since $\lambda > 1$, it follows

$$\log(1+\lambda^2) \le \log(2\lambda^2) = 2\log\lambda + \log 2.$$

Hence

$$\frac{1}{2}\int_\Omega |\nabla v_0|^2\,dx \le 4\pi a^2(1+\varepsilon_1)\cdot(2\log\lambda + \log 2) + \frac{C''(\varepsilon_1)}{2}.$$

Therefore we conclude

$$\frac{1}{2}\int_\Omega |\nabla v_0|^2\,dx \le 8\pi a^2(1+\varepsilon_1)\log\lambda + C(\varepsilon_1),$$

where the constant $C(\varepsilon_1)$ is independent of a and λ due to (2.3). □

Proof of Theorem 1.1. For any $\Lambda \in (4\pi, \infty)$, we have $\Lambda/4\pi > 1$. In view of (2.3), we can fix $\varepsilon_1 > 0$ independently of λ such that $(1-\varepsilon_1)a - 1 > (1-\varepsilon_1)\frac{\Lambda}{4\pi} - 1 > 0$, where $a = a(\lambda)$ is defined in Lemma 2.3. Then it follows that

$$a((1-\varepsilon_1)a - 1) > \frac{\Lambda}{4\pi}\left((1-\varepsilon_1)\frac{\Lambda}{4\pi} - 1\right) > 0, \qquad \text{for all } \lambda > \lambda_*. \tag{2.10}$$

Collecting (2.6), (2.7) and (2.8), we infer that there exists some $C > 0$ such that

$$\mathcal{F}(u_0, v_0) \leq I_1 \cdot \log \lambda + I_2 + C,$$

where

$$\begin{aligned} I_1 &:= 8\pi a - 16\pi a^2 + 8\pi a^2(1+\varepsilon_1) = -8\pi a((1-\varepsilon_1)a - 1), \\ I_2 &:= \frac{64\pi a^2 \log \lambda}{4 + \lambda^{\frac{1}{3}}} + K(R)\lambda^{-\frac{1}{2}}(2\log(1+\lambda^2) + \log 8). \end{aligned}$$

Here (2.10) implies $I_1 < 0$ for all $\lambda > \lambda_*$. On the other hand, we note

$$\lim_{\lambda \to \infty} I_2 = 0.$$

Based on the above discussion, for $\Lambda \in (4\pi, \infty)$ and $M > 0$, we can choose some $\lambda > \lambda_*$ such that

$$\mathcal{F}(u_0, v_0) < -M.$$

We conclude the proof. □

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Conflict of interest

The authors declare no conflict of interest.

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