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Research article

Asymptotic finite-dimensional approximations for a class of extensible elastic systems

Matteo Fogato*

Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci, 32 - 20133 Milano, Italy

* **Correspondence:** Email: matteo.fogato@polimi.it.

Abstract: We consider the equation

$$u_{tt} + \delta u_t + A^2 u + \|A^{\theta/2} u\|^2 A^\theta u = g$$

where A^2 is a diagonal, self-adjoint and positive-definite operator and $\theta \in [0, 1]$ and we study some finite-dimensional approximations of the problem. First, we analyze the dynamics in the case when the forcing term g is a combination of a finite number of modes. Next, we estimate the error we commit by neglecting the modes larger than a given N. We then prove, for a particular class of forcing terms, a theoretical result allowing to study the distribution of the energy among the modes and, with this background, we refine the results. Some generalizations and applications to the study of the stability of suspension bridges are given.

Keywords: nonlinear nonlocal beam equation; stability; asymptotic behavior; dissipative equation; approximation

1. Introduction

Let A^2 be a diagonal, self-adjoint, strictly positive operator, densely defined on a real Hilbert space $(\mathcal{H}, (\cdot, \cdot), \|\cdot\|)$ and we consider the following nonlinear nonlocal evolution equation

$$u_{tt} + \delta u_t + A^2 u + \|A^{\theta/2} u\|^2 A^\theta u = g \quad \text{in } \mathcal{H} \times \mathbb{R}_+ \tag{1.1}$$

where $\theta \in [0, 1]$, $\delta > 0$ and $g \in C^0(\mathbb{R}_+, \mathcal{H})$ is a given forcing term.

The purpose of the present paper is to give a rigorous finite-dimensional approximation of (1.1). To be more precise, we introduce the projection P_N onto the space generated by the first N modes, that is, by the first N eigenvectors of the operator A^2 and we consider the approximated problem

$$u_{tt} + \delta u_t + A^2 u + \|A^{\theta/2}u\|^2 A^\theta u = P_N g \quad \text{in } \mathcal{H} \times \mathbb{R}_+ \tag{1.2}$$

We remark that, by taking $u(0)$ and $u_t(0)$ in $P_N\mathcal{H}$, Eq (1.2) can be interpreted as a system of N ODEs. Therefore, Eq (1.2) actually provides a finite-dimensional approximation of equation (1.1). We aim to prove that any solution of (1.2) is asymptotically finite-dimensional and to estimate, for any $\varepsilon > 0$, the smallest $N = N(\varepsilon)$ such that the asymptotic distance in the phase space between the solution of (1.1) and the corresponding solution of (1.2) is less than ε. An improvement of the result will be studied for a particular class of forcing terms.

The reduction of infinite-dimensional dynamical systems to finite-dimensional systems of ODEs is a technique which has been widely used in the theoretical and numerical study of PDEs. The idea was first stated by Galerkin [28] and it has been used in many different applied frameworks as well as in the theory of finite-dimensional inertial manifolds (see [15,19,21,52,54,55] and the references therein). In particular, it is a fairly common procedure, which we aim to make rigorous, in the study of suspension bridges [3] to approximate the physical system with the dynamics finite number of modes in order to reduce the computational complexity of the model. This approach can be physically justified by observing that *"the higher modes with their shorter waves involve sharper curvature in the truss and, therefore, grater bending moment at a given amplitude and accordingly reflect the influence of the truss stiffness to a greater degree than do the lower modes"* [51, p.11], which means that the dynamics of the higher modes corresponds to a physically irrelevant phenomenon. We remark that our goal would not be achieved just by estimating the dimension of the inertial manifold of our system, since we are interested in providing a finite-dimensional approximation of its asymptotic behavior.

The problem of finding a finite number of natural parameters of a system that uniquely determine its asymptotic behavior was first discussed for the 2D Navier-Stokes equation [24, 43] and to tackle it the concepts of finite-dimensional inertial manifold, determining modes and, later, determining nodes and determining local volume averages were introduced (see [16, Ch. 5], [18] and the references therein). Regarding our problem, Chueshov in [16, Ch. 5, Thm. 7.2] proved that the dynamics of the first N modes of (1.1) completely determines the evolution of the system and Eden and Milani in [22] proved that if the forcing term is $N-$dimensional, then any solution is attracted to an $M-$dimensional manifold with $M \geq N$.

Some particular cases of the damped Eq (1.1) have been widely studied in mathematical literature. An ODE version of the problem was investigated by Loud in [44, 45]. Fitouri and Haraux in [27] improved some of the previous results on the ODE case and in [26] they provided a close-to-optimal ultimate bound in the PDE version of the problem. More recently, some sharp stability criteria for the unimodal version of (1.1) and for a related evolution equation were obtained by Haraux in [37] in the case $g = 0$. The case when $\theta = 1$ was studied in a slightly different framework by Holmes and others in [40, 47] as an example of chaotic dynamics (see also [34]) and some undamped versions of (1.1) were studied in the case $\theta = 0$ by Cazenave, Weissler and Haraux in [11–14] in order to obtain a description of the qualitative behavior of more complicated nonlinearities and by Gazzola and Garrione in [29] to study the dynamics of suspension bridges with multiple intermediate piers.

The considered abstract equation was analyzed by many other authors in an even more general framework. Biler [7] and de Brito [9] investigated the decay properties of the unforced problem with weak damping and a more general nonlinear nonlocal term. Da Silvia and Narciso [49, 50] studied an extensible beam model subject to a nonlocal nonlinear parameter-dependent damping and a forcing term. A lot of different variations of (1.1) with a large variety of damping and nonlinear terms has been studied in mathematical literature (see [16, 17, 20] and the references therein).

In addition to its mathematical relevance, our study also presents a certain physical and engineering interest. In fact, the considered model is suitable to describe both mono-dimensional and multi-dimensional physical systems. More precisely, some particular cases of (1.1) concerning the dynamics of beams and plates was considered by Holmes and Marsden [38, 39] in order to study the problem of flow-induced oscillations (see also [41, 42]) and in order to provide some more information about the nonlinear structural behavior of suspension bridges. In particular, we expect our results to allow some progress in the study of the structural and torsional instability of plates, to which a vast literature is devoted [1, 2, 4, 5, 31, 32].

If we set $A^2 = \Delta^2$, $\theta = 1$ and $\mathcal{H} = L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with the smooth boundary $\partial\Omega$, we obtain the equation

$$u_{tt} + \delta u_t + \Delta^2 u + \left(\int_\Omega |\nabla u|^2\right)\Delta u = g, \qquad \text{in } \Omega \times (0, T).$$

This problem is a special case of the more general model

$$u_{tt} + \Delta^2 u - \phi(\|\nabla u\|^2_{L^2(\Omega)})\Delta u = \mathcal{F}(x, t, u, u_t)$$

that was introduced in 1955 by Berger [6] as a simplification of the von Karman plate equation which describes large deflection of plate. Some related models were later applied to the study of the torsional instability of suspension bridges. In particular, our results apply also to the partially-hinged plate problem discussed in [8, 25]

$$\begin{cases} u_{tt} + \delta u_t + \Delta^2 u + \left(P - S \int_\Omega u_r^2(r, s, t)drds\right) u_{xx} = g & \text{in } \Omega \times (0, T) \\ u = u_{xx} = 0 & \text{on } \{0, \pi\} \times [-l, l] \\ u_{yy} + \sigma u_{xx} = u_{yyy} + (2 - \sigma)u_{xxy} = 0 & \text{on } [0, \pi] \times \{-l, l\} \end{cases}$$

where $S > 0$ depends on the elasticity of the material of the deck of the bridge, $l > 0$ represents the width of the bridge and $\sigma > 0$ is the Poisson's ratio of the structure, which is assumed to be, in the case of suspension bridges, between 0 and 0.5. The term P is called "prestressing constant" and it expresses the buckling loads on the plate. In the case of suspension bridges, the compressive forces along the edges are introduced in order to increase the stability of the structure. The abstract prestressed model reads

$$u_{tt} + \delta u_t + A^2 u - PAu + \|A^{\theta/2}u\|^2 A^\theta u = g \quad \text{in } \mathcal{H} \times \mathbb{R}_+. \tag{1.3}$$

The study of this equation will not be discussed in detail since, under the hypothesis $P < \alpha_1^{1/2}$ (weak prestressing), the prestressing term does not modify the qualitative behavior of the system and in the case when $P \geq \alpha_1^{1/2}$ (strong prestressing) our results do not hold. In fact, in a strongly prestressed suspension bridge the linear part of (1.3), which is given by $A^2 - PA$, is not a strictly positive operator anymore.

Concerning the case where the models describes the dynamics of a mono-dimensional structure, if we take $\mathcal{H} = L^2(I)$ (with $I = [-\pi, \pi]$) and $A = -\partial_{xx}$, we can distinguish three different physically significant cases: $\theta = 0$, $\theta = 1$ and $\theta = 2$.

In the first case, the considered model has been introduced by Garrione and Gazzola [29] in order to describe the behavior of the deck of suspension bridges with two intermediate piers. In the work of Garrione and Gazzola, the deck of the bridge is modeled by a degenerate plate consisting of a beam with a continuum of cross sections free to rotate around the beam. Therefore, the longitudinal dynamics of the bridge is modeled by a beam equation, whose nonlinear term can be interpreted as a representation of *"a stiffened beam where the displacement behaves superquadratically and nonlocally: if the beam is displaced from its equilibrium position in some point, then this increases the resistance to further displacements in all the other points"* [29]. The nonlocal nature of such term is due to the elastic behavior of the components of the bridge, the sustaining cables in particular. This choice of the nonlinear term follows from a comparison between the qualitative behavior of some possible models and the actual behavior of suspension bridges. If we consider $\mathcal{D}(A) = \{v \in H^2(I) \cap H^1_0(I) : v(-\pi) = v(\pi) = v(-a\pi) = v(b\pi) = 0\}$ for $a, b \in (0, 1)$, where a and b model the position of the piers along the deck of the bridge, the system reads

$$\begin{cases} u_{tt} + \delta u_t + u_{xxxx} + \|u\|^2_{L^2(I)} u = g(x,t) & \forall t \geq 0, \forall x \in I \\ u(0) = u_0 \in H^2(I) \cap H^1_0(I), u_t(0) = u_1 \in L^2(I) \\ u(-\pi, t) = u(-\pi b, t) = u(\pi a, t) = u(\pi, t) = 0, & \forall t \geq 0. \end{cases}$$

An analogous equation, in a different functional framework, is involved in the study of the interaction between the cables and the deck of a suspension bridge in the case when the hangers are considered inextensible (see [29, 46]).

The second case ($\theta = 1$) was obtained by Woinowsky-Krieger [53] in 1950 and, independently, by Burgreen [10] in 1951. It models the physical phenomenon that *"if the beam is stretched somewhere, then this increases the resistance to further stretching in all the other points"* [29]. The system has been widely studied in both mathematical and engineering literature (see [22, 33] and the references therein). If we choose $\mathcal{D}(A) = \{v \in H^2(I) \cap H^1_0(I) : v(-\pi) = v(\pi) = v_{xx}(-\pi) = v_{xx}(\pi) = 0\}$, the model becomes

$$\begin{cases} u_{tt} + \delta u_t + u_{xxxx} - \|u_x\|^2_{L^2(I)} u_{xx} = g(x,t) & \forall t \geq 0, \forall x \in I \\ u(0) = u_0 \in H^2(I) \cap H^1_0(I), u_t(0) = u_1 \in L^2(I) \\ u(-\pi, t) = u_{xx}(-\pi, t) = u_{xx}(\pi, t) = u(\pi, t) = 0, & \forall t \geq 0. \end{cases}$$

The case $\theta = 2$ was first introduced in [29]. If we consider $\mathcal{H} = L^2(I)$ and $A = -\partial_{xx}$ as we did before, the nonlinear term $\|u\|^2_\theta A^{\theta/2} u$ reads $\|u_{xx}\|^2_{L^2(I)} u_{xxxx}$ and the corresponding nonlinear equation can be interpreted as a model for *"a stiffened beam with bending energy behaving superquadratically and nonlocally: this means that if the beam is bent somewhere, then this increases the resistance to further bending in all the other points"* [29]. Despite the physical interest of the case $\theta = 2$, due to its technical difficulty, in this paper we decided to restrict ourselves to the cases where $\theta \in [0, 1]$.

The results of the paper are given in three main theorems. First, in Theorem 2.3, we prove that if the forcing term is finite-dimensional, i.e., if g is a combination of a finite number N of modes, then any solution is asymptotically finite-dimensional too in a sense that we specify in Definition 2.2. In the case of small oscillations or large damping, our result improves the one of Eden and Milani [22]. The proof is based on an application of a recent work of Haraux [37]. Next, in Theorem 2.4 we prove that, under suitable smallness conditions on the nonlinearity and on the forcing term, we are able to give an $M-$dimensional approximation of (1.1). More precisely, we prove that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$

such that the asymptotic distance between a solution of (1.1) and a solution of (1.2) is controlled by ε in the phase space norm. The proof relies on a continuous dependence result and on Theorem 2.3. To conclude, in Theorem 2.5, fixed $\theta = 0$, we focus on a particular class of forcing terms and we refine the result of Theorem 2.4. In particular, under suitable smallness conditions on the solution, we improve the ultimate bounds previously given for general forcing terms in [8, 26] and we estimate how much the dynamics changes as we eliminate a single mode from the dynamics. This latter result represents one of the main novelties of the paper since, to the author's knowledge, this is the first statement of this type present in literature.

The paper is organized as follows. In Section 2 we give some definitions and we state the main results of the paper. In Section 3, some technical results are given. The proofs of the main results are contained in Section 4, Section 5 and Section 6, which are devoted to the proof of Theorem 2.3, Theorem 2.4 and Theorem 2.5 respectively. In Section 7, we present some physical conclusions concerning the application of our results to suspension bridges with multiple intermediate piers.

2. Statement of the main results

Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a Hilbert space and consider a diagonal, self-adjoint and positive-definite operator $A^2 : \mathcal{D}(A^2) \subset \mathcal{H} \to \mathcal{H}$, with eigenvalues $0 < \alpha_1 < \cdots < \alpha_j \nearrow \infty$ and eigenfunctions e_n, solutions of the problem

$$(Ae_n, Av) = \alpha_n (e_n, v) \qquad \forall v \in \mathcal{D}(A).$$

The sequence $(e_n)_{n\geq 1}$ is a complete orthonormal system of $\mathcal{H}$. For our convenience, we preferred to use A^2 instead of A to build the functional framework of the problem. The operator A^2 defines a family of Hilbert spaces $\mathcal{H}^\sigma = \mathcal{D}(A^{\sigma/2})$ with $\sigma \geq 0$, endowed with the norms $\|\cdot\|_\sigma$ induced by the scalar products

$$\begin{aligned} u, v \in \mathcal{H}^\sigma \implies (u,v)_\sigma &:= (A^{\frac{\sigma}{2}} u, A^{\frac{\sigma}{2}} v) = \sum_{n=1}^{\infty} \alpha_n^{\sigma/2} u_n v_n, \\ \|u\|_\sigma &:= \sqrt{(u,u)_\sigma} \end{aligned} \tag{2.1}$$

where $u_n = (u, e_n)$ and $v_n = (v, e_n)$. In particular, $\|\cdot\|_0 = \|\cdot\|$. In the context of this work, we consider the cases when $\sigma \in [-2, 2]$, where for negative s the space $\mathcal{H}^s$ is defined as the dual of $\mathcal{H}^{-s}$. Throughout the paper, we denote by $\langle \cdot, \cdot \rangle$ the duality product of $\mathcal{H}^2$. It possible to verify that $\mathcal{H}^\rho \hookrightarrow \mathcal{H}^\sigma$ densely whenever $0 \leq \sigma \leq \rho$ and that

$$u \in \mathcal{H}^\rho, \quad 0 \leq \sigma < \rho \implies \|u\|_\rho \geq \alpha_1^{\frac{\rho-\sigma}{4}} \|u\|_\sigma. \tag{2.2}$$

In this framework, for any family of indices $J = \{j_1, \ldots, j_n\}$, we define the projection

$$\begin{aligned} P_J : \mathcal{H} &\to \langle e_{j_1}, \ldots, e_{j_n} \rangle \\ u = \sum_{h=1}^{\infty} u_h e_h &\mapsto \sum_{r=1}^{n} u_{j_r} e_{j_r}. \end{aligned}$$

In particular, we denote by P_N and $Q_N := I - P_N$ the orthogonal projections onto $\langle e_1, \ldots e_N \rangle$ and onto $\langle e_{N+1}, \ldots \rangle$ respectively. In addition, for any $k \in \mathbb{N}$ we introduce the projection $\sqcap_k$ onto the orthogonal complement of e_k given by

$$\sqcap_k := I - P_k Q_{k-1} : \mathcal{H} \to \langle e_k \rangle^\perp.$$

Since A is a diagonal operator, we remark that

$$\forall s \in [0,2], \forall M = \{m_1, \dots, m_n\}, \quad A^s P_M = P_M A^s \text{ and } A^s Q_M = Q_M A^s. \tag{2.3}$$

Moreover, if $u = Q_N u$ for some $N \in \mathbb{N}$, then the estimate (2.2) can be improved by

$$u \in \mathcal{H}^\rho, \quad 0 \le \sigma < \rho \implies \|u\|_\rho \ge \alpha_{N+1}^{\frac{\rho-\sigma}{4}} \|u\|_\sigma. \tag{2.4}$$

By using the notation in (2.1), problem (1.1) may be rewritten as

$$u_{tt} + \delta u_t + A^2 u + \|u\|_\theta^2 A^\theta u = g \quad \text{ in } \mathcal{H} \times \mathbb{R}_+. \tag{2.5}$$

Let us make clear what is meant by weak solution of (2.5):

Definition 2.1. *Assume that*

$$g \in C_b^0(\mathbb{R}_+, \mathcal{H}) := C^0(\mathbb{R}_+, \mathcal{H}) \cap L^\infty(\mathbb{R}_+, \mathcal{H}). \tag{2.6}$$

A weak solution of (2.5) *is a function*

$$u \in C^0(\mathbb{R}_+, \mathcal{H}^2) \cap C^1(\mathbb{R}_+, \mathcal{H}) \cap C^2(\mathbb{R}_+, \mathcal{H}^{-2})$$

such that

$$\langle u_{tt}, \varphi \rangle + \delta(u_t, \varphi) + (u, \varphi)_2 + \|u\|_\theta^2 (u, \varphi)_\theta = (g, \varphi) \qquad \forall \varphi \in \mathcal{H}^2.$$

We remark that by this definition it follows that $u(0) = u_0 \in \mathcal{H}^2$ and $u_t(0) = u_1 \in \mathcal{H}$. Existence and uniqueness of weak solutions follows from an immediate adaptation of the result in [33, Theorem 2.1] (see Theorem 3.1).

First, we prove that if the forcing term if finite-dimensional, i.e. if $g = P_N g$ for some $N \in \mathbb{N}$, then any weak solution of (2.5) is asymptotically finite-dimensional. Actually, we guarantee the validity of the result for a more general family of forcing terms. We introduce the notion of exponentially $N-$dimensional forcing term.

Definition 2.2. *We say that $g \in C_b^0(\mathbb{R}_+, \mathcal{H})$ is exponentially $N-$dimensional if there exists $\eta > 0$ such that*

$$\lim_{t\to\infty} (\|Q_N g(t)\| + \|Q_N g_t(t)\|) e^{\eta t} = 0.$$

In Section 4, we prove the following statement which describes the asymptotic behavior of the solution in the case when the forcing term is exponentially $N-$dimensional.

Theorem 2.3. *Assume* (2.6) *and let $\delta > 0$. If g is exponentially $N-$dimensional, there exists $M \ge N$ and $\tilde{\eta} > 0$, both depending on δ, $\limsup_{t\to\infty} \|g(t)\|$, θ, N, η and α_1, i.e., the first eigenvalue of A^2, such that*

$$\lim_{t\to\infty} (\|Q_M u(t)\|_2^2 + \|Q_M u_t(t)\|^2) e^{\tilde{\eta} t} = 0,$$

where u is a weak solution of (2.5).

Motivated by physical arguments (see Section 7), we now consider a "separated variables" forcing term such as $g(t) = \mathfrak{g}f(t)$, where $\mathfrak{g} \in \mathcal{H}$ and $f \in C_b^0(\mathbb{R}_+, \mathbb{R})$.

Let us consider a weak solution u of (2.5). Numerical simulations show that for some j we have $\limsup_{t\to\infty} |(u(t), e_j)| \ll \limsup_{t\to\infty} \|u(t)\|$, that is, we have that the asymptotic amplitude of some modes of u seems to be negligible with respect to the overall dynamics (see Figure 3). Hence, we expect to be able to neglect such modes both from the forcing term g and the solution u, thus reducing the numerical complexity of the model. Therefore, for any finite family of indices $J = \{j_1, \dots, j_m\}$, we consider the finite-dimensional approximation of (2.5) given by

$$v_{tt} + \delta v_t + A^2 v + \|v\|_\theta^2 A^\theta v = P_J g. \tag{2.7}$$

We remark that in virtue of Theorem 2.3, any solution of (2.7) is exponentially finite-dimensional. We prove that under suitable smallness conditions on the forcing term, for an appropriate choice of J, (2.7) is a good approximation of (2.5), i.e., for any weak solution u of (2.5), the weak solution v of (2.7) provides a good exponentially finite-dimensional approximation of u. More precisely, in Section 5 we prove the following theorem:

Theorem 2.4. *Assume $\delta > 0$ and $g(t) = \mathfrak{g}f(t)$ with $\mathfrak{g} \in \mathcal{H}$ and $f \in C_b^0(\mathbb{R}_+, \mathbb{R})$. There exists $\bar{g}_\infty = \bar{g}_\infty(\alpha_1, \delta, \theta) > 0$ such that, if*

$$g_\infty := \limsup_{t\to\infty} \|g(t)\| < \bar{g}_\infty,$$

then for every $\varepsilon > 0$ there exists a finite family of indices $J = \{j_1, \dots j_m\}$ depending on α_1, δ, g_∞ and ε such that

$$\limsup_{t\to\infty} (\|u(t) - v(t)\|_2^2 + \|u_t(t) - v_t(t)\|^2) \le \varepsilon$$

where u is a weak solution of (2.5) *and v is a weak solution of* (2.7).

Moreover, if g is exponentially N−dimensional, then there exist $M \ge N$ and $\tilde{\eta} > 0$, both depending on α_1, δ, $\limsup_{t\to\infty} \|g(t)\|$, θ, N and η, such that, if $J = \{1, \dots, M\}$, then

$$\lim_{t\to\infty} (\|P_M u(t) - v(t)\|_2^2 + \|P_M u_t(t) - v_t(t)\|^2) e^{\tilde{\eta} t} = 0.$$

In Section 6 we further restrict ourselves to the case when the forcing term is sinusoidal in time and, for the sake of simplicity, we focus on the case when $\theta = 0$, i.e., we study the problem

$$u_{tt} + \delta u_t + A^2 u + \|u\|^2 u = \mathfrak{g} \sin(\omega t). \tag{2.8}$$

For $\|\mathfrak{g}\|$ small enough, Theorem 2.4 states that if we replace $\mathfrak{g}$ with $P_M \mathfrak{g}$, we commit an error arbitrarily small as M grows. This suggests to consider the case when $\mathfrak{g} = P_M \mathfrak{g}$ for some $M \in \mathbb{N}$. Let v be a solution of

$$v_{tt} + \delta v_t + A^2 v + \|v\|^2 v = \sqcap_k \mathfrak{g} \sin(\omega t). \tag{2.9}$$

Let us now estimate the distance between u and v. The following theorem holds:

Theorem 2.5. *Assume $\delta > 0$ and let $g(t) = \mathfrak{g} \sin(\omega t)$ with $\mathfrak{g} = P_M \mathfrak{g}$ for some $M \in \mathbb{N}$. There exists $\bar{\mathfrak{g}} > 0$ depending on δ, ω and α_j with $j = 1, \dots M$, such that, if $\|\mathfrak{g}\| < \bar{\mathfrak{g}}$, then, for any $k \in \{1, \dots, M\}$ and for any u and v weak solutions of* (2.5) *and* (2.9),

$$\limsup_{t\to\infty} (\|\sqcap_k u(t) - v(t)\|_2^2 + \|\sqcap_k u_t(t) - v_t(t)\|^2) \le \frac{C(\mathfrak{g}, e_k)^4}{((\alpha_k - \omega^2)^2 + \delta^2\omega^2)^2},$$

where $C = C(\alpha_1, \dots, \alpha_M, \mathfrak{g}, \delta, \omega) > 0$.

The results involved in the proof of Theorem 2.5 are the most physically significant in the applications considered (see Section 7). In fact, Theorem 2.5 relies upon an estimate on the asymptotic amplitude of each mode, that allows us to study the distribution of the energy among the modes (see Figures 3 and 5) and to obtain a new bound on the asymptotic $\mathcal{H}^2-$norm of u that improves the estimate given in [8, Lemma 22] (see Figure 2).

Theorems 2.4 and 2.5 are not perturbation statements. Indeed, for any fixed $\delta > 0$, an explicit expression of the smallness conditions on g_∞ and $\|g\|$ required by the statements of Theorems 2.4 and 2.5 is obtained in Sections 5 and 6 respectively. Since the term g models the action of the wind along the deck of the bridge, we physically interpret such smallness conditions on g_∞ as requirements on the aerodynamic load on the structure. In particular, the conditions of Theorems 2.4 and 2.5 are equivalent to require that the speed of the wind v is below a certain threshold $\bar{v}$. Moreover, we remark that such conditions can not be avoided since even in the ODE case large forcing terms lead to a chaotic dynamics [44, 45] and the behavior of the solutions can be quite complicated, even where the forcing term is periodic in time [30, 48].

Our results are adaptable to more general frameworks. In particular, exploiting the abstract results of Haraux [37] and Chueshov [16], the cases with strong damping terms and with more general nonlinearities such as $A^\theta u_t$ and $M(\|u\|_\theta^2)A^{\theta/2}u$ with $0 \le \theta \le 1$ appear to be treatable. On the other hand, our results can not be immediately generalized to evolution equations with nonlinear nonlocal damping terms such as $N(\|u\|_1^2)g(u_t)$, since the linear analysis on which the proof of Theorem 2.5 is based seems not to be easily extendable to such case.

We notice that, if the initial states of (2.5) and (2.7) were close to each other, a uniform estimate on the distance in the phase space between the solutions of the approximated and the exact problem would be expected to hold for any $t \ge 0$. Unfortunately, we were not able to obtain such estimate and the techniques exploited in the proofs of Theorems 2.4 and 2.5 do not seem suitable to get this result.

3. Preliminary results

We start by recalling some basic properties concerning well-posedness and regularity of the solutions.

Theorem 3.1. *Let* (2.6) *hold. Then*

1). (Weak solutions) If $u(0) = u_0 \in \mathcal{H}^2$ *and* $u_t(0) = u_1 \in \mathcal{H}$*, problem* (2.5) *admits a unique global weak solution such that*

$$u \in C(\mathbb{R}_+, \mathcal{H}^2) \cap C^1(\mathbb{R}_+, \mathcal{H}) \cap C^2(\mathbb{R}_+, \mathcal{H}^{-2});$$

2). (Regular solutions) If $u(0) = u_0 \in \mathcal{H}^4$ *and* $u_t(0) = u_1 \in \mathcal{H}^2$*, problem* (2.5) *admits a unique regular solution, that is, a unique global weak solution such that*

$$u \in C(\mathbb{R}_+, \mathcal{H}^4) \cap C^1(\mathbb{R}_+, \mathcal{H}^2) \cap C^2(\mathbb{R}_+, \mathcal{H});$$

3). (Continuous dependence on initial data) Let (u_{0n}, u_{1n}) *be any sequence with*

$$(u_{0n}, u_{1n}) \to (u_0, u_1) \quad \text{in } \mathcal{H}^2 \times \mathcal{H},$$

and let $u_n(t)$ denote the weak solution of (2.5) *with initial data $u_n(0) = u_n$ and $u_t(0) = u_{1n}$. Then for every $T > 0$ we have that*

$$(u_n(t), u_{n,t}(t)) \to (u(t), u_t(t)) \text{ uniformly in } C^0([0, T], \mathcal{H}^2 \times \mathcal{H}).$$

The proof follows from a standard applications of monotone operator theory with locally Lipschitz perturbations. We refer to [20, Theorem 1.5 and Proposition 1.15] and the references therein for a detailed discussion, that we decided to omit. For an alternative approach, see [33, Theorem 2.1] for the global existence and uniqueness of weak solutions and continuous dependence on initial data and [8, Theorem 5] for the global existence and uniqueness of regular solutions.

We remark that in Theorem 3.1 we did not introduce the concept of strong or classical solution. This choice is motivated by the fact that in some applications such formulations are not possible, as in the case of the multiple intermediate piers model discussed in the introduction (see [29, Section 4] for a more detailed discussion).

The following proposition gives some ultimate bounds on the Sobolev norms of u. Since the result comes from a straightforward generalization of the estimates proved in Section 7 of [8], we omit the proof.

Proposition 3.2. *Assume* (2.6) *and let u be a weak solution of* (2.5)*. We introduce the quantities $g_\infty := \limsup_{t\to\infty} \|g(t)\|$ and*

$$E_\infty := g_\infty^2 \max\left(\frac{2}{\delta^2}, \frac{1}{2\alpha_1}\right), \quad \alpha := \begin{cases} \delta/2 & \text{if } \delta^2 < 4\alpha_1, \\ \delta/2 - \sqrt{\delta^2/4 - \alpha_1} & \text{if } \delta^2 \geq 4\alpha_1. \end{cases}$$

Then, the following estimates on u hold:

$$\limsup_{t\to\infty} \|u(t)\|^2 \leq \frac{4E_\infty}{\sqrt{\alpha_1^2 + 4\alpha_1^\theta E_\infty} + \alpha_1} =: \Phi_0;$$

$$\limsup_{t\to\infty} \|u(t)\|_\theta^2 \leq \frac{4E_\infty + 2\alpha^2\Phi_0}{\sqrt{\alpha_1^{2-\theta} + 2(2E_\infty + \alpha^2\Phi_0)} + \alpha_1^{1-\theta/2}} =: \Phi_\theta;$$

$$\limsup_{t\to\infty} \|u(t)\|_2^2 \leq 2E_\infty + \alpha^2\Phi_0 =: \Phi_2;$$

$$\limsup_{t\to\infty} \|u_t(t)\|^2 \leq \min_{\lambda>0} \frac{1+\lambda}{\lambda}\left(2E_\infty + \max_{s\in[0,\Phi_0]}\left((\lambda+1)\alpha^2 - \alpha_1 s - \frac{1}{2}s^2\right)\right) =: \Phi_v.$$

3.1. Continuous dependence on the forcing term

We now prove the continuous dependence of the solutions on the forcing term under suitable smallness conditions on the parameters of the problem.

Proposition 3.3. *Let u and v be weak solutions respectively of the problems*

$$u_{tt} + \delta u_t + A^2 u + \|u\|_\theta^2 A^\theta u = g_1, \quad v_{tt} + \delta v_t + A^2 v + \|v\|_\theta^2 A^\theta v = g_2 \tag{3.1}$$

where $g_1, g_2 \in C_b^0(\mathbb{R}_+, \mathcal{H})$. Let $\Upsilon_\mu := \limsup_{t\to\infty} \|(u(t) + v(t))/2\|_\mu^2$ with μ in $[0, 2]$. There exists $\mathcal{F}_\theta(\alpha_1, \delta, \Upsilon_\theta, \Upsilon_{2\theta})$ such that, if $\mathcal{F}_\theta < 1$ holds, then there exists $C > 0$ depending on δ and g_∞ such *that*

$$\limsup_{t\to\infty}(\|u(t)-v(t)\|_2^2+\|u_t(t)-v_t(t)\|^2)\le C\limsup_{t\to\infty}\|g_1(t)-g_2(t)\|. \tag{3.2}$$

Moreover, if there exists $\eta>0$ *such that* $\limsup_{t\to\infty}\|g_1(t)-g_2(t)\|e^{\eta t}=0$*, then there exists* $\eta_1>0$ *such that*

$$\lim_{t\to\infty}(\|u(t)-v(t)\|_2^2+\|u_t(t)-v_t(t)\|^2)e^{\eta_1 t}=0. \tag{3.3}$$

In particular, we can take

$$\mathcal{F}_\theta := \frac{2\sqrt{\Upsilon_\theta\Upsilon_{2\theta}}\alpha_1^{-\theta/4}+\Upsilon_\theta}{\alpha_1^{(1-\theta)/2}}\max\left(\frac{1}{\delta},\frac{1}{2\sqrt{\alpha_1}}\right). \tag{3.4}$$

Proof. The idea of the proof is standard but, for our purposes, it is mandatory to fully report it since we are interested in making the smallness conditions required from our results explicit.

Let $\alpha>0$. We define

$$\Lambda_\alpha := \frac{1}{2}\|w_t\|^2+\frac{1}{2}\|w\|_2^2+\frac{\alpha\delta}{2}\|w\|^2+\frac{1}{16}\|w\|_\theta^4+\alpha(w_t,w)$$

and let E be the quantity

$$E := \frac{1}{2}\|w_t\|^2+\frac{1}{2}\|w\|_2^2+\frac{1}{4}\|w\|_\theta^4.$$

Remark that, by using the Cauchy-Schwarz inequality, the Young inequality and (2.2), we get

$$\Lambda_\alpha \le \frac{1+\alpha\varepsilon_1^2}{2}\|w_t\|^2+\frac{\alpha\delta}{2}\|w\|^2+\frac{\alpha_1+\alpha/\varepsilon_1^2}{2\alpha_1}\|w\|_2^2+\frac{1}{16}\|w\|_\theta^4\le C_1E,$$

$$\Lambda_\alpha \ge \frac{1-\alpha\varepsilon_2^2}{2}\|w_t\|^2+\frac{\alpha\delta}{2}\|w\|^2+\frac{\alpha_1-\alpha/\varepsilon_2^2}{2\alpha_1}\|w\|_2^2+\frac{1}{16}\|w\|_\theta^4\ge C_2E, \tag{3.5}$$

where C_1 and C_2 are positive numbers, obtainable for suitable choices of the values of α, ε_1 and ε_2. In particular, to get C_2 we have to require

$$1-\alpha\varepsilon_2^2>0, \qquad \alpha_1-\frac{\alpha}{\varepsilon_2^2}>0.$$

Hence, for every α such that $\alpha<\sqrt{\alpha_1}$ we can find ε_2 such that (3.5) holds.

We first consider u and v as regular solutions of the problems in (3.1). We define $w:=v-u$ and $r:=g_2-g_1$. The function w is the regular solution of the problem

$$w_{tt}+\delta w_t+A^2w+\|v\|_\theta^2A^\theta v-\|u\|_\theta^2A^\theta u=r. \tag{3.6}$$

We remark that, if $\xi:=(u+v)/2$, we have

$$\|v(t)\|_\theta^2A^\theta v(t)-\|u(t)\|_\theta^2A^\theta u(t)=2(\xi(t),w)_\theta A^\theta\xi(t)+\|\xi(t)\|_\theta^2A^\theta w+\frac{1}{4}\|w\|_\theta^2A^\theta w. \tag{3.7}$$

From the definition of Λ_α, by using (3.6) and (3.7), since u and v are regular solutions we get

$$\begin{aligned}&\dot{\Lambda}_\alpha + (\delta-\alpha)\|w_t\|^2 + \alpha\|w\|_2^2 + 2(\xi,w)_\theta(A^\theta\xi, w_t) + \|\xi\|_\theta^2(A^\theta w, w_t) + \\ &+ 2\alpha|(\xi,w)_\theta|^2 + \alpha\|\xi\|_\theta^2\|w\|_\theta^2 + \frac{\alpha}{4}\|w\|_\theta^4 = (r, w_t+\alpha w).\end{aligned} \tag{3.8}$$

Let $C_\mu = \sup_{t\geq 0}\|\xi(t)\|_\mu^2$ for any $\mu \in [0,2]$. For a suitable choice of α, by using Cauchy-Schwarz and Young inequality we have that for some positive constants $\bar{\alpha}$ and $\widetilde{\alpha}$

$$\begin{aligned}&(\delta-\alpha)\|w_t\|^2 + \alpha\|w\|_2^2 + 2(\xi,w)_\theta(A^\theta\xi, w_t) + \|\xi\|_\theta^2(A^\theta w, w_t) + 2\alpha|(\xi,w)_\theta|^2 + \\ &+ \alpha\|\xi\|_\theta^2\|w\|_\theta^2 + \frac{\alpha}{4}\|w\|_\theta^4 \geq (\delta-\alpha)\|w_t\|^2 + \\ &+ \alpha\|w\|_2^2 - 2\|\xi\|_\theta\|w\|_\theta\|\xi\|_{2\theta}\|w_t\| - \|\xi\|_\theta^2\|w\|_{2\theta}\|w_t\| + \frac{\alpha}{4}\|w\|_\theta^4 \geq \\ &\geq \left(\delta - \alpha - \frac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}}\right)\|w_t\|^2 + \left(\alpha - \frac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}}\right)\|w\|_2^2 + \\ &+ \frac{\alpha}{4}\|w\|_\theta^4 \geq \bar{\alpha}E \geq \widetilde{\alpha}\Lambda_\alpha.\end{aligned} \tag{3.9}$$

In particular, we choose the parameter α so that

$$\begin{cases} \delta - \alpha - \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}} > 0 \\ \alpha - \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}} > 0, \end{cases} \iff \begin{cases} \delta > \alpha + \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}} \\ \alpha > \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}}. \end{cases}$$

Hence, since $\alpha < \sqrt{\alpha_1}$, if

$$\begin{cases} \delta > \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{\alpha_1^{(1-\theta)/2}}, \\ \sqrt{\alpha_1} > \dfrac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{2\alpha_1^{(1-\theta)/2}} \end{cases}$$

we can find values of α such that (3.9) holds. Therefore we can find α such that (3.9) is satisfied if

$$\frac{2\sqrt{C_\theta C_{2\theta}}\alpha_1^{-\theta/4} + C_\theta}{\alpha_1^{(1-\theta)/2}} \max\left(\frac{1}{\delta}, \frac{1}{2\sqrt{\alpha_1}}\right) < 1. \tag{3.10}$$

Now, for some positive $\widetilde{\alpha}$ and $\widetilde{C}$ we get, from (3.8) and (3.9),

$$\dot{\Lambda}_\alpha + \widetilde{\alpha}\Lambda_\alpha \leq (r, w_t + \alpha w) \leq \widetilde{C}\|r\| =: \widetilde{f}(t). \tag{3.11}$$

By defining

$$M_\alpha(t) = \Lambda_\alpha(t) - \int_{t_0}^t \widetilde{f}(s)e^{\widetilde{\alpha}(s-t)}ds,$$

from (3.11) we obtain

$$\dot{M}_\alpha(t) + \widetilde{\alpha}M_\alpha(t) \leq 0.$$

Hence, from the Gronwall inequality and from the fact that for any $\varepsilon > 0$ there exists $t_0 > 0$ such that $|\widetilde{f}(s)| \leq \widetilde{C}(\varepsilon + \limsup_{t\to\infty} \|r(t)\|)$ for any $s \geq t_0$, we get

$$\begin{aligned}\Lambda_\alpha(t) &\leq \Lambda_\alpha(t_0)e^{-\widetilde{\alpha}(t-t_0)} + \int_{t_0}^{t} \widetilde{f}(s)e^{\widetilde{\alpha}(s-t)}ds \leq \\ &\leq \Lambda_\alpha(t_0)e^{-\widetilde{\alpha}(t-t_0)} + \widetilde{C}(\varepsilon + \limsup_{t\to\infty} \|r(t)\|)e^{-\widetilde{\alpha}t}\frac{e^{\widetilde{\alpha}t} - e^{\widetilde{\alpha}t_0}}{\widetilde{\alpha}}, \quad \forall t \geq t_0.\end{aligned} \tag{3.12}$$

Since we can take ε arbitrarily small as t_0 goes to infinity, from (3.12) we infer that there exists $C > 0$ such that

$$\limsup_{t\to\infty} \Lambda_\alpha(t) \leq C \limsup_{t\to\infty} \|r(t)\|. \tag{3.13}$$

Moreover, if there exists $\eta > 0$ such that $\limsup_{t\to\infty} \|r(t)\|e^{\eta t} = 0$, then (3.12) yields that there exists $\eta_1 > 0$ such that

$$\lim_{t\to\infty} \Lambda_\alpha(t)e^{\eta_1 t} = 0. \tag{3.14}$$

From (3.5), there exists a positive constant C_2 such that $\Lambda_\alpha(t) \geq C_2E(t)$. Therefore, (3.13) and (3.14) imply (3.2) and (3.3) respectively.

We remark that

$$\limsup_{t\to\infty} \|\xi(t)\|_\mu^2 = \Upsilon_\mu.$$

Hence, we can take $C_\mu = \Upsilon_\mu$. Therefore, from (3.10), we get that if

$$\frac{2\sqrt{\Upsilon_\theta\Upsilon_{2\theta}}\alpha_1^{-\theta/4} + \Upsilon_\theta}{\alpha_1^{(1-\theta)/2}} \max\left(\frac{1}{\delta}, \frac{1}{2\sqrt{\alpha_1}}\right) < 1,$$

then the thesis holds for regular solution u and v.

The same conclusions hold for u and v weak solutions of the problems in (3.1) by using a standard density argument. Indeed, since $\mathcal{H}^4$ is dense in $\mathcal{H}^2$ and $\mathcal{H}^2$ is dense in $\mathcal{H}$, setting $(u(0) = u^0, u_t(0) = u^1)$ and $(v(0) = v^0, v_t(0) = v^1)$, there exists two sequences (u_n^0, u_n^1) and (v_n^0, v_n^1) in $\mathcal{H}^4 \times \mathcal{H}^2$ such that

$$(u_n^0, u_n^1) \to (u^0, u^1) \quad \text{and} \quad (v_n^0, v_n^1) \to (v^0, v^1) \qquad \text{in } \mathcal{H}^2 \times \mathcal{H}.$$

Hence, from Theorem 3.1 we have the two sequences of regular solutions u_n and v_n with $(u_n(0) = u_n^0, u_{n,t}(0) = u_n^1)$ and $(v_n(0) = v_n^0, v_{n,t}(0) = v_n^1)$ such that, for any $T > 0$,

$$(u_n, u_{n,t}) \to (u, u_t), \quad (v_n, v_{n,t}) \to (v, v_t) \quad \text{uniformly in } C([0,T], \mathcal{H}^2 \times \mathcal{H}).$$

Therefore, since all the calculations hold for u_n and v_n (and the difference $w_n := u_n - v_n$), we get the thesis for the weak solutions u and v passing to the limit when $n \to \infty$. □

3.2. Some general stability results

In order to prove Theorem 2.3, we give a reformulation of Theorem 4.1 of [37] adapted to our framework.

Proposition 3.4. *Let $(H, (\cdot,\cdot), |\cdot|)$ be a Hilbert space and let A^2 be a self-adjoint and strictly positive linear operator on H with dense domain $\mathcal{D}(A)$. We introduce the Hilbert space $V := \mathcal{D}(A)$ endowed with the norm $\|\cdot\|^2 := (A\cdot, A\cdot)$ and we identify the unbounded operator A^2 with its extension in $\mathcal{L}(V, V')$. The duality pairing in $V' \times V$ will be denoted in the same way as the inner product in H.*

We consider $B(t) \in C^1(\mathbb{R}_+, \mathcal{L}(V, H))$ such that for any $v \in V$

$$0 \le \limsup_{t\to\infty}(B(t)v, v) \le \lambda\|v\|^2, \quad \limsup_{t\to\infty}(B'(t)v, v) \le \lambda'\|v\|^2$$

for some positive numbers λ and λ'.

Let u be a bounded solution of

$$u_{tt} + \delta u_t + (A^2 + B(t))u = g$$

where $\delta > 0$, $g \in C(\mathbb{R}_+, H)$ and $\lim_{t\to\infty} |g(t)|e^{c_0 t} = 0$ for some positive constant c_0.

If

$$\frac{\lambda'}{\delta} < 1$$

then there exists $c > 0$ such that

$$\lim_{t\to\infty}(\|u(t)\|^2 + |u_t(t)|^2)e^{ct} = 0.$$

Proof. We proceed as in the proof of Theorem 4.1 of [37] and we define the quadratic form on $V \times H$ given by

$$\Phi(t) = \frac{1}{2}(|u_t|^2 + \|u\|^2) + \frac{\delta}{2}(u, u_t) + \frac{\delta^2}{4}|u|^2 + \frac{1}{2}(B(t)u, u).$$

For any fixed $t_0 > 0$ we have, if $t \ge t_0$,

$$\begin{aligned}\Phi_t &= \frac{1}{2}(B'(t)u, u) - \frac{\delta}{2}|u_t|^2 - \frac{\delta}{2}(B(t)u + A^2u, u) + (g, u_t + \frac{\delta}{2}u) \le \\ &\le \frac{1}{2}\sup_{t\ge t_0}(B'(t)u, u) - \frac{\delta}{2}|u_t|^2 - \frac{\delta}{2}\|u\|^2 + Ke^{-c_0 t}.\end{aligned}$$

for some positive constant K. Hence, for t_0 large enough

$$\Phi_t(t) \le -\frac{\delta}{2}|u_t(t)|^2 - \frac{\delta - \lambda'}{2}\|u(t)\|^2 + Ke^{-c_0 t}$$

Therefore, if $\lambda' < \delta$ we get, for some positive α,

$$\Phi_t(t) + \alpha\Phi(t) \le Ke^{-c_0 t}$$

for any $t \ge t_0$ and from Gronwall lemma we get the thesis. □

We recall a further stability result due to Haraux for an ODE related to our problem.

Proposition 3.5. *[Theorem 2.1 of [37]] Let $\lambda, \delta > 0$, $a \in L^\infty(\mathbb{R}_+)$ with $a(t) \ge 0$ for any $t \ge 0$. Let $x \in C^2(\mathbb{R}_+)$ be a solution of*

$$\ddot{x} + \delta\dot{x} + (\lambda + a(t))x = 0. \tag{3.15}$$

Assume

$$\limsup_{t\to\infty} a(t) < \delta \max(\delta, 2\sqrt{\lambda}).$$

There there are $\eta_1 > 0$ *and* $M > 0$ *such that any bounded solution* x *of* (3.15) *satisfies*

$$x^2(t) + \dot{x}^2(t) \le M[x^2(s) + \dot{x}^2(s)]e^{-\eta_1(t-s)}$$

for any $s \le t$.

With minimal effort, the same statement can be proven for x solving

$$\ddot{x} + \delta\dot{x} + (\lambda + a(t))x = \widetilde{g}.$$

where $\widetilde{g} \in C(\mathbb{R}_+)$ satisfies $\lim_{t\to\infty} \widetilde{g}(t)e^{\eta t} = 0$ for some $\eta > 0$.

3.3. Linear analysis

Some preliminary results on the behavior of a damped and forced harmonic oscillator are useful in order to simplify the following study. In particular, we study the equation

$$\ddot{y} + \delta\dot{y} + \lambda y = \Psi, \tag{3.16}$$

where we require Ψ to be antiperiodic. We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be antiperiodic of antiperiod τ (i.e. $\tau-$antiperiodic) if

$$f(t + \tau) = -f(t), \quad \forall t \in \mathbb{R}.$$

Proposition 3.6. *Let us consider* $\Psi \in L^2_{loc}(\mathbb{R}_+)$ *antiperiodic of anti-period* π/ω. *We suppose that* $\lambda > 0$ *and* $\delta > 0$. *Then there exists an antiperiodic solution* z *of anti-period* π/ω *of* (3.16) *and we have that for some* $\eta > 0$, *for any* $y(t)$ *solution of* (3.16),

$$\lim_{t\to\infty}(|y(t) - z(t)| + |\dot{y}(t) - \dot{z}(t)|)e^{\eta t} = 0.$$

Proof. Let us consider $\mathcal{A}_\omega \subset L^2([0, \pi/\omega])$ the space of the locally square-integrable antiperiodic functions with anti-period π/ω, endowed with the standard L^2 norm on the interval $[0, \pi/\omega]$. The family $\{e_n = \sqrt{\omega/\pi}e^{(2n+1)i\omega t}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of this space. Hence, we write

$$\Psi(t) = \sqrt{\frac{\omega}{\pi}} \sum_{n\in\mathbb{Z}} \psi_j e^{(2n+1)i\omega t}.$$

Setting

$$z(t) := \sqrt{\frac{\omega}{\pi}} \sum_{n\in\mathbb{Z}} \frac{\psi_n}{-\omega^2(2n+1)^2 + \lambda + i\delta\omega(2n+1)} e^{(2n+1)i\omega t},$$

it is immediate to verify that $z(t)$ is an antiperiodic solution of (3.16). The thesis now follows from the standard theory of ODEs. Indeed, any solution of (3.16) is given by the sum of $z(t)$ with a general solution y_g of the associated homogeneous equation

$$\ddot{y}_g + \delta\dot{y}_g + \lambda y_g = 0,$$

which is given by

$$y_g(t) = e^{-\delta t/2} f(t),$$

with

$$f(t) := \begin{cases} S\,\sin\left(\frac{t}{2}\sqrt{4\lambda - \delta^2} + \varphi\right), & \text{if} \quad 4\lambda > \delta^2, \\ S\,\frac{t}{2}\sqrt{4\lambda - \delta^2}\cos(\varphi) + S\,\sin(\varphi), & \text{if} \quad 4\lambda = \delta^2, \\ S\,\sinh\left(\frac{t}{2}\sqrt{\delta^2 - 4\lambda} + \varphi\right), & \text{if} \quad 4\lambda < \delta^2, \end{cases}$$

where the arbitrary constants S and φ are dependent from the initial conditions. We notice that

$$\max(|f(t)|, |f'(t)|) \le Ce^{\mu t},$$

for some constants $C > 0$ and $0 \le \mu < \delta/2$. Therefore, since $y(t) = z(t) + y_g(t)$, we get that for a suitable choice of $\eta > 0$

$$\lim_{t\to\infty}(|y(t) - z(t)| + |\dot{y}(t) - \dot{z}(t)|)e^{\eta t} = \lim_{t\to\infty}\left(|f(t)| + \left|f'(t) - \frac{\delta}{2}f(t)\right|\right)e^{(\eta-\delta/2)t} \le$$
$$\le \frac{\delta+4}{2}C\lim_{t\to\infty} e^{(\eta+\mu-\delta/2)t} = 0,$$

which is the thesis. □

Proposition 3.7. *Let us consider* $\Psi \in L^2_{loc}(\mathbb{R}_+)$ *antiperiodic of anti-period* π/ω *and let* $y(t)$ *satisfy* (3.16)*. We suppose* $\lambda, \delta > 0$ *and* $2\sqrt{\lambda} \neq \delta$*. We introduce the quantities*

$$w_\lambda^\pm := \frac{\pi^2}{\omega^2}\left(\lambda - \frac{\delta^2}{2} \pm \delta\sqrt{\frac{\delta^2}{4} - \lambda}\right),$$

$$\Omega_\lambda^2 := \frac{\pi^4}{2\omega^4(w_\lambda^+ - w_\lambda^-)}\left(\frac{\tan\left(\frac{\sqrt{w_\lambda^+}}{2}\right)}{\sqrt{w_\lambda^+}} - \frac{\tan\left(\frac{\sqrt{w_\lambda^-}}{2}\right)}{\sqrt{w_\lambda^-}}\right)$$

where, for any $w \in \mathbb{C}$, $\sqrt{w}$ *is the complex number* z *such that*

$$z^2 = w \text{ and } z \in \{\zeta : \Re(\zeta) > 0\} \cup \{\zeta : \Re(\zeta) = 0 \text{ and } \Im(\zeta) \ge 0\}.$$

Then the following estimate holds

$$\limsup_{t\to\infty} y(t) \le \Omega_\lambda \|\Psi\|_{L^\infty([0,\pi/\omega])}. \tag{3.17}$$

Moreover, if $\Psi \in C^2(\mathbb{R}_+)$*, then*

$$\limsup_{t\to\infty} \dot{y}(t) \le \Omega_\lambda \|\dot{\Psi}\|_{L^\infty([0,\pi/\omega])}.$$

Proof. From Proposition 3.6, Eq (3.16) admits an antiperiodic solution $z(t)$ and any solution of $y(t)$ of (3.16) converges exponentially to $z(t)$, which yields that $\limsup_{t\to\infty} y(t) = \limsup_{t\to\infty} z(t)$. Hence, since from the antiperiodicity of $z(t)$ we have that $\limsup_{t\to\infty} z(t) = \|z\|_\infty$, in order to get the result it suffices to estimate the L^∞−norm of $z(t)$. In the notation of Proposition 3.6, we have that

$$z(t) := \sqrt{\frac{\omega}{\pi}} \sum_{n\in\mathbb{Z}} \frac{\psi_n}{-\omega^2(2n+1)^2 + \lambda + i\delta\omega(2n+1)} e^{(2n+1)i\omega t},$$

Then, if $c_n = \sqrt{(-\omega^2(2n+1)^2 + \lambda)^2 + \delta^2\omega^2(2n+1)^2}$, from Cauchy-Schwarz inequality we obtain

$$|z(t)| \leq \sqrt{\frac{\omega}{\pi}} \sum_{n\in\mathbb{Z}} \frac{|\psi_n|}{c_n} \leq \sqrt{\frac{\omega}{\pi}} \sqrt{\sum_{n\in\mathbb{Z}} |\psi_n|^2} \sqrt{2\sum_{n\geq 0} \frac{1}{c_n^2}}. \tag{3.18}$$

Moreover, if $\Psi \in C^2(\mathbb{R}_+)$, we have

$$|\dot{z}(t)| \leq \sqrt{\frac{\omega}{\pi}} \sum_{n\in\mathbb{Z}} \frac{|(2n+1)\omega\psi_n|}{c_n} \leq \sqrt{\frac{\omega}{\pi}} \sqrt{\sum_{n\in\mathbb{Z}} |(2n+1)\omega\psi_n|^2} \sqrt{2\sum_{n\geq 0} \frac{1}{c_n^2}}. \tag{3.19}$$

First, we remark that from Parseval's theorem

$$\begin{aligned} \sqrt{\sum_{n\in\mathbb{Z}} |\psi_n|^2} &= \|\Psi\|_{L^2([0,\pi/\omega])} \leq \sqrt{\frac{\pi}{\omega}} \|\Psi\|_{L^\infty([0,\pi/\omega])}, \\ \sqrt{\sum_{n\in\mathbb{Z}} |(2n+1)\omega\psi_n|^2} &= \|\dot{\Psi}\|_{L^2([0,\pi/\omega])} \leq \sqrt{\frac{\pi}{\omega}} \|\dot{\Psi}\|_{L^\infty([0,\pi/\omega])}. \end{aligned} \tag{3.20}$$

Then, to conclude the proof, we compute a closed form for the serie

$$\sum_{n\geq 0} \frac{1}{c_n^2} = \sum_{n\geq 0} \frac{1}{\omega^4(2n+1)^4 - (2\lambda - \delta^2)(2n+1)^2\omega^2 + \lambda^2}. \tag{3.21}$$

We observe that (3.21) becomes

$$\sum_{n\geq 0} \frac{1}{c_n^2} = \sum_{n\geq 0} \frac{\pi^4}{(w_\lambda^+ - w_\lambda^-)\omega^4} \left[\frac{1}{(2n+1)^2\pi^2 - w_\lambda^+} - \frac{1}{(2n+1)^2\pi^2 - w_\lambda^-} \right]. \tag{3.22}$$

We now recall that the Mittag-Leffler expansion for the cotangent function gives

$$\cot(w) = \frac{1}{w} + \sum_{n=1}^{\infty} \frac{2w}{w^2 - \pi^2 n^2}.$$

Some straightforward computations give

$$\frac{1}{2}\tan\left(\frac{w}{2}\right) = \frac{1}{2}\cot\left(\frac{w}{2}\right) - \cot(w) = \sum_{n=0}^{\infty} \frac{2w}{(2n+1)^2\pi^2 - w^2}.$$

Thus, we can infer that

$$\sum_{n\geq 0} \frac{1}{(2n+1)^2\pi^2 - w_\lambda} = \frac{\tan\left(\frac{\sqrt{w_\lambda}}{2}\right)}{4\sqrt{w_\lambda}}.$$

Hence, from (3.22) we can conclude that

$$\sum_{n\geq 0} \frac{1}{c_n^2} = \frac{\pi^4}{4\omega^4(w_\lambda^+ - w_\lambda^-)} \left(\frac{\tan\left(\frac{\sqrt{w_\lambda^+}}{2}\right)}{\sqrt{w_\lambda^+}} - \frac{\tan\left(\frac{\sqrt{w_\lambda^-}}{2}\right)}{\sqrt{w_\lambda^-}} \right). \tag{3.23}$$

By using (3.20) and (3.23) in (3.18) and (3.19), we obtain the thesis. □

In [36, Theorem 2.1], a result similar to Proposition 3.7 is proven. In particular, the maximum value of $\limsup_{t\to\infty} y(t)$ as the forcing term Ψ varies in the unitary ball of $L^\infty(\mathbb{R})$ is determined. On the other hand, for any fixed antiperiodic forcing term Ψ in $C^2(\mathbb{R})$, in Proposition 3.7 we estimated $\limsup_{t\to\infty} y(t)$ and $\limsup_{t\to\infty} \dot{y}(t)$. As Figure 1 shows, Proposition 3.7 almost always gives a better estimate on $\limsup_{t\to\infty} y(t)$.

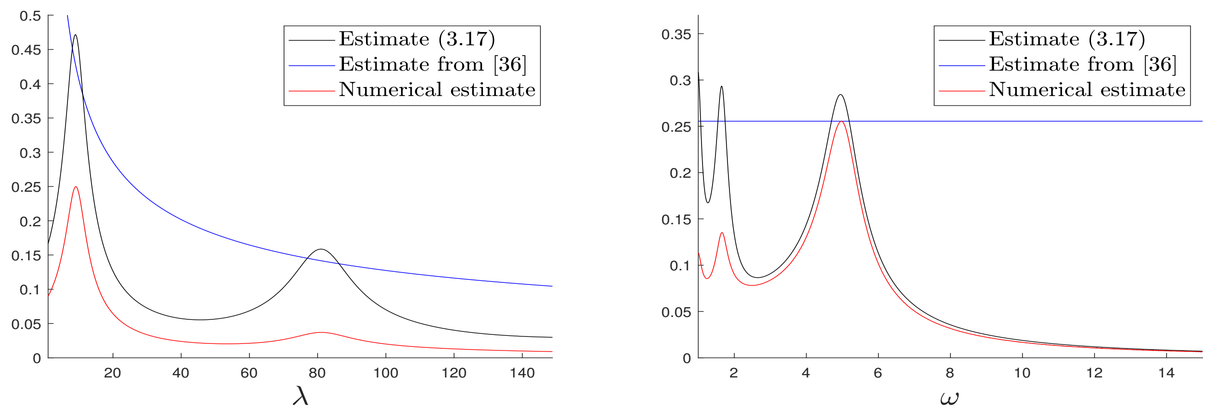

Figure 1. Comparison between the estimates on the $\|\cdot\|_\infty$−norm of y solution of (3.16) given by [36] (blue) and by (3.17) (black) with $\delta = 1$ and $\omega = 3$ as λ vary from 1 to 150 (left) and with $\delta = 1$ and $\lambda = 5$ as ω vary from 1 to 15 (right). In red, we represented the $\|\cdot\|_\infty$−norm of the antiperiodic solution of (3.16) with $\Psi(t) = \text{signum}(\sin(\omega t))$.

3.4. *Structure of the paper*

The remainder of the paper is organized as follows. First, in Section 4 we apply the results of Subsection 3.2 in order to prove Theorem 2.3. In particular, we apply Proposition 3.4 to prove that for N large enough, if g is exponentially N−dimensional, then there exists $\bar{N} \geq N$ such that any solution u of (2.5) is exponentially $\bar{N}$−dimensional (see Lemma 4.1). After that, fixed $n > N$, we study the asymptotic amplitude of $u_n(t) = (u(t), e_n)$ for any u solution of (2.5) and in Lemma 4.2 we determine whether $u_n(t)$ decays exponentially as t goes to infinity. In subsection 4.2 we exploit Lemma 4.1 and Lemma 4.2 in order to get Theorem 2.3. We remark that, even though the thesis of Theorem 2.3 follows from Lemma 4.1, Lemma 4.2 is necessary in order to improve the result of Lemma 4.1. More precisely, Lemma 4.2 provides an improvement of the smallest number $M \geq N$ obtained in Lemma 4.1 such that if g is exponentially $N-$dimensional then any solution u is exponentially $M-$dimensional.

Next, by exploiting the continuous dependence of the solution from the forcing term, that is, Proposition 3.3, and Theorem 2.3, in Section 5 we give the proof of Theorem 2.4.

In Section 5, by proceeding as in a result of Bonheure, Gazzola and Moreira dos Santos [8, Theorem 6], we show that (2.8) admits an antiperiodic solution p. In Lemma 6.2 we use Proposition 3.7 to estimate, for any $n \in \mathbb{N}$, the asymptotic amplitude of $p_n(t) := (p(t), e_n)$. Such result yields an estimate on the H^s-norms of p (see Lemma 6.3) which we numerically verified to be better than the a-priori estimates obtained in [8] (see Figure 2). From Proposition 3.3, we have that under suitable smallness conditions on $\limsup_{t\to\infty} \|g(t)\|$, any solution u of (2.8) converges to p in the phase space norm. Hence, from Lemma 6.2 and Lemma 6.3, in Lemma 6.4 we get an estimate on the asymptotic amplitude of $u_n(t) = (u(t), e_n)$ and on the H^s-norms of u for any u solution of (2.8). Finally, in Lemma 6.5, we exploit the previous results of Section 6 in order to get a results for finite-dimensional systems of ODEs and in Subsection 2.5 we apply Lemma 6.5 and Lemma 6.4 to get Theorem 2.5.

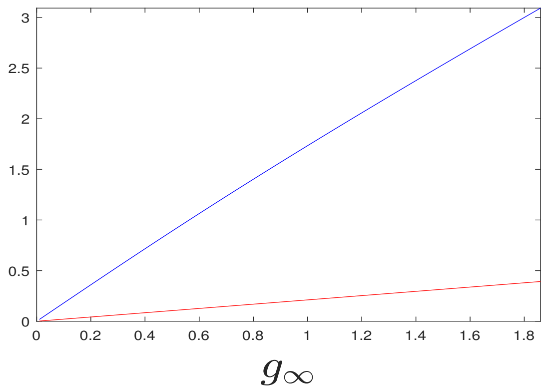

Figure 2. Comparison between the general estimate on $\limsup_{t\to\infty} \|u(t)\|_2$ (blue) and the one obtained by using the antiperiodicity of the forcing term (red).

4. Proof of Theorem 2.3

4.1. Stability of the higher modes

We now apply the results of the previous section to our framework in order to prepare the proof of Theorem 2.3.

Lemma 4.1. *Let u be a weak solution of* (2.5). *Let g be exponentially $N-$dimensional. If there exists $\bar{N} \geq N$ such that*

$$\limsup_{t\to\infty} \left(\frac{1}{\alpha_1^{1-\theta}} \|u(t)\|_2^2 + \|u_t(t)\|^2 \right) < 2\delta \alpha_{\bar{N}+1}^{(2-\theta)/2}$$

then there exists $\widetilde{\eta} > 0$ such that

$$\limsup_{t\to\infty} (\|Q_{\bar{N}} u(t)\|_2^2 + \|Q_{\bar{N}} u_t(t)\|^2) e^{\widetilde{\eta} t} = 0.$$

Proof. Fix $\bar{N} \geq N$ and, for any $s \in [0,2]$, let $\Upsilon_s := \limsup_{t\to\infty} \|u(t)\|_s^2$. We introduce the operator-valued function $B(t) := \|u(t)\|_\theta^2 A^\theta$. By using (2.3), we get that $w = Q_{\bar{N}} u$ solves

$$w_{tt} + \delta w_t + (A^2 + B(t))w = Q_{\bar{N}} g. \tag{4.1}$$

By using (2.4) we remark that for any $v \in \mathcal{H}^2$ such that $Q_{\bar{N}} v = v$

$$\begin{aligned} 0 \leq \limsup_{t\to\infty} (B(t)v, v) &= \limsup_{t\to\infty} \|u(t)\|_\theta^2 \|v\|_\theta^2 \leq \frac{\Upsilon_\theta}{\alpha_{\bar{N}+1}^{(2-\theta)/2}} \|v\|_2^2, \\ \limsup_{t\to\infty} (B'(t)v, v) &= \limsup_{t\to\infty} (u_t(t), A^\theta u(t)) \|v\|_\theta^2 \leq \\ &\leq \frac{1}{2\alpha_{\bar{N}+1}^{(2-\theta)/2}} \limsup_{t\to\infty} \left(\frac{1}{\alpha_1^{1-\theta}} \|u(t)\|_2^2 + \|u_t(t)\|^2 \right) \|v\|_2^2. \end{aligned} \tag{4.2}$$

We introduce

$$\varphi(t) = \frac{1}{2}(\|u_t(t)\|^2 + \|Au(t)\|^2) + \frac{\delta}{2}(u(t), u_t(t)) + \frac{\delta^2}{4}\|u(t)\|^2.$$

By applying Proposition 3.4 to (4.1), from (4.2) we get that if

$$\limsup_{t\to\infty} \left(\frac{1}{\alpha_1^{1-\theta}} \|u(t)\|_2^2 + \|u_t(t)\|^2 \right) < 2\delta \alpha_{\bar{N}+1}^{(2-\theta)/2},$$

then $\varphi(t) \to 0$ exponentially as t goes to infinity. This yields that there exists $\tilde{\eta} > 0$ such that

$$\lim_{t\to\infty} (\|Aw(t)\|^2 + \|w_t(t)\|^2) e^{\tilde{\eta} t} = 0.$$

Therefore, since $\|Aw\|^2 = \|w\|_2^2$, we get the thesis. □

We now apply Proposition 3.5 to the projection of (2.5) on the n−th mode. The following lemma holds.

Lemma 4.2. *Let g be exponentially N−dimensional. For any weak solution u of* (2.5)*, if*

$$\exists n \geq N + 1 \quad \textit{such that} \quad \limsup_{t\to\infty} \|u(t)\|_\theta^2 < \delta \max(2^\theta \delta^{1-\theta}, 2\alpha_n^{(1-\theta)/2}), \tag{4.3}$$

then for any $M \geq n$ there exists $\tilde{\eta} > 0$ such that for any $n \leq \bar{N} \leq M$

$$\lim_{t\to\infty} (|(u(t), e_{\bar{N}})|^2 + |(u_t(t), e_{\bar{N}})|^2) e^{\tilde{\eta} t} = 0.$$

Proof. Fixed $n \geq N + 1$, we consider the projection of u on the n−th mode, i.e., $u_n := (u, e_n)$. The function u_n satisfies

$$\ddot{u}_n + \delta \dot{u}_n + (\alpha_n + \|u(t)\|_\theta^2 \alpha_n^{\theta/2}) u_n = (g, e_n).$$

Since $n \geq N + 1$, for some $\eta > 0$, $\lim_{t\to\infty} (g(t), e_n) e^{\eta t} = 0$. Let us suppose that $\limsup_{t\to\infty} \|u(t)\|_\theta^2 < \delta \max(2^\theta \delta^{1-\theta}, 2\alpha_n^{(1-\theta)/2})$. Since

$$\max(2^\theta \delta^{1-\theta}, 2\alpha_n^{(1-\theta)/2}) \leq \max\left(\frac{\delta}{\alpha_n^{\theta/2}}, 2\alpha_n^{(1-\theta)/2} \right),$$

we have that

$$\alpha_n^{\theta/2} \limsup_{t\to\infty} \|u(t)\|_\theta^2 < \delta \max(\delta, 2\sqrt{\alpha_n}),$$

which yields that, from Proposition 3.5,

$$\lim_{t\to\infty} (|u_n(t)|^2 + |\dot{u}_n(t)|^2) e^{\eta_1 t} = 0.$$

Since $(\alpha_j)_j$ is strictly increasing, $\max(2^\theta \delta^{1-\theta}, 2\alpha_n^{(1-\theta)/2})$ is an increasing sequence. Hence, if (4.3) holds, then for any $\bar{N} \geq n$

$$\limsup_{t\to\infty} \|u(t)\|_\theta^2 \leq \delta \max(2^\theta \delta^{1-\theta}, 2\alpha_{\bar{N}}^{(1-\theta)/2}),$$

that implies that for any $M \geq n$ there exists $\tilde{\eta} > 0$ such that for any $n \leq \bar{N} \leq M$

$$\lim_{t\to\infty} (|u_{\bar{N}}(t)|^2 + |\dot{u}_{\bar{N}}(t)|^2) e^{\tilde{\eta} t} = 0,$$

that is the thesis. □

4.2. Completion of the proof of Theorem 2.3

Let g be exponentially $N-$dimensional and let u be a weak solution of (2.5). We recall that, from Proposition 3.2, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \|u(t)\|_\theta^2 \leq \frac{4E_\infty + 2\alpha^2 \Phi_0}{\sqrt{\alpha_1^{2-\theta} + 2(2E_\infty + \alpha^2\Phi_0)} + \alpha_1^{1-\theta/2}} =: \Phi_\theta; \\
&\limsup_{t\to\infty} \|u(t)\|_2^2 \leq 2E_\infty + \alpha^2 \Phi_0 =: \Phi_2; \\
&\limsup_{t\to\infty} \|u_t(t)\|^2 \leq \min_{\lambda>0} \frac{1+\lambda}{\lambda} \left(2E_\infty + \max_{s\in[0,\Phi_0]} \left((\lambda+1)\alpha^2 - \alpha_1 s - \frac{1}{2} s^2 \right) \right) =: \Phi_v.
\end{aligned} \tag{4.4}$$

We introduce the quantity $\bar{N}$ defined as the smallest integer number greater than N such that

$$\frac{1}{\alpha_1^{1-\theta}} \Phi_2 + \Phi_v < 2\delta \alpha_{\bar{N}+1}^{(2-\theta)/2}. \tag{4.5}$$

From (4.4), (4.5) implies

$$\limsup_{t\to\infty} \left(\frac{1}{\alpha_1^{1-\theta}} \|u(t)\|_2^2 + \|u_t(t)\|^2 \right) < 2\delta \alpha_{\bar{N}+1}^{(2-\theta)/2}.$$

Hence, from Lemma 4.1, if (4.5) holds then there exists $\eta_1 > 0$ such that

$$\lim_{t\to\infty} (\|Q_{\bar{N}} u(t)\|_2^2 + \|Q_{\bar{N}} u_t(t)\|^2) e^{\eta_1 t} = 0.$$

We introduce the set

$$B := \{n \in \mathbb{N} : n \in [N, \bar{N}] \text{ and } \Phi_\theta < \delta \max(2^\theta \delta^{1-\theta}, 2\alpha_{n+1}^{(1-\theta)/2})\}$$

and we define

$$\underline{N} := \begin{cases} \min B & \text{if } B \neq \emptyset \\ +\infty & \text{if } B = \emptyset. \end{cases}$$

From Proposition 3.2 we have that $\limsup_{t\to\infty} \|u(t)\|_\theta^2 \le \Phi_\theta$. Hence, from Lemma 4.2, if $\underline{N} \neq +\infty$, there exists $\eta_2 > 0$ such that

$$\lim_{t\to\infty}(|(u(t), e_{n+1})|^2 + |(\dot{u}(t), e_{n+1})|^2)e^{\eta_2 t} = 0$$

for any $n \in [\underline{N}, \bar{N}] \cap \mathbb{N}$, which yields

$$\lim_{t\to\infty}(\|Q_{\underline{N}} P_{\bar{N}} u(t)\|_2^2 + \|Q_{\underline{N}} P_{\bar{N}} u_t(t)\|^2)e^{\eta_2 t}.$$

Hence, if we set $P_\infty := I$, $Q_\infty := 0$ and $M := \min\{\underline{N}, \bar{N}\}$, for some $\tilde{\eta} > 0$

$$\begin{aligned}\lim_{t\to\infty}(\|Q_M u(t)\|_2^2 + \|Q_M u_t(t)\|^2)e^{\tilde{\eta} t} &= \lim_{t\to\infty}(\|Q_{\underline{N}} P_{\bar{N}} u(t)\|_2^2 + \|Q_{\underline{N}} P_{\bar{N}} u_t(t)\|^2)e^{\tilde{\eta} t} + \\ &+ \lim_{t\to\infty}(\|Q_{\bar{N}} u(t)\|_2^2 + \|Q_{\bar{N}} u_t(t)\|^2)e^{\tilde{\eta} t} = 0.\end{aligned}$$

This concludes the proof of Theorem 2.3.

5. Proof of Theorem 2.4

Let us suppose that

$$\frac{2\sqrt{\Phi_\theta \Phi_2}\,\alpha_1^{(\theta-2)/4} + \Phi_\theta}{2\alpha_1^{(1-\theta)/2}} \max\left(\frac{1}{\delta}, \frac{1}{\sqrt{\alpha_1}}\right) < 1, \tag{5.1}$$

where Φ_θ and Φ_2 are defined in Proposition 3.2. Since Φ_θ and Φ_2 depend on g_∞ and δ, we get that, for any fixed δ, (5.1) translates into $F_\theta(\alpha_1, \delta, g_\infty) < 1$ for some F_θ. Therefore, for any fixed $\delta > 0$, there exists $\bar{g}_\infty > 0$ such that if $g_\infty < \bar{g}_\infty$, then (5.1) holds. We remark that, since the term g models the action of the wind along the deck of the bridge, we physically interpret (5.1) as a requirement on the load exerted on the structure by the wind. In particular, since $\bar{g}_\infty$ in engineering applications (see [23]) is proportional to the speed of the wind v, the relation (5.1) is equivalent to require that $v < \bar{v}$ for some $\bar{v} > 0$.

Let u be a weak solution of (2.5) and for any $J = \{j_1, \ldots, j_m\}$ let v^J be a weak solution of the problem

$$v_{tt}^J + \delta v_t^J + A^2 v^J + \|v^J\|_\theta^2 A^\theta v^J = P_J g.$$

We introduce the quantities $\Upsilon_\mu = \limsup_{t\to\infty} \|(u(t) + v^J(t))/2\|_\mu^2$, where $\mu \in [0, 2]$. From Proposition 3.3 with $g_1 = P_J g$ and $g_2 = g = P_J g + Q_J g$, there exists a function $\mathcal{F}_\theta = \mathcal{F}_\theta(\alpha_1, \delta, \Upsilon_\theta, \Upsilon_{2\theta})$, given by (3.4), such that if $\mathcal{F}_\theta < 1$ then there exists a constant $C > 0$ such that

$$\limsup_{t\to\infty}(\|u(t) - v^J(t)\|_2^2 + \|u_t(t) - v_t^J(t)\|^2) \le C \limsup_{t\to\infty} \|Q_J g(t)\|. \tag{5.2}$$

Since $g = \mathfrak{g} f(t)$, for a suitable choice of J, we have that $C \limsup_{t\to\infty} \|Q_J g(t)\| < \varepsilon$. Hence we can conclude that, for a suitable choice of the family J, (5.2) gives

$$\limsup_{t\to\infty}(\|u(t) - v^J(t)\|_2^2 + \|u_t(t) - v_t^J(t)\|^2) \le \varepsilon. \tag{5.3}$$

From Proposition 3.2 and (2.2), we have that $\Upsilon_\theta \leq \Phi_\theta$ and $\Upsilon_{2\theta} < \alpha_1^{\theta-1}\Phi_2$. Hence, $\mathcal{F}_\theta < 1$ is implied by (5.1). Therefore, fixed δ, if $g_\infty < \bar{g}_\infty$ for some positive constant $\bar{g}_\infty$, where $\bar{g}_\infty$ does not depend by J, then (5.3) holds. This proves the first part of Theorem 2.4.

Let now g be exponentially N-dimensional and let $M \geq N$ be obtained from Theorem 2.3, i.e., let $M \geq N$ be such that for some $\eta > 0$

$$\lim_{t\to\infty}(\|Q_M u(t)\|_2^2 + \|Q_M u_t(t)\|^2)e^{\eta t} = 0. \tag{5.4}$$

Let u and v be, respectively, weak solutions of (2.5) and

$$v_{tt} + \delta v_t + A^2 v + \|v\|_\theta^2 A^\theta v = P_M g.$$

We remark that u is solution of the following problem

$$u_{tt} + \delta u_t + A^2 u + \|u\|_\theta^2 A^\theta u = g = P_M g + Q_M g.$$

Since we supposed g to be exponentially N-dimensional and $M \geq N$, there exists $\eta > 0$ such that

$$\lim_{t\to\infty} \|P_M g(t) + Q_M g(t) - P_M g(t)\|e^{\eta t} = \lim_{t\to\infty} \|Q_M g(t)\|e^{\eta t} = 0.$$

Therefore, from Proposition 3.3 with $g_1 = P_M g$ and $g_2 = g = P_M g + Q_M g$ we have that, fixed δ, if g_∞ is sufficiently small, then there exists $\eta_1 > 0$ such that

$$\lim_{t\to\infty}(\|u(t) - v(t)\|_2^2 + \|u_t(t) - v_t(t)\|^2)e^{\eta_1 t} = 0.$$

Since $v = P_M v$, from (5.4) we get that for some $\tilde{\eta} > 0$

$$\lim_{t\to\infty}(\|P_M u(t) - v(t)\|_2^2 + \|P_M u_t(t) - v_t(t)\|^2)e^{\tilde{\eta} t} = 0.$$

This concludes the proof of Theorem 2.4.

6. Proof of Theorem 2.5

6.1. Some preliminary results

In Theorem 2.5, we restrict ourselves to the case when the forcing term is antiperiodic in time due to the engineering interest of this case (see Section 7). Moreover, for the sake of simplicity, we consider the case $\theta = 0$. The antiperiodicity of the forcing term allows us to provide some more information about the solution of (2.8). In particular, proceeding as in Theorem 6 of [8], where the result was proven in the periodic framework, by using Proposition 3.6, we obtain the following statement:

Proposition 6.1. *If $g(t)$ is a continuous antiperiodic function of anti-period τ, then there exists a solution of* (2.5) *antiperiodic of anti-period τ.*

Proof. The proof proceeds as in [8, Theorem 6]. First, we fix $n \geq 1$ and we prove the existence of a $\tau-$antiperiodic solution for the problem

$$u_{tt} + \delta u_t + A^2 u + \|u\|^2 u = P_n g. \tag{6.1}$$

Hence, we seek a $\tau-$antiperiodic solution u^n in the form

$$u^n(x,t) := \sum_{k=1}^{n} h_k^n(t) e_k(x).$$

We consider the spaces $C^2_\tau(\mathbb{R})$ and $C^0_\tau(\mathbb{R})$ of C^2 and C^2 $\tau-$antiperiodic functions and in the same notations of [8, Theorem 6] we have that (6.1) is equivalent to

$$L_n(\boldsymbol{h}(t)) + \nabla G_n(\boldsymbol{h}(t)) = \boldsymbol{g}(t),$$

where $\boldsymbol{h} := (h_1^n, \dots, h_n^n)$, $\boldsymbol{g} := (g_1, \dots, g_n)$, L_n is a diagonal operator such that

$$L_n^k(\boldsymbol{h}) := \ddot{h}_k + \delta \dot{h}_k + \alpha_k h_k$$

and

$$G_n(\boldsymbol{h}) := \frac{1}{4} \sum_{j,k=1}^{n} h_j^2 h_k^2.$$

We observe that for any $\boldsymbol{q} \in (C^0_\tau(\mathbb{R}))^n$ from Proposition 3.6 there exists a unique $\boldsymbol{h} \in (C^2_\tau(\mathbb{R}))^n$ such that $L_n(\boldsymbol{h}) = \boldsymbol{q}$. Thanks to the compact embedding $(C^2_\tau(\mathbb{R}))^n \subset (C^0_\tau(\mathbb{R}))^n$, we have that the nonlinear map $\Gamma_n : (C^0_\tau(\mathbb{R}))^n \times [0,1] \to (C^0_\tau(\mathbb{R}))^n$ defined by

$$\Gamma_n(\boldsymbol{h}, \nu) = L_n^{-1}(\boldsymbol{g} - \nu \nabla G_n(\boldsymbol{h})), \quad \forall (\boldsymbol{h}, \nu) \in (C^0_\tau(\mathbb{R}))^n \times [0,1]$$

is compact. Moreover, from Proposition 3.2 we have that there exists $H_n > 0$ (independent of ν) such that if $\boldsymbol{h} \in (C^0_\tau(\mathbb{R}))^n$ solves $\boldsymbol{h} = \Gamma_n(\boldsymbol{h}, \nu)$, then

$$\|\boldsymbol{h}\|_{(C^0_\tau(\mathbb{R}))^n} \leq H_n.$$

Hence, since the equation $\boldsymbol{h} = \Gamma_n(\boldsymbol{h}, 0)$ from Proposition 3.6 admits a unique $\tau-$antiperiodic solution, the Leray-Schauder principle ensures the existence of a solution $\boldsymbol{h} \in (C^0_\tau(\mathbb{R}))^n$ of $\boldsymbol{h} = \Gamma_n(\boldsymbol{h}, 1)$. This proves the existence of a $\tau-$antiperiodic solution of (6.1). The proof the result follows from the existence of a $\tau-$antiperiodic solution of (6.1) exactly as in [8, Theorem 6] by showing that the sequence (u^n) converges to a $\tau-$antiperiodic solution u of (2.8). □

In this section we use the quantities

$$\begin{aligned} w_j^{\pm} &:= \frac{\pi^2}{\omega^2}\left(\alpha_j - \frac{\delta^2}{2} \pm \delta \sqrt{\frac{\delta^2}{4} - \alpha_j}\right), \\ \Omega_j^2 &:= \frac{\pi^4}{2\omega^4 (w_j^+ - w_j^-)} \left(\frac{\tan\left(\frac{\sqrt{w_j^+}}{2}\right)}{\sqrt{w_j^+}} - \frac{\tan\left(\frac{\sqrt{w_j^-}}{2}\right)}{\sqrt{w_j^-}} \right) \end{aligned} \tag{6.2}$$

obtained by replacing λ by α_j in Proposition 3.7.

We now apply Proposition 3.7 in order to get an estimate on the $j-$th mode of the antiperiodic solution p of (2.8), which we proved to exist in Proposition 6.1. In the following, whenever a real-valued function $f(t)$ will be antiperiodic, we will write interchangeably $\limsup_{t\to\infty} f(t)$ and $\|f\|_\infty$.

Lemma 6.2. *Let p be an antiperiodic solution of* (2.8). *If*

$$\max_j \Omega_j \limsup_{t\to\infty} \|p(t)\|^2 < 1 \tag{6.3}$$

where Ω_j is defined in (6.2)*, then, if we set $\Upsilon_0 := \limsup_{t\to\infty} \|p(t)\|^2$ and $\Upsilon_v := \limsup_{t\to\infty} \|p_t(t)\|^2$,*

$$\frac{g_j}{(1+\Upsilon_0\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \le \limsup_{t\to\infty}|p_j(t)| \le \frac{g_j}{(1-\Upsilon_0\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$
$$\frac{(\omega(1-\Upsilon_0\Omega_j)-2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j)g_j}{(1-(\Upsilon_0\Omega_j)^2)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \le \limsup_{t\to\infty}|\dot p_j(t)| \le \frac{(\omega(1-\Upsilon_0\Omega_j)+2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j)g_j}{(1-\Upsilon_0\Omega_j)^2\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$

where $p_j := (p, e_j)$ and $g_j := \limsup_{t\to\infty}(g(t), e_j) = (\mathfrak{g}, e_j)$.

Proof. We study the $j-$th component of the problem (2.8), namely

$$\ddot p_j + \delta \dot p_j + \alpha_j p_j + \|p\|^2 p_j = g_j \sin(\omega t). \tag{6.4}$$

We consider the antiperiodic solution v of the problem

$$\ddot v + \delta \dot v + \alpha_j v = g_j \sin(\omega t). \tag{6.5}$$

It is possible to verify that the general solution of (6.5) is given by

$$v(t) = \frac{g_j}{\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \sin\left(\omega t + \arctan\frac{\delta\omega}{\omega^2-\alpha_j}\right) + S\,e^{-\delta t/2}\sin\left(\frac{t}{2}\sqrt{4\alpha_j-\delta^2}+\varphi\right),$$

where the constants S and φ are determined by the initial data of (6.5). Hence, it follows that, for any choice of the initial data of (6.5),

$$\limsup_{t\to\infty} v(t) = \frac{g_j}{\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}}, \qquad \limsup_{t\to\infty} \dot v(t) = \frac{\omega g_j}{\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}}. \tag{6.6}$$

If we subtract (6.5) from (6.4), if $w := p_j - v$ we get

$$\ddot w + \delta \dot w + \alpha_j w = -\|p\|^2 p_j.$$

Hence, from Proposition 3.7 we get, if $\mathfrak{p}_j^{(0)} := \limsup_{t\to\infty} p_j(t)$, $\mathfrak{p}_j^{(1)} := \limsup_{t\to\infty} \dot p_j(t)$, $\Upsilon_0 := \limsup_{t\to\infty} \|p(t)\|^2$ and $\Upsilon_v := \limsup_{t\to\infty} \|p_t(t)\|^2$,

$$\begin{aligned}
&\limsup_{t\to\infty} |w(t)| \le \Upsilon_0 \Omega_j \mathfrak{p}_j^{(0)},\\
&\limsup_{t\to\infty} |\dot w(t)| \le \Omega_j \|2(p(t), p_t(t)) p_j(t) + \|p(t)\|^2 \dot p_j(t)\|_{L^\infty(0,\pi/\omega)} \le \\
&\qquad \le 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j \mathfrak{p}_j^{(0)} + \Upsilon_0\Omega_j \mathfrak{p}_j^{(1)}.
\end{aligned} \tag{6.7}$$

Since p and v are both antiperiodic, w is antiperiodic and (6.7) gives

$$\left|\|v\|_\infty - \|p_j\|_\infty\right| \le \|w\|_\infty \le \Upsilon_0 \Omega_j \mathfrak{p}_j^{(0)},$$

$$\left| \|\dot{v}\|_\infty - \|\dot{p}_j\|_\infty \right| \le \|\dot{w}\|_\infty \le 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j \mathfrak{p}_j^{(0)} + \Upsilon_0\Omega_j\mathfrak{p}_j^{(1)}.$$

We get then

$$\limsup_{t\to\infty} v(t) - \Upsilon_0\Omega_j\mathfrak{p}_j^{(0)} \le \mathfrak{p}_j^{(0)} \le \limsup_{t\to\infty} v(t) + \Upsilon_0\Omega_j\mathfrak{p}_j^{(0)},$$

$$\limsup_{t\to\infty} \dot{v}(t) - 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j\mathfrak{p}_j^{(0)} - \Upsilon_0\Omega_j\mathfrak{p}_j^{(1)} \le \mathfrak{p}_j^{(1)} \le \limsup_{t\to\infty} \dot{v}(t) + \Upsilon_0\Omega_j\mathfrak{p}_j^{(1)} + 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j\mathfrak{p}_j^{(0)}.$$

Hence, from (6.6) we get, since hypothesis (6.3) holds,

$$\frac{g_j}{(1+\Upsilon_0\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \le \mathfrak{p}_j^{(0)} \le \frac{g_j}{(1-\Upsilon_0\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$

which yields

$$\frac{(\omega(1-\Upsilon_0\Omega_j) - 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j)g_j}{(1-(\Upsilon_0\Omega_j)^2)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \le \mathfrak{p}_j^{(1)} \le \frac{(\omega(1-\Upsilon_0\Omega_j) + 2\sqrt{\Upsilon_0\Upsilon_v}\Omega_j)g_j}{(1-\Upsilon_0\Omega_j)^2\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}}$$

that is the thesis. □

We now apply the results of Lemma 6.2 in order to get an estimate on the $\mathcal{H}-$norm and $\mathcal{H}^2-$norm of an antiperiodic solution p of (2.8).

Lemma 6.3. *Let p be an antiperiodic solution of* (2.8). *Let us suppose that*

$$\max_j \Omega_j\Phi_0 < 1,$$

where Φ_0 is defined in Proposition 3.2. Then the following estimates hold:

$$\limsup_{t\to\infty} \|p(t)\|^2 \le \sum_{j=1}^{\infty} \frac{g_j^2}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)} =: \varphi < \infty, \tag{6.8}$$

$$\limsup_{t\to\infty} \|p_t(t)\|^2 \le \sum_{j=1}^{\infty} \frac{(\omega(1-\Phi_0\Omega_j)+2\sqrt{\Phi_0\Phi_v}\Omega_j)^2 g_j^2}{(1-\Phi_0\Omega_j)^4((\alpha_j-\omega^2)^2+\delta^2\omega^2)} =: \varphi_v < \infty, \tag{6.9}$$

$$\limsup_{t\to\infty} \|p(t)\|_2^2 \le \sum_{j=1}^{\infty} \frac{\alpha_j g_j^2}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)} =: \varphi_2 < \infty. \tag{6.10}$$

Proof. We prove (6.10) only, since the proofs of (6.8) and (6.9) are completely analogous. From Lemma 6.2, by using that from Proposition 3.2 $\Upsilon_0 := \limsup_{t\to\infty} \|p(t)\|^2 \le \Phi_0$,

$$\limsup_{t\to\infty} \|p(t)\|_2^2 \le \sum_{j=1}^{\infty} \alpha_j \|p_j\|_\infty^2 \le \sum_{j=1}^{\infty} \frac{\alpha_j g_j^2}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)}.$$

We recall that the sequence $(\alpha_j)_j$ is divergent. Therefore, for j large enough, $w_j^- = \overline{w_j^+}$ and $|w_j^+ - w_j^-| = 2\pi^2\delta\sqrt{\alpha_j - \delta^2/4}/\omega^2 \ge \pi^2\delta\sqrt{\alpha_j}/\omega^2$. Hence

$$\left|\Omega_j^2\right| \le \frac{\pi^2}{\delta\omega^2\sqrt{\alpha_j}} \left| \Im\left(\frac{\tan\left(\frac{\sqrt{w_j^+}}{2}\right)}{\sqrt{w_j^+}} \right) \right| \le \frac{\pi^2}{\delta\omega^2\sqrt{\alpha_j}} \frac{\left|\tan\left(\frac{\sqrt{w_j^+}}{2}\right)\right|}{\sqrt{|w_j^+|}}.$$

We remark that

$$|\tan(a+ib)| \leq \sqrt{\frac{\sin^2(2a)+\sinh^2(2b)}{(\cos(2a)+\cosh(2b))^2}}.$$

Moreover, from the definition of w_j^+ (see (6.2)), we have that $\Im(w_j^+) \to +\infty$. Hence, we conclude that $\lim_{j\to\infty} |\tan(\sqrt{w_j^+}/2)| = 1$ and consequently

$$\lim_{t\to\infty} \Omega_j = 0.$$

Then, since $\lim_{j\to\infty} \alpha_j = +\infty$ and $\max_j \Omega_j \Phi_0 < 1$, we have that, for some positive constant C, for any $j \in \mathbb{N}$

$$\frac{\alpha_j}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)} < C.$$

Therefore, by using that

$$\sum_{j=1}^{\infty} g_j^2 = \|g\|^2 < \infty,$$

we get that

$$\sum_{j=1}^{\infty} \frac{\alpha_j g_j^2}{(1-\Phi_0\Omega_j)^2((\alpha_j-\omega^2)^2+\delta^2\omega^2)} \leq \sum_{j=1}^{\infty} C g_j^2 = C\|g\|^2 < \infty,$$

that is the thesis. □

We observe that, from Proposition 3.3, any solution u of (2.8) exponentially converges to p under suitable smallness conditions on $\|g\|$. Hence, Lemma 6.2 and Lemma 6.3 hold for any weak solution u of (2.8). More precisely, the following lemma holds.

Lemma 6.4. *Let u be a weak solution of* (2.8)*. If*

$$\max_j \Omega_j \Phi_0 < 1, \qquad F(\xi_\infty) < 1,$$

where $F(\xi) = 3\xi \max(1/\delta, 1/(2\sqrt{\alpha_1}))/\sqrt{\alpha_1}$ and $\xi_\infty := \left((\sqrt{\Phi_0}+\sqrt{\varphi})/2\right)^2$, then

$$\limsup_{t\to\infty} \|u(t)\|^2 \leq \varphi, \quad \limsup_{t\to\infty} \|u(t)\|_2^2 \leq \varphi_2, \quad \limsup_{t\to\infty} \|u_t(t)\|^2 \leq \varphi_v,$$

and

$$\frac{g_j}{(1+\varphi\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \leq \limsup_{t\to\infty} |(u(t), e_j)| \leq \frac{g_j}{(1-\varphi\Omega_j)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$
$$\frac{(\omega(1-\varphi\Omega_j)-2\sqrt{\varphi\varphi_v}\Omega_j)g_j}{(1-(\varphi\Omega_j)^2)\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}} \leq \limsup_{t\to\infty} |(u_t(t), e_j)| \leq \frac{(\omega(1-\varphi\Omega_j)+2\sqrt{\varphi\varphi_v}\Omega_j)g_j}{(1-\varphi\Omega_j)^2\sqrt{(\alpha_j-\omega^2)^2+\delta^2\omega^2}},$$

where φ, φ_v and φ_2 are defined in (6.8)*,* (6.9) *and* (6.10) *respectively.*

Proof. Let p be an antiperiodic solution of (2.8). We define $w = p - u$. The function w solves

$$w_{tt} + \delta w_t + A^2 w + \|p\|^2 p - \|u\|^2 u = 0.$$

We proceed as in Proposition 3.3 and we get that if

$$F(\limsup_{t\to\infty} \|\xi(t)\|^2) < 1$$

where $\xi = (u + p)/2$, then

$$\lim_{t\to\infty}(\|u(t) - p(t)\|_2^2 + \|u_t(t) - p_t(t)\|^2) = 0. \tag{6.11}$$

Since

$$\limsup_{t\to\infty} \|\xi(t)\| \leq \frac{\limsup_{t\to\infty} \|u(t)\| + \limsup_{t\to\infty} \|p(t)\|}{2} \leq \frac{\sqrt{\Phi_0} + \sqrt{\varphi}}{2},$$

from the monotonicity of F we get that $F(\xi_\infty) < 1$ implies (6.11). Hence, the thesis follows from Lemma 6.2 and Lemma 6.3. □

6.2. The role of a single mode in the dynamics

Let us consider the finite-dimensional problem

$$\ddot{\underline{x}} + \delta\dot{\underline{x}} + \Lambda\underline{x} + \|\underline{x}\|^2\underline{x} = \underline{g}(t) \tag{6.12}$$

where $\underline{x}(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$, $\underline{g}(t) = (g_1(t), \ldots, g_n(t))$, $\Lambda = \text{diag}(\alpha_j)_{j=1}^n$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. This problem is a finite-dimensional approximation of (2.8).

Here, we estimate how much the evolution of the system changes as we eliminate a single mode from the dynamics. For the sake of simplicity, in the following we consider the case when the higher mode is the one we choose to neglect. We observe that

$$P_{n-1}\ddot{\underline{x}} + \delta\dot{\underline{x}} + \Lambda_{n-1}P_{n-1}\underline{x} + \|P_{n-1}\underline{x}\|^2 P_{n-1}\underline{x} + x_n^2 P_{n-1}\underline{x} = P_{n-1}\underline{g}(t) \tag{6.13}$$

where $P_{n-1}(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1})$, $\Lambda_{n-1} = \text{diag}(\alpha_j)_{j=1}^{n-1}$. We consider now the function $\underline{y}(t)$, solution of

$$\ddot{\underline{y}} + \delta\dot{\underline{y}} + \Lambda_{n-1}\underline{y} + \|(\underline{y}, 0)\|^2\underline{y} = P_{n-1}\underline{g}(t) \tag{6.14}$$

At this point, the question is reduced to estimate the (asymptotic) distance between the solution $\underline{x}$ of (6.12) and the solution $\underline{y}$ of (6.14). To this end, with a slight abuse of notations, we introduce the $\mathbb{R}^n$−norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined by $\|\underline{x}\|_1 = |x_1| + \cdots + |x_n|$ and $\|\underline{x}\|_2 = \sqrt{\alpha_1|x_1|^2 + \cdots + \alpha_n|x_n|^2}$. We remark that the result is completely independent of the choice of the mode neglected. The following lemma holds.

Lemma 6.5. *Let* $\underline{x}$ *and* $\underline{y}$ *be solutions of Eqs* (6.12) *and* (6.14) *respectively. Let* $g = \underline{g}\sin(\omega t)$ *with* $\underline{g} \in \mathbb{R}^n$ *and we suppose that* $F(\xi_\infty) < 1$, *where* ξ_∞ *is defined in Lemma 6.4 and* $F(\xi) = 3\xi\max(1/\delta, 1/(2\sqrt{\alpha_1}))/\sqrt{\alpha_1}$. *Moreover, we suppose that*

$$\max_j \Omega_j\Phi_0 < 1, \qquad \max_j \Omega_j\varphi < 1.$$

Then there exists a function S of the parameters of the problem such that if $S < 1$ then we have that

$$\limsup_{t\to\infty} \|P_{n-1}\underline{x}(t) - \underline{y}(t)\|_2 \le C(\underline{\chi})\chi_n^2,$$
$$\limsup_{t\to\infty} \|P_{n-1}\underline{\dot{x}}(t) - \underline{\dot{y}}(t)\| \le C_1(\underline{\chi}, \underline{\chi}_v)\chi_n^2 + C_2(\underline{\chi}, \underline{\chi}_v)\chi_{n,v}\chi_n$$

where $\underline{\chi} = (\chi_1, \dots \chi_n)$, $\chi_j := \limsup_{t\to\infty} \max(|x_j(t)|, |y_j(t)|)$, $\underline{\chi}_v = (\chi_{1,v}, \dots \chi_{n,v})$ *and* $\chi_{j,v} := \limsup_{t\to\infty} \max(|\dot{x}_j(t)|, |\dot{y}_j(t)|)$.

Proof. First, we remark that as in Lemma 6.4, since $F(\xi_\infty) < 1$, we have that there exist two antiperiodic functions $\underline{p_1} \in C^2(\mathbb{R}_+, \mathbb{R}^n)$ and $\underline{p_2} \in C^2(\mathbb{R}_+, \mathbb{R}^{n-1})$ such that

$$\lim_{t\to\infty} \|\underline{x}(t) - \underline{p_1}(t)\|_2^2 + \|\underline{\dot{x}}(t) - \underline{\dot{p}_1}(t)\|^2 = 0,$$
$$\lim_{t\to\infty} \|\underline{y}(t) - \underline{p_2}(t)\|_2^2 + \|\underline{\dot{y}}(t) - \underline{\dot{p}_2}(t)\|^2 = 0.$$

Therefore, since we are interested in the asymptotic behavior of our system, we can restrict ourselves to the case when $\underline{x}$ and $\underline{y}$ are both antiperiodic without loss of generality.

Let us consider the difference between Eqs (6.13) and (6.14). If we set $\underline{w} := P_{n-1}\underline{x}$ and $\underline{z} := \underline{w} - \underline{y}$, we get

$$\underline{\ddot{z}} + \delta\underline{\dot{z}} + \Lambda_{n-1}\underline{z} = \underline{\Psi}$$

where $\underline{\Psi} = -x_n^2\underline{w} - (\|\underline{w}\|^2 - \|\underline{y}\|^2)\underline{y} - \|\underline{w}\|^2\underline{z}$ and for the sake of simplicity, abusing the notations, we wrote $\|\underline{w}\|$ and $\|\underline{y}\|$ instead of $\|(\underline{w}, 0)\|$ and $\|(\underline{y}, 0)\|$ respectively.

We focus on one component, say j, in order to treat only scalar quantities. Hence, we consider the equation

$$\ddot{z}_j + \delta\dot{z}_j + \alpha_j z_j = \Psi_j \tag{6.15}$$

where $\Psi_j = -x_n^2 x_j - (\|\underline{w}\|^2 - \|\underline{y}\|^2)y_j - \|\underline{w}\|^2 z_j = -x_n^2 x_j - (\underline{w} - \underline{y}, \underline{w} + \underline{y})y_j - \|\underline{w}\|^2 z_j$. The fact that $\underline{x}$ and $\underline{y}$ are antiperiodic implies that $\underline{\Psi}$ is antiperiodic too. Hence, we can apply Proposition 3.7 to (6.15) and, if we introduce the quantities

$$\varphi := \max_{t\ge0} \max(\|\underline{x}(t)\|^2, \|\underline{y}(t)\|^2), \qquad \varphi_v := \max_{t\ge0} \max(\|\underline{\dot{x}}(t)\|^2, \|\underline{\dot{y}}(t)\|^2),$$
$$\chi_j := \max(\|x_j\|_\infty, \|y_j\|_\infty), \qquad \chi_{j,v} := \max(\|\dot{x}_j\|_\infty, \|\dot{y}_j\|_\infty) \quad \text{for } j = 1, \dots, n,$$

then, set $\mathcal{Z} := \max_{t\ge0} \|\underline{z}(t)\|$, we have

$$\|z_j\|_\infty \le \Omega_j \|\Psi_j\|_\infty \le \Omega_j(\chi_n^2\chi_j + 2\sqrt{\varphi}\chi_j\mathcal{Z} + \varphi\|z_j\|_\infty).$$

Therefore, set $Z_j := \|z_j\|_\infty$ and $C_j := \Omega_j\varphi$, by requiring that $C_j < 1$ for any $j = 1, \dots n$ we get

$$Z_j \le \frac{C_j\chi_j}{1 - C_j}\left(\frac{\chi_n^2 + 2\sqrt{\varphi}\mathcal{Z}}{\varphi}\right). \tag{6.16}$$

We define the quantity

$$S := \sum_{j=1}^{n-1} \frac{2C_j\chi_j}{(1 - C_j)\sqrt{\varphi}}$$

and we suppose $S < 1$.

We remark that for any $\underline{x} \in \mathbb{R}^n$, $\|\underline{x}\| \leq \|\underline{x}\|_1 := |x_1| + \ldots |x_n|$ and, for any bounded function $\underline{f} : \mathbb{R} \to \mathbb{R}^n$, $\sup_t \|\underline{f}(t)\|_1 \leq \|f_1\|_\infty + \cdots + \|f_n\|_\infty$. Hence we have that $\mathcal{Z} \leq \sum_{j=1}^{n-1} Z_j$. Therefore, by summing (6.16) over j and solving in $\mathcal{Z}$ we get

$$\mathcal{Z} \leq \frac{S}{1-S} \frac{\chi_n^2}{2\sqrt{\varphi}}. \tag{6.17}$$

Next, we remark that for any bounded function $\underline{f} : \mathbb{R} \to \mathbb{R}^n$ we have that $\sup_t \|\underline{f}(t)\|_2 \leq \sqrt{\alpha_1}\|f_1\|_\infty + \cdots + \sqrt{\alpha_n}\|f_n\|_\infty$. Hence $\mathcal{Z}_2 := \max_{t \geq 0} \|\underline{z}(t)\|_2 \leq \sum_{j=1}^{n-1} \sqrt{\alpha_j} Z_j$ and from (6.16) and (6.17) it follows that

$$\mathcal{Z}_2 \leq \sum_{j=1}^{n-1} \sqrt{\alpha_j} Z_j \leq \sum_{j=1}^{n-1} \frac{C_j \chi_j \sqrt{\alpha_j}}{1 - C_j} \left(\frac{\chi_n^2 + 2\sqrt{\varphi}\mathcal{Z}}{\varphi} \right) \leq \frac{1}{\varphi(1-S)} \sum_{j=1}^{n-1} \frac{C_j \chi_j \sqrt{\alpha_j}}{1 - C_j} \chi_n^2. \tag{6.18}$$

In particular, from (6.17) and (6.18) we conclude that there exist two positive constants b and c such that

$$\mathcal{Z} \leq b\chi_n^2, \qquad \mathcal{Z}_2 \leq c\chi_n^2. \tag{6.19}$$

Moreover, from (6.19) and (6.16), there exist constants a_j such that

$$Z_j \leq a_j \chi_n^2 \quad \text{for any } j = 1, \ldots, n-1. \tag{6.20}$$

We now define $Z_j^{(1)} := \|\dot{z}_j\|_\infty$ and $\mathcal{Z}^{(1)} := \max_{t \geq 0} \|\dot{\underline{z}}(t)\|$. By applying Proposition 3.7 to (6.15) we get

$$Z_j^{(1)} = \limsup_{t \to \infty} |\dot{z}_j(t)| \leq \Omega_j \limsup_{t \to \infty} |\dot{\Psi}_j(t)|. \tag{6.21}$$

Since $\|\underline{w}\|^2 - \|\underline{y}\|^2 = (\underline{w} + \underline{y}, \underline{w} - \underline{y}) = (\underline{w} + \underline{y}, \underline{z})$, we have

$$\begin{aligned} \dot{\Psi}_j = &- 2x_n \dot{x}_n x_j - x_n^2 \dot{x}_j - (\dot{\underline{w}} + \dot{\underline{y}}, \underline{z}) y_j + \\ &- (\underline{w} + \underline{y}, \dot{\underline{z}}) y_j - (\underline{w} + \underline{y}, \underline{z}) \dot{y}_j - 2(\underline{w}, \dot{\underline{w}}) z_j - \|\underline{w}\|^2 \dot{z}_j. \end{aligned} \tag{6.22}$$

Therefore from (6.22) and (6.21) we get

$$Z_j^{(1)} \leq \Omega_j (2\chi_n \chi_{n,v} \chi_j + \chi_n^2 \chi_{j,v} + 2\sqrt{\varphi_v} \chi_j \mathcal{Z} + 2\sqrt{\varphi} \chi_j \mathcal{Z}^{(1)} + 2\sqrt{\varphi} \chi_{j,v} \mathcal{Z} + 2\sqrt{\varphi_v \varphi} Z_j + \varphi Z_j^{(1)}).$$

Hence, by using (6.19) and (6.20), if $L_j := \chi_{j,v} + 2\sqrt{\varphi_v \varphi} a_j + 2(\sqrt{\varphi} \chi_{j,v} + \sqrt{\varphi_v} \chi_j) b$ and C_j is defined as before, then

$$Z_j^{(1)} \leq \frac{C_j}{1 - C_j} \frac{2\chi_n \chi_{n,v} \chi_j + L_j \chi_n^2 + 2\sqrt{\varphi} \chi_j \mathcal{Z}^{(1)}}{\varphi}.$$

By reasoning as before we conclude that, if $S < 1$, then

$$\mathcal{Z}^{(1)} \leq \frac{1}{1-S} \left(\frac{S}{\sqrt{\varphi}} \chi_n \chi_{n,v} + L\chi_n^2 \right)$$

where L is a suitable constant.

We are now able to estimate the asymptotic distance between $\underline{x}$ and $\underline{y}$, since

$$\limsup_{t\to\infty} \|\sqcap_n \underline{x}(t) - \underline{y}(t)\|_2 \leq c\chi_n^2,$$
$$\limsup_{t\to\infty} \|\sqcap_n \underline{\dot{x}}(t) - \underline{\dot{y}}(t)\| \leq \frac{S}{(1-S)\sqrt{\varphi}}\chi_n\chi_{n,v} + \frac{L}{1-S}\chi_n^2. \tag{6.23}$$

We remark that, since we can estimate φ and φ_v in function of $\underline{\chi}$ and $\underline{\chi}_v$, S and L are dependent by $\chi_1, \ldots \chi_n$ and $\chi_{v,1}, \ldots \chi_{v,n}$ only. Therefore, from (6.23) we get the thesis. □

6.3. *Completion of the proof of Theorem 2.5*

Since $\mathrm{g} = P_M \mathrm{g}$, from Lemma 6.4 we get that, if $F(\xi_\infty) < 1$,

$$\lim_{t\to\infty} |(u(t), e_j)| = 0, \quad \lim_{t\to\infty} |(u_t(t), e_j)| = 0 \quad \text{for } j > M.$$

Therefore, we can rewrite (2.8) and (2.9) as finite-dimensional dynamical systems of the form (6.12) and (6.14) respectively.

We introduce the quantities

$$\chi_j := \limsup_{t\to\infty} |(u(t), e_j)|, \quad \chi_{j,v} := \limsup_{t\to\infty} |(u_t(t), e_j)| \quad \text{for } j \leq M.$$

From Lemma 6.5, we have that if $\Omega_j \Phi_0 < 1$, $C_j = \Omega_j \varphi < 1$ for any $j \leq M$ and

$$S = \sum_{j=1}^{M} \frac{2C_j\chi_j}{(1-C_j)\sqrt{\varphi}} < 1$$

where Φ_0 and φ are defined in Proposition 3.2 and in Lemma 6.3, then

$$\limsup_{t\to\infty} \|\sqcap_k u(t) - v(t)\|_2 \leq \frac{1}{\varphi(1-S)} \sum_{j=1}^{M} \frac{C_j\chi_j\sqrt{\alpha_j}}{1-C_j}\chi_k^2,$$
$$\limsup_{t\to\infty} \|\sqcap_k u_t(t) - v_t(t)\| \leq \frac{S}{(1-S)\sqrt{\varphi}}\chi_k\chi_{k,v} + \frac{L}{1-S}\chi_k^2, \tag{6.24}$$

where L is obtained in the proof of Lemma 6.5. Fixed δ, we recall that S and L are constants depending on $\chi_1, \ldots \chi_n$ and $\chi_{v,1}, \ldots \chi_{v,n}$. Hence, since from Lemma 6.4 we have that

$$\chi_j \leq \frac{g_j}{(1-\varphi\Omega_j)\sqrt{(\alpha_j - \omega^2)^2 + \delta^2\omega^2}}, \qquad \chi_{v,j} \leq \frac{(\omega(1-\varphi\Omega_j) + 2\sqrt{\varphi\varphi_v}\Omega_j)g_j}{(1-\varphi\Omega_j)^2\sqrt{(\alpha_j-\omega^2)^2 + \delta^2\omega^2}},$$

from (6.24) we obtain that

$$\limsup_{t\to\infty} (\|\sqcap_k u(t) - v(t)\|_2^2 + \|\sqcap_k u(t) - v(t)\|^2) \leq \frac{Cg_k^4}{((\alpha_k - \omega^2)^2 + \delta^2\omega^2)^2},$$

where C is a constant depending on A^2, g and ω, that is the thesis.

7. The intermediate piers model

In this section we show how the analysis performed in this paper can be useful in order to get some more information about the stability of real world structures such as suspension bridges.

While in the first part of the paper (Theorem 2.3 and Theorem 2.4) we study the general case given by (2.5), in the second part (Theorem 2.5) we focus in particular on the case when $\theta = 0$ and

$$g = \mathfrak{g} \sin(\omega t).$$

In particular, taking $\mathcal{H} = L^2(I)$ with $I = [-\pi, \pi]$, $A = -\partial_{xx}$ and $\mathcal{D}(A) = \{v \in H^2(I) \cap H_0^1(I) : v(-\pi) = v(\pi) = v(-a\pi) = v(b\pi) = 0\}$ for $a, b \in (0, 1)$, the results of Section 6 apply to the system

$$\begin{cases} u_{tt} + \delta u_t + u_{xxxx} + \|u\|^2_{L^2(I)} u = g(x) \sin(\omega t) & \forall t \geq 0, \forall x \in I \\ u(0) = u_0 \in H^2(I) \cap H_0^1(I), u_t(0) = u_1 \in L^2(I) & \\ u(-\pi, t) = u(-\pi b, t) = u(\pi a, t) = u(\pi, t) = 0, & \forall t \geq 0. \end{cases} \tag{7.1}$$

This choice of the forcing term comes from the fact that, in engineering literature (see [35]), the load due to the vortex shedding of the wind along the structure of the bridge is usually modeled in this way with $g(x) \equiv g_\infty \in \mathbb{R}$. The coefficient g_∞ depends on the wind speed and on the geometry of the structure and ω is the frequency at which vortex shedding occurs. More precisely, we have that in engineering applications $g(x, t) = W^2 \sin(\omega t)$, where W is the scalar velocity of the wind blowing on the deck of the bridge and ω can be expressed in terms of the structural constants of the bridge and the aerodynamic parameters of the air. We refer to the European Eurocode [23] (see also [8]) for a more detailed discussion.

The peculiar expression of the forcing term allows us to improve the estimate on the asymptotic $\mathcal{H}^2-$norm of the solution of (7.1) that one is able to obtain with no other information on g than the value of $\limsup_{t\to\infty} \|g(t)\|$. A comparison between the general estimate on $\limsup_{t\to\infty} \|u\|_2$ (see Proposition 3.2) obtained by using the methods of [8, Lemma 22] and the one obtained by using the antiperiodicity of the forcing term (see Lemma 6.4) is given in Figure 2. The data considered are $a = b = 14/25$, $\delta = 1.5$, and $\omega = 20$. The maximum value of g_∞ considered represents the largest value of g_∞ such that Lemma 6.4 can be applied.

The improvement in the estimates on the asymptotic $\mathcal{H}^2-$norm is obtained by using also ultimate bounds of the asymptotic amplitude of each mode. We represent in Figure 3 a comparison between these estimates, obtained in Lemma 6.4, and a numerical estimate on the asymptotic amplitude of each of the first 20 modes. Fixed $\delta = 1.5$ and $g_\infty = 1.5$, we considered the cases when $\omega = 5$ (left) and $\omega = 10$ (right). We considered different positions of the piers, namely we chose $a = b = 14/25$ (up) and $(a, b) = (0.51, 0.67)$ (down). Each of these choices respect the hypothesis of Lemma 6.4. We remark that the mode with largest amplitude is such that $\sqrt{\alpha_j}/\omega \approx 1$.

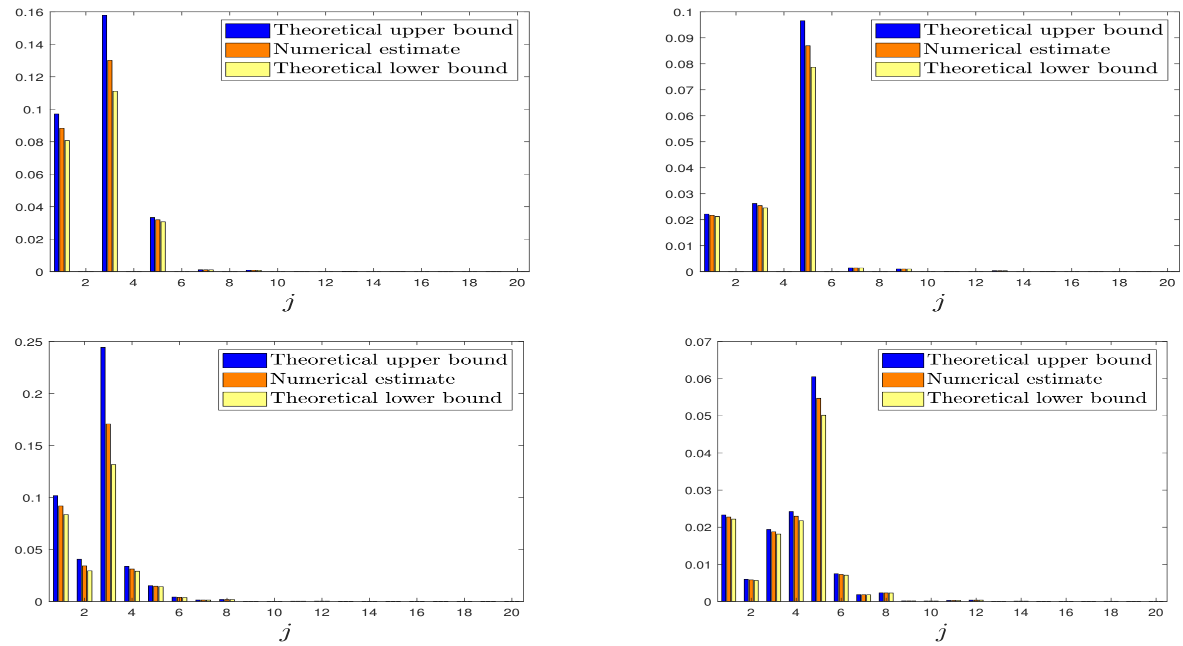

Figure 3. Comparison between the asymptotic estimate on the amplitude of the first 20 modes for different values of ω and for different configurations of the piers.

The estimates on each single mode of u allow us to study more precisely how the asymptotic $\mathcal{H}^2-$norm of u varies as the position of the piers vary, i.e., as a and b varies (see Lemma 6.4). Since most suspension bridges have symmetrical piers with $a = b \in [1/2, 2/3]$, we restrict ourselves to the case where $(a, b) \in [1/2, 2/3] \times [1/2, 2/3]$. We represent in Figure 4 the estimate on the asymptotic $\mathcal{H}^2-$norm given by Lemma 6.4 in function of a and b, with $\delta = 1.5$, $g_\infty = 1.5$ and $\omega = 10$ fixed. We remark that this figure does not give any information about the stability of the bridge as a and b vary. In fact, the stability of a bridge is more endangered by the concentration of the energy on a single mode than by the generalized oscillation of the structure.

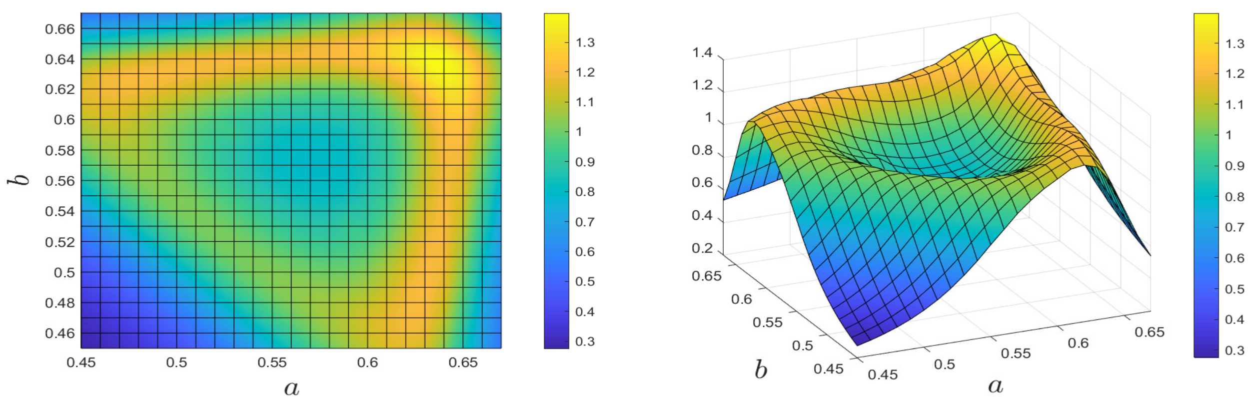

Figure 4. Plot of a theoretical estimate of the asymptotic $\mathcal{H}^2-$norm in function of a and b.

In order to study the distribution of the $\mathcal{H}^2-$norm among the modes, we introduce the concept of family of asymptotic $\eta-$prevailing modes.

Definition 7.1. *Let* $0 < \eta < 1$*. We say that a weak solution of* (2.5) *has a family* $S = \{j_1, \dots j_n\}$ *of asymptotic* $\eta-$*prevailing modes if*

$$\limsup_{t\to\infty} \|Q_S u\|_2^2 < \eta^4 \limsup_{t\to\infty} \|P_S u\|_2^2.$$

In Figure 5 we plot the number of $\eta-$prevailing modes for $\eta = 0.1$. The value of the parameters is the same as in Figure 4, namely $\delta = 1.5$, $g_\infty = 1.5$ and $\omega = 10$. We can observe that the asymptotic $\mathcal{H}^2-$norm concentrates on few modes as $a = b$. Moreover, we notice how the energy turns out to be more dispersed among the modes when $a \neq b$.

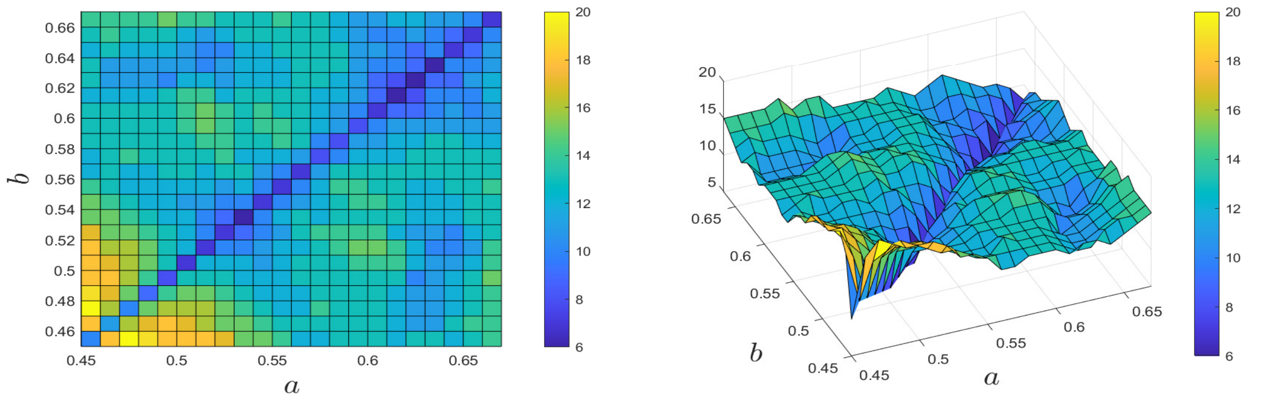

Figure 5. Number of 0.1−prevailing modes in function of a and b.

In conclusion, we are able to assert that under suitable smallness conditions on the asymptotic amplitude of the forcing term and on the nonlinearity, we are able to perform a rather accurate modal analysis for the nonlinear nonlocal beam equations considered. In particular, Figure 5, allows us to conclude that the more stable configurations are achieved when $a \neq b$. This suggests that, according to the model considered, asymmetric suspension bridges are more stable than suspension bridges where the piers are symmetric with respect to the center of the deck.

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Conflict of interest

The author declares no conflict of interest.

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