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Research article

On the boundary Harnack principle in Hölder domains†

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Abstract: We investigate the boundary Harnack principle for uniformly elliptic operators in divergence form in Hölder domains of exponent $\alpha > 0$. We also deal with operators in nondivergence form with coefficient that remain constant in the graph direction.

Keywords: boundary Harnack principle; Hölder domains; divergence operators; non-divergence operators; Carleson estimate

1. Introduction

In this paper we continue the study of the boundary Harnack principle for solutions to elliptic equations, based on the method developed in [7]. The classical boundary Harnack principle states that two positive harmonic functions that vanish on a portion of the boundary of a Lipschitz domain must be comparable up to a multiplicative constant, see for example [1,6,11,14]. Further extensions to more general operators and more general domains were obtained in several subsequent works [2, 3, 5, 10, 12, 13].

In particular, Bass and Burdzy [3, 4] and Banuelos, Bass and Burdzy [2] provided sharp versions using probabilistic methods. They established the boundary Harnack principle for nondivergence elliptic operators in Hölder domains (or more general twisted Hölder domains) of exponent $\alpha > \frac{1}{2}$, and for divergence operators in Hölder domains of arbitrary exponent $\alpha > 0$. For the case of divergence operators, an analytical proof based on Green's function was given by Ferrari in [8].

To state precisely the boundary Harnack principle in Hölder domains, first we introduce some notation. Let $g : \overline{B}_1' \to \mathbb{R}$ be a C^α Hölder function of $n-1$ variables with $g(0) = 0$, and $\alpha \in (0, 1)$.

Denote by $\Gamma \subset \mathbb{R}^n$ the graph of g,

$$\Gamma := \{x_n = g(x')\}, \qquad 0 \in \Gamma,$$

and by C_r the cylinder above B'_r and at height r on top of Γ

$$C_r := \{x' \in B'_r, \quad g(x') < x_n < g(x') + r\}.$$

We say that C_1 is a C^α-Holder domain in a neighborhood of Γ.

The following version of the boundary Harnack principle is due to Banuelos, Bass and Burdzy, see [2].

Theorem 1.1. *Let $Lu = div(A(x)\nabla u)$ be a uniformly elliptic linear operator, and assume that u, v are two positive solutions to*

$$Lu = Lv = 0, \quad \textit{in } C_1,$$

which vanish on Γ. Then

$$\frac{u}{v}(x) \leq C \, \frac{u}{v}\left(\frac{1}{2}e_n\right) \quad \textit{for all } x \in C_{1/2},$$

with C depending on n, α, $\|g\|_{C^\alpha}$ and the ellipticity constants of L.

The assumption that $u = v = 0$ in Γ is understood in the H^1 sense, i.e., $u, v \in H^1_0(C_1)$ in a neighborhood of Γ.

Recently, in [7] we found a direct analytical method of proof of the boundary Harnack principle based on an iteration scheme and Harnack inequality. In particular we established the corresponding results in Hölder domains of exponent $\alpha > \frac{1}{2}$ for general equations either in divergence or nondivergence form.

In the present paper we discuss further the case of Hölder domains of arbitrary exponent $\alpha > 0$, and give a proof of Theorem 1.1 using the same ideas from [7]. We also consider some novel extensions of Theorem 1.1 to non-divergence equations whose coefficients remain constant in the vertical direction (see Section 4).

The paper is self-contained and is organized as follows. In Section 2 we give two lemmas concerning Harnack inequality outside domains of small capacity. In Section 3 we use these lemmas and employ the arguments from [7] to prove Theorem 1.1. Finally in Section 4 we provide some extensions of Theorem 1.1 to more general divergence operators, and certain non-divergence or fully nonlinear operators.

2. Two lemmas

In this section we present two lemmas concerning solutions to divergence equations in domains whose complement in the unit cube Q_1 has small capacity.

Given a domain Ω and a compact set $K \subset \Omega$, we say that two functions $u, v \in H^1(\Omega)$ agree on K, and write $u = v$ in K, if $u - v \in H^1_{0,loc}(K^c)$. Here K^c denotes the complement of K in $\mathbb{R}^n$.

In particular, if L is a uniformly elliptic operator in divergence form

$$Lu = div(A(x)\nabla u),$$

with

$$A(x) \text{ measurable}, \qquad \Lambda\,|\xi|^2 \geq \xi^T A(x)\,\xi \geq \lambda\,|\xi|^2, \qquad \lambda > 0,$$

then the statement that u solves

$$Lu = 0 \quad \text{in } \Omega \setminus K, \quad \text{and } u = 0 \text{ on } \partial\Omega,\ u = 1 \text{ in } K, \tag{2.1}$$

means that $Lu = 0$ in the open set $\Omega \setminus K$, and

$$u - \eta \in H_0^1\,(\Omega \setminus K)\,,$$

where $\eta \in C_0^\infty(\Omega)$, and $\eta = 1$ in a neighborhood of K.

Notice that the solution u to (2.1) is a supersolution in Ω, i.e., $Lu \leq 0$ in Ω.

Let Q_1 denote the unit cube in $\mathbb{R}^n$ centered at 0, and $E \subset Q_1$ a closed set. Set,

$$cap_{3/4}(E) := cap_{Q_1}(E \cap Q_{3/4}) = \inf_{w \in \mathcal{A}} \int_{Q_1} |\nabla w|^2 dx,$$

where

$$\mathcal{A} := \{w \in H_0^1(Q_1), \quad w = 1 \quad \text{in } E \cap \overline{Q}_{3/4}\}.$$

The first lemma states that a solution to $Lv = 0$ in $Q_1 \setminus E$ satisfies the Harnack inequality in measure if E has small capacity. Positive constants depending on the dimension n and the ellipticity constants λ, Λ are called universal.

Lemma 2.1. *Assume $v \geq 0$ is defined in $Q_1 \setminus E$ and satisfies*

$$Lv = 0.$$

Let

$$Q^i := Q_{1/8}(x_i) \subset Q_{1/2}, \quad i = 1, 2$$

be two cubes of size $1/8$ included in $Q_{1/2}$. Assume that

$$cap_{3/4}(E) \leq \delta \quad \text{and} \quad \frac{|\{v \geq 1\} \cap Q^1|}{|Q^1|} \geq 1/2,$$

for some δ small, universal. Then

$$\frac{|\{v \geq c_0\} \cap Q^2|}{|Q^2|} \geq 1/2$$

for some c_0 small.

The second lemma is standard and states that the weak Harnack inequality holds for a subsolution $v \geq 0$ which vanishes on a set E of positive capacity.

Lemma 2.2. *Assume that $v \geq 0$ in Q_1, and*

$$Lv \geq 0 \text{ in } Q_1,\ v = 0 \text{ in } E \cap \overline{Q}_{3/4}.$$

If

$$cap_{3/4}(E) \geq \delta,$$

then

$$v(0) \leq (1 - c(\delta))\|v\|_{L^\infty}.$$

Proof of Lemma 2.1. Let ψ be the solution to

$$L\psi = 0 \text{ in } Q_{3/4} \setminus K, \qquad \psi = 1 \text{ in } K, \quad \psi = 0 \text{ on } \partial Q_{3/4},$$

where K is a compact subset of $\{v \geq 1\} \cap (Q^1 \setminus E)$ with $|K| \geq \frac{1}{4}|Q^1|$ (see Figure 1). By hypothesis and weak Harnack inequality (see Theorem 8.18 and Theorem 9.22 in [9]) we find

$$\psi \geq c_0 \quad \text{in} \quad Q_{1/2}, \tag{2.2}$$

for some small c_0.

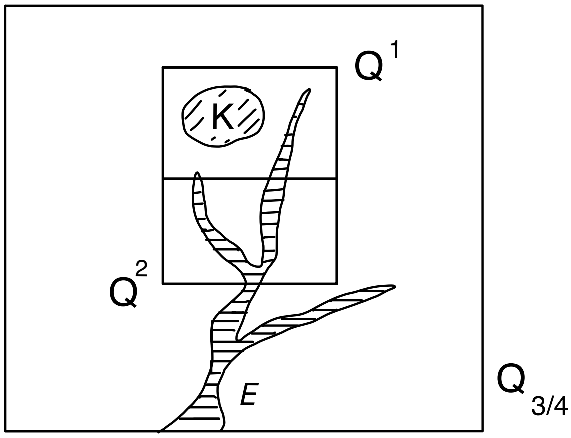

Figure 1. Lemma 2.1.

Similarly as above we define ϕ to be the solution to

$$L\phi = 0 \text{ in } Q_1 \setminus (E \cap Q_{3/4}), \qquad \phi = 1 \text{ in } E \cap Q_{3/4}, \phi = 0 \text{ on } \partial Q_1. \tag{2.3}$$

We claim that if δ is chosen sufficiently small then,

$$|\{\phi > \frac{1}{4}c_0\} \cap Q^2| \leq \frac{1}{2}|Q^2|. \tag{2.4}$$

For this we let w be the solution to (2.3) when $L = \triangle$. The Dirichlet energies of ϕ and w are comparable since

$$\int (\nabla(\phi - w))^T A \nabla\phi \, dx = 0,$$

hence

$$cap_{3/4}(E) \leq \int |\nabla\phi|^2 \, dx \leq C \int (\nabla\phi)^T A \nabla\phi dx \leq C \int |\nabla w|^2 \, dx = C cap_{3/4}(E).$$

By Poincaré inequality we find

$$\int \phi^2 \, dx \leq C \int |\nabla\phi|^2 \, dx \leq C\delta,$$

which gives the claim (2.4).

Next we compare $2v$ with $\psi - \phi$ in $Q_{3/4} \setminus E$.

They satisfy the same equation in $Q_{3/4} \setminus (E \cup K)$, and in a neighborhood of K by the continuity of v we have

$$2v \geq 1 \geq \psi - \phi.$$

On the other hand

$$\psi - \phi \leq 0 \quad \text{on} \quad \partial(Q_{3/4} \setminus E)$$

in the sense that $(\psi - \phi)^+ \in H_0^1(Q_{3/4} \setminus E)$. Since $v \geq 0$, the maximum principle gives

$$2v \geq \psi - \phi,$$

which by (2.2), (2.4) yields the desired conclusion.

□

Proof of Lemma 2.2. Assume that $\|v\|_{L^\infty} = 1$. Then, by the maximum principle we have

$$1 - v \geq \phi,$$

with ϕ as in (2.3) above. It suffices to show that $\phi \geq c(\delta)$ on $\partial Q_{7/8}$ which by the maximum principle implies the desired conclusion $\phi(0) \geq c(\delta)$ small. Since all the values of ϕ are comparable near $\partial Q_{7/8}$ by the Harnack inequality, we need to show that $\phi \geq c'(\delta)$ at some point on $\partial Q_{7/8}$.

Assume by contradiction that $|\phi| \leq \mu$ is very close to 0 on $\partial Q_{7/8}$. The Caccioppoli inequality (we think that ϕ is extended to 0 outside Q_1) implies

$$\|\nabla\phi\|_{L^2(Q_1 \setminus Q_{15/16})} \leq C\|\phi\|_{L^2(Q_1 \setminus Q_{7/8})} \leq C\mu. \tag{2.5}$$

On the other hand if $\eta \in C_0^\infty(Q_1)$ with $\eta = 1$ in $Q_{15/16}$ then

$$\int \nabla[\eta^2(1 - \phi)]A\nabla\phi \, dx = 0,$$

hence

$$\int \eta^2 \nabla\phi A \nabla\phi \, dx \leq C \int |\nabla\phi|^2 |\nabla\eta|^2 \, dx \leq C\mu^2.$$

This together with (2.5) implies that the Dirichlet energy of ϕ in Q_1 is bounded above by $C\mu^2$, and we reach a contradiction. □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We recall that Γ denotes the graph of a C^α function g, with $\alpha \in (0, 1)$,

$$\Gamma := \{x_n = g(x')\}, \qquad 0 \in \Gamma,$$

and C_r denotes the cylinders of size r on top of Γ

$$C_r := \{x' \in B_r', \quad g(x') < x_n < g(x') + r\}.$$

The main idea of the proof is to show through an iterative procedure that a solution w which vanishes on Γ and is mostly positive in C_r, becomes positive near the origin.

We denote by

$$\mathcal{A}_r := \left\{x \in B'_r | \quad g(x') + r^\beta \leq x_n < g(x') + r\right\},$$

the points in the cylinder C_r at height greater than r^β on top of Γ, for some $\beta > 1$.

We divide the proof in three steps.

Step 1. We show that, there exist $C_0, \beta > 1$ depending on n, α, $\|g\|_{C^\alpha}$, and the ellipticity constants of L, such that if w is a solution to $Lw = 0$ in C_r (possibly changing sign) which vanishes on Γ,

$$w \geq f(r) \qquad \text{on} \quad \mathcal{A}_r,$$

and

$$w \geq -1 \quad \text{on } C_r,$$

where

$$f(r) := e^{C_0 r^\gamma}, \quad \gamma := \beta(1 - \frac{1}{\alpha}) < 0,$$

then,

$$w \geq f(\frac{r}{2}) \; a \quad \text{on } \mathcal{A}_{\frac{r}{2}}, \tag{3.1}$$

and

$$w \geq -a \quad \text{on } C_{\frac{r}{2}}, \tag{3.2}$$

for some small $a = a(r) > 0$, as long as $r \leq r_0$ universal.

The conclusion can be iterated and we obtain that if the hypotheses are satisfied in C_{r_0} then

$$w > 0 \quad \text{on the line segment } \{te_n, \quad 0 < t < r_0\}.$$

Since g is Hölder continuous, we can apply interior Harnack inequality to $w + 1$ in a chain of balls and need

$$C(r^\beta)^{1-\frac{1}{\alpha}} = Cr^\gamma \quad \text{balls}$$

to connect a point in $\mathcal{A}_{r/2}$ with a point in $\mathcal{A}_r$. We conclude that

$$w \geq (f(r) + 1)e^{-C_1 r^\gamma} - 1 \quad \text{in } \mathcal{A}_{r/2}, \tag{3.3}$$

for some C_1 universal, hence $w \geq 1$ in $\mathcal{A}_{r/2}$ if C_0 is sufficiently large.

Next we take a point on $\Gamma := \{x_n = g(x')\}$, say 0 for simplicity, and consider the cubes of size $r^{\beta/\alpha}$ centered on the e_n axis, i.e., $Q_{r^{\beta/\alpha}}(te_n)$ (see Figure 2).

When $t > Cr^\beta$ the cube is in the interior of the domain and when $t < -Cr^\beta$ the cube is in the complement. There are Cr^γ stacked cubes which connect the domain with its complement. The graph property of the domain implies that the capacity of the complement

$$E = \{x_n \leq g(x')\}$$

in $Q_{r^{\beta/\alpha}}(te_n)$ is decreasing with t. By continuity we can find a cube centered at $t_0 e_n$ such that, after a rescaling of factor $r^{-\beta/\alpha}$, $cap_{3/4}(E) = \delta$ in that cube, with δ as in Lemma 2.1.

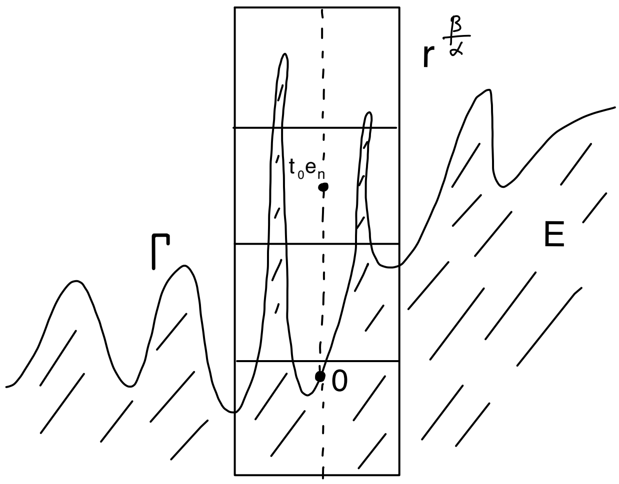

Figure 2. Step 1, Theorem 1.1.

For all cubes centered at te_n with $t \geq t_0$ we can apply Lemma 2.1 repeatedly for $w + 1$ and obtain an inequality in measure as in (3.3),

$$|\{w \geq 1\} \cap Q^i(te_n)| \geq \frac{1}{2}|Q^i|, \qquad Q^i(te_n) := Q_{\frac{1}{8}r^{\beta/\alpha}}(te_n).$$

Thus $\{w^- = 0\}$ has positive density in all cubes centered at te_n with $t \geq t_0$.

Now we notice that we can apply weak Harnack inequality for w^- in all cubes, in the top cubes with $t \geq t_0$ because of the density property, and in the bottom ones with $t \leq t_0$ because of Lemma 2.2.

Hence w^- decreased by a fixed factor on the e_n axis passing through a point on Γ, with respect to its maximum over all cubes of size $r^{\beta/\alpha}$ centered on that axis. As we move each time a $r^{\beta/\alpha}$ distance inside the domain from the sides of C_r, $\sup w^-$ decreases geometrically hence,

$$w \geq -e^{-c_0 r^{1-\beta/\alpha}} \quad \text{in } C_{r/2},$$

for c_0 small universal. We choose

$$a(r) := e^{-c_0 r^{1-\beta/\alpha}},$$

and in view of (3.3), our claim

$$w \geq 1 \geq a(r) f\left(\frac{r}{2}\right),$$

is satisfied for all r small.

Step 2. [Carleson estimate] We show that,

$$u, v \leq C_2 \quad \text{in } C_{1/2},$$

with C_2 universal. We apply an iterative argument similar to the one in Step 1. Since $u(e_n/2) = 1$, the interior Harnack inequality gives that

$$u \leq e^{C_1 h_\Gamma^{1-1/\alpha}} \qquad \text{in } C_{3/4}, \qquad h_\Gamma(x) := x_n - g(x'), \tag{3.4}$$

with C_1 universal. With the same notation as Step 1, we wish to prove that if r is smaller than a universal r_0 and

$$u(y) \geq f(r),$$

for some $y \in C_{1/2}$, then we can find

$$z \in S := \{|y' - z'| = r, \quad 0 < h_\Gamma(z) < r^\beta\},$$

such that

$$u(z) \geq f\left(\frac{r}{2}\right).$$

Since $|z - y| \leq Cr^\alpha$, we see that for r small enough, we can build a convergent sequence of points $y_k \in C_{3/4}$ with $u(y_k) \geq f(2^{-k}r) \to \infty$. On the other hand the extension of u by 0 below Γ is a subsolution in a neighborhood of Γ. Therefore u is bounded above, and we reach a contradiction.

To show the existence of the point z we let

$$w := \left(u - \frac{1}{2}e^{C_0 r^\gamma}\right)^+, \qquad \text{with} \quad C_0 \gg C_1.$$

By (3.4) we know that

$$w = 0 \quad \text{when} \quad h_\Gamma(x) \geq r^\beta.$$

By Lemma 2.1 this estimate can be extended in measure for the cubes of size $r^{\beta/\alpha}$ with $t \geq t_0$ since the capacity of the complement is bounded above. More precisely, as in Step 1, in each cube of size $r^{\beta/\alpha}$ we have that either $\{w = 0\}$ has positive density (for the cubes with $t \geq t_0$), or positive capacity (for the cubes with $t \leq t_0$).

Moreover, if our claim is not satisfied then we apply Weak Harnack inequality for w repeatedly as in Step 1 above. As we move inside the domain from the sides of $C_r(y', g(y'))$ we obtain

$$w \leq f\left(\frac{r}{2}\right)\ e^{-c_0 r^{1-\beta/\alpha}} \quad \text{in } C_{r/2}(y', g(y')).$$

In particular

$$\frac{1}{2}f(r) \leq w(y) \leq f\left(\frac{r}{2}\right)e^{-c_0 r^{1-\beta/\alpha}},$$

and we reach a contradiction.

Step 3. We prove the theorem using the Steps 1 and 2 above. After multiplication by a constant we may assume that $u = v = 1$ at $\frac{1}{2}e_n$. It suffices to show that for a large constant $C_3 > 0$ universal,

$$w := C_3 u - C_3^{-1} v \geq 0 \quad \text{in } C_{1/2}.$$

By Step 2 we know that $v \leq C_3$ hence $w \geq -1$ in $C_{3/4}$. Moreover, since $u(e_n/2) = 1$, we conclude by interior Harnack for u that

$$w \geq f(r_0) \quad \text{in} \quad C_{3/4} \cap \{x_n \geq r_0^\beta\},$$

provided that C_3 is chosen sufficiently large. Here f, r_0 and β are as in Step 1.

We conclude by Step 1 that $w \geq 0$ on the line $\{te_n, 0 < t < 3/4\}$. We can repeat the argument at all points on $\Gamma \cap \overline{C}_{1/2}$, and the theorem is proved. $\square$

4. Some extensions

In this section we state a few variants of the Theorem 1.1. First we remark that the proof applies to operators involving lower order terms.

Theorem 4.1. *The statement of Theorem* 1.1 *holds for general uniformly elliptic operators*

$$Lu = div(A(x)\nabla u) + b(x)\cdot\nabla u + d(x)u, \qquad b \in L^q,\ d \in L^{q/2}, \quad q > n,$$

with the constant C depending also on q, $\|b\|_{L^q}$, $\|d\|_{L^{q/2}}$.

Indeed, we only need to check that the statements of Section 2 continue to hold in the small cubes of size $r^{\beta/\alpha}$. After a dilation this corresponds to proving Lemmas 2.1 and 2.2 for operators L as above with $\|b\|_{L^q}$, $\|d\|_{L^{q/2}}$ sufficiently small. The proofs are identical since the presence of such lower order terms does not affect the energy estimates.

A counterexample of Bass and Burdzy in [4] shows that Theorem 1.1 does not hold in general for nondivergence equations when $\alpha < \frac{1}{2}$. Here we remark that Theorem 1.1 remains valid with $\alpha > 0$ for nondivergence linear operators which are translation invariant in the vertical direction.

Theorem 4.2. *The statement of Theorem* 1.1 *holds for linear nondivergence uniformly elliptic operators of the form*

$$Lu = tr\ (A(x')D^2u).$$

In this theorem we assume that the coefficient matrix A depends continuously on its argument, although the estimates do not depend on its modulus of continuity. Since u, v might not be continuous at all points on Γ, the hypothesis that u, v vanish on the boundary is understood in the sense that their extensions with 0 below Γ are bounded subsolutions for L, see [7].

In this case we provide the corresponding lemmas of Section 2 by defining the capacity (with respect to L) as

$$cap_{3/4}(E) = \inf_{Q_{1/4}} \phi,$$

where ϕ solves

$$L\phi = 0 \text{ in } Q_1 \setminus (E \cap Q_{3/4}),\ \phi = 1 \text{ in } E \cap Q_{3/4},\ \phi = 0 \text{ on } \partial Q_1.$$

Then Lemma 2.2 follows directly from the definition of the capacity, with $c(\delta) = \delta$. For Lemma 2.1 we see that (2.4) is satisfied since by the Weak Harnack inequality the set $\{\phi > c_0\}$ must have small measure in Q_1 if δ is sufficiently small. The rest of the proof is the same.

The arguments of Section 3 can be repeated in the same way. The invariance of the operator L with respect to the vertical direction and the graph property of the boundary imply that the capacity of the complement E in the cubes $Q_{r^{\beta/\alpha}}(te_n)$ is monotone in t. We can apply again Lemma 2.1 for the top cubes with $t \geq t_0$ and Lemma 2.2 for the bottom cubes with $t \leq t_0$, and carry on as before.

We also discuss the case of fully nonlinear operators

$$F(D^2u) = 0 \quad \text{in} \quad C_1, \tag{4.1}$$

with F uniformly elliptic with constants λ, Λ, and homogenous of degree 1.

We can prove the lemmas of Section 2 for the operator F by using as capacity the definition above with $L\phi = F(-D^2\phi)$. Then Lemma 2.2 follows again directly from the definition. For the proof of Lemma 2.1 we choose the function ψ to satisfy $\mathcal{M}^-_{\lambda/n,\Lambda}(\psi) = 0$. Here as usual, $\mathcal{M}^-$ denotes the extremal Pucci operator,

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_A tr(AM),$$

with A a symmetric matrix whose eigenvalues belong to $[\lambda, \Lambda]$.

Then $\psi - \phi$ is a subsolution

$$F(D^2(\psi - \phi)) \geq F(D^2(-\phi)) + \mathcal{M}^-_{\lambda/n,\Lambda}(\psi) \geq 0,$$

and the rest of proof remains as before. However in the proof of the main Theorem 1.1 only Step 2, the Carleson estimate, can be carried out in this setting, since for Step 1 we need the lemmas of Section 2 to hold not only for solutions of the operator F but for the difference of two such solutions as well.

Theorem 4.3 (Carleson estimate)**.** *Assume that $u \geq 0$ satisfies* (4.1) *and u vanishes on Γ. Then*

$$u \leq Cu\left(\frac{1}{2}e_n\right) \quad in \quad C_{1/2},$$

with C depending on n, α, $\|g\|_{C^\alpha}$, λ and Λ.

Finally we mention that in $\mathbb{R}^2$ Theorem 1.1 holds under very mild assumptions on the domain and the operator. Here we state a version for L^∞ graphs and linear operators.

Theorem 4.4. *Assume $\Gamma \subset \mathbb{R}^2$ is the closure of the graph of a function g with $\|g\|_{L^\infty} \leq 1/4$. Then the statement of Theorem 1.1 holds for uniformly elliptic linear operators L in divergence or nondivergence form with constant C depending only on the ellipticity constants of L.*

We only sketch Steps 1 and 2 of Section 3 in this setting which can be adapted to more general situations. They are based on topological considerations and do not require an iterative argument.

Step 1. If $Lw = 0$ and $w \geq -1$ in C_1, and w vanishes continuously on Γ, then $w > 0$ in $C_{1/2}$ provided that $u(\frac{1}{2}e_2)$ is large.

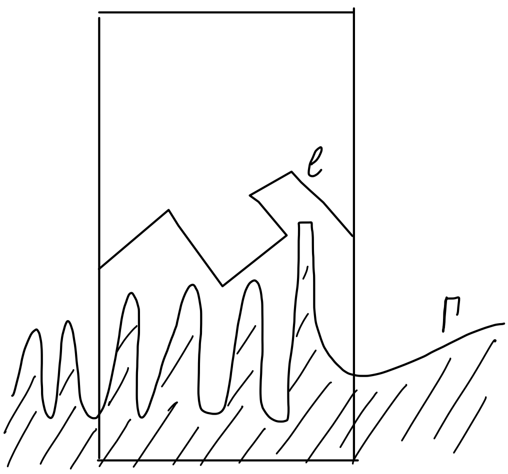

Figure 3. Proof of Theorem 4.4.

To prove this, we assume by contradiction that there is a connected component U of $\{w < 0\}$ which intersects $C_{1/2}$. This connected component must exit C_1 and since $u(\frac{1}{2}e_2) \gg 1$, U must stay close to Γ. Thus we can find a nonintersecting polygonal line ℓ, say included in

$$l \subset \left\{\frac{1}{2} \le x_1 \le \frac{3}{4}\right\} \cap U,$$

which connects the two lateral sides $x_1 = \frac{1}{2}$ and $x_1 = \frac{3}{4}$ (see Figure 3). The line ℓ splits the cylinder

$$D := \left(\frac{1}{2}, \frac{3}{4}\right) \times \left(\frac{1}{2}, \frac{1}{2}\right)$$

into two disjoint sets, and we define $\tilde{w}$ to be equal to w on the set "above" ℓ and $\tilde{w} = \min\{w, 0\}$ on the set "below" ℓ. Then $\tilde{w}$ is a supersolution of L in D. Since $\tilde{w}$ is sufficiently large in a ball above ℓ, and $\tilde{w} \ge -1$ in D we find that $\tilde{w} \ge 0$ on the segment

$$\left\{x_1 = \frac{5}{8}\right\} \cap \left\{|x_2| \le \frac{3}{8}\right\}.$$

We reached a contradiction at the point where ℓ intersects this segment.

Step 2. Assume $Lu = 0$ and $u \ge 0$ in C_1, and u vanishes continuously on Γ, with $u(\frac{1}{2}e_2) = 1$. Then $u \le C$ in $C_{1/2}$, for some large C.

This follows similarly as in Step 1. If $\{u > C\}$ has a connected component that intersects $C_{1/2}$, then we can find a polygonal line ℓ as above where u is large. Thus $\min\{u, C\}$ extended by C below ℓ is a supersolution for L in D, and the maximum principle implies that u is large at the point $(5/8, 1/2)$. Therefore by Harnack inequality $u(0, 1/2)$ is large as well, and we reach a contradiction.

Conflict of interest

The authors declare no conflict of interest.

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