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Research article

A remark on the first p-buckling eigenvalue with an adhesive constraint†

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Abstract: We consider a fourth order nonlinear eigenvalue problem with an adhesive constraint. The problem is regarded as a generalization of the buckling eigenvalue problem with the clamped boundary condition. We prove the existence of the first eigenvalue of the problem and show that the corresponding eigenfunction does not have "flat core of adhesion type".

Keywords: nonlinear fourth order eigenvalue problem; gradient nonlinearity

1. Introduction

In this paper we are interested in a fourth order nonlinear eigenvalue problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = -\lambda\nabla\cdot(|\nabla u|^{p-2}\nabla u) & \text{in} \quad \Omega, \\ u = \partial_\nu u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

with the adhesive constraint

$$|\mathcal{O}(u)| = \omega_0, \tag{1.2}$$

where $p > 1$, $0 < \omega_0 < |\Omega|$ and $\mathcal{O}(u) := \{x \in \Omega \mid u(x) \neq 0\}$. Here Ω, ν and $|\Omega|$ denote a smooth bounded domain in $\mathbb{R}^N$, the unit outer normal of $\partial\Omega$ and the Lebesgue measure of Ω, respectively. The eigenvalue problem (1.1) is regarded as a generalization of the buckling eigenvalue problem with the clamped boundary condition

$$\begin{cases} \Delta^2 u = -\lambda\Delta u & \text{in} \quad \Omega, \\ u = \partial_\nu u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

The first eigenvalue of (1.3) with $N = 2$ is called the buckling load of a clamped plate and characterized by

$$\mu_1(\Omega) = \inf_{H_0^2(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta u|^2\, dx}{\displaystyle\int_\Omega |\nabla u|^2\, dx}.$$

In 1951, Polya and Szegö [9] conjectured that the disk minimizes the buckling load μ_1 among domains of given measure. The conjecture has been attracted a great interest and studied by many researchers (e.g., see [2, 10], [6, Section 3.2] and references therein). In particular, recently Stollenwerk [10] considered problem (1.3) with constraint (1.2) to study the Polya-Szegö conjecture for $N = 2, 3$. Although nonlinear eigenvalue problems for the p-biharmonic operator of the type

$$\Delta(|\Delta u|^{p-2}\Delta u) = \lambda |u|^{p-2}u \quad \text{in} \quad \Omega$$

have been well studied in the mathematical literature (e.g., see [3, 5, 7] and references therein), to the best of our knowledge, there is few result on the eigenvalue problem of the type (1.1).

The purpose of this paper is to study the first eigenvalue of (1.1) with constraint (1.2) for $N = 1$:

$$\begin{cases} (|u''|^{p-2}u'')'' = -\lambda(|u'|^{p-2}u')' & \text{in} \quad I, \\ u = u' = 0 & \text{on} \quad \partial I, \end{cases} \tag{1.4}$$

with

$$|O(u)| = \omega_0, \tag{1.5}$$

where

$$O(u) := \{x \in I \mid u(x) \neq 0\}.$$

Here $I \subset \mathbb{R}$ denotes a bounded open interval and $0 < \omega_0 < |I|$ is a given constant.

One of our motivations is to prove the existence of the first eigenvalue and corresponding eigenfunctions of problem (1.4) with (1.5). To this end, we consider the minimization problem

$$\min_{v \in \mathcal{A}_{\omega_0}} E(v), \tag{P}$$

where

$$E(v) := \frac{\displaystyle\int_I |v''|^p dx}{\displaystyle\int_I |v'|^p dx},$$

$$\mathcal{A}_{\omega_0} := \{v \in W_0^{2,p}(I) \mid |O(v)| = \omega_0\}.$$

As we prove in Lemma 3.2, solutions to problem (P) satisfy problem (1.4) with (1.5) in a weak sense. Thus problem (P) gives us the first p-buckling eigenvalue and corresponding eigenfunctions.

The second motivation is to show a property of the eigenfunction corresponding to the first p-buckling eigenvalue. In 2014, Watanabe [12] studied the p-elastic curves which are critical points of the p-elastic energy

$$\int_\gamma |\kappa|^p\, ds,$$

where γ, κ and s respectively denote a planar curve, the curvature of γ and the arc length parameter of γ, and proved the existence of solutions with 'flat core'. Here, we say that $u : I \to \mathbb{R}$ has 'flat core' if the graph $(x, u(x))$ contains a part where the graph is parallel to the x-axis (more precisely, see [11,12]). In order to state our second motivation precisely, we define

$$\mathcal{J}^0 := \{x \in \partial\mathcal{O}(u) \mid u'(x) = 0\}, \qquad \mathcal{J}^1 := \{x \in \partial\mathcal{O}(u) \mid |u'(x)| > 0\},$$

and

$$\mathcal{I}(u) := \mathcal{O}(u) \cup \mathcal{J}^1.$$

We say that $u : I \to \mathbb{R}$ has *flat core of adhesion type* if the set $\mathcal{I}(u)$ is not connected. Our second motivation is to ask whether constraint (1.5) can induce the eigenfunction corresponding to the first p-buckling eigenvalue to have flat core of adhesion type or not.

The main result of this paper is stated as follows:

Theorem 1.1. *Let $I \subset \mathbb{R}$ be an open interval. Let $p > 1$ and $0 < \omega_0 < |I|$. Then problem* (P) *possesses a solution $u \in \mathcal{A}_{\omega_0}$. Moreover, $\mathcal{I}(u)$ is connected.*

We deduce from Theorem 1.1 that the eigenfunction corresponding to the first p-buckling eigenvalue does not have flat core of adhesion type. Due to adhesive constraint (1.5), it is difficult to solve problem (P) by the direct method of calculus of variations. To overcome the difficulty, we employ an idea by Alt and Caffarelli [1] as in [10]. More precisely, considering a penalized problem, once we remove adhesive constraint (1.5) from (P). Studying the regularity of the penalized solution u_ε, we prove the relation $|\mathcal{O}(u_\varepsilon)| = \omega_0$ for sufficiently small $\varepsilon > 0$. Then we obtain a minimizer of problem (P). We note that, if we employ the same strategy to find the first p-buckling eigenvalue for $N \geq 2$, then one of the arising difficulties is the lack of regularity of the penalized solution u_ε.

This paper is organized as follows: In Section 2, we collect notations and inequalities which are used in this paper; In Section 3, we define a penalized problem and prove the existence and the regularity of the penalized solutions; In Section 4, we prove Theorem 1.1.

2. Preliminary

In this section, we collect function spaces and inequalities used in this paper.

The space $W^{2,p}_0(I)$ is the closure of $C^\infty_c(I)$ in $W^{2,p}(I)$. In this paper, we employ $\|v\|_{2,p} := \|v''\|_{L^p(I)}$ as the norm in $W^{2,p}_0(I)$. Here we note that the norm $\|\cdot\|_{2,p}$ is equivalent to the standard $W^{2,p}$ norm. Indeed, by the Poincaré inequality we find a positive constant C such that

$$\|v\|_{L^p(I)} + \|v'\|_{L^p(I)} + \|v''\|_{L^p(I)} \leq C\|v\|_{2,p} \quad \text{for all} \quad v \in W^{2,p}_0(I).$$

This clearly implies that the norm $\|\cdot\|_{2,p}$ is equivalent to the standard $W^{2,p}$ norm.

In order to treat L^p norms, we employ the following inequality (see [8]):

$$|b|^p \geq |a|^p + p\langle |a|^{p-2}a, b-a\rangle \quad \text{for all} \quad a, b \in \mathbb{R}^N \quad \text{and} \quad p \geq 1, \tag{2.1}$$

which expresses the convexity of the function $x \mapsto |x|^p$ for $p \geq 1$.

3. Penalized problem

In this section we consider a penalized problem. We define the function f_ε and the functional E_ε by

$$f_\varepsilon(s) := \begin{cases} \dfrac{s-\omega_0}{\varepsilon} & \text{if} \quad s \geq \omega_0, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

$$E_\varepsilon(u) := E(u) + f_\varepsilon(|O(u)|), \tag{3.2}$$

for $\varepsilon > 0$. Then the penalized problem corresponding to (P) is written as follows:

$$\min_{v \in W_0^{2,p}(I)} E_\varepsilon(v). \tag{P$_\varepsilon$}$$

To begin with, we prove the existence of solutions of penalized problem (P$_\varepsilon$).

Lemma 3.1. *Problem* (P$_\varepsilon$) *possesses a nontrivial solution for each* $\varepsilon > 0$.

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{2,p}(I)$ be a minimizing sequence for E_ε, i.e.,

$$\lim_{k\to\infty} E_\varepsilon(u_k) = \inf_{v\in W_0^{2,p}(I)} E_\varepsilon(v).$$

We note that E_ε is nonnegative. Extracting a subsequence, we find a constant $C > 0$ such that

$$E_\varepsilon(u_k) \leq C \quad \text{for all} \quad k \in \mathbb{N}, \tag{3.3}$$

where we denote by $\{u_k\}$ this subsequence, for short. Since E_ε is homogeneous of degree 0, we are able to normalize the minimizing sequence $\{u_k\}_{k\in\mathbb{N}}$ as follows:

$$\int_I |u_k'(x)|^p \, dx = 1 \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.4}$$

Then problem (P$_\varepsilon$) is reduced into

$$\min_{v\in\mathcal{A}} E_\varepsilon(v),$$

where

$$\mathcal{A} := \{v \in W_0^{2,p}(I) \mid \|v'\|_{L^p(I)}^p = 1\}.$$

By (3.1), (3.2), (3.3) and (3.4) we have

$$\|u_k\|_{2,p}^p = E(u_k) \leq E_\varepsilon(u_k) \leq C \quad \text{for all} \quad u_k \in \mathcal{A}.$$

Thus we find a function $u_\varepsilon \in W_0^{2,p}(I)$ such that

$$u_k \rightharpoonup u_\varepsilon \quad \text{weakly} \quad \text{in} \quad W_0^{2,p}(I) \quad \text{as} \quad k \to \infty, \tag{3.5}$$

up to a subsequence. Since the embedding $W_0^{2,p}(I) \subset C^{1,\alpha}(\bar{I})$ is compact for each $\alpha \in (0, 1-1/p)$, it follows from (3.5) that

$$u_k \to u_\varepsilon \quad \text{in} \quad C^{1,\alpha}(\bar{I}) \quad \text{as} \quad k \to \infty. \tag{3.6}$$

This together with $\{u_k\} \subset \mathcal{A}$ implies that

$$\int_I |u_\varepsilon'(x)|^p dx = 1,$$

and then $u_\varepsilon \in \mathcal{A}$. Moreover, this clearly implies that u_ε is nontrivial.

Next we show that $u_\varepsilon \in \mathcal{A}$ is the desired minimizer of E_ε. First it follows from (3.5) that

$$E(u_\varepsilon) = \|u_\varepsilon\|_{2,p}^p \le \liminf_{k\to\infty} \|u_k\|_{2,p}^p = \liminf_{k\to\infty} E(u_k). \tag{3.7}$$

We can prove the relation

$$f_\varepsilon(|O(u_\varepsilon)|) \le \liminf_{k\to\infty} f_\varepsilon(|O(u_k)|) \tag{3.8}$$

along the same line as in [10, Theorem 2.1]. Indeed, since f_ε is non-decreasing, it suffices to prove the relation

$$|O(u_\varepsilon)| \le \liminf_{k\to\infty} |O(u_k)|. \tag{3.9}$$

By the Banach–Alaoglu theorem we find a function $\rho \in L^\infty(I)$ with $0 \le \rho(x) \le 1$ for a.e. $x \in I$ such that

$$\lim_{k\to\infty} \int_I \chi_{O(u_k)} \varphi \, dx = \int_I \rho\varphi \, dx \tag{3.10}$$

for all $\varphi \in L^1(I)$ up to a subsequence, where

$$\chi_{O(u_k)}(x) := \begin{cases} 1 & \text{if} \quad x \in O(u_k), \\ 0 & \text{if} \quad x \in I \setminus O(u_k). \end{cases}$$

Then we observe from (3.6) and (3.10) that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \Big[\int_I u_k^+ [1 - \chi_{O(u_k)}] \, dx + \int_I u_k^- [1 - \chi_{O(u_k)}] \, dx \Big] \\ &= \int_I u_\varepsilon^+ [1-\rho] \, dx + \int_I u_\varepsilon^- [1-\rho] \, dx, \end{aligned}$$

where $g^+ := \max\{g, 0\}$ and $g^- := \max\{-g, 0\}$. This together with $0 \le \rho \le 1$ implies that $\rho = 1$ a.e. in $O(u_\varepsilon)$. Thus we obtain (3.8) as follows:

$$|O(u_\varepsilon)| = \int_{O(u_\varepsilon)} 1 \, dx \le \int_I \rho \, dx = \liminf_{k\to\infty} \int_I \chi_{O(u_k)} \, dx = \liminf_{k\to\infty} |O(u_k)|.$$

Combining (3.7) with (3.8), we obtain

$$\begin{aligned} E_\varepsilon(u_\varepsilon) = E(u_\varepsilon) + f_\varepsilon(|O(u_\varepsilon)|) &\le \liminf_{k\to\infty} E(u_k) + \liminf_{k\to\infty} f_\varepsilon(|O(u_k)|) \\ &\le \liminf_{k\to\infty} [E(u_k) + f_\varepsilon(|O(u_k)|)] = \liminf_{k\to\infty} E_\varepsilon(u_k). \end{aligned}$$

Therefore Lemma 3.1 follows. □

From now on, we set

$$\Lambda_\varepsilon := \int_I |u_\varepsilon''|^p dx. \tag{3.11}$$

Moreover, we define

$$\mathcal{J}_\varepsilon^0 := \{x \in \partial\mathcal{O}(u_\varepsilon) \mid u_\varepsilon'(x) = 0\}, \qquad \mathcal{J}_\varepsilon^1 := \{x \in \partial\mathcal{O}(u_\varepsilon) \mid |u_\varepsilon'(x)| > 0\},$$

and

$$\mathcal{I}(u_\varepsilon) := \mathcal{O}(u_\varepsilon) \cup \mathcal{J}_\varepsilon^1.$$

Lemma 3.2. *Let u_ε be a solution of* (P$_\varepsilon$)*. Then $(u_\varepsilon, \Lambda_\varepsilon)$ satisfies the following eigenvalue problem in the weak sense:*

$$\begin{cases} (|u_\varepsilon''|^{p-2}u_\varepsilon'')'' = -\Lambda_\varepsilon(|u_\varepsilon'|^{p-2}u_\varepsilon')' & \text{in} \quad \mathcal{I}(u_\varepsilon), \\ u_\varepsilon = u_\varepsilon' = 0 & \text{on} \quad \partial\mathcal{I}(u_\varepsilon). \end{cases}$$

Proof. It suffices to prove that $(u_\varepsilon, \Lambda_\varepsilon) \in W_0^{2,p}(I) \times \mathbb{R}$ satisfies

$$\int_I [|u_\varepsilon''|^{p-2}u_\varepsilon''\varphi'' - \Lambda_\varepsilon|u_\varepsilon'|^{p-2}u_\varepsilon'\varphi']\, dx = 0 \quad \text{for all} \quad \varphi \in W_0^{2,p}(\mathcal{I}(u_\varepsilon)). \tag{3.12}$$

Fix $\varphi \in W_0^{2,p}(\mathcal{I}(u_\varepsilon))$ arbitrarily. Since $u_\varepsilon + \delta\varphi_\varepsilon \in W_0^{2,p}(\mathcal{I}(u_\varepsilon))$, we deduce from the minimality of u_ε that

$$\frac{d}{d\delta}E_\varepsilon(u_\varepsilon + \delta\varphi_\varepsilon)\Big|_{\delta=0} = 0.$$

Moreover, it follows from $|\mathcal{O}(u_\varepsilon + \delta\varphi_\varepsilon)| = |\mathcal{O}(u_\varepsilon)|$ that

$$f_\varepsilon(|\mathcal{O}(u_\varepsilon + \delta\varphi_\varepsilon)|) = f_\varepsilon(|\mathcal{O}(u_\varepsilon)|)$$

for sufficiently small δ, and then

$$\frac{d}{d\delta}E(u_\varepsilon + \delta\varphi_\varepsilon)\Big|_{\delta=0} = \frac{d}{d\delta}E_\varepsilon(u_\varepsilon + \delta\varphi_\varepsilon)\Big|_{\delta=0} = 0. \tag{3.13}$$

By a direct calculation we have

$$\frac{d}{d\delta}E(u_\varepsilon + \delta\varphi_\varepsilon)\Big|_{\delta=0} = \frac{p\displaystyle\int_I |u_\varepsilon''|^{p-2}u_\varepsilon''\varphi''\, dx \int_I |u_\varepsilon'|^p\, dx - p\int_I |u_\varepsilon''|^p dx \int_I |u_\varepsilon'|^{p-2}u_\varepsilon'\varphi'\, dx}{\left[\displaystyle\int_I |u_\varepsilon'|^p\, dx\right]^2}.$$

Recalling that $\|u_\varepsilon'\|_{L^p(I)} = 1$ and $\|u_\varepsilon''\|_{L^p(I)}^p = \Lambda_\varepsilon$, we obtain

$$\frac{d}{d\delta}E(u_\varepsilon + \delta\varphi_\varepsilon)\Big|_{\delta=0} = p\int_I [|u_\varepsilon''|^{p-2}u_\varepsilon''\varphi'' - p\Lambda_\varepsilon|u_\varepsilon'|^{p-2}u_\varepsilon'\varphi']dx.$$

This together with (3.13) implies (3.12). Therefore Lemma 3.2 follows. □

We prove the regularity of the minimizer u_ε. To begin with, we show some properties of the support of u_ε.

Lemma 3.3. *Let u_ε be a solution of* (P_ε). *Then*

$$|\mathcal{O}(u_\varepsilon)| \geq \omega_0 \quad \textit{for all} \quad \varepsilon > 0. \tag{3.14}$$

Proof. Assume that (3.14) does not hold. Then we find $\varepsilon_* > 0$ such that

$$|\mathcal{O}(u_{\varepsilon_*})| < \omega_0. \tag{3.15}$$

Then there exist $x_0 \in I$ and $0 < r < 1$ such that

$$\begin{cases} B(x_0, r) \subset I, \\ B(x_0, r) \cap \mathcal{O}(u_{\varepsilon_*}) = \emptyset, \\ |\mathcal{O}(u_{\varepsilon_*}) \cup B(x_0, r)| \leq \omega_0, \end{cases} \tag{3.16}$$

where $B(y, \rho) := \{x \in I \mid |x - y| < \rho\}$. Fix $v \in C_{\mathrm{c}}^\infty(B(0,1))$ arbitrarily. We define $v_r : B(x_0, r) \to \mathbb{R}$ by $v_r(x) := v(x_0 + rx)$. Since

$$u_{\varepsilon_*} + v_r \in W_0^{2,p}(B(x_0, r) \cup \mathcal{I}(u_{\varepsilon_*})) \subset W_0^{2,p}(I)$$

and

$$|\mathcal{O}(u_{\varepsilon_*} + v_r)| \leq \omega_0, \tag{3.17}$$

we observe from (3.15) and (3.17) that

$$f_{\varepsilon_*}(|\mathcal{O}(u_{\varepsilon_*})|) = f_{\varepsilon_*}(|\mathcal{O}(u_{\varepsilon_*} + v_r)|) = 0.$$

This together with the minimality of u_{ε_*} implies that

$$E(u_{\varepsilon_*}) = E_{\varepsilon_*}(u_{\varepsilon_*}) \leq E_{\varepsilon_*}(u_{\varepsilon_*} + v_r) = E(u_{\varepsilon_*} + v_r). \tag{3.18}$$

Recalling the definition of Λ_ε, we deduce from (3.18) that

$$\Lambda_{\varepsilon_*} \int_{B(x_0,r)\cup\mathcal{O}(u_{\varepsilon_*})} |(u_{\varepsilon_*} + v_r)'|^p \, dx \leq \int_{B(x_0,r)\cup\mathcal{O}(u_{\varepsilon_*})} |(u_{\varepsilon_*} + v_r)''|^p \, dx. \tag{3.19}$$

Thanks to (3.16), we reduce (3.19) into

$$\Lambda_{\varepsilon_*} \Big[\int_{\mathcal{O}(u_{\varepsilon_*})} |u'_{\varepsilon_*}|^p \, dx + \int_{B(x_0,r)} |v'_r|^p \, dx \Big] \leq \int_{\mathcal{O}(u_{\varepsilon_*})} |u''_{\varepsilon_*}|^p \, dx + \int_{B(x_0,r)} |v''_r|^p \, dx. \tag{3.20}$$

Since

$$\int_{B(x_0,r)} |v'_r|^p \, dx = r^{p-1} \int_{B(0,1)} |v'|^p \, dx, \qquad \int_{B(x_0,r)} |v''_r|^p \, dx = r^{2p-1} \int_{B(0,1)} |v''|^p \, dx,$$

recalling that $\|u'_\varepsilon\|^p_{L^p(I)} = 1$ and $\|u''_\varepsilon\|^p_{L^p(I)} = \Lambda_\varepsilon$ for all $\varepsilon > 0$, we observe from (3.20) that

$$\Lambda_{\varepsilon_*} \leq \frac{\|v''\|^p_{L^p(B(0,1))}}{\|v'\|^p_{L^p(B(0,1))}} r^p. \tag{3.21}$$

On the other hand, combining $\|u'_\varepsilon\|^p_{L^p(I)} = 1$ and $\|u''_\varepsilon\|^p_{L^p(I)} = \Lambda_\varepsilon$ with Poincaré's inequality, we find a constant $C > 0$ being independent of r such that

$$0 < \frac{1}{C} \le \Lambda_\varepsilon \quad \text{for all} \quad \varepsilon > 0,$$

This together with (3.21) implies that

$$0 < \frac{1}{C} \le \frac{\|v''\|^p_{L^p(B(0,1))}}{\|v'\|^p_{L^p(B(0,1))}} r^p.$$

Taking $0 < r < 1$ small enough, we lead a contradiction. Thus Lemma 3.3 follows. □

Lemma 3.3 implies that the 'size' of the support of minimizer u_ε is uniformly bounded from below with respect to $\varepsilon > 0$. Next we prove that the support of u_ε is connected.

Lemma 3.4. *Let u_ε be a solution of* (P$_\varepsilon$)*. Then $\mathcal{I}(u_\varepsilon)$ is connected for all $\varepsilon > 0$.*

Proof. Suppose not, we find $\varepsilon_* > 0$ such that $\mathcal{I}(u_{\varepsilon_*})$ is not connected. Then there exist an open interval I_1 and an open set I_2 such that $I_1 \cap I_2 = \emptyset$ and

$$\mathcal{I}(u_\varepsilon) = I_1 \cup I_2. \tag{3.22}$$

We define U_i by

$$U_i := \begin{cases} \dfrac{u_\varepsilon}{\|u'_\varepsilon\|_{L^p(I_i)}} & \text{in} \quad I_i, \\ 0 & \text{in} \quad I \setminus I_i, \end{cases} \qquad \text{for} \quad i = 1, 2. \tag{3.23}$$

Then it holds that $U_i \in W^{2,p}_0(I)$ for $i = 1, 2$. If $|I_1| \ge \omega_0$, then we deduce from (3.22) that

$$E_\varepsilon(u_\varepsilon) = \Lambda_\varepsilon + f_\varepsilon(O(u_\varepsilon)) = E(U_1) + E(U_2) + f_\varepsilon(O(u_\varepsilon)) > E(U_1).$$

This clearly contradicts to the minimality of u_ε. If $|I_1| < \omega_0$, then Lemma 3.3 implies that

$$E_\varepsilon(u_\varepsilon) < E_\varepsilon(U_1). \tag{3.24}$$

Since it follows from $|I_1| < \omega_0$ that $f_\varepsilon(U_1) = 0$, we observe from (3.24) that $\Lambda_\varepsilon < E(U_1)$, and then

$$\Lambda_\varepsilon \int_{I_1} |U'_1|^p \, dx < \int_{I_1} |U''_1|^p \, dx. \tag{3.25}$$

On the other hand, it follows from (3.23) that

$$\int_{I_2} |U''_2|^p \, dx = \int_I |u''_\varepsilon|^p \, dx - \int_{I_1} |U''_1|^p \, dx = \Lambda_\varepsilon - \int_{I_1} |U''_1|^p \, dx. \tag{3.26}$$

Plugging (3.25) into (3.26), we have

$$\begin{aligned} \int_{I_2} |U''_2|^p \, dx &< \Lambda_\varepsilon - \Lambda_\varepsilon \int_{I_1} |U'_1|^p \, dx \\ &= \Lambda_\varepsilon \Big(1 - \int_{I_1} |U'_1|^p \, dx\Big) = \Lambda_\varepsilon \Big(\int_I |u'_\varepsilon|^p \, dx - \int_{I_1} |U'_1|^p \, dx\Big) = \Lambda_\varepsilon \int_{I_2} |U'_2|^p \, dx, \end{aligned} \tag{3.27}$$

where we used (3.22) again in the last equality of (3.27). Then (3.27) implies that

$$E(U_2) < \Lambda_\varepsilon = E(u_\varepsilon). \tag{3.28}$$

Since $f_\varepsilon(|O(U_2)|) \le f_\varepsilon(|O(u_\varepsilon)|)$, we deduce form (3.28) that

$$E_\varepsilon(U_2) < E_\varepsilon(u_\varepsilon).$$

This contradicts to the minimality of u_ε. Therefore Lemma 3.4 follows. □

Lemma 3.5. *There exists a constant* $\Lambda_{\max}$ *such that*

$$\Lambda_\varepsilon \le \Lambda_{\max} \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. Let $r_0 := \omega_0/4$ and set $x_0 \in I$ such that $B(x_0, r_0) \subset I$, where $B(y,\rho) := \{x \in I \mid |x-y| < \rho\}$. We consider the problem

$$\min_{v \in W_0^{2,p}(B(x_0,r_0))} E(v). \tag{3.29}$$

Along the same line as in the proof of Lemma 3.1, we find a solution $\varphi_0 \in W_0^{2,p}(B(x_0,r_0))$ of problem (3.29) satisfying the following:

$$\int_{B(x_0,r_0)} |\varphi_0'|^p \, dx = 1. \tag{3.30}$$

Since $\varphi_0 \in W_0^{2,p}(B(x_0,r_0))$, we can extend φ_0 as a function in $W_0^{2,p}(I)$. Recalling that

$$|O(\varphi_0)| \le |B(x_0,r_0)| \le \omega_0/2,$$

we deduce from Lemma 3.3 that

$$E_\varepsilon(u_\varepsilon) < E_\varepsilon(\varphi_0) = E(\varphi_0).$$

This together with (3.30) that

$$\Lambda_\varepsilon < \int_{B(x_0,r_0)} |\varphi_0''|^p \, dx =: \Lambda_{\max}.$$

Therefore Lemma 3.5 follows. □

Here we employ the idea in [4, Proof of Theorem 3.9]:

Lemma 3.6. *Let* $a_1, a_2 \in I$ *with* $a_1 < a_2$. *Fix* $\eta \in C_c^\infty((a_1,a_2))$ *and set*

$$\varphi_1(x) := \int_{a_1}^{x} \int_{a_1}^{y} \eta(s)\, ds dy + \alpha(x-a_1)^2 + \beta(x-a_1)^3, \tag{3.31}$$

$$\alpha := \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \eta(y)\, dy - \frac{3}{(a_2-a_1)^2}\int_{a_1}^{a_2}\int_{a_1}^{y} \eta(s)\, ds dy, \tag{3.32}$$

$$\beta := -\frac{\alpha}{a_2-a_1} - \frac{1}{(a_2-a_1)^3}\int_{a_1}^{a_2}\int_{a_1}^{y} \eta(s)\, ds dy. \tag{3.33}$$

Then $\varphi_1 \in W_0^{2,p}((a_1,a_2))$ *and there exist* $C_1, C_2, C_3 > 0$ *depending only on* a_1 *and* a_2 *such that*

$$\|\varphi_1\|_{W^{1,\infty}((a_1,a_2))} \le C_1\|\eta\|_{L^1((a_1,a_2))}, \; |\alpha| \le C_2\|\eta\|_{L^1((a_1,a_2))}, \; |\beta| \le C_3\|\eta\|_{L^1((a_1,a_2))}.$$

Proof. By (3.31), (3.32) and (3.33) we have $\varphi_1(a_1) = \varphi_1(a_2) = 0$. Since

$$\varphi_1'(x) = \int_{a_1}^{x} \eta(s)\,ds + 2\alpha(x - a_1) + 3\beta(x - a_1)^2,$$

it follows from (3.32) and (3.33) that $\varphi_1'(a_1) = \varphi_1'(a_2) = 0$. Thus we see that $\varphi_1 \in W_0^{2,p}((a_1, a_2))$. Moreover, we have

$$|\alpha| \le \frac{1}{a_2 - a_1}\|\eta\|_{L^1((a_1,a_2))} + \frac{3}{(a_2 - a_1)^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} |\eta(s)|\,ds dy = \frac{4}{a_2 - a_1}\|\eta\|_{L^1((a_1,a_2))},$$
$$|\beta| \le \frac{|\alpha|}{a_2 - a_1} + \frac{1}{(a_2 - a_1)^3}\int_{a_1}^{a_2}\int_{a_1}^{a_2} |\eta(s)|\,ds dy \le \frac{5}{(a_2 - a_1)^2}\|\eta\|_{L^1((a_1,a_2))}.$$

Similarly we obtain

$$\|\varphi_1\|_{L^\infty((a_1,a_2))} \le (a_2 - a_1)\int_{a_1}^{a_2} |\eta(s)|\,ds + |\alpha|(a_2 - a_1) + |\beta|(a_2 - a_1)^2 \le 10(a_2 - a_1)\|\eta\|_{L^1((a_1,a_2))},$$
$$\|\varphi_1'\|_{L^\infty((a_1,a_2))} \le \|\eta\|_{L^1((a_1,a_2))} + 2|\alpha|(a_2 - a_1) + 3|\beta|(a_2 - a_1)^2 \le 24\|\eta\|_{L^1((a_1,a_2))}.$$

Thus Lemma 3.6 follows. □

Theorem 3.7. *Let $u_\varepsilon \in W_0^{2,p}(I)$ be a solution to (P_ε). Then there exists a constant $M > 0$ such that*

$$\|u_\varepsilon''\|_{L^\infty(\mathcal{I}(u_\varepsilon))} \le M \quad \text{for all} \quad \varepsilon > 0.$$

Proof. Fix $\varepsilon > 0$ arbitrarily. Since $u_\varepsilon \in W_0^{2,p}(I)$ is a solution to (P_ε), by Lemma 3.2 we have

$$\int_{\mathcal{I}(u_\varepsilon)} [|u_\varepsilon''|^{p-2}u_\varepsilon''\varphi'' - \Lambda_\varepsilon|u_\varepsilon'|^{p-2}u_\varepsilon'\varphi']\,dx = 0 \quad \text{for all} \quad \varphi \in W_0^{2,p}(\mathcal{I}(u_\varepsilon)), \tag{3.34}$$

where the constant Λ_ε is defined by (3.11). By Lemmas 3.3 and 3.4 we find $a_1^\varepsilon, a_2^\varepsilon \in I$ such that

$$\mathcal{I}(u_\varepsilon) = (a_1^\varepsilon, a_2^\varepsilon), \quad \omega_0 \le |a_2^\varepsilon - a_1^\varepsilon| \le |I|, \quad \text{for all} \quad \varepsilon > 0. \tag{3.35}$$

Fix $\eta \in C_c^\infty(\mathcal{I}(u_\varepsilon))$ arbitrarily. Taking $(a_1^\varepsilon, a_2^\varepsilon)$ as (a_1, a_2) in Lemma 3.6, we observe from (3.35) that the constants C_1, C_2 and C_3 in Lemma 3.6 depends only on ω_0 and I. Taking $\varphi = \varphi_1$ in (3.34), where φ_1 is the function defined in Lemma 3.6, we have

$$\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^{p-2}u_\varepsilon''\eta\,dx = \int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^{p-2}u_\varepsilon''[-2\alpha - 6\beta(x - a_1)]\,dx + \Lambda_\varepsilon\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon'|^{p-2}u_\varepsilon'\varphi_1'\,dx.$$

This together with Lemma 3.6 implies that

$$\begin{aligned}\Big|\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^{p-2}u_\varepsilon''\eta\,dx\Big| &= \int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^{p-1}[2|\alpha| + 6|\beta||I|]\,dx + \Lambda_\varepsilon\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon'|^{p-1}|\varphi_1'|\,dx \\ &\le [2C_2\|\eta\|_{L^1(\mathcal{I}(u_\varepsilon))} + 6C_3|I|\|\eta\|_{L^1(\mathcal{I}(u_\varepsilon))}]\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^{p-1}\,dx \\ &\quad + C_1\|\eta\|_{L^1(\mathcal{I}(u_\varepsilon))}\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon'|^{p-1}\,dx.\end{aligned} \tag{3.36}$$

By Hölder's inequality we have

$$\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^{p-1}\,dx \le \Big(\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^p\,dx\Big)^{\frac{p-1}{p}} (a_2^\varepsilon - a_1^\varepsilon)^{\frac{1}{p}} \le \Lambda_\varepsilon^{\frac{p-1}{p}} |I|^{\frac{1}{p}}. \tag{3.37}$$

Similarly we obtain

$$\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon'|^{p-1}\,dx \le \Big(\int_I |u_\varepsilon'|^p\,dx\Big)^{\frac{p-1}{p}} |I|^{\frac{1}{p}} \le C\Lambda_\varepsilon^{\frac{p-1}{p}} |I|^{\frac{1}{p}}. \tag{3.38}$$

Plugging (3.37) and (3.38) into (3.36) we see that

$$\Big|\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^{p-2} u_\varepsilon'' \eta\,dx\Big| \le C\Lambda_\varepsilon^{\frac{p-1}{p}} \|\eta\|_{L^1(\mathcal{I}(u_\varepsilon))},$$

where the constant $C > 0$ depends only on ω_0 and I. This together with Lemma 3.5 implies that

$$\Big|\int_{a_1^\varepsilon}^{a_2^\varepsilon} |u_\varepsilon''|^{p-2} u_\varepsilon'' \eta\,dx\Big| \le C\|\eta\|_{L^1(\mathcal{I}(u_\varepsilon))}, \tag{3.39}$$

where $C > 0$ depends only on ω_0 and I. Using the fact that $(L^1(\mathcal{I}(u_\varepsilon)))^* = L^\infty(\mathcal{I}(u_\varepsilon))$ and Riesz's representation theorem, we deduce from (3.39) that

$$\||u_\varepsilon''|^{p-1}\|_{L^\infty(\mathcal{I}(u_\varepsilon))} \le C.$$

Therefore Theorem 3.7 follows. □

4. Proof of Theorem 1.1

In order to prove Theorem 1.1, it suffices to show the following:

Theorem 4.1. *Let u_ε be a solution to* (P$_\varepsilon$) *obtained by Lemma 3.1. Then there exists a constant $\varepsilon_0 > 0$ such that*

$$|\mathcal{O}(u_\varepsilon)| = \omega_0 \quad \textit{for all} \quad 0 < \varepsilon < \varepsilon_0. \tag{4.1}$$

Proof. Assume that (4.1) does not hold. Then, by Lemma 3.3 we find a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$|\mathcal{I}(u_{\varepsilon_j})| = |\mathcal{O}(u_{\varepsilon_j})| > \omega_0 \quad \text{for all} \quad j \in \mathbb{N}. \tag{4.2}$$

Fix $j \in \mathbb{N}$ arbitrarily, and set $\varepsilon_j = \varepsilon$ for short. By (4.2) we find $x_\varepsilon \in \mathcal{I}(u_\varepsilon)$ such that

$$|\mathcal{I}(u_\varepsilon)| - |B(x_\varepsilon, r_\varepsilon)| > \omega_0,$$

where

$$r_\varepsilon := \frac{1}{2}\mathrm{dist}(x_\varepsilon, \partial\mathcal{I}(u_\varepsilon)).$$

Let $\eta \in C_c^\infty(B(x_\varepsilon, 2r_\varepsilon))$ be a cut-off function with

$$\begin{cases} 0 \le \eta \le 1 \quad \text{in} \quad I, \\ \eta \equiv 1 \quad \text{in} \quad B(x_\varepsilon, r_\varepsilon), \\ \|\eta'\|_{L^\infty(I)} \le \dfrac{C}{r_\varepsilon}, \quad \|\eta''\|_{L^\infty(I)} \le \dfrac{C}{r_\varepsilon^2}, \end{cases} \tag{4.3}$$

where C is a positive constant. Let $v_\varepsilon := u_\varepsilon - \eta u_\varepsilon$. Then we see that $v_\varepsilon \in W_0^{2,p}(I)$ and

$$v_\varepsilon = \begin{cases} u_\varepsilon & \text{in} \quad I \setminus B(x_\varepsilon, 2r_\varepsilon), \\ 0 & \text{in} \quad B(x_\varepsilon, r_\varepsilon). \end{cases} \tag{4.4}$$

Recalling that

$$|\mathcal{O}(v_\varepsilon)| = |\mathcal{I}(u_\varepsilon)| - |B(x_\varepsilon, r_\varepsilon)| > \omega_0,$$

by the minimality of u_ε we have $E_\varepsilon(u_\varepsilon) \le E_\varepsilon(v_\varepsilon)$, i.e.,

$$\Lambda_\varepsilon + f_\varepsilon(|\mathcal{O}(u_\varepsilon)|) \le \frac{\displaystyle\int_I |v_\varepsilon''|^p \, dx}{\displaystyle\int_I |v_\varepsilon'|^p \, dx} + f_\varepsilon(|\mathcal{O}(v_\varepsilon)|). \tag{4.5}$$

Since it follows from (3.1) that

$$f_\varepsilon(|\mathcal{O}(u_\varepsilon)|) - f_\varepsilon(|\mathcal{O}(v_\varepsilon)|) = \frac{2r_\varepsilon}{\varepsilon}, \tag{4.6}$$

plugging (4.6) into (4.5), we obtain

$$\Lambda_\varepsilon \int_I |v_\varepsilon'|^p \, dx + \frac{2r_\varepsilon}{\varepsilon} \int_I |v_\varepsilon'|^p \, dx \le \int_I |v_\varepsilon''|^p \, dx. \tag{4.7}$$

From (4.4) we see that

$$\begin{aligned} \int_I |v_\varepsilon'|^p \, dx &= \int_{B(x_\varepsilon, 2r_\varepsilon)} |((1-\eta)u_\varepsilon)'|^p \, dx + \int_{I \setminus B(x_\varepsilon, 2r_\varepsilon)} |u_\varepsilon'|^p \, dx \\ &= \int_{B(x_\varepsilon, 2r_\varepsilon)} \left[|((1-\eta)u_\varepsilon)'|^p - |u_\varepsilon'|^p \right] dx + 1. \end{aligned} \tag{4.8}$$

Similarly we have

$$\int_I |v_\varepsilon''|^p \, dx = \int_{B(x_\varepsilon, 2r_\varepsilon)} \left[|((1-\eta)u_\varepsilon)''|^p - |u_\varepsilon''|^p \right] dx + \Lambda_\varepsilon. \tag{4.9}$$

Combining (4.7) with (4.8) and (4.9), we find

$$\Lambda_\varepsilon \int_{B(x_\varepsilon, 2r_\varepsilon)} \left[|((1-\eta)u_\varepsilon)'|^p - |u_\varepsilon'|^p \right] dx + \frac{2r_\varepsilon}{\varepsilon} \int_I |v_\varepsilon'|^p \, dx \le \int_{B(x_\varepsilon, 2r_\varepsilon)} \left[|((1-\eta)u_\varepsilon)''|^p - |u_\varepsilon''|^p \right] dx. \tag{4.10}$$

Since it follows from (2.1) that

$$\int_{B(x_\varepsilon,2r_\varepsilon)} [|((1-\eta)u_\varepsilon)'|^p - |u_\varepsilon'|^p]\,dx \geq -p\int_{B(x_\varepsilon,2r_\varepsilon)} |u_\varepsilon'|^{p-2}u_\varepsilon'(\eta u_\varepsilon)'\,dx$$

and

$$\int_{B(x_\varepsilon,2r_\varepsilon)} [|((1-\eta)u_\varepsilon)''|^p - |u_\varepsilon''|^p]\,dx \leq p\int_{B(x_\varepsilon,2r_\varepsilon)} |((1-\eta)u_\varepsilon)''|^{p-2}((1-\eta)u_\varepsilon)''(\eta u_\varepsilon)''\,dx,$$

we reduce (4.10) into

$$\begin{aligned}\frac{2r_\varepsilon}{\varepsilon}\int_I |v_\varepsilon'|^p\,dx &\leq p\Lambda_\varepsilon\int_{B(x_\varepsilon,2r_\varepsilon)} |u_\varepsilon'|^{p-2}u_\varepsilon'(\eta u_\varepsilon)'\,dx \\ &\quad + p\int_{B(x_\varepsilon,2r_\varepsilon)} |((1-\eta)u_\varepsilon)''|^{p-2}((1-\eta)u_\varepsilon)''(\eta u_\varepsilon)''\,dx \\ &=: K_1 + K_2.\end{aligned} \tag{4.11}$$

By Theorem 3.7 we have

$$|u_\varepsilon'(x)| = |u_\varepsilon'(x) - u_\varepsilon'(y)| \leq C|x-y| \leq 2Cr_\varepsilon \quad \text{for all} \quad x \in \bar{B}(x_\varepsilon, 2r_\varepsilon), \tag{4.12}$$

where $y \in \partial\mathcal{I}(u_\varepsilon)$, and the constant C is independent of ε. Moreover, we deduce from (4.12) that

$$|u_\varepsilon(x)| = \left|\int_y^x u_\varepsilon'(\xi)\,d\xi\right| \leq 2Cr_\varepsilon|x-y| \leq 4Cr_\varepsilon^2, \tag{4.13}$$

where $y \in \partial\mathcal{I}(u_\varepsilon)$. It follows from (4.3), (4.12) and (4.13) that

$$K_1 \leq C\Lambda_\varepsilon r_\varepsilon^p|B(x_\varepsilon, 2r_\varepsilon)| \leq C\Lambda_\varepsilon r_\varepsilon^{p+1}, \tag{4.14}$$

where $C > 0$ is independent of ε. Similarly, we infer from (4.3), (4.12), (4.13) and Theorem 3.7 that

$$K_2 \leq C|B(x_\varepsilon, 2r_\varepsilon)| \leq Cr_\varepsilon, \tag{4.15}$$

where $C > 0$ is independent of ε. Plugging (4.14) and (4.15) into (4.11), we obtain

$$\frac{2r_\varepsilon}{\varepsilon}\int_I |v_\varepsilon'|^p\,dx \leq Cr_\varepsilon(1 + r_\varepsilon^p), \tag{4.16}$$

where $C > 0$ is independent of ε. Since $\|u_\varepsilon'\|_{L^p(I)} = 1$, by (2.1) we have

$$\int_I |v_\varepsilon'|^p\,dx = 1 + \int_I [|v_\varepsilon'|^p - |u_\varepsilon'|^p]\,dx \geq 1 - p\int_{B(x_\varepsilon,2r_\varepsilon)} |u_\varepsilon'|^{p-2}u_\varepsilon'(\eta u_\varepsilon)'\,dx \geq 1 - Cr_\varepsilon^{p+1}.$$

Taking $x_\varepsilon \in \mathcal{I}(u_\varepsilon)$ sufficiently close to $\partial\mathcal{I}(u_\varepsilon)$, we see that

$$\int_I |v_\varepsilon'|^p\,dx \geq \frac{1}{2}.$$

This together with (4.16) implies that

$$\frac{1}{\varepsilon} \leq C(1 + r_\varepsilon^p).$$

Letting $\varepsilon \to 0$, we lead a contradiction. Therefore Theorem 4.1 follows. □

We are in a position to prove Theorem 1.1:

proof of Theorem 1.1. By Lemma 3.1 we see that there exists a solution u_ε of (P_ε) for each $\varepsilon > 0$. Thanks to Theorem 4.1, we find $\varepsilon_0 > 0$ such that

$$|O(u_\varepsilon)| = \omega_0 \quad \text{for all} \quad 0 < \varepsilon < \varepsilon_0.$$

This implies that u_ε is a solution of problem (P), providing that $\varepsilon > 0$ is small enough. Moreover, it follows from Lemma 3.2 that $(u_\varepsilon, \Lambda_\varepsilon)$ satisfies problem (1.4) in a weak sense. Finally, we deduce from Lemma 3.4 that $\mathcal{I}(u_\varepsilon)$ is connected. Therefore Theorem 1.1 follows. □

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Conflict of interest

The authors declare no conflict of interest.

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